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Indefinite metric and its application to quantum mechanics

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§1. **Introduction.** Finite dimensional complex linear spaces with indefinite metrices will be constructed by using Weyl's method[1]. The triangular inequality does not hold valid in these spaces. Especially, in the Minkowski space, the straight line is longest among curves connecting two fixed points in the time-like region.

An n-dimensional linear space is metrizable by using an arbitrary positive definite or indefinite hermitian form as the metric fundamental form. The unitarity and the selfadjointness of an operator in an indefinite metric space must be defined in a proper way. The spectral theory of the selfadjoint operator in a finite or infinite dimensional space was extensively studied by Nevanlinna and others[2].

Some remarks on the group representation in a indefinite metric space will be given. The spinor representation of the general Lorentz group is one of the examples of such a representation. It seems that only the irreducible subspace of the spinor space plays an essential role in the beta-interaction. The success of the theory on the beta-interaction[3], which concludes that only vector and axial-vector couplings are effective, seems to support such a thought. An analogy of the spinor representation will be considered in spaces of pion wave functions.

§2. Introduction of Indefinite Metric Space.

1. An n-dimensional vector x is an ordered set $(x_1, x_2, \cdots, x_n)$ of complex numbers $x_1, x_2, \cdots, x_n$. The n-dimensional vector space is a linear set of n-dimensional vectors. Here the addition of two vectors and the multiplication of a vector by a complex number are defined as usual. In the n-dimensional vector space $\mathfrak{R}_n$, there are n linearly independent vectors called basis vectors such that every vector in $\mathfrak{R}_n$ is expressed as a linear combination of them. A linear transformation A in $\mathfrak{R}_n$ is represented by a matrix when a set of basis vector is introduced. When a set $\mathfrak{G}$ of linear transformations A_i $(i=1, 2, \cdots)$ has an invariant subspace in $\mathfrak{R}_n$, $\mathfrak{G}$ is said to be reducible. Here the invariant subspace must not coincide with $\mathfrak{R}_n$, nor with the space consisting of only the null vector. $\mathfrak{G}$ is said to be irreducible when it has no invariant subspace. Schur's lemma on the irreducibility of $\mathfrak{G}$ holds valid, irrespective of the nature of metric.

2. Let η be a non-singular hermitian matrix, and $\boldsymbol{\alpha}$ and $\boldsymbol{x}$ be two vectors. $L(\boldsymbol{x})=\boldsymbol{\alpha}^{+}\eta\boldsymbol{x}$ is a linear functional of $\boldsymbol{x}$. If $L(\boldsymbol{x})$ is invariant under a pair of linear transformations $\boldsymbol{\beta}=B\boldsymbol{\alpha}$ and $\boldsymbol{y}=A\boldsymbol{x}$, i.e., $\boldsymbol{\beta}^{+}\eta\boldsymbol{y}=\boldsymbol{\alpha}^{+}\eta\boldsymbol{x}$, the relation $B=\eta^{-1}(A^{+})^{-1}\eta$ must hold valid. A linear space P of vectors $\boldsymbol{\alpha}$ which are, corresponding to A, transformed by B, is called η-dual space of $\mathfrak{R}$, the space $\mathfrak{R}$ of $\boldsymbol{x}$ being conversely η-dual to P.

Let the dual space P coincide with $\mathfrak{R}$ itself, then we obtain the inner product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ belonging to the same space $\mathfrak{R}$:

$$(\boldsymbol{x}, \boldsymbol{y})_{\eta}=\boldsymbol{x}^{+}\eta\boldsymbol{y}\,, \qquad \boldsymbol{x}, \boldsymbol{y}\in\mathfrak{R}\,.$$

A transformation U which makes this inner product invariant is said to be η-unitary: $U^{*}U=1$, $U^{*}=\eta^{-1}U^{+}\eta$. The square of η-norm $\|\boldsymbol{x}\|_{\eta}^{2}$ is defined by $(\boldsymbol{x}, \boldsymbol{x})_{\eta}$. The length of a vector is measured by using the surface $\|\boldsymbol{x}\|^{2}=\pm1$ as the "Eichfläche". The following theorem is then obtained, which was stated in a geometrical language by Nevanlinna in his first paper[2].

