Journal of Mathematics,
Osaka City University,
Vol. 12, No. 1-2.

On the realizabilities of integral homology classes by orientable submanifolds

By Yoshihiro SHIKATA

(Received September 6, 1961)

The present paper is concerned with the problem: which integral homology class of a compact orientable differentiable manifold is realizable by a non-singularly imbedded submanifold? As was shown by Thom [5], this problem is reduced to the mapping problem of the manifold into the Thom complex MSO for the rotation group SO; he solved this problem for homology classes of dimensions≤ 7 provided the dimension of manifold is less than 17.

By making use of the results due to Milnor [3] and Wall [6], we shall in the present paper determine the stable homotopy type of MSO. Then this leads us to the following:

MAIN THEOREM. Let V be a compact orientable C^∞-manifold of dimension m, and let n be an integer such that $2n\geq m$ if n is even, $2n-1\geq m$ if n is odd. Then the homology class x_* dual to $x\in H^n(V; Z)$ is realisable by an orientable C^∞-submanifold of V if and only if

$$(*)\qquad \delta_p \mathcal{P}_p^1 \mathcal{P}_p^p \cdots \mathcal{P}_p^{p^q}(x) = 0$$

for any odd prime p and for any integer $q\geq 0$.

Since the Pontrjagin classes of V satisfy the condition (*), the homology classes dual to the Pontrjagin classes are realizable by orientable submanifolds under some restrictions on dimensionality.

By the analogous methods, we determine the stable homotopy type of MU and study the realizability by submanifolds whose normal bundles have unitary structures.

1. Preliminaries

For any prime p, we denote by A_p the Steenrod algebra mod p, and by δ_p, $\mathcal{P}_p$ the mod p Bockstein, the mod p Steenrod reduced power operation respectively.

According to Milnor [2, 3], there exist the elements

$$Q_i \in A_p, \qquad i = 0, 1, 2, \cdots$$

satisfying the following conditions (1. 1)-(1. 4):

$$(1.1)\qquad Q_0 = \delta_p, \quad Q_{i+1} = Q_i \mathcal{P}_p^{p^i} - \mathcal{P}_p^{p^i} Q_i,$$

(1.2) The subalgebra A_0 generated by $[Q_i,\ i=0, 1, 2, \cdots]$ in A_p is a Grassman algebra over Z_p.

(1.3) A_p is a free right A_0-module,

(1.4) The quotient algebra of A_p by the left A_p-ideal

$$A_pQ_0+A_pQ_1+\cdots$$

is isomorphic, as a left A_p-module, to the quotient algebra of A_p by the both sided ideal (Q_0) generated by Q_0 in A_p, i.e.

$$A_p/A_pQ_0+A_pQ_1+\cdots = A_p/(Q_0) .$$

From these conditions, we have

(1.5) Let $x\in H^*(X;\ Z_p)$ satisfy $Q_ix=0$ for $i<n$. Then

$$Q_nx = \delta_p\mathcal{P}_p^1 \cdots \mathcal{P}_p^{p^{n-1}}x .$$

Proof. By (1.1) we have

$$\begin{aligned} Q_nx &= Q_{n-1}\mathcal{P}_p^{p^{n-1}}x-\mathcal{P}_p^{p^{n-1}}Q_{n-1}x = Q_{n-1}\mathcal{P}_p^{p^{n-1}}x \\ &= Q_{n-2}\mathcal{P}_p^{p^{n-2}}\mathcal{P}_p^{p^{n-1}}x-\mathcal{P}_p^{p^{n-2}}Q_{n-2}\mathcal{P}_p^{p^{n-1}}x . \end{aligned}$$

Using a free base A^0 of the free right A_0-module A_p (see (1.3)), write

$$Q_{n-2}\mathcal{P}_p^{p^{n-1}} = \mathcal{P}^{R_0}Q_0+ \cdots +\mathcal{P}^{R_l}Q_l \quad (\mathcal{P}^{R_i}\in A^0) .$$

Then, since the degrees of $Q_{n-2}\mathcal{P}_p^{p^{n-2}}$ and Q_l are equal to $2p^n-2p^{n-1}+2p^{n-2}-1$ and $2p^l-1$ respectively, it follows that $l<n$ so that $Q_{n-2}\mathcal{P}_p^{p^{n-1}}x=0$. Therefore we have

$$Q_nx = Q_{n-1}\mathcal{P}_p^{p^{n-1}}x = Q_{n-2}\mathcal{P}_p^{p^{n-2}}\mathcal{P}_p^{p^{n-1}}x .$$

Repeating this process, we can get (1.5).

