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Positive functionals and representation theory on Banach algebras. I

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Introduction. This paper is devoted essentially to two main subjects lying in the study of involutive Banach algebras; the first is an investigation about the positive boule of the dual space, while the second is the general Banach representation theory of Banach algebras, the greater part of whose interest however might be found in connection with the group algebra of a locally compact (abbrev. LC) group.

Here the *positive boule* is a regularly convex subset of the dual unit sphere consisting of positive linear functionals; in the case of a real commutative algebra possessing the unit of norm 1, it is well known that the positive boule coincides with the weakly closed linear convex hull of all multiplicative linear functionals, Propositions K and K^{bis}.

In Chapter II, these circumstances are extensively investigated in more general situations, where the algebras are assumed throughout to be over the complex field, not necessarily commutative, and without unit but with the approximate indentity. Thus Theorem 2 is the most general form of the above fact, Proposition K^{bis}, and under some additional conditions, α) and β) in §**5**, this result is made more precise of in Theorem 3, while Theorem 4 determines the complete structure of the positive boule.

Although these results have their own interests, one may easily and directly reproduce them in the theory of topological groups, e.g. the representation theory or the analysis of positive definite functions, without considering the underlying groups. Therefore, our theorems would be considered as a generalization of the group representation theorem.

Chapter I is a basic preliminary; above all Theorem 1 plays an important rôle in later discussions in respect of the real restriction method.

Chapter I. Preliminary study of the functionals on B-algebras

1. Preliminaries about B-algebras. We shall recall some fundamental notions and fix our notations at first. Let R or K be the real or complex number field respectively.

$A(S)$ is a Banach algebra (B-algebra) over the scalar field S which is either R

or K; if $A(S)$ has an algebraic unit 1 with norm 1, we say $A(S)$ to be *unitary*. If $A(S)$ has an approximate identity $\{v^\lambda\}$ such that $av^\lambda \to a$ and $v^\lambda a \to a$ (uniformly) in $A(S)$, we call $A(S)$ *semi-unitary*[(1)].

An *involutive* (or *self-adjoint*) B–algebra (B_*–algebra) over S is a B–algebra which admits such a *–operation that is a conjugate linear, involutely, anti-automorphism of $A(S)$.[(2)] Further if a B_*–algebra $A^*(S)$ has the norm condition

$$\|aa^*\| = \|a\| \cdot \|a^*\|, \tag{1.1}$$

$A^*(S)$ is called a *B*–algebra over S*: it has sometimes a stronger norm condition, $\|aa^*\| = \|a\|^2$; in the case, it must be $\|a\| = \|a^*\|$ [(3)].

If $A^*(S)$ is commutative, it is easily proved that the latter norm condition is equal to (1.1).

Throughout this paper, we shall assume $A(S)$ or $A_*(S)$ to be over K, *i.e.* $S=K$, unless otherwise specified by adding the term "real": then we abbreviate sometimes $A(S)$ (or $A_*(S)$) to A (or A^*) only, if there is no confusion.

An element a with $a=a^*$ is called *hermitian* (or *self-adjoint*) and the collection of all hermitian elements in A is called the *hermitian kernel* of it, denoting by $H(A_*)$.

It is easy to see $H(A)$ being a B–algebra (necessarily over the reals R), if and only if A is commutative; in other words, a commutative real B–algebra is characterized by a B_*–algebra A_* with $A_*=H(A_*)$, and if it forms moreover a B*–algebra, it is essentially a continuous function algebra $C(X)$ on a compact X[(4)].

If A_* is unitary, unit must be hermitian.

Proposition 1. *Denoting by $D(A_*)$ the subset of $H(A_*)$ consisting of all such elements as in the form h^*h, $h \in A_*$, every product ab in A_* may be written as follows*;

$$ab = \sum_{k=1}^{4} i^{k-1} d_k \qquad (i=\sqrt{-1}) \tag{1.2}$$

for $d_k \in D(A_*)$, and if A_* is semi-unitary, A_* is uniformly approximated by the complex linear envelope of $D(A_*)$.

In fact, (1.2) is easily deduced, putting $d_k=h_k^*h_k$ and $h_k=\frac{1}{2}(f^*+(-i)^{k-1}g)$ for $k=1, 2, 3, 4$: the rest is somewhat manifest.

As mentioned above, $H(A_*)$ is not always an algebra, but there exist two definable manners of introducing products in it, that is,

Proposition 2. *$H(A_*)$ is a special Jordan and besides special Lie algebra with respect to the products*;

1) It should be noticed that the word "unitary" or "semi-unitary" is sometimes used in different senses; e.g. see J. Diximer [2].

2) That is ; $(\alpha a+b)^*=\bar{\alpha}a^*+b^*$, $a^{**}=a$, $(ab)^*=b^*a^*$.

3) For example, C. Rickart, *Banach algebra with an adjoint operation*, Ann. of Math., **47** (1946), *pp.* 528–550; I. Kaplansky, *Normec algebra*, Duke Math. J., vol. **16** (1949), *pp.* 339–418.

4) R. V. Kadison [4], Theorem 6.8; S. Matsushita, [6] Theorem 4.

(1.3) $$a \circ b = \frac{1}{2}(ab+ba)$$

and

(1.4) $[a, b] = (ab - ba)$ (special Poisson's product) *respectively.*

The former is always commutative, distributive, but not associative, while the latter skew-symmetric, distributive, and satisfies the Jacobi's equality

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0\,.$$

Then $H(A_*)$ is a non-associative real normed algebra with respect to the product (1.3), satisfying usual norm condition

(1.5) $$\| a \circ b \| \leqq \| a \| \cdot \| b \|$$

and $a \circ a = a^2$. If A_* is semi-unitary, it is clear that $v^\lambda \circ a = a \circ v^\lambda \to a$ for every $a \in H(A_*)$ no matter when the approximate identity $\{v^\lambda\}$ is in $H(A_*)$ or not.

If an approximate identity is wholly contained in $H(A_*)$, it is called an hermitian approximate identity; for example, this is the case when $\| a^* \| = \| a \|$ for every a of A^*.

Proposition 3. *In $H(A^*)$, every Jordan product $a \circ b$ may be written in the form*;

(1.6) $$f \circ g = \frac{1}{2}[d_1 + d_1' - (d_3 + d_3')]\,,$$

d_i, $d_i' \in D(A^*)$, so that if A_* has an hermitian approximate identity, $H(A^*)$ is uniformly approximated by the real linear envelope of $D(A^*)$.

According to the decomposition (1.2), assume now $ab = \sum_k i^{k-1} d_k$ and $ba = \sum_k i^{k-1} d_k'$; then by simple calculation, we have

$$i^{k-1}(f + i^{k-1} \cdot g)(f + (-i)^{k-1} \cdot g) + i^{k+1}(f + i^{k+1} \cdot g)(f + (-i)^{k+1} \cdot g) = 0$$

for $k = 2n$, so that

$$i d_2 + (-i) d_4' = i d_2' + (-i) d_4 = 0\,,$$

from which follows (1.6).

The set of all w such that $w = [a, b]$ for all pairs a, b in $H(A_*)$ is denoted by $W(A^*)$.

2. Functionals on B_*-algebras. To study the functionals defined on B_*-algebras, the following symbolical conventions are adopted:

$\Gamma(\cdot)$ = the dual space of a normed vector space $(\cdot)$; $\Gamma(A)$ and $\Gamma(H(A_*))$ are mainly considered. $\Gamma(\cdot)$ is over K or R, according to $(\cdot)$'s circumstances; for example $\Gamma(A_*)$ is over K, while $\Gamma(H(A_*))$ over R.

$\Xi(A_*)$ = the sub-space of $\Gamma(A_*)$, over R, whose elements are characterized by

(2.1) $$\varphi(a^*) = \overline{\varphi(a)} \quad \text{for every} \quad a \in A_*\,.$$

$\Pi(A_*)$ = the convex subset of $\Gamma(A_*)$, whose elements called *positive functionals*, are characterized by the property;

(2.2) $\varphi(a^*a) \geqq 0$, *i.e.* $\varphi(d) \geqq 0$ for every $d \in D(A_*)$.

$\Phi(A_*)$=the set of all *multiplicative linear functionals*;

(2.3) $$\varphi(ab) = \varphi(a)\varphi(b).$$

$\hat{\Pi}(A_*)=\Pi(A^*) \cap \Xi(A_*)$, which is clearly a convex subset of $\Pi(A_*)$.

$\hat{\Phi}(A_*)=\Phi(A_*) \cap \Xi(A_*)$.

As immediate consequences of the definition, we hold i) $\hat{\Phi}(A_*) \subset \hat{\Pi}(A_*)$ (hence $\subset \Pi(A_*)$), ii) if $a \in H(A_*)$, then $\varphi(a^2) \geqq 0$ for every $\varphi \in \Pi(A^*)$.