Theorem. The end points of two vectors ψ_1 and ψ_2 are on the Eichfläche $\|\psi\|_{\eta}^{2}=1$ or $\|\psi\|_{\eta}^{2}=-1$ simultaneously. The end point of the vector $\psi_3=\lambda\psi_1+\mu\psi_2$ is on the surface $\|\psi\|_{\eta}^{2}=1$ or $\|\psi\|_{\eta}^{2}=-1$. Then the end point of the vector $\psi_4=\bar{\mu}\psi_1+\bar{\lambda}\psi_2$ is on the same surface as in the case of ψ_3. Here λ and μ are arbitrary complex numbers.

Proof.

$$\|\psi_3\|_{\eta}^{2}=|\lambda|^{2}\|\psi_1\|_{\eta}^{2}+2\Re e(\bar{\lambda}\mu\psi_1^{+}\eta\psi_2)+|\mu|^{2}\|\psi_2\|_{\eta}^{2}\,,$$
$$\|\psi_4\|_{\eta}^{2}=|\mu|^{2}\|\psi_1\|_{\eta}^{2}+2\Re e(\bar{\lambda}\mu\psi_1^{+}\eta\psi_2)+|\lambda|^{2}\|\psi_2\|_{\eta}^{2}\,.$$

Subtracting both sides respectively, we obtain

$$\|\psi_3\|_{\eta}^{2}-\|\psi_4\|_{\eta}^{2}=(|\lambda|^{2}-|\mu|^{2})(\|\psi_1\|_{\eta}^{2}-\|\psi_2\|_{\eta}^{2})\,.$$

From the assumption, the right hand side vanishes, and we obtain

$$\|\psi_4\|_{\eta}^{2}=\|\psi_3\|_{\eta}^{2}=1 \qquad \text{or} \qquad -1\,.$$

§3. Inequalities of Schwartz and Minkowski.

1. As a special case, we assume that the matrix η determining the metric in an n-dimensional real vector space $\mathfrak{R}$ has the form

$$\eta=\begin{pmatrix} I_l & 0 \\ 0 & -I_m \end{pmatrix}, \tag{1}$$

where I_l and I_m are unit matrices of l- and m-dimensions, respectively, and $l+m=n$. Let arbitrary two vectors in $\mathfrak{R}$ be

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad \text{and} \qquad \psi'=\begin{pmatrix}\psi_1'\\ \psi_2'\end{pmatrix},$$

where ψ_1 and ψ_1' are l-dimensional, and ψ_2 and ψ_2' are m-dimensional vectors.

The norms and inner product of ψ and ψ' are given by

$$||\psi||_\eta^2 = ||\psi_1||^2 - ||\psi_2||^2, \qquad ||\psi'||_\eta^2 = ||\psi_1'||^2 - ||\psi_2'||^2,$$
$$(\psi, \psi')_\eta = \psi_1^+\psi_1' - \psi_2^+\psi_2',$$

where $||\psi||^2$ is the ordinary Euclidean norm of ψ. For simplicity, we put

$$||\psi_1|| = r, \quad ||\psi_2|| = s, \quad ||\psi_1'|| = r', \quad ||\psi_2'|| = s', \quad \angle\psi_1\psi_1' = \theta, \quad \angle\psi_2\psi_2' = \tau.$$

The difference D of the product $||\psi||_\eta^2||\psi'||_\eta^2$ and $(\psi, \psi')_\eta^2$ is then written as follows.

$$D = ||\psi||_\eta^2||\psi'||_\eta^2 - (\psi, \psi')_\eta^2 = (r^2-s^2)(r'^2-s'^2) - (rr'\cos\theta - ss'\cos\tau)^2$$
$$= r^2r'^2\sin^2\theta + s^2s'^2\sin^2\tau - \{(rs'-r's)^2 + 2rr'ss'(1-\cos\theta\cos\tau)\}.$$

Concerning the sign of D, the following three cases occur.

1. $l=0$ or $m=0$. In this case, we always have $D\geqq 0$. That is, ordinary Schwarz's inequality holds valid.

