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On the exact sequence for a special cofibre space and its dual

By Noboru YAMAMOTO

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0. Introduction

Let X, Y be two spaces and $f: X \to Y$ be a map. Then, there are a fibre space and a cofibre space such that the projection and the injection are equivalent to f, respectively. Hence, for any spaces U and V, we have the well-known exact sequences of sets of homotopy classes:

$$\pi(U, F_f) \longrightarrow \pi(U, X) \xrightarrow{f_*} \pi(U, Y)$$

and

$$\pi(C_f, V) \longrightarrow \pi(Y, V) \xrightarrow{f^*} \pi(X, V)$$

where F_f and C_f are the fibre and the cofibre, respectively.

The main purpose of this paper is to extend these exact sequences by one term, under the assumption that F_f and C_f are homotopy equivalent to the loop space and the suspension of a space, commuting with operators and cooperators (in the sense of Eckmann- Hilton [2]), respectively.

In §§ 1-2, we shall deal with the notion of cofibre spaces following Eckmann and Hilton, [1], [2], [3]. In § 3, the theorem for cofibre spaces is proved, and in §§ 4-5, it is dualized for fibre spaces and the main theorem is proved.

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1. Cofibre spaces

Throughout this paper, unless otherwise stated, spaces will be arcwise connected and have the homotopy type of a CW-complex. On each space a base point is given, each map takes base point to base point and each homotopy leaves base point fixed.

The following definition of the cofibre space is due to Eckmann-Hilton [1].

DEFINITION (1. 1) A triple (A, q, B) of two spaces A, B and a map $q: A \to B$ is called a *cofibre space* (or a *cofibration*), if the following condition is satisfied: Let V be any space (which is not necessarily of the same homotopy type as a CW-complex), and let $g_0: A \to V$, $h_0: B \to V$ be maps such that $g_0 = h_0 q$. Then, for any given homotopy g_t ($t \in I=[0, 1]$) of g_0, there exists a homotopy h_t of h_0 such that $g_t = h_t q$.

For a cofibre space (A, q, B), the identifying space $C=B/qA$ of B, shrinking qA to the base point, is called the *cofibre*; A and B are called the *cobase* and the *total space*, respectively. We shall denote the identification map $B \to C$ by γ.

DEFINITION (1.2) Let (A, q, B) and (A', q', B') be two cofibre spaces whose cofibres have the same homotopy type. A *cofibre map* f: $(A, q, B) \to (A', q', B')$ is a triple of maps $(f, \bar{f}, \tilde{f})$ such that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{q} & B & \xrightarrow{\gamma} & C \\ {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ A' & \xrightarrow{q'} & B' & \xrightarrow{\gamma'} & C' \end{array}$$

is commutative and $\tilde{f}$ is a homotopy equivalence.

DEFINITION (1.3) Let A, A', B be spaces and ϕ; $A \to A'$, ψ: $A \to B$ be maps. We shall define $M(\phi, \psi)$ to be the space obtained from the disjoint union $A' \cup B$ by identifying $\phi(a) \in A'$ and $\psi(a) \in B$ for each $a \in A$. The natural maps $A' \to M(\phi, \psi)$ and $B \to M(\phi, \psi)$ are denoted by $i_{A'}$, and i_B, respectively, and $i_{A'}(a_0') = i_B(b_0)$ is taken as the base point of $M(\phi, \psi)$ where $a_0' \in A'$, $b_0 \in B$ are the base points.

DEFINITION (1.4) Let A, B be space and f: $A \to B$ be a map. The *mapping cylinder B_f of f* is the space obtained from the disjoint union $A \times I \cup B$ by identifying $(a, 0) \in A \times 0$ with $f(a) \in B$ for each $a \in A$ and shrinking $a_0 \times I$ to the base point. The mapping cone C_f of f is the identifying space $B_f/A \times 1$. In particular, the space C_i for the identity map i: $A \to A$ is the *cone over* A and denoted by TA. The *suspension* ΣA of A is obtained from TA by shrinking $A \times 0$ to the base point.

