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Homotopy of two-fold symmetric products of spheres

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Homological structure of the 2-fold symmetric products $S^n * S^n$ of an n-sphere S^n is well known. (See R. Bott [2], S. K. Stein [12] and the recent paper [5] of S. D. Liao.)[1] In the present note, we shall calculate some homotopy groups of $S^n * S^n$ by making use of the results on homology. If we denote by π_i the (stable) homotopy group $\pi_i(S^n * S^n)$ for $i \leqq 2n-2$, our results are as follows:

$$\text{(A)} \qquad \begin{array}{lll} \pi_{n+1} = 0, & \pi_{n+2} = 0, & \pi_{n+3} \approx Z_3, \\ \pi_{n+4} = 0, & \pi_{n+5} \approx Z_2, & \pi_{n+6} = 0, \\ \pi_{n+7} \approx Z_{15}, & \pi_{n+8} \approx Z_2, & \pi_{n+9} \approx Z_2. \end{array}$$ [2]

Two different methods are explained. One of these is the method employed by J-P. Serre in [10] for calculation of homotopy groups of spheres.[3] The other starts with a construction of a reduced complex of the same $(n+6)$-homotopy type as $S^n * S^n$, in which the homotopy boundaries in dimensions $\leqq n+7$ are well defined.

In the last section, we state some results on the following: i) homotopy of $S^n * S^n$ for $n \leqq 5$, ii) the homotopy groups of the p-fold cyclic product of a sphere, iii) the homology and homotopy of the 2-fold symmetric product of the suspended projective plane.

1. Homological properties

We shall first recall some homological properties of $S^n * S^n$ (see [2], [5], [12]). The i-dimensional homology group $H_i = H_i(S^n * S^n; Z)$[2] is as follows:

$$\text{(1.1)} \qquad \begin{array}{rll} H_0 \approx Z, & H_i = 0 \ \ \textit{for} \ 0 < i < n, & H_n \approx Z, \\ H_{n+j} = 0 & \textit{for} \ 1 \leqq j < n \ \textit{with odd} \ j, & \\ \approx Z_2 & \textit{for} \ 1 \leqq j < n \ \textit{with even} \ j, & \\ H_{2n} = 0 & \textit{for odd} \ n, & \\ \approx Z & \textit{for even} \ n, & \\ H_i = 0 & \textit{for} \ i > 2n. & \end{array}$$

Thus the i-dimensional cohomology group $H^i(S^n * S^n; Z_2)$ is Z_2 for $i = 0$, n and $n+2 \leqq i \leqq 2n$, and is zero for other i.

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1) Numbers in brackets refer to the bibliography at the end of this paper.
2) We denote by Z and Z_p the additive groups of integers, of integers mod p respectively.
3) The author is indebted to Prof. H. Toda for pointing out the use of this method.

As for the Steenrod square $Sq^i: H^{n+j}(S^n*S^n; Z_2)\longrightarrow H^{n+i+j}(S^n*S^n; Z_2)$, we have

(1.2)
$$Sq^i H^n(S^n*S^n; Z_2) = H^{n+i}(S^n*S^n; Z_2),$$
$$Sq^i H^{n+j+1}(S^n*S^n; Z_2) = \binom{j}{i} H^{n+i+j+1}(S^n*S^n; Z_2) \quad (j\geqq 0)$$

where $\binom{j}{i}$ *is the binomial coefficient with the usual conventions.*

Let $K(\pi, n)$ be an Eilenberg-MacLane complex with the only non-vanishing homotopy group $\pi_n(K(\pi, n))\approx\pi$, where π is an abelian group. Denote by u the generator of the n-dimensional cohomology group $H^n(Z, n; Z_2)$ or $H^n(Z_2, n; Z_2)$[4]. Then it is well known [10] that

(1.3) $H^{n+j}(Z, n; Z_2)$ *(resp.* $H^{n+j}(Z_2, n; Z_2)$*) for* $j<n$ *is a vector space having as a base the all iterated Steenrod squares* $Sq^{i_r}Sq^{i_{r-1}}\cdots Sq^{i_1} u$ *which satisfy the following conditions* i), ii) *and* iii) *(resp.* i) *and* ii)).

i) $i_1+i_2+\cdots+i_r=j$, ii) $i_{k+1}\geqq 2i_k$ *for* $k=1, 2, \cdots, r-1$, iii) $i_1>1$.

The following relations (1.4) among the iterated Steenrod squares, which are found by J. Adem [1], are very useful in later part.

(1.4)
$$Sq^{2t}\, Sq^s = \sum_{j=0}^{t}\binom{s-t+j-1}{2j} Sq^{t+s+j}\, Sq^{t-j}.$$

2. Some general properties

Let K_n be a cellular decomposition of S^n*S^n given by Steenrod, and let $E(S^n*S^n)$ be the suspended space of S^n*S^n. Then $E(S^n*S^n)$ is imbedded in $S^{n+1}*S^{n+1}$ naturally, and forms the $(2n+1)$-skelton of K_{n+1} [5]. Thus we have

$$i_{\#}:\ \pi_{i+1}(E(S^n*S^n))\approx\pi_{i+1}(S^{n+1}*S^{n+1})$$

for $i\leqq 2n-1$, where $i: E(S^n*S^n)\subset S^{n+1}*S^{n+1}$ is the inclusion. Let

$$E:\ \pi_i(S^n*S^n)\longrightarrow\pi_{i+1}(E(S^n*S^n))$$

be the suspension homomorphism. Since S^n*S^n is $(n-1)$-connected from (1.1), E is isomorphic for $i\leqq 2n-2$, and is onto for $i\leqq 2n-1$ [13]. Therefore we have

(2.1) *The homomorphism*

$$i_{\#}\circ E:\ \pi_i(S^n*S^n)\longrightarrow\pi_{i+1}(S^{n+1}*S^{n+1})$$

is isomorphic for $i\leqq 2n-2$, *and onto for* $i\leqq 2n-1$.