2. $l=1$, $m\geqq 1$ (or $m=1$, $l\geqq 1$). In this case, we have $\theta=0$ or π, so long as ψ_1 and ψ_1' are not null.

When $\theta=0$, D is written as $D=-s^2s'^2B$, where

$$B = (\cos\tau - zz')^2 - z^2z'^2 + z^2 + z'^2 - 1$$

and $z=r/s$ and $z'=r'/s'$. B becomes minimum for $\cos\tau=1$ when $zz'\geqq 1$, and we obtain

$$B_{\min} = (z-z')^2 \geqq 0.$$

That is, $D\leqq 0$, when $zz'\geqq 1$. When $zz'<1$, we have $B_{\min}=(z^2-1)(1-z'^2)$ whose sign being indefinite, so that the sign of D is indefinite.

For the case $\theta=\pi$, similar reasoning gives the same conclusion as above.

3. $l\geqq 2$, $m\geqq 2$. As seen from the expression

$$D = s^2s'^2\{(z^2-1)(z'^2-1) - (zz'\cos\theta - \cos\tau)^2\},$$

D becomes positive when $(z^2-1)(z'^2-1)>0$ and $zz'\cos\theta - \cos\tau = 0$, and negative for $(z^2-1)(z'^2-1)<0$. The sign of D is therefore not definite.

Definition. The domains in a complex vector space whose vectors satisfy

$$||\psi||_\eta^2 > 0, \qquad ||\psi||_\eta^2 = 0, \qquad \text{and} \qquad ||\psi||_\eta^2 < 0$$

are called the time-like, the light-like, and space-like domains, respectively.

It has been proved that the inequality

$$||\psi||_\eta^2||\psi'||_\eta^2 \leqq |(\psi, \psi')|_\eta^2 \tag{2}$$

holds valid when $l=1$ and both vectors ψ and ψ' are in the time-like or light-like region of a real vector space. This inequality holds valid also in a complex vector space when $l=1$. *Proof*: Let two vectors in the time-like or light-like region be $\psi=(t, x_1, x_2, \cdots)$ and $\psi'=(t', x_1', x_2', \cdots)$. From (2) we have

$$\|\psi\|_\eta^2\|\psi'\|_\eta^2 = (|t|^2-\sum|x_i|^2)(|t'|^2-\sum|x_i'|^2)$$
$$\leqq (|t|\,|t'|-\sum|x_i|\,|x_i'|)^2 .$$

From $|t|^2\geqq\sum|x_i|^2$ and $|t'|^2\geqq|x_i'|^2$, the inequality $|t|\,|t'|\geqq\sum|x_i|\,|x_i'|$ is obtained. The complex number $\bar{t}t'$ is on a circle of radius $|\bar{t}t'|$, and obviously the inequality $|\sum\bar{x}_ix_i'|\leqq\sum|x_i|\,|x_i'|$ holds valid. Accordingly the inequality

$$0\leqq|t|\,|t'|-\sum|\bar{x}_i|\,|x_i'|\leqq|\bar{t}t'-\sum\bar{x}_ix_i'|$$

is obtained, so that we have

$$\|\psi\|_\eta^2\|\psi'\|^2\leqq|\bar{t}t'-\sum\bar{x}_ix_i'|^2 = |(\psi, \psi')_\eta|^2$$

which proves (2) for a complex vector space.

We summarize the above results by the

Theorem. When the matrix η has the form (1) and $l=1$, the inequality (2) holds valid for time-like or light-like vectors ψ and ψ'. For $l\geqq2$ and $m\geqq2$ we have no definite inequality. For $l=0$ or $m=0$, ordinary Schwarz's inequality holds valid.

2. In a similar way, we obtain an inequality corresponding to that of Minkowski:

Theorem. η has the form (1) and $l=1$. ψ and ψ' are time-like or light-like vectors in a real vector space, and are simultaneously in the upper half or the lower half of the light-cone. Then the inequality

$$\|\psi+\psi'\|_\eta\geqq\|\psi\|_\eta+\|\psi'\|_\eta$$

holds valid. When $m=0$ in (1), ordinary Minkowski's inequality holds valid. Except for these cases, no definite inequality holds valid.