Lemma 1. (generalized Cauchy-Schwarz's Lemma) *If φ is in $\hat{\Pi}(A_*)$, we have*

(2.4) $$|\varphi(a^*b)|^2 = |\varphi(b^*a)|^2 \leqq \varphi(a^*a)\varphi(b^*b).$$

In fact, for an arbitrary α in K, we hold $\varphi(\alpha a+b)^* \cdot (\alpha a+b)) = |\alpha|^2\varphi(a^*a) + \alpha\varphi(a^*b) + \bar{\alpha}\varphi(b^*a) + \varphi(b^*b) \geqq 0$: if $\varphi(a^*a)=0$ and $\varphi(a^*b) \neq 0$, we may put α (real) $\leqq -\varphi(b^*b)/2\Re(\varphi(a^*b))$ and get a contradiction; if $\varphi(a^*a) \neq 0$, putting $\alpha = -\varphi(b^*a)/\varphi(a^*a)$, we can easily obtain (2.4).

Now denoting the zero functional by θ, which is evidently contained in all the classes enumerated above, we shall make use of Γ^0, $\hat{\Pi}^0$, $\hat{\Phi}^0$, etc. instead of $\Gamma-\theta$, $\hat{\Pi}-\theta$, $\hat{\Phi}-\theta$, etc. respectively.

Proposition 4. *For any $\varphi \in \Gamma(A)$, the set $'I_\varphi$, I_φ', or I_φ, defined by*

$$'I_\varphi = \{a;\ \varphi(\alpha a+xa)=0 \quad \text{for every } x \text{ of } A\},$$
$$I'_\varphi = \{a;\ \varphi(\alpha a+ax)=0 \quad \text{for every } x \text{ of } A\},$$
$$I_\varphi = \{a;\ \varphi(\alpha a+xay)=0 \quad \text{for every } x,\ y \text{ of } A\},$$

where α being an arbitrary number of S, forms a closed left, right, or two-sided ideal of $A(S)$ respectively; further if φ is in $\Xi(A_*)$, then $'I_\varphi^* = I'_\varphi$ and I_φ is self-adjoint, i.e. $I_\varphi^*=I_\varphi$.

If $\varphi \in \Gamma^0(A)$, these ideals are proper.

Proposition 4[bis]. *Analogously as above, φ being in $\Gamma(A)$, the set $'\mathring{I}_\varphi$, $\mathring{I}'_\varphi$,* or *$\mathring{I}_\varphi$, defined by*

$$'\mathring{I}_\varphi = \{a;\ \varphi(xa)=0 \quad \text{for every } x \text{ of } A\},$$
$$\mathring{I}'_\varphi = \{a;\ \varphi(ax)=0 \quad \text{for every } x \text{ of } A\},$$
$$\mathring{I}_\varphi = \{a;\ \varphi(xay)=0 \quad \text{for every } x, y \text{ yof } A\},$$

forms respectively a closed left, right, or two-sided ideal of A, too. The assertions mentioned in the latter half of Proposition 4 are also valid for these ideals, except the last assertion.

The proof follows immediately from the definitions themselves. As a matter of course, $'I_\varphi \subset {'\mathring{I}_\varphi}$, $I'_\varphi \subset \mathring{I}'_\varphi$, $I_\varphi \subset \mathring{I}_\varphi$; but we shall make a further remark:

Proposition 5. *If A is semi-unitary, or especially unitary, these two kinds of ideals must be identical; that is, $'I_\varphi = {'\mathring{I}_\varphi}$, $I'_\varphi = \mathring{I}_\varphi'$ and $\mathring{I}_\varphi = I_\varphi$.*

Proposition 6. *For any* $\varphi \in \hat{\Pi}(A_*)$, *the quotient B-space* $A_*/'\mathring{I}_\varphi$ (*or* $A^*/\mathring{I}'_\varphi$) *forms a pre-Hilbert space with the inner product;*

$$(2.5) \qquad (X_a, X_b)_\varphi = \varphi(b^*a) \quad (or \ = \varphi(ab^*)),$$

where X_a *is an element of* $A_*/'\mathring{I}_\varphi$ (*or* $A_*/\mathring{I}'_\varphi$) *which contains a of A. Completing* $A_*/'\mathring{I}_\varphi$ ($A_*/\mathring{I}'_\varphi$) *by the norm* $\| X_a \|_\varphi = (X_a, X_a)_\varphi^{1/2}$, *we have Hilbert space* $'H_\varphi$ *which contains the dense subspace* $A_*/'\mathring{I}_\varphi$ (*or resp.* H'_φ).

By the same reasoning, for a two-sided ideal $\mathring{I}_\varphi$, *the completion of* $A_*/\mathring{I}_\varphi$ *with respect to the norm* $\| \cdot \|_\varphi$ *forms a Hilbert algebra. Owing to Proposition 5, if* A_* *is semi-unitary,* $'\mathring{I}_\varphi$ ($\mathring{I}'_\varphi$, *or* $\mathring{I}_\varphi$) *may be replaced by* $'I_\varphi$ (I'_φ, *or* I_φ).

To prove this proposition, we have only to prove that $\| X_a \|_\varphi = 0$ if and only if $a \in '\mathring{I}_\varphi$ (or resp. $\in \mathring{I}'_\varphi$); indeed, $a \in '\mathring{I}_\varphi$ implies $\varphi(a^*a) = \| X_a \|_\varphi^2 = 0$ and conversely if $\varphi(a^*a) = 0$, $\varphi(b^*a) = 0$ for all b in virtue of Lemma 1.

Proposition 7. *For any* $\varphi \in \Phi^0(A)$, *the set*

$$(2.6) \qquad I_\varphi = \{a\,;\ \varphi(a) = 0\},$$

forms a maximal two-sided regular ideal, for which A/I_φ *is isomorphic to the scalar field S of A(S). If* $A_*(K)$ *is considered, then* $\Phi^0(A_*) = \hat{\Phi}^0(A_*)$.

Here, the notion of regularity in ideals follows a costomary manner; that is, a left (right, or two-sided) ideal is said to be regular if there exists such an element u, called right (left, or resp. two-sided) identity modulo the ideal, that $au - a$ ($ua - a$, resp. $aub - ab$) in the ideal for every a, b in the algebra.

We shall now prove the above Proposition: but the first half is somewhat trivial, and to prove $\Phi^0(A_*) = \hat{\Phi}^0(A_*)$, we need only to show $\Phi^0(A_*) \subset \Xi^0(A_*)$. For this purpose, we prepare

Lemma 2. *If u is a (two-sided) identity modulo* I_φ, *then* $\varphi(u) = \varphi(u^*) = 1$.

In fact, $u^*u - u \in I_\varphi$ implies that $(u^*u - u^*)^* = u^*u - u \in I_\varphi^* = I_\varphi$, so that $u^* - u \in I_\varphi$ or equivalently $\varphi(u^*) = \varphi(u)$, which must be equal to 1 from $A/I_\varphi \cong K$.

Proof of $\Phi^0(A_*) \subset \Xi^0(A_*)$: Putting $z = \varphi(a)$ ($\in K$), $zu - a$ is in I_φ, and so $(zu - a)^* = \bar{z}u^* - a^*$, from which it holds $\varphi(a^*) = \bar{z}\varphi(u^*) = \bar{z} = \overline{\varphi(a)}$. This completes the proof of Proposition 7.

Remark I: The iverse problem of Proposition 7, which will estimate profoundly the existence of non-trivial (*i.e.* $\neq \theta$) multiplicative functionals, is solved in such a manner that if I_φ is a two-sided regular ideal which is maximal as a left as well as right ideal, then $\varphi \in \Phi^0(A)$.

Remark II: Owing to the latter half of Proposition 7, in the case of B_*-algebras there needs no distinction between Φ and $\hat{\Phi}$ at all.

3. Functional on $H(A_*)$. We shall begin with some notations: $\Gamma(H(A_*))$ is the same as mentioned above, § **2.**

$\tilde{\Pi}(H(A_*))$=the convex subset of $\Gamma(H(A_*))$ consisting of all such functionals φ defined on $H(A_*)$;

$$2|\varphi([a, b])| \leqq \varphi(a^2)+\varphi(b^2)\,. \tag{3.1}$$

$\tilde{\Phi}(H(A_*))$=all of multiplicative linear functionals with respect to the product (1.3), *i.e.* $\varphi(a\circ b)=\varphi(a)\cdot\varphi(b)$, vanishing on $W(A_*)$.