Since S^n*S^n is $(n-1)$-connected and $H_n(S^n*S^n; Z)\approx Z$ from (1.1), the Hurewicz theorem implies $\pi_n(S^n*S^n)\approx Z$. Let $f: S^n\longrightarrow S^n*S^n$ be a map which represents

4) As usual, we denote $H^i(K(\pi, n); G)$ by $H^i(\pi, n; G)$ simply.

a generator of $\pi_n(S^n*S^n)$, and let $k[p]$ be a field of characteristic p. Then, for the homomorphism $f_*: H_i(S^n; k[p]) \longrightarrow H_i(S^n*S^n; k[p])$, we have from (1.1) that i) if n is odd, f_* is isomorphic onto for any i and any $p \neq 2$, ii) if n is even, f_* is isomorphic onto for any $i<2n$ and any $p \neq 2$. Thus the following result is obvious from the generalized J. H. C. Whitehead theorem due to J-P. Serre [9]. (See also [6].)[5]

(2.2) *If n is odd, then $\pi_i(S^n*S^n)$ is finite for any $i \neq n$, and $C(\pi_i(S^n*S^n), p) \approx C(\pi_i(S^n), p)$ for any odd prime p, where $C(\pi, p)$ denotes the p-primary subgroup. If n is even, the same properties are true for $i \leqq 2n-2$.*

Let $p: S^n \times S^n \longrightarrow S^n*S^n$ be the projection (i.e. the identification map), and let $f: S^n \longrightarrow S^n*S^n$ be a map defined by

$$f(y) = p(y \times y_0) = p(y_0 \times y), \quad y \in S^n,$$

where $y_0 \in S^n$ is a base point. Since it is obvious that $f_*: H_n(S^n; Z) \approx H_n(S^n*S^n; Z)$, we see that f represents a generator ι'_n of $\pi_n(S^n*S^n)$. Thus p is a map of type (ι'_n, ι'_n). Therefore it follows from the well known theorem [13] that the Whitehead product $[\iota'_n, \iota_n']$ is zero. Thus we have

$$(2.3) \qquad [\alpha, \beta] = 0 \quad \text{for } \alpha, \beta \in \pi_n(S^n*S^n).$$

3. Proof of (A)

Let $(S^n*S^n, n+j)$ $(j=0, 1, 2, \cdots)$ be the Cartan-Serre sequence of the space S^n*S^n [4]. Then, by the definition, $\pi_{n+i}(S^n*S^n, n+j)=0$ for $i<j$ and $\pi_{n+i}(S^n*S^n) \approx \pi_{n+i}(S^n*S^n, n+j)$ for $i \geqq j$. Moreover there exists a fiber space for each j such that i) the total space is of the same homotopy type as $(S^n*S^n, n+j)$, ii) the base space is an Eilenberg-MacLane complex $K(\pi_{n+j}(S^n*S^n), n+j)$, and the fiber is $(S^n*S^n, n+j+1)$. (For brevity of the notation, we use $(S^n*S^n, n+j)$ to denote the total space of the above fiber space.) Thus we have for $i \leqq n+2j$ the exact sequence [8]:

$$(3.1) \qquad \cdots \xrightarrow{i^*} H^{n+i-1}(S^n*S^n, n+j+1; Z_2) \xrightarrow{\tau} H^{n+i}(\pi_{n+j}(S^n*S^n), n+j; Z_2) \xrightarrow{p^*} H^{n+i}(S^n*S^n, n+j; Z_2) \xrightarrow{i^*} H^{n+i}(S^n*S^n, n+j+1; Z_2),$$

where p^*, i^* are the homomorphisms induced by the projection and the inclusion respectively, and τ is the transgression.

Throughout this section, we assume that n is sufficiently large (for example $n \geqq 13$).

I) Let $j=0$ in (3.1), and consider the homomorphism $p^*: H^{n+i}(\pi_n(S^n*S^n), n; Z_2) \longrightarrow H^{n+i}(S^n*S^n, n; Z_2)$. Since $\pi_n(S^n*S^n) \approx Z$, if we denote by u the generator

5) We assume throughout this paper that $n \geqq 2$.

of $H^n(\pi_n(S^n*S^n), n; Z_2)$, then we see from (1.3) that $H^*(\pi_n(S^n*S^n), n; Z_2)$ has a base

$$u,\ Sq^2u,\ Sq^3u,\ \cdots,\ Sq^6u,\ Sq^4Sq^2u,\ \cdots,\\ Sq^{13}u,\ Sq^{11}Sq^2u,\ Sq^{10}Sq^3u,\ Sq^9Sq^4u, \tag{3.2}$$

in dimensions $\leqq n+13$. On the other hand, since $(S^n*S^n, n)=S^n*S^n$, we see from (1.2) that $H^*(S^n*S^n, n; Z_2)$ has a base

$$v,\ Sq^2v,\ Sq^3v,\ \cdots,\ Sq^{12}v,\ Sq^{13}v \tag{3.3}$$

in dimensions $\leqq n+13$, where v is the generator of $H^n(S^n*S^n, n; Z_2)$. Furthermore, since $H^n(S^n*S^n, n+1; Z_2)=0$, p^* is onto in dimension n, and so we have $p^*u=v$. Thus we see from (3.2) and (3.3) by making use of the naturality of Sq^i that

$(3.4)_1$ p^* is isomorphic onto for $i\leqq 5$, and

$(3.4)_2$ p^* is onto for $i\leqq 13$.