It is easy to prove the following lemma.

LEMMA (1.5) Let A, B be spaces and f: $A \to B$ be a map. Then, the triple (A, i_f, B_f) is a cofibre space whose cofibre is the mapping cone C_f of f, where i_f: $A \to B_f$ is the natural injection. (We shall call such a triple the *cofibre space associated with the map f*.)

LEMMA (1.6) Let (A, q, B) be a cofibre space whose cofibre is C, and f: $A \to A'$ be a map. Then, $(A', i_{A'}, M(f, q))$ is a cofibre space having C as the cofibre, and there is a cofibre map f: $(A, q, B) \to (A', i_{A'}, M(f, q))$, i.e., the following diagram is commutative:

$$\begin{array}{ccccc} A & \xrightarrow{q} & B & \xrightarrow{\gamma} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}=i_B} & & \downarrow{\scriptstyle \tilde{f}=id_C} \\ A' & \xrightarrow{q'=i_{A'}} & M(f, q) & \xrightarrow{\gamma'} & C\,. \end{array}$$

Proof. As easily seen, $M(f, q)/i_{A'}A'=B/qA=C$, and the above diagram is commutative.

Let g'_t: $A'\to V$ be a homotopy and h'_0: $M(f, q)\to V$ be a map such that $g'_0=h'_0 i_{A'}$. Then, $g'_t f$: $A\to V$ is a homotopy of $h'_0 i_B q$: $A\to V$. Therefore, we have a homotopy h_t: $B\to V$ such that

$$h_0 = h'_0 i_B \text{ and } h_t q = g'_t f\,,$$

because (A, q, B) is a cofibre space. These relations show that the homotopy h'_t: $M(f, q)\to V$ of h'_0, defined by

$$h'_t i_{A'} = g'_t\,, \quad h'_t i_B = h_t\,,$$

is well-defined. Hence, $(A', i_{A'}, M(f, q))$ is a cofibre space. q.e.d.

DEFINITION (1.7) The triple $(A', i_{A'}, M(f, q))$ of the above lemma is called the *cofibre space induced from* (A, q, B) *by* f, and its total space $M(f, q)$ is denoted by $f_{\#}(B)$. Also, the above triple of maps (f, i_B, id_C) is called the *cofibre map induced by* f. (See Hilton [3], §6.)

LEMMA (1.8) Let $(f, \bar{f}, \bar{\bar{f}})$: $(A, q, B)\to(A', q', B')$ be a cofibre map, and (A', q'', B'') and $(f, \bar{g}, \bar{\bar{g}})$: $(A, q, B)\to(A', q'', B'')$ be the cofibre space and the cofibre map induced by f, respectively. Then, there is a cofibre map $(id_{A'}, \bar{f}_0, \bar{\bar{f}}_0)$: $(A', q'', B'')\to(A', q', B')$ such that $\bar{f}_0\bar{g}=\bar{f}$ and $\bar{\bar{f}}_0\bar{\bar{g}}=\bar{\bar{f}}$.

Proof. By the definition of the induced cofibre space, $B''=M(f, q)$, $q''=i_{A'}$, $\bar{g}=i_B$ and $\bar{\bar{g}}=id_C$. Define the map $\bar{f}_0$: $B''\to B'$ by

$$\bar{f}_0 i_{A'} = q' \text{ and } \bar{f}_0 i_B = f\,.$$

By the definition of $M(f, q)$ and $q'f=\bar{f}q$, $\bar{f}_0$ is well-defined and $\bar{f}_0\bar{g}=\bar{f}_0 i_B=\bar{f}$, $\bar{f}_0 q''=\bar{f}_0 i_{A'}=q'$. Hence $\bar{f}_0$ induces a map $\bar{\bar{f}}_0$: $C=B''/q''A\to C'=B'/q'A'$. Since $\bar{\bar{f}}$ and $\bar{\bar{g}}$ are induced by $\bar{f}$ and $\bar{g}$, respectively, $\bar{f}_0\bar{g}=\bar{f}$ shows that $\bar{\bar{f}}_0\bar{\bar{g}}=\bar{\bar{f}}$. Therefore, $\bar{\bar{f}}_0$ is a homotopy equivalence, because $\bar{\bar{f}}$ is so and $\bar{\bar{g}}=id_C$. q.e.d.