Then we can establish;

Theorem 1. i) *In a B_*-algebra $A_*(K)$, any functional of $\Gamma(A_*)$ is entirely determined on $H(A_*)$.* ii) *Especially, we have*

$$\alpha) \qquad \Xi(A_*) \approx \Gamma(H(A_*))\,,$$
$$\beta) \qquad \hat{\Pi}(A^*) \approx \tilde{\Pi}(H(A_*))\,,$$
$$\gamma) \qquad \Phi(A_*) \approx \tilde{\Phi}(H(A_*))\,,$$

where $\Delta(A_*)\approx\tilde{\Delta}(H(A_*))$ means that Δ and $\tilde{\Delta}$ are topologically isomorphic one another withe respect to the weak topologies (w^*-topology) as functionals in respective dual spaces, $\Gamma(A^*)$ and $\Gamma(H(A_*))$, in such a fashion that if $\tilde{\varphi}\leftrightarrow\varphi$ for $\tilde{\varphi}\in\tilde{\Delta}$, $\varphi\in\Delta$, then

$$\varphi(a) = \tilde{\varphi}\Big(\frac{a+a^*}{2}\Big)+i\tilde{\varphi}\Big(\frac{a-a^*}{2i}\Big)\,. \tag{3.2}$$

Proof of i) is immediate; if φ coincides with ψ on $H(A_*)$, both of which being in $\Gamma(A_*)$, then $\varphi(a)=\tilde{\varphi}(s)+i\tilde{\varphi}(t)=\tilde{\psi}(s)+i\tilde{\psi}(t)=\tilde{\psi}(a)$, where $s=\dfrac{a+a^*}{2}$ and $t=\dfrac{a-a^*}{2i}$.

Proof of ii): α). Since φ, $\varphi\in\Xi(A^*)$, is real on $H(A_*)$, the restricted $\tilde{\varphi}$ of φ on $H(A_*)$ is in $\Gamma(H(A_*))$; conversely, any extended φ of $\tilde{\varphi}$, $\tilde{\varphi}\in\Gamma(H(A_*))$ by the relation (3.2), is clearly contained in $\Xi(A_*)$. The algebraic isomorphism between them is somewhat trivial, but the topological one shall be proved afterwards in a lump.

β). We shall first prove that each functional of $\hat{\Pi}(A_*)$ becomes an element of $\tilde{\Pi}(H(A_*))$. For the decomposition $a=s+it$ as above, we have

$$\begin{cases} a^*a = a^2+t^2+2[s,t] \\ aa^* = s^2+t^2-2[s,t]\,, \end{cases} \tag{3.3}$$

so that if φ is in $\hat{\Pi}(A_*)$, $\varphi(s^2)+\varphi(t^2)\geqq 2\varphi([s, t])$ and $-2\varphi([s, t])$, from which the inequality (3.1). The converse assertion is also clear on account of (3.3).

γ). For $\varphi\in\Phi^0(A_*)$, it is easy to see that the restricted $\tilde{\varphi}$ is also in $\tilde{\Phi}(H(A_*))$, since $\tilde{\varphi}([a, b]) = \dfrac{1}{2i}(\varphi(ac)-\varphi(ba)) = 0$. Thus, it is sufficient to prove that every $\tilde{\varphi}\in\tilde{\Phi}(H(A_*))$ is extensible to the whole A_* preserving its multiplicativity.

Let φ be the extension of $\tilde{\varphi}$ as in the form (3.2), them

$$\varphi(st)-\varphi(ts) = 2i\Big(\frac{\varphi(st)-\varphi(ts)}{2i}\Big) = 2i\cdot\varphi([s, t]) = 2i\cdot\tilde{\varphi}([s, t]) = 0\,,$$

for every a, b in $H(A_*)$; putting $a=s_1+it_1$ and $b=s_2+it_2$, $a, b\in A_*$, s_1, s_2, t_1, $t_2\in H(A_*)$, it holds

$$\begin{cases} ab = s_1s_2 - t_1t_2 + i\cdot(s_1t_2 + t_1s_2) , \\ ba = s_2s_1 - t_2t_1 + i\cdot(t_2s_1 + s_2t_1) , \end{cases}$$

and

$$\varphi(ab) - \varphi(ba) = [\varphi(s_1s_2) - \varphi(s_2s_1)] - [\varphi(t_1t_2) - \varphi(t_2t_1)] \\ + i[\varphi(s_1t_2) - \varphi(t_2s_1)] + t[\varphi(t_1s_2) - \varphi(s_2t_1)] = 0 ,$$

that is, $\varphi(ab) = \varphi(ba)$. Therefore, we can conclude the multiplicativity as follows;

$$\begin{aligned} \varphi(ab) &= \frac{1}{2}(\varphi(ab) + \varphi(ba)) \\ &= \frac{1}{2}[\varphi(s_1s_2) + \varphi(s_2s_1)] - \frac{1}{2}[\varphi(t_1t_2) + \varphi(t_2t_1)] \\ &\quad + \frac{i}{2}[\varphi(s_1t_2) + \varphi(t_2s_1)] + \frac{i}{2}[\varphi(t_1s_2) + \varphi(s_2t_1)] \\ &= \tilde{\varphi}(s_1 \circ s_2) - \tilde{\varphi}(t_1 \circ t_2) + i(\tilde{\varphi}(s_1 \circ t_2) + \tilde{\varphi}(t_1 \circ s_2)) \\ &= \tilde{\varphi}(s_1)\tilde{\varphi}(s_2) - \tilde{\varphi}(t_1)\tilde{\varphi}(t_2) + i(\tilde{\varphi}(s_1)\tilde{\varphi}(t_2) + \tilde{\varphi}(t_1)\tilde{\varphi}(s_2)) \\ &= (\tilde{\varphi}(s_1) + i\tilde{\varphi}(t_1))(\tilde{\varphi}(s_2) + i\tilde{\varphi}(t_2)) = \varphi(a)\varphi(b) . \end{aligned}$$

iii) Finally, it remains only to prove the topological equivalency of $\approx$ in α), β), γ): but the continuity $\tilde{\Delta} \to \Delta$ is evident; in fact, for a w*-nighborhood of φ_0 in Δ, $U_{\varphi_0}(a_1, \cdots\cdots, a_n; \varepsilon)$, $a_k \in A_*$, $\varepsilon > 0$, we see easily that the w*-neighborhood $U_{\tilde{\varphi}_0}(s_1, \cdots\cdots, s_n, t_1, \cdots\cdots, t_n; \varepsilon/\sqrt{2})$ of $\tilde{\varphi}_0$ in $\tilde{\Delta}$ is mapped into U_{φ_0}, since we have evidently

$$\begin{aligned} |\varphi(a_k) - \varphi_0(a_k)| &= |(\tilde{\varphi}(s_k) + i\tilde{\varphi}(t_k)) - (\tilde{\varphi}_0(s_k) + i\tilde{\varphi}_0(t_k))| \\ &= (|\tilde{\varphi}(s_k) - \tilde{\varphi}_0(s_k)|^2 + |\tilde{\varphi}(t_k) - \tilde{\varphi}_0(t_k)|^2)^{1/2} \\ &< \sqrt{\varepsilon^2/2 + \varepsilon^2/2} = \varepsilon , \end{aligned}$$

for $k = 1, \cdots\cdots, n$. The inverse continuity $\Delta \to \tilde{\Delta}$ is clear, since $H(A_*) \subset A_*$ and consequently the w_*-topology of Δ is stronger than that of $\tilde{\Delta}$.

Thus Theorem 1 is completely proved.

In a commutative A_*, (3.1) is naturally replaced by $\tilde{\varphi}(a \circ a) = \tilde{\varphi}(a^2) \leqq 0$ for $a \in H(A_*)$, and $\tilde{\Phi}(H(A_*))$ is characterized by the singule condition of multiplicativity.

Moreover if A is real and commutative, *i.e.* $A = A_*$ with $A_* = H(A_*)$, the both sides of $\approx$ in Theorem 1 must be identical and of course $\Gamma(A) = \Xi(A)$, so that the sign $\wedge$ would be unnecessary.

In passing, we shall touch on the *-operation defined in $\Gamma(A_*)$: putting

$$(\varphi, a) = \varphi(a) ,$$

we may consider it as a bilinear functional defined on the direct space $\Gamma(A_*) \times A_*$: there exists in $\Gamma(A_*)$ such an element φ^* that $(\varphi^*, a) = \overline{(\varphi, a)} (= \overline{\varphi(a^*)})$, for which the following properties are verified; i) $(\alpha\varphi_1 + \varphi_2)^* = \bar{\alpha}\varphi_1^* + \varphi_2^*$, ii) $\varphi^{**} = \varphi$, iii) $\varphi^* = \varphi$ for $\varphi \in \Xi(A_*)$.

Thus we comprehend that $\Xi(A_*)$ is the hermitian kernel of $\Gamma(A_*)$ with respect to such *-operation and (φ, a) is real on the product of both hermitian kernels $\Xi(A_*) \times H(A_*)$.