Then it follows from $(3.4)_1$ by the generalized Whitehead theorem that $p_\#$: $C(\pi_{n+i}(S^n*S^n), 2)\longrightarrow C(\pi_{n+i}(K(\pi_n(S^n*S^n), n)), 2)$ is isomorphic onto for $i\leqq 4$, and so we have

$$C(\pi_{n+i}(S^n*S^n), 2)=0 \quad for \quad 1\leqq i\leqq 4. \tag{3.5}$$

Let N_0 be the kernel of p^*, then (3.1) and $(3.4)_2$ imply that τ: $H^{n+i-1}(S^n*S^n, n+1; Z_2)\longrightarrow N_0$ is isomorphic onto for $i\leqq 13$. Furthermore we see from (1.2) and (1.4) that N_0 has a base

$$\begin{aligned}&Sq^4Sq^2u,\quad Sq^5Sq^2u=Sq^1Sq^4Sq^2u,\quad Sq^6Sq^2u=Sq^2Sq^4Sq^2u,\\ &Sq^7Sq^2u=Sq^3Sq^4Sq^2u,\quad Sq^6Sq^3u=Sq^2Sq^1Sq^4Sq^2u,\\ &Sq^8Sq^2u,\quad Sq^7Sq^3u=Sq^3Sq^1Sq^4Sq^2u,\quad Sq^9Sq^2u=Sq^1Sq^8Sq^2u,\\ &Sq^8Sq^3u=Sq^9Sq^2u+Sq^4Sq^1Sq^4Sq^2u,\\ &Sq^{10}Sq^2u=Sq^2Sq^8Sq^2u+Sq^5Sq^1Sq^4Sq^2u,\quad Sq^9Sq^3u=Sq^5Sq^1Sq^4Sq^2u\\ &Sq^8Sq^4u,\quad Sq^{11}Sq^2u=Sq^3Sq^8Sq^2u,\quad Sq^{10}Sq^3u=Sq^2Sq^1Sq^8Sq^2u,\\ &Sq^9Sq^4u=Sq^1Sq^8Sq^4u.\end{aligned} \tag{3.6}$$

Let a (dim $a=n+5$), b (dim $b=n+9$) and c (dim $c=n+11$) be the elements such that

$$\tau a=Sq^4Sq^2u,\quad \tau b=Sq^8Sq^2u,\quad \tau c=Sq^8Sq^4u$$

respectively. Then, since $\tau Sq^i=Sq^i\tau$, it follows from (3.1) and (3.6) that $H^*(S^n*S^n, n+1; Z_2)$ has a base

$$\begin{aligned}&a,\ Sq^1a,\ Sq^2a,\ Sq^3a,\ Sq^2Sq^1a,\\ &b,\ Sq^3Sq^1a,\ Sq^1b,\ Sq^1b+Sq^4Sb^1a,\\ &Sq^2b+Sq^5Sq^1a,\ Sq^5Sq^1a,\ c,\ Sq^3b,\ Sq^2Sq^1b,\ Sq^1c\end{aligned} \tag{3.7}$$

in dimensions $\leqq n+12$.

We have from (3.5) that $H^*(\pi_{n+j}(S^n*S^n), n+j; Z_2)=0$ for $1\leqq j\leqq 4$. Therefore it follows from (3.1) for $j=1, 2, 3$ and 4 that

$$H^*(S^n*S^n, n+5; Z_2)\approx H^*(S^n*S^n, n+1; Z_2)$$

in dimensions $\leqq n+12$, under the composition of the inclusions and homotopy equivalences. Thus we may consider (3.7) as a base of $H^*(S^n*S^n, n+5; Z_2)$ in dimensions $\leqq n+12$. Especially we see that $H^{n+5}(S^n*S^n, n+5; Z_2)\approx \mathrm{Hom}\ (H_{n+5}(S^n*S^n, n+5), Z_2)\approx Z_2$, and so we have $C(H_{n+5}(S^n*S^n, n+5), 2)\approx Z_{2^r}$ for some $r\geqq 1$. However, since $Sq^1a\neq 0$, r must be $=1$. Thus we obtain by the generalized Hurewicz isomorphism [9] that $C(\pi_{n+5}(S^n*S^n, n+5), 2)\approx C(H_{n+5}(S^n*S^n, n+5), 2)\approx Z_2$. Namely we have

(3.8) $$C(\pi_{n+5}(S^n*S^n), 2)\approx Z_2 .$$

II) Let $j=5$ in (3.1), and consider the homomorphism p^*: $H^{n+i}(\pi_{n+5}(S^n*S^n), n+5; Z_2)\longrightarrow H^{n+i}(S^n*S^n, n+5; Z_2)$. Then it follows from (3.8) that $H^{n+i}(\pi_{n+5}(S^n*S^n), n+5; Z_2)\approx H^{n+i}(Z_2, n+5; Z_2)$. Therefore if we denote by α the generator of $H^{n+5}(\pi_{n+5}(S^n*S^n), n+5; Z_2)$, we see from (1.3) that $H^*(\pi_{n+5}(S^n*S^n), n+5; Z_2)$ has a base