These circumstances make difficult the study of the topology of indefinite metric spaces. Though the triangular inequality in the ordinary sense does not hold valid, it should be noted[4)] that the principle of variation

$$\delta\int ds = 0$$

gives always the straight line.

§4. Relation between η-unitary and η-selfadjoint Matrices. The position vector ψ of a point in a complex vector space is assumed to move with time t according to the equation

$$\dot{\psi} = C\psi .$$

If the η-norm of ψ is invariant in the course of time, it is easily shown that $\sqrt{-1}\,C$ must be η-selfadjoint. That is, the above equation can be written in the form

$$i\frac{\partial\psi}{\partial t}=H\psi\,,\qquad H^*=H\,,\qquad H^*\equiv\eta^{-1}H^+\eta\,.$$

The solution of this equation is given by $\psi(t)=U(t)\psi_0$, where $U(t)=exp\,(-iHt)$ and ψ_0 is an arbitrary constant vector. If Q is an η-selfadjoint matrix and is independent of t, then $Q(t)=U^{-1}QU$ is η-selfadjoint and satisfies the equation

$$\frac{dQ(t)}{dt}=i[H,\,Q(t)]\,,$$

where $[A, B]=AB-BA$.

Though these equations have the same forms as those of the fundamental equations of motion in quantum mechanics, the implications of both cases are widely different. This is mainly due to the facts that the eigenvalues of an η-selfadjoint matrix are not always real, the absolute values of eigenvalues of an η-unitary matrix are not always equal to 1, and so on.

§5. **Remarks on Group Representation.** From now on, the matrix η used for the metrization is always assumed to satisfy $\eta^2=1$. Matrices β and γ are also used for the metrization and satisfy the same conditions as η, i.e., $\beta^+=\beta$ and $\beta^2=1$ etc. In the first place, we shall classify the matrices used in the metrization.

Lemma 1. When β satisfies the conditions $\beta^+=\beta$ and $\beta^2=1$, matrix $\gamma=U^{-1}\beta U$ satisfies the same conditions, where U is an arbitrary unitary matrix.

Lemma 2. U_1 and U_2 are arbitrary unitary matrices, and $\gamma_1=U_1^{-1}\beta U_1$ and $\gamma_2=U_2^{-1}\beta U_2$. Then $\gamma_2=V^{-1}\gamma_1 V$, where $V=U_1^{-1}U_2$ is unitary.

Selfadjoint matrices whose squares are equal to 1 are therefore classified according to the unitary equivalence, i.e., arbitrary two matrides γ_1, γ_2 of the same class are in the relation $\gamma_2=U^{-1}\gamma_1 U$, where U is a suitably chosen unitary matrix. The unit matrix makes a class by itself. The eigenvalues of the matrices are $+1$ or -1. When two matrices are in the same class, the multiplicities of $+1$ and -1 are respectively equal, and vice versa. So that we obtain the

Theorem. When and only when two matrices β and γ belong to the same class, the multiplicities of eigenvalues $+1$ and -1 are respectively equal, and the metric fundamental forms $\psi^+\beta\psi$ and $\phi^+\gamma\phi$ are unitary equivalent with each other.

As pointed out in the introduction, the group representation in an indefinite metric space is of great use. Let us here examine the β-unitary representation in a two dimentional space, where the eigenvalues of β are ±1. β is unitary equivalent to $\sigma=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ according to the above theorem, i.e., $\beta=U\sigma U^+$. Let u be a β-unitary matrix, then the β-unitarity condition $u^*u=1$ is written as

$$\beta = U\sigma U^+ = u^+\beta u = u^+ U\sigma U^+ u\,,$$

so that we have

$$w^+\sigma w = \sigma\,,$$

where $w=U^+uU$. The general expression of w is then obtained from the above equation:

$$w = \begin{pmatrix} \cosh\theta e^{i\theta'} & \sinh\theta e^{i(\theta'+\tau'-\theta'')} \\ \sinh\theta e^{i\theta''} & \cosh\theta e^{i\tau'} \end{pmatrix},$$

where θ, θ', θ'' and τ' are all real and $\theta \geqq 0$. The expression of u is obtained from that of w by the unitary transformation U.