Chapter II. Structure of positive boule

In the preceding Chapter, we have investigated the space of functionals defined on a B– (or particularly B_*–) algebra, but we shall now pursure the further investigation about it, correlating with ideals of the algebra or with the reducibility of quotient spaces (or algebras) relative to the ideals; from such a standpoint, the following well-known assertion plays a fundamental rôle in the case of a real commutative unitary B-algebra A, *i.e.* unitary A with $A_*=H(A_*)$:

Proposition K. *Denoting the unit sphere (boule) of $\Gamma(A)$ and its surface by E and $S(E)$ respectively, $S(\hat{E})=\Pi(A)\cap S(E)$ is a w*-compact convex subset of $\Gamma(A)$ and if $S(\hat{E})$ is non-null, each of its extreme points, whose set shall be denoted by extr. $S(\hat{E})$ is multiplicative and vice versa, that is,*

$$\text{(K)} \qquad \textit{extr. } S(\hat{E}) = \Phi^0(A)\,,$$

so that $\varphi^{-1}(0)$ is a maximal ideal of A for every extreme φ of $S(\hat{E})$. (See R. V. Kadison [9], *pp.* 23–24).

We should remark that $\varphi \in S(\hat{E})$ *if and only if* $\varphi \in \Pi(A)$ *with* $\varphi(1)=1$ (1 being the unit of A).

In this Chapter, this result shall be extended to the case that A is a non-commutative, non-unitary (but semi-unitary), complex B_*-algebra $A=A_*(K)$ in a certain situation. We shall begin with some preparatory discussions.

4. Extremity and irreducibility.

Denoting the unit sphere of $\Gamma(H(A_*))$ by E_0, which is a w*-compact convex set, due to S. Kakutani and J. Dieudonné, the intersection (called the *positive boule*)

$$\text{(4.1)} \qquad \tilde{E}_0 = E_0 \cap \tilde{\Pi}(H(A_*))$$

is also w*-compact and convex, *i.e.* regularly convex in the sense of M. Krein–V. Smulian; since $\tilde{E}_0$ is bounded, the Krein–Milman's theorem is also applicable to it, from which it follows that $\tilde{E}_0$ has sufficiently many extreme points whose convex hull is w*-dense in E.[5]

As we have seen before, for each closed one-sided (two-sided) ideal I, the quotient space A/I turns to be a Banach or Hilbera space (algebra) by suitable norming and, if necessary, completing; then it is easy to see that $\overline{A/I}$ (completion of A/I, or $=A/I$ if it is complete in itself) admits the left, right or two-sided translation operator L_a, R_a or T_a respectively according as I is a left, right or two-sided ideal;

$$L_aX=\{ax\,;\ x\in X\}\,, \qquad R_aX=\{xa\,;\ x\in X\}$$
$$\text{and} \quad T_aX=L_aX=R_aX \qquad \text{for} \quad X\in\overline{A/I},\ a\in A\,.$$

Obviously, $L_aX_x=X_{ax}$ and $R_aX_x=X_{xa}$, so that $'(A)_I\equiv\{L_a\,;\ a\in A\}$, $(A)'_I$

5) See N. Bourbaki [1], Chap. II, §4.

$\equiv\{R_a;\ a\in A\}$ or $(A)_I\equiv\{T_a;\ a\in A\}$ forms a bounded, continuous Banach or Hilbert representation of A on $\overline{A/I}$ according to I's kind, with respect to the usual operator norming. Then we arrange

Definition 1. Let I be a closed left ideal of A such that $H=\overline{A/I}$ forms a Hilbert space. If there exists no such projective operator P on H relative to any closed proper subspace of H, as reduces (or equivalently commutes with) every element L_a of Hilbert representation $'(A)_I$, then $H=\overline{A/I}$ is said to be *irreducible*, otherwise *reducible*. In right or two-sided cases, these notions are analogously defined.

Proposition 8. *$L_a{}^*=L_{a^*}$, $R_a{}^*=R_{a^*}$ and $T_a{}^*=T_{a^*}$, where S^* is the conjugate operator of S on $H=\overline{A/I}$; therefore, for each $a\in H(A_*)$, L_a, R_a and T_a are hermitian operators.*

A bounded hermitian operator A is reduced by the projective operator P relative to a closed sub-space M if and only if $AM\subset M$; then decomposing every L_a as in the form

$$L_a=L_s+i\cdot L_t, \tag{4.2}$$

where $s=\frac{1}{2}(a+a^*)$ and $t=\frac{1}{2i}(a-a^*)$, both L_s and L_t are hermitian and hence we see directly

Proposition 9. *$H=\overline{A/I}$ is irreducible if and only if there is no closed linear manifold which is invariant under $\{L_a\}$, $\{R_a\}$ or $\{T_a\}$ according to I's kind.*

It is easy to see that I is never maximal but A/I is irreducible, while the converse is not necessarily true. We appreciate however that the converse is also true in the case of unitary commutative B_*-algebras A_*;

Lemma 3. *If A_* is unitary and commutative, the following four conditions are mutually equivalent;* i) *$H_\varphi=\overline{A_*/I_\varphi}$ is irreducible,* ii) *H_φ is simple,* iii) *I_φ is maximal,* iv) *φ is multiplicative.*

The equivalency of ii) iii) and iv) is clear, so that we have only to show that of i) and ii): in fact, if Hilbert algebra H is supposed not to be simple, then there would exist an ideal I_0 such that $I\subset I_0$, for which I_0/I is a proper ideal of H and so contained in a maximal ideal M of H which is invariant under $\{T_a\}$. This yields the reducibility of H_φ, that is, i)$\to$ii). Next if H_φ is simple, then I_φ is maximal, so that H is irreducible from Theorem 3 mentioned later, since A satisfies the conditions α) and β) in §**5**.

Thus we can replace, in beforesaid Proposition K, the terms "$\varphi^{-1}(0)$ is maximal" by those of "$\overline{A/I_\varphi}$ is irreducible", and get

Proposition K^{bis}. *Under the same conditions as in Propositon K, $H_\varphi=\overline{A/I_\varphi}$, $\varphi\in S(\hat{E})$, is irreducible, if and only if φ is extreme of $S(\hat{E})$ or equivalently I_φ is maximal,* In the following, we shall extend this assertion to more general cases, as is already announced at the outset of this Chapter. To begin with, some conventions and notation for later use;

1) For $\varphi\in\Gamma(H(A_*))$, $\hat{\varphi}$ means the extend functional of φ over the whole A^*, *i.e.* $\hat{\varphi}(a)=\varphi\left(\frac{a+a^*}{2}\right)+i\varphi\left(\frac{a-a^*}{2i}\right)$, while for $\varphi\in\Xi(A_*)$, $\tilde{\varphi}$ the restricted one of φ to $H(A_*)$; these are really reasonable owing to Theorem 1.

We often use, however, the same φ instead of $\hat{\varphi}$ or of $\tilde{\varphi}$, if there exists no confusion.

2) $'I_\varphi(I_\varphi', I_\varphi)$, $\varphi\in\tilde{E}_0$, is the ideal $'I_{\hat{\varphi}}(I'_{\hat{\varphi}}, I_{\hat{\varphi}})$ defined by the extended $\hat{\varphi}$ of φ.

3) If $'\mathring{I}_\varphi={}'\mathring{I}_\psi$ for φ, $\psi\in\tilde{E}_0$, we call such φ and ψ (*left*) *equivalent*, writing $\varphi\sim\psi$; it is easily possible to define *right, or two-sided equivalency* in an analogical manner.

Although we are going to consider mainly about the case of left ideals (and hence of left equivalency) hereafter, the other cases are still well treated by analogy; then we abbreviate "left equivalent" to the term "equivalent" merely, unless otherwise specified.

Definition 2. If $\varphi\in\tilde{E}_0$ is equivalent to no linear convex combination of ψ_1 and ψ_2, each of which is in $\tilde{E}_0$ and not equivalent to φ, then φ is said to be *weakly extreme* (or abbrev. *w. extr.*) *in* $\tilde{E}_0$.

From the definition itself, it follows immediately:

i) $\varphi\sim\psi$ implies $\varphi\sim(\alpha\varphi+\beta\varphi)$ for $\alpha,\beta>0$ with $\alpha+\beta=1$,

ii) if φ is *w. extr.* in $\tilde{E}_0$ and $\varphi\sim\psi$, then ψ is also *w. extr.* in it,

iii) $'\mathring{I}_\varphi\cap{}'\mathring{I}_\psi={}'\mathring{I}_{\alpha\varphi+\beta\psi}$ for $\alpha,\beta>0$ with $\alpha+\beta=1$.

It should be noticed that the collection of all mutually equivalent elements in $\tilde{E}_0$ is not always w*-closed: for a counter-example, it is sufficient to consider $'\mathring{I}_\varphi$ and $'\mathring{I}_\psi$ such that $'\mathring{I}_\psi\subset{}'\mathring{I}_\varphi$ properly; indeed, $\alpha\varphi+\beta\psi\sim\varphi$ for $\alpha>0$ but $\alpha\varphi+\beta\psi\sim\psi$ for $\alpha=0$.