(3.9) $$\begin{array}{l}\alpha,\ Sq^1\alpha,\ Sq^2\alpha,\ Sq^3\alpha,\ Sq^2Sq^1\alpha,\ Sq^4\alpha,\ Sq^3Sq^1\alpha,\\ Sq^5\alpha,\ Sq^4Sq^1\alpha,\ Sq^6\alpha,\ Sq^5Sq^1\alpha,\ Sq^4Sq^2\alpha,\ Sq^7\alpha,\ Sq^6Sq^1\alpha,\\ Sq^5Sq^2\alpha,\ Sq^4Sq^2Sq^1\alpha .\end{array}$$

in dimensions $\leqq n+12$. Since p^* is onto in dimension $n+5$, we see $p^*\alpha=a$, and so p^* is isomorphic onto for $i\leqq n+8$. Thus, by the similar arguments as in the proof of (3.5), we obtain

(3.10) $$C(\pi_{n+i}(S^n*S^n), 2)=0 \quad \textit{for} \quad i=6 \textit{ and } 7 .$$

We have by (1.4)

$$\begin{aligned}
\tau p^*Sq^4\alpha &= Sq^4Sq^4Sq^2u = Sq^3Sq^1Sq^4Sq^2u = \tau Sq^3Sq^1a ,\\
\tau p^*Sq^5\alpha &= Sq^5Sq^4Sq^2u = Sq^1Sq^7Sq^3u = 0 ,\\
\tau p^*Sq^6\alpha &= Sq^6Sq^4Sq^2u = Sq^7Sq^1Sq^3Sq^1u = 0 ,\\
\tau p^*Sq^4Sq^2\alpha &= Sq^4Sq^2Sq^4Sq^2u = Sq^5Sq^1Sq^4Sq^2u + Sq^2Sq^8Sq^2u = \tau(Sq^5Sq^1a + Sq^2b) ,\\
\tau p^*Sq^7\alpha &= Sq^7Sq^4Sq^2u = Sq^1Sq^6Sq^4Sq^2u = 0 ,\\
\tau p^*Sq^6Sq^1\alpha &= Sq^6Sq^1Sq^4Sq^2u = Sq^2Sq^5Sq^4Sq^2u = 0 ,\\
\tau p^*Sq^5Sq^2\alpha &= Sq^5Sq^2Sq^4Sq^2u = Sq^3Sq^8Sq^2u = \tau Sq^3b ,\\
\tau p^*Sq^4Sq^2Sq^1\alpha &= Sq^4Sq^2Sq^1Sq^4Sq^2u = Sq^1Sq^8Sq^4u + Sq^2Sq^1Sq^8Sq^2u = \tau(Sq^2Sq^1b + Sq^1c) ,
\end{aligned}$$

and τ is isomorphic into by $(3.4)_1$. Therefore it follows from (3.7) and (3.9) that the kernel N_5 of p^* has a base

$$Sq^4\alpha + Sq^3Sq^1\alpha,\quad Sq^5\alpha,\quad Sq^6\alpha,\quad Sq^7\alpha,\quad Sq^6Sq^1\alpha$$

in dimensions $\leqq n+12$. Since p^* is onto in dimension $n+8$, we see that τ: $H^{n+8}(S^n*S^n,$

$n+6\,;\,Z_2)\approx N_5$. Let γ be the element such that

$$\tau\gamma = Sq^4\alpha + Sq^3Sq^1\alpha\,.$$

Then, since we have by (1.4)

$$\begin{aligned}\tau Sq^1\gamma &= Sq^1Sq^4\alpha + Sq^1Sq^3Sq^1\alpha = Sq^5\alpha\,,\\ \tau Sq^2\gamma &= Sq^2Sq^4\alpha + Sq^2Sq^3Sq^1\alpha = Sq^6\alpha\,,\\ \tau Sq^3\gamma &= Sq^1\tau Sq^2\gamma = Sq^7\alpha\,,\\ \tau Sq^2Sq^1\gamma &= Sq^2\tau Sq^1\gamma = Sq^2Sq^5\alpha = Sq^6Sq^1\alpha\,,\end{aligned}$$

it follows from (3.7) and (3.9) that $H^*(S^n*S^n,\ n+6\,;\,Z_2)$ has a base

$$(3.11) \qquad \gamma,\ \ Sq^1\gamma,\ \ b',\ \ Sq^2\gamma,\ \ Sq^1b',\ \ Sq^2\gamma,\ \ Sq^2Sq^1\gamma,\ \ Sq^1c' = Sq^2Sq^1b'$$

in dimensions $\leqq n+11$, where $b'=i^*(b)$ and $c'=i^*(c)$. We have from (3.10) that $H^*(\pi_{n+j}(S^n*S^n),\ n+j\,;\,Z_2)=0$ for $j=6$ and 7. Therefore if we consider (3.1) for $j=6$ and 7, we have under a natural map

$$H^*(S^n*S^n,\ n+8\,;\,Z_2) \approx H^*(S^n*S^n,\ n+6\,;\,Z_2)\,.$$

Thus we may consider (3.11) as a base of $H^*(S^n*S^n,\ n+8\,;\,Z_2)$ in dimensions $\leqq n+11$. Especially $H^{n+8}(S^n*S^n,\ n+8\,;\,Z_2)\approx Z_2$ and $Sq^1\gamma\neq 0$, and so we have

$$(3.12) \qquad C(\pi_{n+8}(S^n*S^n),\ 2) \approx Z_2\,,$$

by the similar arguments as in the proof of (3.8).