It is well known [1] that the stable cohomology group of the Eilenberg-MacLane complex $\boldsymbol{K}(\boldsymbol{Z}, \boldsymbol{n})$ is isomorphic to $A_p/[Q_0]$, where $[Q_0]$ stands for the left ideal of A_p generated by Q_0.

Let X be a complex, and let x_i $(i=1, 2, \cdots)$ be any $(n+1)$-dimensional integral cohomology classes of X. Then there exists a fibre space over X such that the fibre is $\boldsymbol{K}(\boldsymbol{Z}, \boldsymbol{n})\times \cdots \times \boldsymbol{K}(\boldsymbol{Z}, \boldsymbol{n})$ and the transgression of the fundamental cohomology class of the fibre is x_i [4].

The homotopy type of the total space of this fibre space is determined uniquely. Such a construction is usually called the Postonikov construction and will be used frequently in the following.

LEMMA 1. *Let p be a fixed odd prime, and n any integer >2. Then there exists a space $\mathcal{K}^p(n)$ satisfying the following conditions* (1.6)-(1.8):

(1.6)
$$H^*(\mathcal{K}^p(n)\ ;\ Z) = 0, \quad \textit{for} \quad 0<k<n,$$
$$H^0(\mathcal{K}^p(n)\ ;\ Z) = H^n(\mathcal{K}^p(n)\ ;\ Z) = Z .$$

(1.7) *$H^*(\mathcal{K}^p(n); Z_p)$ is a free $A_p/(Q_0)$-module in dimensions $<2n$, and is generated by elements of dimensions $n+4j$ $(j=0,1,2,\cdots)$.*

(1.8) *The free base of dimension $n+4j$ is in 1−1 correspondence with the set of partitions $\omega(j)$ of j in which any number of the form $(p^*-1)/2$ does not appear. The element corresponding to a partition $\omega(j)$ is also denoted by $\omega(j)$.*

Proof. Starting with $\boldsymbol{K}_0^p=\boldsymbol{K}(\boldsymbol{Z}, \boldsymbol{n})$, assume inductively that $\boldsymbol{K}_i^p$ satisfying the following two conditions $(1.9)_i$, $(1.10)_i$ is constructed.

Decompose a partition $\omega(j)$ of j in the form

$$\omega(j) = (\omega_p'(\omega(j)),\ \omega_p''(\omega(j)))$$

where $\omega_p'(j)=\omega_p'(\omega(j))$ is of the form $((p^{w_1}-1)/2\cdots(p^{w_r}-1)/2)$, $0<w_1\leq\cdots\leq w_r$, and $\omega_p''(j)=\omega_p''(\omega(j))$ does not contain any integer of the form $(p^*-1)/2$.

$(1.9)_i$ $H^*(\boldsymbol{K}_i^p; Z_p)$ is a Z_p-vector space whose base is given by the set of symbols of the following types in dimensions $<2n$:

A) $\mathcal{P}^R\omega(j)$ with $\omega_p'(j)=\phi$ (empty set) $(0\leq j\leq i,\ \mathcal{P}^R\in A_0)$,

B) $\mathcal{P}^R Q_{l_1}\cdots Q_{l_k}\omega(j)$ with $4j+2p^{l_k}-1>4(i+1)$, $l_1>\cdots>l_k>0$ and $\omega_p'(j)=\phi$ $(0\leq j\leq i,\ \mathcal{P}^R\in A^0)$,

C) $\mathcal{P}^R Q_{l_1}\cdots Q_{l_k}\omega(j)$ with $4j+2p^{l_k}-1>4(i+1)$, $l_1>\cdots>l_k>0$, $w_1>l_k$ and $\omega(j)\neq\phi$ $(0\leq j\leq i,\ \mathcal{P}^R\in A^0)$.

$(1.10)_i$ In the complex $\boldsymbol{K}_i^p$, Q_l operates trivially on $\omega(j)$ $(0\leq j\leq i)$ if and only if

$$l=0 \quad\text{or}\quad 4j+2p^l-1\leq 4(i+1)\ ,$$

and no element of A^0 operates trivially on $\omega(j)$ $(0\leq j\leq i)$.