The theorem enlarging Proposition K^{bis} is:

Theorem 2. *For a semi-unitary B_*-algebra A_*, a necessary and sufficient condition $'H=\overline{A_*/'I_\varphi}$, $\varphi\in\tilde{E}_0$, would be irreducible is that φ is w. extr. in $\tilde{E}_0$; thus if $'I_\varphi$ is maximal, φ is w. extr. in $\tilde{E}_0$.*

Proof. Suppose first φ not to be *w. extr.* in $\tilde{E}_0$, then there exists such $\varphi_0\in\tilde{E}_0$ that $\varphi\sim\varphi_0$ and $\varphi_0=\frac{1}{2}(\varphi_1+\varphi_2)$, where $'I\varphi_1\neq{}'I\varphi_2$, $\varphi_1,\varphi_2\in E_0$. By iii) above, we have $'I\varphi_1\cap{}'I\varphi_2={}'I\varphi_0$. Denoting the completion of $M^0\equiv{}'I\varphi_1/'I\varphi_0$ by M, we see that M is a closed proper sub-space of $'H_\varphi={}'H\varphi_0$, which is evidently invariant under $\{L_a\}$; in fact, if M^0 were dense in $'H_\varphi$, there would exist for each given $\varepsilon>0$ an element X_ε lying in M^0 such that $\|X_{v^\lambda}-X_\varepsilon\|_\varphi<\varepsilon$, v^λ being an arbitrarily fixed element of the approximate identity $\{v^\lambda\}$, then since $\|\dot{X}_\varepsilon\|_{\varphi_1}=0$ by the assumption, it holds

$$\varphi_1(v^{\lambda*}v^{\lambda})^{1/2} = \| \dot{X}_{v^\lambda} \|_{\varphi_1} = | \, \| \dot{X}_{v^\lambda} \|_{\varphi_1} - \| \dot{X}_\varepsilon \|_{\varphi_1} | \leqq \| \dot{X}_{v^\lambda} - \dot{X}_\varepsilon \|_{\varphi_1} \leqq \sqrt{2} \cdot \| X_{v^\lambda} - X_\varepsilon \|_\varphi < \sqrt{2}\, \varepsilon ,$$

and hence $\varphi_1(v^{\lambda*}v^{\lambda}) \to 0$ along with $\varepsilon \to 0$, so that every v^λ would be contained in $'I_{\varphi_1}$, where $\dot{X}$ means the corresponding element of $\overline{A_*/'I_{\varphi_1}}$ to $X \in {'H}$. This is absurd and hence M is proper, or in other words, $'H_\varphi$ is reducible, which completes the proof of necessity.

To prove the sufficiency, we employ the orthogonal decomposition of $'H_\varphi$: assume that $H={'H_\varphi}$ is reducible, then $H=M_1 \oplus M_2$, both of which are invariant under $\{L_a\}$, whence each X of H is also decomposed into X^1+X^2 for $X^i \in M_i (i=1, 2)$.

Then putting $\varphi_i(b^*a) = (X^i_a, X^i_b)_\varphi$, both φ_1 and φ_2 are well defined on A_* by virtue of the approyimate identity; in fact,

$$(X^i_{v^\lambda}, X^i_{a^*b})_\varphi = (X^i_{v^\lambda}, T_{a^*}X^i_b)_\varphi = (X^i_{v^\lambda}, T^*_a X^i_b)_\varphi = (T_a X^i_{v^\lambda}, X^i_b)_\varphi = (X^i_{av^\lambda}, X^i_b)_\varphi ,$$

so that $\varphi^i(b^*a) = \lim\limits_\lambda \varphi^i(b^*av^\lambda) = (X^i_a, X^i_b)_\varphi$ and hence φ_i is uniquely determined for $i=1, 2$. Clearly, $\hat{\varphi}_1$ and φ_2 are in $\hat{\Pi}(A_*)$ and, from $(X, Y)_\varphi = (X^1, X^1)_\varphi + (X^2, X^2)_\varphi$, it follows

$$\varphi = \tilde{\varphi}_1 + \tilde{\varphi}_2 . \tag{4.3}$$

As is easily seen, both $\| \tilde{\varphi}_1 \|$ and $\| \tilde{\varphi}_2 \|$ are $\leqq \| \varphi \| \leqq 1$ and hence setting

$$\varphi_0 = \alpha\tilde{\varphi}_1 + \beta\tilde{\varphi}_2 \quad \text{for } \alpha, \beta > 0 \text{ with } \alpha+\beta = 1 , \tag{4.4}$$

we see that $\varphi \sim \varphi_0$ and $\varphi_0 \in \tilde{E}_0$, since $\| \varphi_0 \| \leqq \alpha + \beta = 1$. This completes the proof of Theorem 2 entirely.

5. Extremity and irreducibility, cont. One may make precise of Theorem 2 in some special cases: the first is the case when A_* is semi-unitary again and, moreover, satisfies the following two conditions;

α) $\| a^*a \| \leqq \| a \|^2$ for every $a \in A_*$,

β) $\| v^\lambda \| = 1$ for each element of the approximate identity $\{v^\lambda\}$.

The condition α) is fulfiled in the case of a B*–algebra with stronger norm condition $\| a^*a \| = \| a \|^2$ (*e.g.* C*–algebra) or of the group-algebra $L(G)$ on a locally compact group G, while the latter β) is so in the case of every unitary B–algebra or $L(G)$ again. The algebra discussed in Proposition K and $\mathrm{K}^{\mathrm{bis}}$ is also the one which satisfies these two conditions, so that the following Theorem 3 is applicable to it.

In such cases, each element $X_\lambda = X_{v^\lambda}$, being contained in $A_*/'I_\varphi$ for a fixed $\varphi \in \tilde{E}_0$, is bounded and $(X_a, X_\lambda)_\varphi = \hat{\varphi}(v^{\lambda*}a)$ converges to $\hat{\varphi}(a)$; since X_a, $a \in A_*$, is dense in $'H_\varphi$, we see consequently that $\{X_\lambda\}$ converges weakly to a certain element X_φ in $'H_\varphi$.

Proposition 10. *Under the above conditions* α) *and* β), *it holds that* $\|\varphi\| = \|\tilde{\varphi}\|$ *for* $\varphi \in \hat{\Pi}(A_*)$ *with restricted* $\tilde{\varphi} \in \tilde{\Pi}(H(A_*))$.

In fact, by Lemma 1 we have

$$|\varphi(a^*v^\lambda)|^2 \leqq \varphi(a^*a)\varphi(v^{\lambda*}v^\lambda) \leqq \varphi(a^*a)\,\|\varphi\|,$$

from which $|\varphi(a)|^2 \geqq \tilde{\varphi}(a^*a)\,\|\varphi\|$ and hence, for $a \in A_*$ with $\|a\| \geqq 1$, $\|\varphi\|^2 \leqq \tilde{\varphi}(a^*a)\,\|\varphi\| \leqq \|\tilde{\varphi}\| \cdot \|\varphi\|$, *i.e.* $\|\varphi\| \leqq \|\tilde{\varphi}\|$. On the other hand, it is clear that $\|\tilde{\varphi}\| \leqq \|\varphi\|$, which proves the Proposition.

This shows that, in such cases, it holds

$$\tilde{E}_0 \approx E \cap \hat{\Pi}(A_*)\,, \tag{5.1}$$

$$S(\tilde{E}_0) \approx S(\hat{E})\,, \tag{5.2}$$

where the sign $\approx$ designates the same equivalency as is mentioned in Theorem 1, and $S(\tilde{E}_0)$ means the surface of $\tilde{E}_0$, that is, $=S(E) \cap \tilde{\Pi}(H(A_*))$, while $S(\hat{E}) = S(E) \cap \hat{\Pi}(A_*)$; about $S(E)$, refer also to the outset of this Chapter.

Now, we define the subset $\tilde{\tilde{E}}_0$ of $\tilde{E}_0$ as the collection of such functionals lying in $\tilde{E}_0$ that $(X_\varphi, X_\varphi)_\varphi = 1$. We call $\tilde{\tilde{E}}_0$ *normalized positiue boule.*

Proposition 11. *The following three conditions are mutually equivalent*;

i) $\varphi \in \tilde{\tilde{E}}_0$,

ii) $\hat{\varphi} \in \hat{\Pi}(A_*)$ *and* $(X_{\hat{\varphi}}, X_{\hat{\varphi}})_{\hat{\varphi}} = 1$ *for* φ*'s extended* $\hat{\varphi}$,

iii) $\varphi \in S(\tilde{E}_0)$ *or its extended* $\hat{\varphi} \in S(\hat{E})$, *owing to* (5.2).