The secular equation $|w-\lambda I|=0$ is written as

$$\lambda^2 - \cosh\theta(e^{i\theta'}+e^{i\tau'})\lambda + e^{i(\theta'+\tau')} = 0\,.$$

Let the two roots of this equation be $\lambda_1=r_1e^{i\alpha_1}$ and $\lambda_2=r_2e^{i\alpha_2}$. In order to get the eigenvalues whose absolute values are equal to 1, we put $\lambda=e^{i\alpha}$. Then the above equation becomes

$$\cos\left(\alpha - \frac{\theta'+\tau'}{2}\right) = \cosh\theta\cos\frac{1}{2}(\theta'-\tau')\,,$$

and the following four cases occur:

i) $\cosh\theta\cos\frac{\theta'-\tau'}{2}=1$, $\alpha_1=\alpha_2=\frac{\theta'+\tau'}{2}$. The eigenvalues are $e^{i\alpha_1}=e^{i\alpha_2}=e^{i\frac{1}{2}(\theta'+\tau')}$,

ii) $\cosh\theta\cos\frac{\theta'-\tau'}{2}=-1$, $\alpha_1=\alpha_2=\frac{\theta'+\tau'}{2}+\pi$.

The eigenvalues are $e^{i\alpha_1}={}^{i\alpha_2}=-e^{i\frac{1}{2}(\theta'+\tau')}$,

iii) $\left|\cosh\theta\cos\frac{\theta'-\tau'}{2}\right|<1$, $\alpha_1=\frac{\theta'+\tau'}{2}+\alpha_0$, $\alpha_2=\frac{\theta'+\tau'}{2}-\alpha_0$, where

$\alpha_0=\mathrm{Cos}^{-1}\left[\cosh\theta\cos\frac{\theta'-\tau'}{2}\right]$. The eigenvalues are $e^{i\alpha_1}$ and $e^{i\alpha_2}$.

iv) $\left|\cosh\theta\cos\frac{\theta'-\tau'}{2}\right|>1$. There is no real α, i.e., $r_1>1$ and $r_2<1$.

Case iv) means that the β-unitary matrices u^n are not always bounded, where n's are integers. Such a matrix u does not represent a group element of finite order.

It seems that it is not always possible to give an η-unitary representation of an arbitrary group. It is however possible to prove some theorems concerning the group representation. For example, the orthogonality relations[5] of the coefficients of irreducible η-unitary representations can be derived. Here the orthogonality is defined by our η-inner-product. The details will be given in a subsequent paper.

§6. Applications to Quantum Mechanics. One of the most important applications of the group representation in an indefinite metric space is the spinor representation of the Lorentz group. We shall give some remarks on this problem.

The most general homogeneous Lorentz transformation may be written as

$$x'^{\mu} = \sum_{\nu=0}^{3} a^{\mu}_{\nu} x^{\nu} , \tag{3}$$

where the coefficients a^{μ}_{ν} are real and satisfy the orthogonality condition

$$\sum a^{\nu}_{\mu} a^{\mu}_{\lambda} = \delta^{\nu}_{\lambda} = \sum a^{\nu\mu} a_{\lambda\mu} .$$

Let ψ be the electron wave function satisfying the Dirac equation. The ψ', which represents ψ in the new frame of reference, will be obtained from ψ by a linear tranformation

$$\psi' = S\psi .$$

The transformotion S is a 4×4 matrix and satisfies the condition[6]

$$SS^* = bI , \qquad S^* = \beta S^+ \beta ,$$

where β is the usual Dirac matrix. The constant b is equal to $+1$ or -1 according as the Lorentz transformation (3) does not or does reverse the direction of the time, respectively. As well known, the spinor representation of the proper Lorentz group is reducible. That is, if the representation

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

is used, the general expression of S is given by

$$S = \begin{pmatrix} A & 0 \\ 0 & (A^+)^{-1} \end{pmatrix} , \tag{5}$$

where A is an element of the special linear complex group $SL(2, C)$. Since the group $SL(2, C)$ is simply connected and non-compact, and has no normal subgroup except for the discrete subgroup $\{I, -I\}$, the group of S has the same properties.