It is obvious that $\boldsymbol{K}_0^p=\boldsymbol{K}(\boldsymbol{Z}, n)$ satisfies the conditions $(1.9)_0$, $(1.10)_0$. (We regard that (0), the only partition of 0, is not of the form $(p^*-1)/2$).

Now we proceed to the construction of $\boldsymbol{K}_{i+1}^p$. Let $\pi(i+1)$ denote the number of all partitions of $i+1$. Then we shall construct the complex $\boldsymbol{K}_{i+1}^p$ as a fibre space over $\boldsymbol{K}_i^p$ such that the fibre is

$$\boldsymbol{K}(\boldsymbol{Z}, n+4(i+1))\times\cdots\times\boldsymbol{K}(\boldsymbol{Z}, n+4(i+1)),\ \pi(i+1)\text{-times}$$

and the transgression is given by (1.11) as follows: Index the fundamental class of each component of the fibre by the corresponding partition $\omega(i+1)$.

$$(1.11)\qquad \tau_p\omega(i+1)=0 \quad\text{if}\quad \omega_p'(i+1)=\phi$$
$$\tau_p\omega(i+1)=Q_{w_1}((p^{w_2}-1)/2,\cdots,(p^{w_r}-1)/2,\ \omega_p'') \quad\text{if}\quad \omega_p'\neq\phi$$

Here the following should be noted:

(1.12) the dimension of $\tau_p\omega(i+1)$ is equal to $n+4(i+1)+1$;

(1.13) $\tau_p\omega(i+1)$ is in the integral cohomology of $\boldsymbol{K}_i^p$.

Since the degree of Q_{w_1} is $2p^{w_1}-1$, (1.12) is obvious. (1.13) follows from the assumption $(1.10)_i$ and (1.5), more precisely we have in K_i^p

$$\tau_p \omega(i+1) = Q_{w_1}\omega = \delta_p \mathcal{P}_p^1 \cdots \mathcal{P}_p^{p^{w_1-1}}\omega$$

with $\omega = ((p^{w_2}-1)/2, \cdots, (p^{w_r}-1)/2, \omega_p''(i+1))$.

We give next a proof of that K_{i+1}^p satisfies $(1.9)_{i+1}$ and $(1.10)_{i+1}$. For $(1.10)_{i+1}$: It is sufficient to show that

$$n+4j+2p^l-1 = n+4(i+1)+1$$

implies for any partition $\omega(j)$ of j

$$Q_l\omega(j) = 0 \quad \text{in } K_{i+1}^p .$$

This follows from that

$$\tau_p((p^l-1)/2, \omega(j)) = Q_l\omega(j) .$$

For $(1.9)_{i+1}$: From the assumptions $(1.9)_i$ and $(1.10)_{i+1}$ it follows that it is sufficient to show the following (1.14) and (1.15):

(1.14) the elements of $H^*(K_{i+1}; Z_p)$ which correspond to $\omega(i+1)$ with $\omega_p'(\omega(i+1))=\phi$ and to $Q_l\omega(i+1)$ with $\omega_p'(\omega(i+1))\neq\phi$ and $l\leq w_1$, do not vanish.

(1.15) the elements of the type (A), (B) and (C) in $(1.9)_{i+1}$ are linearly independent.

By the definition (1.11) the first part of (1.14) is obvious. From the fact

$$\begin{aligned}\tau_p(Q_l\omega(i+1)) &= Q_l\tau_p(w(i+1)) \\ &= Q_lQ_{w_1}(\omega) = -Q_{w_1}(Q_l(\omega))\end{aligned}$$

where ω is the partition of $(i+1)-(p^{w_1}-1)/2$ in (1.13), it follows that

$$\tau_p(Q_l\omega(i+1)) = 0$$

if and only if $l=w_1$ or $4((i+1)-(p^{w_1}-1)/2)+2p^l-1<4(i+2)$ ((1.2), $(1.10)_{i+1}$), equivalently, $w_1\geq l$.

Thus the element $Q_l\omega(i+1)$ in the cohomology of the fibre does not vanish in $H^*(K_{i+1}^p; Z_p)$.