III) Let $j=8$ in (3.1), and consider $p^*\colon H^i(\pi_{n+8}(S^n*S^n),\ n+8\,;\,Z_2) \longrightarrow H^i(S^n*S^n,\ n+8\,;\,Z_2)$. Then it follows from (1.3) and (3.12) that $H^*(\pi_{n+8}(S^n*S^n),\ n+8\,;\,Z_2)$ has a base

$$(3.13) \qquad \nu,\ \ Sq^1\nu,\ \ Sq^2\nu,\ \ Sq^3\nu,\ \ Sq^2Sq^1\nu$$

in dimensions $\leqq n+11$, where ν is the generator of $H^{n+8}(\pi_{n+8}(S^n*S^n),\ n+8\,;\,Z_2)$. Since p^* is onto in dimensions $n+8$, we have $p^*\nu=\gamma$, and so it follows from (3.11) and (3.13) that p^* is isomorphic onto in dimensions $\leqq n+8$, and is isomorphic into in dimensions $\leqq n+11$. Thus we see from (3.1) that $H^*(S^n*S^n,\ n+9\,;\,Z_2)$ has a base

$$b'',\ \ Sq^1b''$$

in dimensions $\leqq n+10$, where $b''=i^*(b')\in H^{n+9}(S^n*S^n,\ n+9\,;\,Z_2)$. Then we have by the same arguments in the proof of (3.8) that

$$(3.14) \qquad C(\pi_{n+9}(S^n*S^n),\ Z_2) \approx Z_2\,.$$

Since $C(\pi_{n+i}(S^n),\ p)\approx Z_3$ for $i=3$, 7 and $p=3$, $\approx Z_5$ for $i=7$ and $p=5$, and is zero otherwise for $i\leqq 9$ and any odd prime p [13, 14], our main result (A) follows from (2.1), (2.2), (3.5), (3.8), (3.10), (3.12) and (3.14).

4. Reduced complex M'_n

Let $e_i^{r_i}$ $(i=1, 2, \cdots, s)$ be s disjoint r_i-cells, and let f_i: $(\dot{e}_i^{r_i}, y_i)\longrightarrow(X, x_0)$ be s maps of the boundary $\dot{e}_i^{r_i}$ in a 1-connected space X, where $y_i\in\dot{e}_i^{r_i}$ and $x_0\in X$ are base points. Then we shall denote by $\{X\cup e_1^{r_1}\cup\cdots\cup e_s^{r_s}; f_1, \cdots, f_s\}$ a space obtained by identifying each point $y\in\dot{e}_i^{r_i}$ to $f_i(y)\in X$ in the union $X\cup e_1^{r_1}\cup\cdots\cup e_s^{r_s}$. Let E^r be the r-cube in the Cartesian space, and let g_i: $(E^r, \dot{E}^r, z_0)\longrightarrow(e_i^{r_i}, \dot{e}_i^{r_i}, y_i)$ $(i=1, 2, \cdots, t\leqq s)$ be maps such that $\sum_{i=1}^t f_i\circ(g_i|\dot{E}^r)$ is null-homotopic, where $z_0=(0, 0, \cdots, 0)$ and $\sum$ denotes the addition used in the usual definition of homotopy group $\pi_r(X, x_0)$. Then we can construct a map h of an r-sphere $\dot{E}^{r+1}=S^r$ in $\{X\cup e_1^{r_1}\cup\cdots\cup e_s^{r_s}; f_1, \cdots, f_s\}$ as follows:

Let $\mathcal{E}_i^r$ $(i=1, 2, \cdots, t)$ be t disjoint r-cells in S^r which have a single point z_0 in common, and which are oriented in agreement with the orientation of S^r. Define first h in $\mathcal{E}_i^r$ by $h|\mathcal{E}_i^r=g_i\circ\xi_i$ $(i=1, 2, \cdots, t)$, where ξ_i: $(\mathcal{E}_i^r, \dot{\mathcal{E}}_i^r)\longrightarrow(E^r, \dot{E}^r)$ is a homeomorphism of degree 1 Then it follows from our assumption that $h|\bigcup_{i=1}^t\dot{\mathcal{E}}_i^r$ of a singular $(r-1)$-sphere $\bigcup_{i=1}^t\dot{\mathcal{E}}_i^r$ in X is null-homotopic in X. Choose now such a null-homotopy arbitrarily, and define h in $S^r-\bigcup_{i=1}^t\operatorname{Int}\mathcal{E}_i^r$ by this null-homotopy. This completes the definition of h.

In the following, a map obtained by such a construction from $g_1, g_2, \cdots, g_t$ will be denoted by $<g_1, g_2, \cdots, g_t|X>$.

As for spherical maps, we use the following notations: ι_r; $S^r\longrightarrow S^r$ $(r\geqq 1)$ is the identity; η_r: $S^{r+1}\longrightarrow S^r$ $(r\geqq 2)$ and ν_r: $S^{r+3}\longrightarrow S^r$ $(r\geqq 4)$ are the iterated suspensions of the Hopf fiber maps η_2 and ν_4 respectively. Let ∂_n: $\pi_{n+1}(e^{r+1}, \dot{e}^{r+1})\approx\pi_n(\dot{e}^{r+1})$ be the homotopy boundary, then we refer to maps in the homotopy classes $\partial_r^{-1}\{\iota_r\}$, $\partial_{r+1}^{-1}\{\eta_r\}$ and $\partial_{r+3}^{-1}\{\nu_r\}$ as $\bar{\iota}_{r+1}$, $\bar{\eta}_{r+1}$ and $\bar{\nu}_{r+1}$ respectively.[6)]