Proof. We shall prove this in rotation; iii) $\leftarrow$ i) $\leftarrow$ ii) $\leftarrow$ iii). At first, i) $\rightarrow$ iii). Since $\hat{\varphi}(v^\lambda) = (X_{\hat{\varphi}}, X_\lambda)_{\hat{\varphi}} \rightarrow (X_{\hat{\varphi}}, X_{\hat{\varphi}})_{\hat{\varphi}} = 1$ and $|\hat{\varphi}(v^\lambda)| \leqq \|\hat{\varphi}\| \leqq 1$ for $\varphi \in \tilde{\tilde{E}}_0$, it follows $1 \leqq \|\hat{\varphi}\| \leqq 1$, that is, $\|\hat{\varphi}\| = 1$ and hence $\hat{\varphi} \in S(\hat{E})$ or equivalently $\varphi \in S(\tilde{E}_0)$.

ii) $\rightarrow$ i). For every a with $\|a\| \leqq 1$, we have by Schwarz's inequality that

$$|\hat{\varphi}(a)|^2 = |(X_a, X_{\hat{\varphi}})_{\hat{\varphi}}|^2 \leqq (X_{\hat{\varphi}}, X_{\hat{\varphi}})_{\hat{\varphi}} \cdot (X_a, X_a)_{\hat{\varphi}} = \hat{\varphi}(a^*a) \leqq \|\hat{\varphi}\|$$

from which $\|\hat{\varphi}\|^2 \leqq \|\hat{\varphi}\|$ and so $\|\hat{\varphi}\| \leqq 1$. This shows that $\varphi \in \tilde{\tilde{E}}_0$.

iii) $\rightarrow$ ii). Clearly $|(X_\lambda, X_{\hat{\varphi}})_{\hat{\varphi}}| = |\hat{\varphi}(v^\lambda)| \leqq \|\varphi\| = 1$, from which $(X_{\hat{\varphi}}, X_{\hat{\varphi}})_{\hat{\varphi}} = \alpha \leqq 1$: suppose now $\alpha < 1$, then putting $\hat{\varphi}/\alpha = \hat{\varphi}_0$, we see immediately $'I_{\hat{\varphi}_0} = 'I_{\hat{\varphi}}$, $X_{\hat{\varphi}_0} = X_{\hat{\varphi}}$ and $\hat{\varphi}_0 \in \hat{\Pi}(A_*)$, then ii) and hence successively i) and iii) are valid for such $\hat{\varphi}_0$, so that $\|\hat{\varphi}_0\| = 1$ and $\|\hat{\varphi}\| = \alpha < 1$, which is contradictory with the assumption $\hat{\varphi} \in S(E)$. Hence $(X_\varphi, X_\varphi)_\varphi = 1$, which make sure of ii).

Corollary P. 11. $\|\varphi\| = \|\hat{\varphi}\| = (X_{\hat{\varphi}}, X_{\hat{\varphi}})_{\hat{\varphi}}$.

Thus we conclude that $\tilde{\tilde{E}}_0$ is nothing but a w*-closed convex subset of $\hat{\Pi}(H(A_*))$ with the property $(X_\varphi, X_\varphi)_\varphi = 1$, which coincides with the surface $S(\tilde{E}_0)$ of $\tilde{E}_0$ entirely; indeed, the fact that $\tilde{\tilde{E}}_0$ is w*-closed and convex is easily verified. These make us confirm the

Theorem 3. *Provided that* A_* *satisfies the conditions* α) *and* β) *above,* $'H_\varphi$ *is irreducible if and only if* φ *is an extreme point of* $\tilde{\tilde{E}}_0$.

Proof. Assume that $'H_\varphi$ is reducible, then in the same manner as in the proof of Theorem 2, we get two functionals φ_1 and φ_2, *cf.* (4.3), both of which belong to $\hat{\Pi}(A_*)$. As is easily varified, $(X\varphi_i, X\varphi_i)\varphi_i=(X^i_\varphi, X^i_\varphi)_\varphi$ for $i=1, 2$, and we have

$$(5.3) \qquad (X\varphi_1, X\varphi_1)\varphi_1+(X\varphi_2, X\varphi_2)\varphi_2=(X_\varphi, X_\varphi)_\varphi=1:$$

putting $(X\varphi_i, X\varphi_i)\varphi_i=\alpha_i>0$ and $\hat{\varphi}_i/\alpha_i=\psi_i\in\hat{\Pi}(A_*)$ for $i=1, 2$, we see that $\|\psi_i\|=\|\hat{\varphi}_i\|/\alpha_i=(X\varphi_i, X\varphi_i)\varphi_i/\alpha_i=1$, so that $\tilde{\psi}_i\in\tilde{\tilde{E}}_0$ for $i=1, 2$. On the other hand, it holds

$\varphi=\alpha_1\tilde{\psi}_1+\alpha_2\tilde{\psi}_2$, where $\alpha_1+\alpha_2=1$ by (5.3), which shows φ being a midpoint of a segment in $\tilde{\tilde{E}}_0$. This proves the sufficiency.

Conversely, let $'H_\varphi$ be irreducible; if $\varphi=\frac{1}{2}(\varphi_1+\varphi_2)$ for suitable φ_1 and $\varphi_2\in\tilde{\tilde{E}}_0$, then $\varphi_0=\frac{1}{2}\varphi_1$ is in $\tilde{\tilde{E}}_0$ and $(X_a, X_a)\varphi_0=\varphi_0(a^*a)$ $(\leqq(X_a, X_a)_\varphi$ for every $a\in A_*)$ defines an hermitian form in $'H_\varphi$ and hence an hermitian operator A such that

$$(AX, X)_\varphi=(X, X)\varphi_0 \qquad \text{for all} \quad X\in {'H_\varphi}.$$

As is easily seen, A is commutative with every L_a and, since $'H_\varphi$ is irreducible, it must be in the form $(AX, X)_\varphi=\alpha(X, X)_\varphi$ for a fixed number α; from $\varphi, \varphi_1\in\tilde{\tilde{E}}_0$, it follows $\alpha=\frac{1}{2}$ and so $\varphi_0=\varphi_1/2=\varphi_2/2$, that is, $\varphi_1=\varphi_2=2\varphi$. This shows φ is extreme in $\tilde{\tilde{E}}_0$.

Thus, Theorem 3 is completely proved.

Summerizing all these arguments, we can determine the complete structure of $\tilde{E}_0$:

Theorem 4. *If A_* satisfies the condition α) and β), the set of extreme points, extr. $\tilde{E}_0$, consists of that of $\tilde{\tilde{E}}_0$, extr. $\tilde{\tilde{E}}_0$, and the origin (zero functional) θ. Every w. extr. φ of $\tilde{E}_0$ is either extreme of it or a inner point of a segment combining the origin θ with an element of extr. $\tilde{\tilde{E}}_0$, φ_0 say, i.e.*

$$(5.4) \qquad \varphi=\alpha\varphi_0 \qquad \textit{for} \quad 0<\alpha<1.$$

In fact, an exreme φ_0 is either of norm 1 or identical with θ, since φ_0 with $\|\varphi_0\|<1$, >0 is necessarily a midpoint of a segment. Let φ be *w. extr.* in $\tilde{E}_0$ and $\|\varphi\|\neq 0$, then $\varphi_0=\varphi/\|\varphi\|$ lies on $\tilde{\tilde{E}}_0$ and $\overline{A_*/'I_\varphi}$ is irreducible by Theorem 2; φ_0 is hence extreme by Theorem 3, $\varphi=\alpha\varphi_0$ for $\alpha=\|\varphi_0\|$ which is >0. If $\|\varphi\|=1$, φ itself is extreme. Thus Theorem 4 is completely proved.

Next we shall define another notion: in our present considerations, we need not the assumptions α) and β), which are assumed in the preceding arguments for A_*.

Assume that an element φ of $\Gamma(H(A_*))$ has the following properties;

1°) $'I_\varphi(I'_\varphi, I_\varphi)$ is a closed regular ideal,

2°) $\hat{\varphi}(j)=1$ for an identity j modulo the corresponding ideal cited in 1°) above.

Then φ (or $\hat{\varphi}$) is called *left* (*right*, *tow-sided*) *regular*; there need however only two regularities of φ, *one-sided and two-sided*, since φ is also right regular, having j^* modulo $I'_\varphi=('I_\varphi)^*$, for each left regular φ.