By the construction, for the proof of (1.15), it is sufficient to see that, if

$$(*) \qquad \mathcal{P}^{R_1}Q_{L_1}\omega(i+1)+\cdots+\mathcal{P}^{R_\nu}Q_{L_\nu}\omega(i+1) = 0 \quad \text{in } H^*(K_{i+1}^p; Z_p)$$

then

$$\mathcal{P}^{R_\nu} = 0 \qquad (\mu = 1, \cdots, \nu) ,$$

where $Q_{l_\mu}=Q_{l_{\mu 1}}\cdots Q_{l_{\mu_\rho}}$ with $L_\mu=(l_{\mu_1}, \cdots, l_{\mu_\rho})$ satisfying $(1.9)_{i+1}$ and $\mathcal{P}^{R_\mu}\in A^0$.

On the other hand, by $(1.10)_{i+1}$ the relation (*) can be written in A_p

$$\mathcal{P}^{R_1}Q_{L_1}+\cdots+\mathcal{P}^{R_\nu}Q_{L_\nu} = \mathcal{P}^{R_1'}Q_{L_1'}+\cdots+\mathcal{P}^{R_\nu'}Q_{L_\nu'}$$

where $L_\mu'=(l'_1, \cdots, l'_{\mu_\rho})$ satisfy the conditions of $(1.10)_{i+1}$.

From this and (1. 3), it follows that $\mathcal{P}^{R\mu}=0$.

Thus the construction of K_i^p, $i=0,1,2,\cdots$ is completed.

Let $\mathcal{K}^p(n)=K_{n-1}^p$ then it follows from (1. 4) that $(1.9)_{n-1}$ and $(1.10)_{n-1}$ turn out to the desired properties (1. 8) and (1. 7) respectively. And (1. 6) is obviously satisfied by K_i^p.

This completes the proof of Lemma.

LEMMA 2[1]. *For any integer* $n>2$ *there exists a complex* $\mathcal{K}(n)$ *such that* A_p-*modules* $H^*(\mathcal{K}(n)\,;\,Z_p)$ *and* $H^*(\mathcal{K}^p(n)\,;\,Z_p)$ *are isomorphic for each odd prime* p.

Proof. The construction of $\mathcal{K}(n)$ is done by induction. Let $K_0=K(Z,n)$ and let K_{i+1} be a fibre space over K_i with the fibre

$$K(Z, n+4(i+1))\times\cdots\times K(Z, n+4(i+1)),\ \pi(i+1)\text{-times}$$

and with the transgression $\tau=\sum_{p\,:\,\text{odd prine}}\tau_p$.

This definition is legitimate, since the image of τ_p is in the integral cohomology classes by (1. 13) and since $\tau_p=0$ for sufficiently large prime p in vertue of the definition (1. 11).

Observe that the image of τ_q is in the q-torsion which is zero mod p ($p\neq q$, p, q ; odd prime), i.e. $\tau=\tau_p$ mod p. Then it follows that $H^*(K_i\,;\,Z_p)\approx H^*(K_i^p\,;\,Z_p)$ as A_p-modules. Thus $\mathcal{K}(n)=K_{n-1}$ has the desired properties.

2. *SO*-realizabilities

By the Thom's fundamental theorem [5], the realisability of an integral homology class x of an orientable C^∞-manifold V by an orientable submanifold is equivalent to the existence of a continuous map $q: V\to MSO(n)$ such that

$$q^*(U)=x^* \tag{2.1}$$

Where U is the fundamental class of the Thom complex $MSO(n)$ of $SO(n)$ and x^* is the n-dimensional cohomology class dual to x. (The existence of such a map is called the SO-realizability of x.)

By this fact, the realisability problem is reduced to the analysis of the homotopy type of $MSO(n)$.

On the other hand, Milnor proved in [3] that the mod $p(\geq 3)$ stable cohomology group of $MSO(2k)$ is generated as a free $A_p/(Q_0)$-module by the elements $X_{\omega_p(j)}$ of $(4j+2k)$-dimension. Here $\omega_p(j)$ is a partition of j which does not contain any number of the form $(p^*-1)/2$ and identifying $H^*(MSO(2k)\,;\,Z_p)$ with the ideal of $H^*(PC(\infty)\times\cdots\times PC(\infty)\,;\,Z_p)=Z_p[t_1,\cdots,t_k]$ generated by $t_1\cdots t_k$, is expressed as follows:

1) (*Added in proof*) The author was communicated by prof. Shimada that the essential part of Lemmas 1 and 2 was also obtained independently by B. G. Averbuch, Doklady Akademii Nauk SSSR., **125** (1959).