Let, instead of (4), the representation of β be

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

and the ψ be represented as

$$\psi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} , \qquad \varphi_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \qquad \varphi_2 = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} , \tag{7}$$

then S has a form derived from (5) by a unitary transformation $V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$:

$$S = \frac{1}{2} \begin{pmatrix} A + A^{+^{-1}} & A - A^{+^{-1}} \\ A - A^{+^{-1}} & A + A^{+^{-1}} \end{pmatrix} , \tag{8}$$

and the invariant form is given by $\bar{\psi}\psi=\psi^{+}\beta\psi=|\psi_1|^2+|\psi_2|^2-|\psi_3|^2-|\psi_4|^2$.

Clearly, the invariant subspaces in the representation (4) are the sets of vectors of the forms

$$\begin{pmatrix}\varphi\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\\varphi\end{pmatrix} \tag{9}$$

which correspond to the vectors of the forms

$$\begin{pmatrix}\varphi\\\varphi\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\varphi\\-\varphi\end{pmatrix} \tag{10}$$

in the representation (6), respectively. Vectors (10) are obtained by operating $a=\frac{1}{2}(1-i\gamma_5)$ and $\bar{a}=\frac{1}{2}(1+i\gamma_5)$ on an arbitrary state vector, where $i\gamma_5=\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$. Feynman and Gell-Man[3] assumed that the beta-interaction term has the form

$$\sum C_i\,(\overline{a\psi_n}O_i a\psi_p)(\overline{a\psi_\nu}O_i a\psi_e)\,.$$

This assumption implies that the beta-interaction takes place only through one of the subspaces invariant with respect to the spinor representation of the proper Lorentz group.

Let us next examine the meson field. Kemmer[7] proposed a particle aspect of the meson theory. The wave function of the meson is given by

$$\left(-i\sum\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}+\kappa\right)\psi=0\,, \tag{11}$$

where γ^0 is hermitian and γ^k is anti-hermitian. In order that the equation is invariant under the Lorentz transformation, the linear transformation S defined by $\psi'=S\psi$ must satisfy

$$S^{-1}\gamma^{\lambda}S=\sum_{\lambda}a^{\lambda}_{\mu}\gamma^{\mu}\,, \tag{12}$$

the proof of which is carried out in the same way as in the electron theory[6]. Kemmer used the notations β_4, $i\beta_1$, $i\beta_2$, and $i\beta_3$ in places of γ^0, γ^1, γ^2 and γ^3, respectively.

The commutation rules for the operators β_μ are given by

$$\beta_\mu\beta_\nu\beta_\rho+\beta_\rho\beta_\nu\beta_\mu=\beta_\mu\delta_{\nu\rho}+\beta_\rho\delta_{\nu\mu}\,. \tag{13}$$

As shown by Kemmer, we have three inequivalent irreducible represntations of β_μ, whose dimensions being 1, 5 and 10. The 1-dimensional case is trival because of $\beta_1=\beta_2=\beta_3=\beta_4=(0)$. The 5- and 10-dimensional cases are considered here. The operator $\eta_4=2\beta_4^2-1$ plays a similar role as that of β in the electron theory. From (13), the relations $\eta_4\beta_k\eta_4=-\beta_k$, $k=1, 2, 3$ and $\eta_4\beta_4\eta_4=\beta_4$ are easily derived. By use of the notations γ, these relations are unified in the form

$$(\gamma^{\mu})^{+}=\eta_4\gamma^{\mu}\eta_4\,. \tag{15}$$

It follows from (12) and the reality of the a^{μ}_{γ}'s that

$$(\sum a^{\lambda}_{\nu}\gamma^{\nu})^{+} = (a^{\lambda}_{0}\gamma^{0}-\sum a^{\lambda k}\gamma^{k})^{+}$$
$$= a^{\lambda}_{0}\gamma^{0}+\sum a^{\lambda k}\gamma^{k} = (S^{-1}\gamma^{\lambda}S)^{+}. \tag{15}$$