The set intersection of $\tilde{E}_0$ and of all one-sided (or two-sided) regular functionals is denoted by $\dot{E}_0$ (or resp. $\mathring{E}_0$), which is evidently w*-closed and convex. $\dot{E}_0(\mathring{E}_0)$ is called *one-sided (tow-sided) regular positive boule.*

Lemma 4. *Let $\varphi \in \tilde{\Pi}(H(A_*))$ be one-sided regular; then it holds that $\varphi(a)^2 \leqq \varphi(a^2)$ and hence $\|\varphi\| \leqq 1$, where $a \in H(A_*)$.*

Indeed, in the inequality (2.4) we have only to put $b=j$ and get $\varphi(a)^2=\hat{\varphi}(a)^2 =\hat{\varphi}(aj)^2 \leqq \hat{\varphi}(a^2)\hat{\varphi}(j^*j)=\hat{\varphi}(a^2)=\varphi(a^2)$ for $a \in H(A_*)$, since $\hat{\varphi}(j^*j)=\hat{\varphi}(j)=1$. The proof of the latter is immediate, hence shall be omitted. Then we assert

Proposition 12. *The following two conditions are mutually equivalent*; i) $\varphi \in \dot{E}_0(\mathring{E}_0)$, ii) $\varphi \in \tilde{\Pi}(H(A_*))$ *and it is one-sided (two-sidbd) regular.*

Theorem 5. *For $\varphi \in \dot{E}_0(\mathring{E}_0)$, if $H_\varphi=\overline{A_*/'I_\varphi}(\overline{A_*/I_\varphi})$ is reducible, then there exists a segment of $\dot{E}_0(\mathring{E}_0)$ just in which φ is an inner point.*

Proof. As is easily verified, an identity j modulo $'I_\varphi$ is also the one both modulo $'I_{\varphi_1}$ and modulo $'I_{\varphi_2}$, where φ_1 and φ_2 are just the same as in (4.3), *i.e.* $\varphi=\varphi_1+\varphi_2$. As $\varphi_i(j)=\varphi_i(j^*j) \geqq 0$ and $\hat{\varphi}(j)=\varphi_1(j)+\varphi_2(j)=1$, we see that $1 \geqq \varphi_i(j) > 0$ for $i=1, 2$; putting $\alpha_i=\varphi_i(j)$ and $\dot{\varphi}_i=\tilde{\varphi}_i/\alpha_i$, we see that

$$\varphi(\cdot) = \alpha_1\dot{\varphi}_1(\cdot)+\alpha_2\dot{\varphi}_2(\cdot)\,. \tag{5.5}$$

Since $\hat{\dot{\varphi}}_i(j)=1$ and $'I_{\varphi_i}='I_{\dot{\varphi}_i}$, $\dot{\varphi}_i$ is one-sided regular and clearly $\dot{\varphi}_i \in \tilde{\Pi}(H(A_*))$, from which results $\dot{\varphi}_i \in \dot{E}_0$ by Proposition 12. This shows that φ is an inner point.

Corollary T. 5. 1. *For an extreme φ in $\dot{E}_0$ (or $\mathring{E}_0$), each of $\overline{A_*/'I_\varphi}$ and $\overline{A_*/I'_\varphi}$ (or resp. $\overline{A_*/I_\varphi}$) is irreducible.*

Corollary T. 5. 2. *Every extreme point of $\dot{E}_0$ (or, if the algebra is unitary, of $\tilde{E}_0$) is w. exter. in $\tilde{E}_0$.*

As in Proposition 7, §2, every φ in $\Phi^\circ(A_*)$ causes a maximal two-sided regular ideal I_φ and clearly $\varphi(j)=1$ for an identity j modulo I_φ. Moreover by the multiplicativity of φ, it must be $\|\varphi\| \leqq 1$ (if A_* is unitary, $=1$). All these make us confident that

$$\tilde{\Phi}(H(A_*)) \subset \mathring{E}_0\,. \tag{5.6}$$

More precisely we have

Theorem 6. *Every φ in $\tilde{\Phi}^0(H(A_*))$ is an extreme point of $\tilde{E}_0$ and hence that of $\dot{E}_0$ and of $\mathring{E}_0$.*

Proof. Suppose that $\varphi=\frac{1}{2}(\varphi_1+\varphi_2)$ for $\varphi_1, \varphi_2 \in \tilde{E}_0$: we have $\varphi(a^2)=\frac{1}{2}[\varphi_1(a^2) +\varphi_2(a^2)]=(\varphi(a))^2=\frac{1}{4}[\varphi_1(a)^2+2\cdot\varphi_1(a)\cdot\varphi_2(a)+\varphi_2(a)^2]$; therefore it follows that

$$0 = 4[\varphi(a^2)-\varphi(a)^2] = [\varphi_1(a^2)-\varphi_1(a)^2]+[\varphi_2(a^2)-\varphi_2(a)^2]$$
$$+ [\varphi_1(a^2)-2\varphi_1(a)\varphi_2(a)+\varphi_2(a^2)] \geqq \varphi_1(a^2)-2\varphi_1(a)\varphi_2(a)+\varphi_2(a^2)$$
$$\geqq [\varphi_1(a)-\varphi_2(a)]^2 \geqq 0 \quad \text{(owing to Lemma 3).}$$

Consequently, $\varphi_1(a)=\varphi_2(a)$ and so φ is never a midpoint of any segment in $\tilde{E}_0$. This completes the proof. This skilful way of proving Theorem 6 is essentially due to R. V. Kadison [3], Lemma 3.1.

6. Summerizing and group algebra. We now summerize the considerations of the preceding paragraph and realize those results to the special case of group-algebra on a locally compact group.

First we present a convenient proposition :

Proposition 13. *If A_* is semi-unitary, satisfying the conditions α) and β), then it holds for each $\varphi \in \dot{E}_0$ (or $\mathring{E}_0$) $X_\varphi = X_j$ as elements of $'H_\varphi$ (resp. H_φ), where j is an identity modulo $'I_\varphi$ (resp. I_φ). Therefore, $\mathring{E}_0 \subset \dot{E}_0 \subset \tilde{\tilde{\tilde{E}}}_0$.*

In fact, we have $(X_a, X_{\hat{\varphi}})_{\hat{\varphi}} = \hat{\varphi}(a) = \hat{\varphi}(aj) = (X_a, X_j)_{\hat{\varphi}}$ for $\varphi \in \dot{E}_0$ and for every $a \in A_*$, so that $X_\varphi = X_{\hat{\varphi}} = X_j$; since $(X_j, X_j)_{\hat{\varphi}} = \hat{\varphi}(j^*j) = \hat{\varphi}(j) = 1$, we obtain $(X_{\hat{\varphi}}, X_{\hat{\varphi}})_{\hat{\varphi}} = 1$ and so $\varphi \in \tilde{\tilde{\tilde{E}}}_0$, *i.e.* $\dot{E}_0 \subset \tilde{\tilde{\tilde{E}}}_0$.

Then, in the case of such A_* as is semi-unitary and satisfies α) and β), we establish the following "*inclusion schema,*" in which we write $\mathring{V}_0$, $\dot{V}_0$, $\tilde{\tilde{\tilde{V}}}_0$ or $\tilde{V}_0$ for *extr.* $\mathring{E}_0$, *extr.* $\dot{E}_0$, *extr.* $\tilde{\tilde{\tilde{E}}}_0$ *extr.* $\tilde{E}_0$ respectively and "$X \rightarrow Y$" reads "X is the set of all extreme points of Y":

$$\text{(6.1)} \qquad \begin{array}{ccccccc} \tilde{\Phi}^0(H(A_*)) \subset \mathring{V}_0 & \subset & \dot{V}_0 & \subset & \tilde{\tilde{\tilde{V}}}_0 & & \subset \tilde{V}_0 = \theta \cup \tilde{\tilde{\tilde{V}}}_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathring{E}_0 & \subset & \dot{E}_0 & \subset & \tilde{\tilde{\tilde{E}}}_0 = S(\tilde{E}_0) & & \subset \tilde{E}_0 \ , \end{array}$$

and if A_* is *unitary,*

$$\text{(6.2)} \qquad \begin{array}{ccccccc} \tilde{\Phi}^0(H(A_*)) \subset \mathring{V}_0 & = & \dot{V}_0 & = & \tilde{\tilde{\tilde{V}}}_0 & & \subset \tilde{V}_0 = \theta \cup \tilde{\tilde{\tilde{V}}}_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathring{E}_0 & = & \dot{E}_0 & = & \tilde{\tilde{\tilde{E}}}_0 = S(\tilde{E}_0) & & \subset \tilde{E}_0 \ . \end{array}$$

Moreover, if A_* is *commutative,* then

$$\text{(6.3)} \qquad \tilde{\Phi}^0(H(A_*)) = \mathring{V}_0 = \dot{V}_0 = \tilde{\tilde{\tilde{V}}}_0 = \textit{extr. } S(\tilde{E}_0) \, .$$

When A_* is *unitary* in addition, (6.3) comes from Proposition 3 and Theorem 3 as is mentioned before.