(2.2) $$X_{\omega_p(j)} = X_{(\omega_1,\cdots,\omega_r)} = \sum^* t_1^{2\omega_1+1}\cdots t_r^{2\omega_r+1}\cdot t_{r+1}\cdots t_k\,.$$

where Σ^* means the symmetric sum, e.g. $\Sigma^* t_1 = t_1 + \cdots + t_k$.

LEMMA 3. *There exists a complex*

$$K' = K(Z_2,\ 2k+i_1)\times\cdots\times K(Z_2,\ 2k+i_l)$$

such that the complexes $\mathcal{K}(2k)\times K'$ and $MSO(2k)$ are of the same $4k$-homotopy type, where $\mathcal{K}(2k)$ in the complex in Lemma 2.

Proof. There exists a map $f\colon MSO(2k)\to\mathcal{K}(2k)$ satisfying the relations:

(2.3) $$(f^*)_p(\omega_p(j)) = X_{\omega_p(j)} \quad \text{for any odd prime } p\,,$$
$$(f^*)_2(\omega(j)) = (X_{\omega(j)})_2\,,$$

where $(f^*)_p$ is the reduction mod p of f^*, and $(X_{\omega(j)})_2$ is the reduction mod 2 of the integral cohomology class corresponding to

$$\sum^* t_1^{2\omega_1+1}\cdots t_r^{2\omega_r+1}\cdot t_{r+1}\cdots t_k \in H^*(MSO(2k)\ ;\ Z)\,.$$

In fact, Wall [6] proved that the set of $(X_{\omega(j)})_2$ forms a base of an $A_2/[Sq^1]$-free module M of $H^*(MSO(2k)\ ;\ Z_2)$ and that A_2-module $H^*(MSO(2k)\ ;\ Z_2)$ is a direct sum of M and A_2-free module N in the dimensions $<4k$.

From this and the construction of $\mathcal{K}(2k)$, it follows that the obstructions to define such a map f are the p-torsion elements ($p\geq 3$)

$$\delta_p \mathcal{P}_p^1 \cdots \mathcal{P}_p^{p^q} X_{\omega_p(j)}\,.$$

On the other hand, $H^*(MSO(2k)\ ;\ Z)$ has no p-torsion ($p\geq 3$), hence the obstruction must vanish.

Let i_j, $j=1,\cdots,l$ be the dimension of A_2-free base of the A_2-free module N. Put

$$K' = K(Z_2,\ 2k+i_1)\times\cdots\times K(Z_2,\ 2k+i_l)$$

Then it is easy to define a map $f'\colon MSO(2k)\to K'$ such that the map $(f\times f')\colon MSO(2k)\to\mathcal{K}(2k)\times K'$ induces an isomorphism

$$(f\times f')^*\colon H^*(\mathcal{K}(2k)\times K'\ ;\ Z_p)\to H^*(MSO(2k)\ ;\ Z_p)$$

for any prime $p\geq 2$ in dimensions $<4k$. Now the assertion is a direct consequence of the theorem of J.H.C. Whitehead.

LEMMA 4. *$MSO(2k-1)$ is of the same $(4k-3)$-homotopy type as the complex*

$$\mathcal{K}(2k-1)\times K(Z_2,\ (2k-1)+i_1)\times\cdots\times K(Z_2,\ (2k-1)+i_l)\,.$$

Proof. It is known [5] that $MSO(2k)$ is of the same homotopy type as the suspension $S(MSO(2k-1))$ of $MSO(2k-1)$. On the other hand, since $MSO(2k-1)$ is $2k+2$ connected [5], the loop space $\Omega S(MSO(2k-1))$ of $S(MSO(2k-2))$ is of the same $(4k-3)$-homotopy type as $MSO(2k-1)$ [4]. Therefore the $(4k-3)$-

homotopy type of $MSO(2k-1)$ is that of $\Omega(MSO(2k))$ which is of the same $4k$-homotopy type as $\Omega\mathcal{K}(2k)\times\Omega\boldsymbol{K}'$ by Lemma 3. From the constructions of $\mathcal{K}(2k)$, it is easy to see

$$\Omega(\mathcal{K}(2k)) = \mathcal{K}(2k-1)\,.$$

Thus we have the lemma.

THEOREM 5. *Let V be a compact orientable C^∞-manifold of dimension m, and let n be an integer such that*

$$2n\geq m \quad \text{if } n \text{ is even}, \qquad 2n-1\geq m \quad \text{if } n \text{ is odd}.$$

Then the homology class x_* dual to $x\in H^n(V;Z)$ is realizable by an orientable C^∞-submanifold (without singularities) of V, if and only if

$$\delta_p\mathcal{P}^1_a\cdots\mathcal{P}^{p^q}_p(x) = 0 \tag{2.4}$$

for any odd prime p and for any integer $q\geq 0$.