Multiplying Eq. (15) by η_4 on the left and right, we obtain

$$\eta_4(a^{\lambda}_{0}\gamma^{0}+\sum a^{\lambda k}\gamma^{k})\eta_4 = a^{\lambda}_{0}\gamma^{0}+\sum a^{\lambda k}(\gamma^{k})^{+}$$
$$= a^{\lambda}_{0}\gamma^{0}-\sum a^{\lambda k}\gamma^{k} = \sum a^{\lambda}_{\nu}\gamma^{\nu} = S^{-1}\gamma^{\lambda}S\,.$$

On the other hand we have

$$\eta_4(a^{\lambda}_{0}\gamma^{0}+\sum a^{\lambda k}\gamma^{k})\eta_4 = \eta_4(a^{\lambda}_{0}\gamma^{0^{+}}-\sum a^{\lambda k}\gamma^{k^{+}})\eta_4$$
$$= \eta_4(a^{\lambda}_{0}\gamma^{0^{+}}+\sum a^{\lambda}_{k}\gamma^{k^{+}})\eta_4 = \eta_4(\sum a^{\lambda}_{\nu}\gamma^{\nu})^{+}\eta_4$$
$$= \eta_4(S^{-1}\gamma^{\lambda}S)^{+}\eta_4 = \eta_4 S^{+}\gamma^{\lambda^{+}}(S^{-1})^{+}\eta_4$$
$$= \eta_4 S^{+}\eta_4\gamma^{\lambda}\eta_4(S^{-1})^{+}\eta_4 = (\eta_4 S^{+}\eta_4)\gamma^{\lambda}(\eta_4 S^{+}\eta_4)^{-1}\,.$$

The above results give

$$(\eta_4 S^{+}\eta_4)\gamma^{\lambda}(\eta_4 S^{+}\eta_4)^{-1} = S^{-1}\gamma^{\lambda}S\,,$$

i.e.,

$$(S\eta_4 S^{+}\eta_4)\gamma^{\lambda}(S\eta_4 S^{+}\eta_4)^{-1} = \gamma^{\lambda}\,.$$

We put $S\eta_4 S^{+}\eta_4=A$, then A commute with all γ's. Owing to the irreducibility of Kemmer's representations, A must be a constant multiple of the unit matrix in each of the 5- and 10-dimensional spaces. We put $A=bI$. Since η_4 is hermitian, b must be real: $b^{*}=b$. Furthermore, we may prescribe a normalization for S such that $\det S=1$, in which case we must have $b^{5}=1$ and $b^{10}=1$, i.e., $b=1$ for the 5-dimensional case, and $b=\pm 1$ for the 10-dimensional case.

Consider the following relations:

$$T_r(S^{+}S) = T_r(b\eta_4 S^{-1}\eta_4 S) = Tr(b(2a^{0^2}_{0}\eta_4\gamma^{0}\gamma^{0}-\eta_4))$$
$$= \begin{cases} b(4a^{0^2}_{0}+1)>0 & \text{for 5-dimensional case} \\ b(12a^{0^2}_{0}-2)>0 & \text{for 10-dimensional case.} \end{cases}$$

Here we have used $Tr(\eta_4\gamma^{0}\gamma^{0})=1$ and $=6$ for the 5- and 10-dimensional cases, respectively. Since it is concluded that $b=1$ for both cases, it can be concluded that the time reversal of the Lorentz frame does not give any effect to the representation S, contrary to the case of electron.

The relation $A=I$, i.e., $S\eta^{4}S^{+}\eta_4=1$ means that the representation S of the Lorentz transformation is η_4-unitary: $S^{*}S=1$, $S^{*}=\eta_4 S^{+}\eta_4$.

If the idea that the particle-interaction takes place only through the subspaces invariant with respect to the spinor representations, is correct universally, then the pseudo-scalar coupling between the nucleon and the pseudo-scalar pion should vanish. This conclusion is, however, quite problematical at the present time. The strong coupling will have to be treated separately.

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