Until the end of this section, we assume that $n\geqq 7$. Consider the following $(n+k)$-dimensional cell complexes $M_n^k (k=1, 2, \cdots, 7)$ defined inductively by

$$\begin{aligned}
M_n^1 &= S^n,\\
M_n^2 &= \{M_n^1\cup e^{n+2}\ ;\ \eta_n\},\\
M_n^3 &= \{M_n^2\cup e^{n+3}\ ;\ <2\bar{\iota}_{n+2}|S^n>\},\\
M_n^4 &= \{M_n^3\cup e^{n+4}\ ;\ 3\nu_n\},\\
M_n^5 &= \{M_n^4\cup e^{n+5}\ ;\ <2\bar{\iota}_{n+4}, \bar{\eta}_{n+3}|M_n^2>\},\\
M_n^6 &= \{M_n^5\cup e^{n+6}\ ;\ <\bar{\eta}_{n+4}|S^n>\},\\
M_n^7 &= \{M_n^6\cup e^{n+7}\ ;\ <2\bar{\iota}_{n+6}|N_n^4>\},^{7)}
\end{aligned}$$

where N_n^4 is the subcomplex $\{S^n\cup e^{n+4}; 3\nu_n\}$ of M_n^4. The justificatian of above definitions follows from

6) Let f: $S^r\rightarrow X$, then $\{f\}$ denotes the element of $\pi_r(X)$ containing f.

7) It can be easily seen that the homotopy type of M_n^i does not depend on the choice of null-homotopy in the definition of identification map.

(4.1) i) $2\{\eta_n\}=0$ *in* S^n, ii) $\{<2\bar{\iota}_{n+2}|S^n>\circ\eta_{n+2}\}=\pm 6\{\nu_n\}$ *in* M_n^2, iii) $\{(3\nu_n)\circ\eta_{n+3}\}=0$ *in* S^n, iv) $\{<\bar{\eta}_{n+4}|N_n^4>\circ 2\iota_{n+5}\}=0$ *in* N_n^4.

In fact, i), ii) and iii) are well known (See [13]). In the notation of H. Toda, we have $<2\iota_{n+2}|S^n>\circ\eta_{n+2}\in\{\eta_n, 2\iota_{n+2}, \eta_{n+2}\}$.) iv) is obtained as follows: Consider the homotopy exact sequence of pair (N_n^4, S^n), then it follows from $\pi_{n+4}(S^n)\approx\pi_{n+5}(S^n)=0$ that $\pi_{n+5}(N_n^4)=\pi_{n+5}(N_n^4, S^n)$ under the inclusion map. This and $\pi_{n+5}(N_n^4, S^n)\approx Z_2$ imply iv).

Let M_n' be a $2n$-dimensional cell complex such that i) the $(n+7)$-skelton of M_n' is M_n^7, ii) $\pi_i(M_n')=0$ for $n+7\leqq i<2n$ (Such a complex does exist). Then we have

(4.2) $\pi_n(M_n')\approx Z$, $\pi_{n+1}(M_n')=0$, $\pi_{n+2}(M_n')=0$, $\pi_{n+3}(M_n')\approx Z_3$, $\pi_{n+4}(M_n')=0$, $\pi_{n+5}(M_n')\approx Z_2$, $\pi_{n+6}(M_n')=0$.

(4.3) *Let* $\iota: S^n\longrightarrow M_n^1$ *be the identity map, then* $\pi_{n+3}(M_n')$ *is generated by* $\{\iota\circ\nu_n\}$. $\pi_{n+5}(M_n')$ *is generated by* $\{<\bar{\nu}_{n+2}|S^n>\}$.

(4.4) $\pi_{n+5}(M_n^5)\approx Z_2+Z_2$ *and is generated by* $\{<\bar{\nu}_{n+2}|S^n>\}$ *and* $\{<\bar{\eta}_{n+4}|S^n>\}$.

These can be proved by the similar arguments used in the proof of (8.4) in [7]. Therefore we will note here only the principle and basic tools used, and omit to record the complete calculation.

Since $\pi_{n+i}(S^n)$ $(i\leqq 6)$ is well known (see ii) below), starting with $\pi_{n+i}(M_n^1)$, we determine $\pi_{n+i}(M_n^j)$ inductively with respect to i and j by making use of the homotopy sequence of pair (M_n^j, M_n^{j-1}). In this consideration, the following i) and ii) play essential rôles: i) Let f: $(E^{n+j}, \dot{E}^{n+j})\longrightarrow(M_n^j, M_n^{j-1})$ be the characteristic map of the cell e^{n+j}, then $f_\#$ is isomorphic onto for $i\leqq n+j-3$ in the commutative diagram

$$\begin{array}{ccc}\pi_{n+i}(E^{n+j}, \dot{E}^{n+j}) & \xrightarrow{\partial} & \pi_{n+i-1}(\dot{E}^{n+j}) \\ \downarrow f_\# & & \downarrow (f|\dot{E}^{n+j})_\# \\ \pi_{n+i}(M_n^j, M_n^{j-1}) & \xrightarrow{\partial} & \pi_{n+i-1}(M_n^{j-1})\end{array}$$

[7]. ii) $\pi_{n+1}(S^n)\approx Z_2$, $\pi_{n+2}(S^n)\approx Z_2$, $\pi_{n+3}(S^n)\approx Z_{24}$, $\pi_{n+6}(S^n)\approx Z_2$; they are generated by $\{\eta_n\}$, $\{\eta_n\circ\eta_{n+1}\}$, $\{\nu_n\}$ and $\{\nu_n\circ\nu_{n+3}\}$ respectively; $\pi_{n+4}(S^n)\approx\pi_{n+5}(S^n)=0$. [13].

If we take in consideration that the homotopy boundary of any $(n+8)$-cell of M_n' is in M_n^6, the following can be easily proved [7].