REMARK 1. For a subalgebra B of A_p and a complex X, we shall denote by "*Ker B*" the set of all elements x of $H^*(X;Z_p)$ such that $b\cdot x=0$ for any $b\in B$. Under this convention, (2.4) can be written as;

$$x\in Ker\,(\delta_p) \qquad \text{for any odd prime } p\,. \tag{2.4$'$}$$

Proof of THEOREM 5. Necessecity: It is obvious from the existence of a map $q: V\to MSO(n)$ satisfying (2.1), since $H^*(MSO(n);Z)$ has no odd torsion, [5]. (For this part the restrictions on dimensionality is unnecessary.)

Sufficiency: To obtain a map $q: V\to MSO(n)$ satisfying (2.1), it is sufficient by Lemma 3 and 4 to show that there exists a map

$$F: V\to\mathcal{K}(n)\,;\ F^*((0)) = x\,.$$

The construction of this map is done by induction.

Let $F_0: V\to\boldsymbol{K}_0=\boldsymbol{K}(Z,n)$ be a map such that

$$F_0^*((0)) = x\,,$$

and assume inductively that a map $F_i: V\to\boldsymbol{K}_i$ with the following property is defined:

$$\begin{aligned} &F_i^*(\omega(j)_Z) = 0 \qquad \text{for any}\quad 0<j\leq i\,,\\ &F_i^*((0)) = x\,, \end{aligned} \tag{2.5$_i$}$$

where $\omega(j)_Z$ is an integral cohomology class of $H^*(\boldsymbol{K}_i;Z)$ corresponding to the Z_p-class $\omega(j)$ which does not contain any number of the form $(p^*-1)/2$.

The fibre space $F_i^*\boldsymbol{K}_{i+1}\to V$ of $\boldsymbol{K}_{i+1}\to\boldsymbol{K}_i$ induced by the map $F_i: V\to\boldsymbol{K}_i$ is trivial, by (2.4) and $(2.5)_i$. Therefore $F_i^*\boldsymbol{K}_{i+1}$ and $V\times\boldsymbol{K}(Z,n+4(i+1))\times\cdots\times\boldsymbol{K}(Z,n+4(i+1))$ are of the same homotopy type, hence we can define a cross section $s: V\to F_i^*\boldsymbol{K}_{i+1}$ by

$$s(v) = (v, *, \cdots, *), \qquad v \in V$$

where $*$ is the base point of $\boldsymbol{K}(Z, n+4(i+1))$.

Let $\hat{F}_i : F_i^* \boldsymbol{K}_{i+1} \to \boldsymbol{K}_i$ be the covering map of F_i, and set

$$F_{i+1} = \hat{F}_i \circ s : V \to \boldsymbol{K}_{i+1} .$$

Then, by the definition of s, we have $F_{i+1}^*(\omega(i+1)_Z) = 0$, and from $(2.5)_i$ it follows that $F_{i+1}^*(\omega(j)_Z)=0$ for $0<j\leq i$ and $F_{i+1}^*((0))=x$. Therefore F_{i+1} satifies $(2.5)_{i+1}$, consequently $F=F_{n-1}: V \to \boldsymbol{K}_{n-1}=\mathcal{K}(n)$ can be constructed by induction.

COROLLARY 6. *Let V be a $4m$-dimensional compact orientable manifold. Then for any $k \geq [m/2]$*[2] *the homology class dual to the Pontrjagin class p_k of V is realizable by an orientable C^∞-submanifold (without singularities) of V.*

Proof. On the cohomology of classifying space $B_{SO(4m)}$ of $SO(4m)$, (δ_p) operates trivially for any odd prime p. Hence

$$p_k \in Ker\ (\delta_p) \qquad \text{for any odd prime } p .$$

Thus the assertion follows from Theorem 5.

REMARKS 2. Corollary 6 can be stated in the following slightly different form:

Under the assumption of Corollary 6, a mononial of the Pontrjagin classes is SO-realizable, if the dimension of the monomial is not less than $2m$.

REMARKS 3. If we use the homotopy group of $\mathcal{K}(2k) \times \boldsymbol{K}'$ which is isomorphic to that of $MSO(k)$ in dimensions $<4k$, the result of Milnor [3] on cobordism group Ω^* can be easily obtained without using the spectral sequence of Adams.