In non-unitary case, A_* can be immedded as a maximal ideal of a unitary $\bar{A}_* \equiv \{(\alpha, a)\,;\ \alpha \in K,\ a \in A_*\}$, having the product, norm, involution as follows; $(\alpha, a)(\beta, b) = (\alpha\beta, a+\alpha b+ab)$, $\|(\alpha, a)\| = |\alpha| + \|a\|$, $(\alpha, a)^* = (\bar{\alpha}, a^*)$. Putting $\bar{\varphi}(\alpha, a) = \alpha + \varphi(a)$ for $\varphi \in S(\hat{E})$, it is easily seen that $\bar{\varphi} \in \hat{E}_1 \equiv$ the unit sphere of $\Gamma'(\bar{A}_*) \cap \hat{\Pi}(\bar{A}_*)$ and moreover $\|\bar{\varphi}\| = 1$, that is, $\bar{\varphi} \in S(\hat{E}_1)$, since $\|\varphi\| \leqq \|\bar{\varphi}\|$, $\bar{\varphi}(1, 0) = 1$ and

$$\bar{\varphi}((\alpha, a)^*(\alpha, a)) = |\alpha|^2+2\Re(\bar{\alpha}\varphi(a))+\varphi(a^*a) \geqq 0\,.$$ [6]

Conversely, putting $\varphi(a)=\bar{\varphi}(0, a)$ for $\bar{\varphi}\in S(\hat{E}_1)$, we see that $\varphi\in\hat{E}$ and $\bar{\varphi}\neq\bar{\psi}$ for $\bar{\varphi}, \bar{\psi}\in S(\hat{E}_1)$, implies $\varphi\neq\psi$ because $\bar{\varphi}(1, 0)=\bar{\psi}(1, 0)=1$. Then, suppose that $\bar{\varphi}=\frac{1}{2}(\bar{\varphi}_1+\bar{\varphi}_2)$ for $\tilde{\varphi}\in extr.\ S(\tilde{E}_0)$ and $\bar{\varphi}_1, \bar{\varphi}_2\in S(\hat{E}_1)$ with $\bar{\varphi}_1\neq\bar{\varphi}_2$, thein it follows that $\tilde{\varphi}=\frac{1}{2}(\tilde{\varphi}_1+\tilde{\varphi}_2)$ with $\tilde{\varphi}_2\neq\tilde{\varphi}_2$, both of which lying in $\tilde{E}_0$, so that $\tilde{\varphi}$ cannot be extreme in $\tilde{E}_0$ and so in $S(\tilde{E})$, which is impossible. Therefore $\varphi\in extr.\ S(E_0)$ implies $\bar{\varphi}\in extr.\ S(\tilde{E}_1)$ and, by above argument concerning about unitary A_*, it concludes that $\bar{\varphi}\in\tilde{\Phi}^0(H(\bar{A}_*))$, so that $\varphi\in\tilde{\Phi}^0(H(A_*))$. This proves (6.3) completely.

Proposition 14. *If* $\varphi\in S(\tilde{E}_0)$ *and* $A_*/'I_\varphi$ *(or* A_*/I'_φ*) is finite dimensional as a quotient Banach space, then* φ *is in* $\dot{E}_0$.

If the assumption is held, owing to a noted theorem of Riesz, the unit sphere of $A_*/'I_\varphi$ (or A_*/I'_φ) is compact and hence a suitable sub-family $\{X_\mu\}$ of $\{X_{v\lambda}\}$, which is entirely contained in the unit sphere, converges to a certain X_0 for which $X_aX_\mu \to X_aX_0=X_a$ (resp. $X_\mu X_a\to X_0X_a=X_a$), so that each element $u\in X_0$ is a let (resp. right) identity modulo $'I_\varphi$ (resp. I'_φ).

Now we know that there exist two intersting and important objects to which we could realize the foregoing considerations; one is the C*-algebra, in which $\varphi\in S(\tilde{E}_0)$ is called a "*state*", while $\varphi\in extr.\ S(\tilde{E}_0)$ a "*pure*" one[7], and another is the group algebra, in which every $\varphi\in S(\tilde{E}_0)$ corresponds to a continuous positive definite function, while $\varphi\in extr.\ S(\tilde{E}_0)$ to a "*elementary*" one[8].

The following is devoted exclusively to study of the latter, *i.e.* the group algebra. Let G be a locally compact (LC) group with the unit e, and $L(G)$ the group algebra of G with respect to the left-invariant Haar mesure dx, with elements $f, g, \cdots\cdots$ and the approximate identity $\{v^\lambda\}$.

$L(G)$ forms a B_*-algebra with respect to the involution

$$f^*(x) = \rho(x)\overline{f(x^{-1})}\,, \tag{6.4}$$

where $\rho(x)$ is the density of right-invariant Haar measure, $dx^{-1}=\rho(x)dx$.

In the case, every v^λ is helmitian and of norm 1 and $\|f^*\|=\|f\|$ for every $f\in L(G)$, so that $L(G)$ *satisfies the conditions* α) *and* β) *of* **§5.**

Lemma 5. *Putting* $f_x(\cdot)=f(x^{-1}\cdot)$ *(left translation), we have*

$$(f_x)^*\cdot g = f^*\cdot(g_{x-1})\,. \tag{6.5}$$

Proof. By a direct calculation as follows;

6) Consuliting the Cauchy-Schwarz's lemma (2.4), the second term $\geqq|\alpha|^2-2|\alpha|\cdot\sqrt{\varphi(a^*a)}+\varphi(a^*a)=(|\alpha|-\sqrt{\varphi(a^*a)})^2\geqq 0$.

7) Cf. I. E. Segal [7].

$$\begin{aligned}(f_x)^*\cdot g(\cdot) &= \int_G \overline{f_x(y^{-1})}\,\rho(y)\,g(y^{-1}\cdot)\,dy \\ &= \int_G \overline{f(x^{-1}y^{-1})}\,\rho(y)\,g(y^{-1}\cdot)\,dy = \int_G \overline{f(z^{-1})}\,\rho(z)\,g(xz^{-1}\cdot)\,dz \\ &= \int_G f^*(z)\,g_{x^{-1}}(z^{-1}\cdot)\,dz = f^*\cdot(g_{x^{-1}})\,(\cdot)\,.\end{aligned}$$

We can arrange this Lemma in the following manner, using the operator U_x on the Hilbert space $'H \equiv \overline{L(G)/'I_\varphi}$, $\varphi \in \tilde{\tilde{E}}_0 = S(\tilde{E}_0)$, defined by $U_x X_f = X_{fx}$;

$$(U_x X_f,\, X_g)_\varphi = (X_f,\, U_{x^{-1}} X_g)_\varphi\,.$$

Thus, we see that U_x is unitary and $\{U_x;\, x \in G\}$ forms a *continuous unitary representation* of G on $'H_\varphi$. Then $(X, U_x X)_\varphi$ is a continuous positive definite (*c. p. d.*) function on G. Denoting the collection of all *c. p. d.* functions with norms less than 1 by $P(G)$ and that of those with norms $=1$ by $P_0(G)$, we see that $\xi_\varphi(x) = (X_\varphi, U_x X_\varphi)_\varphi$ for $\varphi \in \tilde{E}_0$ (or $\in \tilde{\tilde{E}}_0$) is an element of $P(G)$ (resp. of $P_0(G)$). Conversely, for each $\xi \in P(G)$ (or $\in P_0(G)$) the functional on $L(G)$ defined by

$$\varphi_\xi(f) = \int_G \overline{\xi(x)}\,f(x)\,dx\,. \tag{6.7}$$

is clearly contained in $\hat{E}$ (resp. in $S(\hat{E})$) and moreover

$$\xi(x) = (X_{\varphi_\xi},\, U_x X_{\varphi_\xi})_{\varphi_\xi}\,. \tag{6.8}$$

Thus we can establish:

Proposition 15.[8] *$\tilde{E}_0$ (or $\tilde{\tilde{E}}_0 = S(\tilde{E}_0)$) is one-to-one corresponding to $P(G)$ (resp. to $P_0(G)$) by $\varphi \to \xi_\varphi$ and $\xi \to \varphi_\xi$ as defined above, where*

i) $\varphi_{\xi_\varphi} = \varphi\,,\quad \xi_{\varphi_\xi} = \xi\,,$

ii) $\varphi_{\alpha\xi_1+\beta\xi_2} = \bar{\alpha}\varphi_{\xi_1} + \bar{\beta}\varphi_{\xi_2}\,,\quad \xi_{\alpha\varphi_1+\beta\varphi_2} = \bar{\alpha}\xi_{\varphi_1} + \bar{\beta}\xi_{\varphi_2}\,,$

iii) $\|\xi\| = \|\varphi_\xi\|\,,\quad \|\varphi\| = \|\xi_\varphi\|\,,$

iv) $\hat{\varphi}(f) = \int_G \overline{\xi_\varphi(x)}\,f(x)\,dx,\quad \hat{\varphi}_\xi(f) = \int_G \overline{\xi(x)}\,f(x)\,dx\,.$

In fact iii) comes from $\|\xi\| = \xi(e) = (X_{\varphi_\xi}, X_{\varphi_\xi})_{\lambda_\xi} = \|\varphi_\xi\|$ by Corollary P. 11.

Proposition 16. *A continuous positive definite function ξ is elementary if and only if $\varphi_\xi \in \tilde{\tilde{V}} =$ exter. $S(\tilde{E}_0)$.*

From (6.3) above, in the case where G is commutative (and so is $L(G)$), $\tilde{\tilde{V}}_0$ coincides with $\tilde{\Phi}^0(H(A_*))$, from which it follows immediately that ξ_φ, $\varphi \in$ *extr.* $S(\tilde{E}_0)$, is a continuous character of G and vice versa; $\tilde{\tilde{V}}_0 \cong \hat{G}$.

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8) See e.g. L. H. Loomis [5], R. Godement [3].

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