(4.5) $H^i(M_n'; Z)\approx Z$ *for* $i=n$, $\approx Z_2$ *for* $n+3$, $n+5$ *and* $n+7$, *and vanishes for other* $i\leqq n+7$

(4.6) *Let* $\{e^n\}\in H^n(M_n'; Z)$ *and* $\{e^{n+j}\}\in H^{n+j}(M_n'; Z)$ $(j=3, 5$ *and* $7)$ *be generators, then we have*

$$Sq^j\{e^n\}=\{e^{n+j}\} \qquad (j=3, 5, 7),$$

(We may consider Sq^j with respect to the integer coefficient, because j is odd [11].)

Let K be any cellular decomposition of S^n*S^n, and K^{n+j} its $(n+j)$-skelton. Take a map $f: K^{n+1}\longrightarrow M_n^6$ such that

(4.7) $$f^*\{\bar{e}^n\}=\{\bar{u}\},$$

where $\{\bar{u}\}\in H^n(K;\ \pi_n(M_n^6))$ and $\{\bar{e}^n\}\in H^n(M_n^6;\ \pi_n(M_u^6))$ are generators. It is well known that such a map exist. Then we have

(4.8) *f can be extended to a map $\tilde{f}: K^{n+7}\longrightarrow M_n^7$.*

This is proved as follows: Since $\pi_{n+1}(M_n^6)$, $\pi_{n+2}(M_n^6)$ and $\pi_{n+4}(M_n^6)$ are trivial from (4.2), and since $H^{n+4}(K;\ \pi_{n+3}(M_n^6))\approx \mathrm{Hom}\ (H_{n+4}(K;\ Z),\ Z_3)+\mathrm{Ext}\ (H_{n+3}(K;Z),\ Z_3)=0$ from (4.2) and (1.1), it follows from the classical obstruction theory [11] that f can be extended to a map $\bar{f}: K^{n+5}\longrightarrow M_n^6$. Consider now a new cell complex

$$L=\{M_n^6\cup e'^{n+6};\ <\bar{\nu}_{n+2}|S^n>\}.$$

Then we have $\pi_{n+5}(L)=0$ from (4.4), and so $\bar{f}$ has an extension $f': K^{n+6}\longrightarrow L$. Thus, for the obstruction $\{c^{n+6}(\bar{f})\}\in H^{n+6}(K,\ \pi_{n+5}(M_n^6))$, we have [11]

(4.9) $$\{c^{n+6}(\bar{f})\}=f'^*\{c^{n+6}(k)\},$$

where $k: L^{n+5}\longrightarrow M_n^6$ is the inclusion. It is obvious from the definition that $\{c^{n+6}(k)\}$ is represented by a cocycle which takes 0 on e^{n+6} and takes $\{<\bar{\nu}_{n+2}|S^n>\}$ on e'^{n+6}. Therefore if we define

$$Sq^4Sq^2:\quad H^n(X;\ \pi_n(M_n^6))\longrightarrow H^{n+6}(X;\ \pi_{n+5}(M_n^6)),$$

$(X=L$ or $K)$ using the unique non-trivial homomorphism of $\pi_n(M_n^6)$ to $\pi_{n+5}(M_n^6)$, then we have

(4.10) $$\{c^{n+6}(k)\}=Sq^4Sq^2\{\bar{e}^n\}.$$

Thus it follows from (4.9), (4.10) and (4.7) that

$$\{c^{n+6}(\bar{f})\}=Sq^4Sq^2\{\bar{u}\}.$$

However it is seen from (1.2) that $Sq^4Sq^2\{\bar{u}\}=0$ in K. Therefore we have $\{c^{n+6}(\bar{f})\}=0$, and so f has an extension $\bar{\bar{f}}: K^{n+6}\longrightarrow M_n^7$. Since $\pi_{n+6}(M_n^7)=0$ from (4.2), $\bar{\bar{f}}$ has also an extension $\tilde{f}: K^{n+7}\longrightarrow M_n^7$. This completes the proof of (4.8).

Since $\pi_i(M_n')=0$ for $n+7\leqq i<2n$ by the definition, and K is $2n$-dimensional, $\tilde{f}$ can be extended to a map $g: K\longrightarrow M_n'$. Let $u\in H^n(K;\ Z)$ be the generator, then it is obvious from (4.7) that $g^*\{e^n\}=f^*\{e^n\}=u$. Therefore it follows from (1.2) and (4.6) by the naturality of Sq^i that $g^*: H^i(M_n';\ Z)\longrightarrow H^i(K;\ Z)$ is isomorphic onto for $i=n,\ n+3,\ n+5$ and $n+7$. Furthermore, since $H^i(M_n';Z)=H^i(K;Z)$ for $i<n$ and for $i=n+1,\ n+2,\ n+4$ and $n+6$, we conclude that $g^*: H^i(M_n';\ Z)\longrightarrow H^i(K;\ Z)$ is isomorphic onto for $i\leqq n+7$. Thus, in virtue of the well known theorem [15], we have

(4.11) *$S^n * S^n$ and M'_n is of the same $(n+6)$-homotopy type. Especially we have $\pi_i(S^n*S^n) \approx \pi_i(M'_n)$ for $i \leqq n+6$.*

This together with (4.2) proves (A) for $i \leqq n+6$.