3. Remarks on *MU*.

We can construct, as in No. 1, a complex $\hat{\mathcal{K}}(n)$ which is of the same homotopy type as $MU(n)$ in dimensions $<4k$. The constructions are similar to that in No. 1. A rough account of constructions of $\hat{\mathcal{K}}(n)$ is as follows:

Let $\hat{\boldsymbol{K}}_0=\boldsymbol{K}(Z, 2n)$ and let $\hat{\boldsymbol{K}}_{i+1}$ be a fibre space over $\hat{\boldsymbol{K}}_i$ with the fibre

$$\boldsymbol{K}(Z, 2n+2(i+1)) \times \cdots \times \boldsymbol{K}(Z, 2n+2(i+1)) \ ; \ \pi(i+1)\text{-times}$$

and with the transgression

$$\hat{\tau} = \textstyle\sum_{p\geq 2:\ \text{prime}} \hat{\tau}_p$$

where the image of $\hat{\tau}_p$ is in the p-torsion and is expressed in terms of the Z_p-cohomology, identifying a partition $\hat{\omega}(j)$ with a $(2n+2j)$-dimensional cohomology class,

$$\hat{\tau}_p(\hat{\omega}(i+1)) = 0$$

2) [ ] means the Gaussian symbol.

if $\hat{\omega}(i+1)$ does not contain any number of the form p^*-1,

$$\hat{\tau}_p(\hat{\omega}(i+1)) = Q_{w_1}(p^{w_2}-1, \cdots, p^{w_r}-1, \hat{\omega}) \qquad \text{if} \quad \hat{\omega}_p''(i+1)$$

is of the form $(p^{w_1}-1, \cdots, p^{w_r}-1, \hat{\omega}_p'')$ where $\hat{\omega}_p''$ does not contain any number of the form p^*-1.

As is shown by Milnor [3], the Z_p-vector space $H^*(MU(n); Z_p)$ is generated in dimensions $<4n$ by the elements $\mathcal{P}^R X_{\hat{\omega}(j)}$, $\mathcal{P}^R \in A^0$ where $X_{\hat{\omega}(j)}$ is a corresponding element of $H^{2n+2j}(MU(n); Z_p)$ to a partition $\hat{\omega}(j)$ not containing any number of the form p^*-1.

From this, we can see $\hat{\mathcal{K}}(n)$ and $MU(n)$ are of the same homotopy type in dimensions $<4n$, hence we can deduce from the Thom's fundamental theorem the following

THEOREM 7. *Let V^m be a compact m-dimensional orientable C^∞-manifold and let n be an integer such that $4n>m$. Then the dual homology class to a cohomology class $u \in H^{2n}(V; Z)$ is realizable by a C^∞-submanifold of V whose normal bundle in V has $U(n)$ as its structure group, if and only if*

$$u \in Ker\ (\delta_p) \qquad \textit{for any prime } p \geq 2\,.$$

Again (δ_p) operates trivially on $H^*(B_{U(2m)}; Z_p)$ for any prime $p \geq 2$, we have

COROLLARY 8. *Let V^{2m} be a 2m-dimensional almost complex manifold, and let n be an integer such that $n \geq [m/2]$. Then the dual homylogy class to the Chern class c_n of V^{2m} is realizable by an orientable submanifold of V^{2m}. (or, more strongly, by a submanifold whose normal bundle in V has an $U(n)$-structure.)*

REMARK 4. As in Remark 2, the result of Milnor [3] on the complex cobordism group Γ follows directly from the homotopy structure of $\hat{\mathcal{K}}(n)$.

References

[1] H. Cartan, Seminaire de topologie, E. N. S. 1954–55.

[2] J. Milnor, The Steenrod algebra and its dual, Ann. of Math., 67 (1958), 150–171.

[3] J. Milnor, On the cobordism ring Ω^* and a complex analogue, PART 1, Amer. J. Math., (1960), 505–521.

[4] J. P. Serre, Homologie singulière des espaces fibrés, Ann. of Math., 54 (1951), 425–505.

[5] R. Thom, Quelques propriétés globales des variétés différentiables, Comment. Math. Helv., 28 (1954), 17–86.

[6] C. T. C. Wall, Determination of the cobordism ring, Ann. of Math., 72 (1960), 292–311.