5. Supplementary remark

I) (5.1) *$S^n*S^n (2 \leqq n \leqq 5)$ is of the same homotopy type as a reduced complex M_n defined as follows: $L_2 = \{S^2 \cup e^4 ; \eta_3\}$, $L_3 = \{EL_2 \cup e^6 ; <2\bar{\iota}_5 | S^3>\}$, $L_4 = \{EL_3 \cup e^8 ; \nu_4 + \omega_4\}$, $L_5 = \{EL_4 \cup e^{10} ; <2\bar{\iota}_5, \eta_8 | \{S^5 \cup e^7 ; \eta_5\}>\}$, where EL_i is the suspended space of L_i, and ω_4 is the suspension of a map $S^6 \longrightarrow S^3$ introduced by Blaker-Massey.*

In fact, since S^2*S^2 is the complex projective plane, i) and ii) are a direct consequence of the cellular decomposition of S^n*S^n due to Steenrod. Thus S^4*S^4 is of the same homotopy type as $\{EL_3 \cup e^8 ; g\}$ with a suitable map g. However, since $\pi_7(EL_3) \approx Z + Z_3$ and is generated by $\{\nu_4\}$ and $\{\omega_4\}$ [13], we may assume that

$$\{g\} = l_1\{\nu_4\} + l_2\{\omega_4\}$$

with some integer l_1 and some integer l_2 mod 3. We saw in (2.3) that $[\iota_4, \iota_4] = 0$ for the inclusion map $\iota_4 : S^4 \longrightarrow S^4*S^4$, and know [10] that $[\iota_4, \iota_4] = 2\{\nu_4\} - \{\omega_4\}$. Therefore we must have

$$2\{\nu_4\} - \{\omega_4\} = k(l_1\{\nu_4\} + l_2\{\omega_4\})$$

with some integer k, and this implies that $\{g\} = \pm(\{\nu_4\} + \{\omega_4\})$ or $\pm(2\{\nu_4\} - \{\omega_4\})$. If the latter holds, we have the cup product of the generator of $H^4(S^4*S^4 ; Z)$ with itself is $2v^8$, where $v^8 \in H^8(S^4*S^4 ; Z)$ is a generator. This contradicts (1.2). Thus we may take $\nu_4 + \omega_4$ in place of g. This proves iii). iv) is obvious. (Note that $E(\{\nu_4\} + \{\omega_4\}) = 3\{\nu_5\}$).

The homotopy group of $S^n*S^n (2 \leqq n \leqq 5)$ can be calculated by making use of L_n. For example, we have easily

(5.2) $$\pi_7(S^4*S^4) \approx Z_3$$

II) Recently H. Cartan [3] has given the structure of $H^*(Z, n ; Z_p)$ and $H^*(Z_p, n ; Z_p)$ for any odd prime p by making use of the reduced cyclic power and the Bockstein homomorphism. On the other hand, S. D. Liao explained the cohomology structure of the p-fold cylic product ϑ_{np} of an n-sphere (See especially (5.4) and (9.7) in [5]). If we apply these results, we can obtain the results with respect to the homotopy of ϑ_{np} by the arguments similar to those in above sections. For example, we have

(5.3) *Let p be an odd prime, and let $n \geqq 2p+2$. Then $C(\pi_i(\vartheta_{np}), p) \approx Z_p$ for $i = n+2j$ $(j = 1, 2, \cdots, p-2)$ and $n+2(p-1)+1$, and vanishes for other $i \leqq n+2(p-1)$.*

III) Let Y be the $(n-1)$-fold suspended space of the real projective plane.

Namely Y is a cell complex $S^n \cup e^{n+1}$ such that e^{n+1} is attached to S^n by a map of degree 2. Then the Stein's formulas [12, p. 582] give the integral homology groups of the symmetric product $Y*Y$ as follows:

(5.4) $H_0(Y*Y;\, Z)\approx Z$; $H^{n+i}(Y*Y;\, Z)\approx Z_2$ *for* $i=0,\ n+1$ *and* $2\leqq i\leqq n-1$; $H_{2n}(Y*Y;\, Z)\approx Z_4$ *for even* n, $\approx Z_2$ *for odd* n; $H_i(Y*Y;\, Z)=0$ *for other* i.

Thus the cohomology group $H^{n+i}(Y*Y;\, Z_2)$ is Z_2 for $i=0$, 1, 2 and $n+2$, and is Z_2+Z_2 for $3\leqq i\leqq n+1$. Let a be the generator of $H^n(Y*Y;\, Z_2)$. Then we have

(5.5) *We can take as a base of* $H^*(Y*Y;\, Z_2)$ *the following*: $Sq^i a\,(0\leqq i\leqq n)$, $Sq^iSq^1a\,(2\leqq i\leqq n+1)$ *and* $a\cup Sq^1\, a$. *Furthermore we have the relations*:

$$Sq^iSq^{j+1}a=\binom{j}{i}Sq^{i+j+1}a+\binom{j-1}{i-2}Sq^{i+j}Sq^1a\,,$$
$$Sq^iSq^{j+1}Sq^1a=\binom{j}{i}Sq^{i+j+1}Sq^1a\,, \qquad (j\geqq 1)\,.$$

Applying the methods similar to those by which R. Bott [2] gives a proof of (1.2) in this paper, (5.5) can be proved easily. (The basic tools of this method are the Smith-Richardson sequence and the Theorem 2 in [2]).

Now we can calculate the (stable) homotopy groups $\pi_i(Y*Y)$ for $i\leqq 2n-2$ by the method explained in §3. The results are as follows:

(5.6) $\pi_i(Y*Y)=0$ *for* $0\leqq i<n$, $n+1\leqq i\leqq n+4$ *and* $n+7$.
$\pi_i(Y*Y)\approx Z_2$ *for* $i=n$, $n+5$, $n+6$ *and* $n+8$, *and* $\pi_{n+9}(Y*Y)$ *is not cyclic.*

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