LEMMA (1.9) Let (A, q, B) and (A, q', B') be cofibre spacs over the same space A. Assume that there is a cofibre map f: $(A, q, B)\to(A, q', B')$ such that f: $A\to A$ is the identity map, and the total spaces B and B' are simply connected. Then, B and B' are homotopy equivalent.

Proof. From the exactness of the homology sequence of cofibre spaces and Five Lemma, it follows that f_*:$H_i(B)\approx H_i(B')$, for $i\geqq 0$. Since B and B' are simply connected, $f_*:\pi_i(B)\approx\pi_i(B')$, for $i\geqq 1$. Therefore, B and B' are homotopy equivalent, because they have the same homotopy type of a CW-complex. q.e.d.

From (1.8), (1.9) and van Kampen's Theorem [6], the next corollary follows immediately.

COROLLARY (1.10) Let (A, q, B) and (A', q', B') be cofibre spaces such that there is a cofibre map $(f, \bar{f}, \tilde{f}): (A, q, B) \to (A', q', B')$. If B, A' and B' are simply connected, then B' and $f_{\#}(B)$ are homotopy equivalent.

2. Existance of cofibre maps

Let (A, q, B) be a cofibre space and $\bar{B} = C_q = TA \cup_q B$ be the mapping cone of $q: A \to B$.

Let $g_t: A \to \bar{B}$ be the homotopy defined by $g_t(a) = (a, t)$, $a \in A$, and $f_0: B \to \bar{B}$ be the map defined by $f_0(b) = b$, $b \in B$. Since $f_0 q = g_0$ and (A, q, B) is a cofibre space, there is a homotopy $f_t: B \to \bar{B}$ such that $f_t q = g_t$, in particular, $f_1 q(A) = b_0$. Hence, f_1 defines a map

$$\varepsilon: C \to \bar{B}. \tag{2.1}$$

Next, let $\bar{\gamma}: \bar{B} \to \Sigma A \vee C$ be the map shrinking $A \times 0 = TA \cap B$ to the base point, and $p_{\Sigma A}: \Sigma A \vee C \to \Sigma A$ be the projection. Define maps

$$\phi: C \to \Sigma A \vee C, \qquad \delta: C \to \Sigma A \tag{2.2}$$

by

$$\phi = \bar{\gamma}\varepsilon, \qquad \delta = p_{\Sigma A}\phi = p_{\Sigma A}\bar{\gamma}\varepsilon.$$

REMARK. The map ϕ is unique up to a homotopy and defines the cooperator in the sense of Eckmann-Hilton [2].

The following lemma is clear.

LEMMA (2.3) For the cofibre space (Z, i_Z, TZ) with cofibre ΣZ, the map $\phi_Z: \Sigma Z \to \Sigma Z \vee \Sigma Z$, in (2.2), may be taken as the map defined by

$$\begin{aligned} \phi_Z(z, t) &= (z, t/t_0), \qquad 0 \leqq t \leqq t_0, \\ &= \left(z, \frac{t-t_0}{1-t_0}\right), \qquad t_0 \leqq t \leqq 1, \end{aligned}$$

for a fixed number t_0, $0 < t_0 < 1$.

Therefore, ϕ_Z defines an H'-structure on ΣZ, i.e., $p_i \phi_Z \sim id_{\Sigma Z}$, where $p_i: \Sigma Z \vee \Sigma Z \to \Sigma Z$ is the projection onto the i-th component, $i = 1, 2$. (See Eckmann-Hilton [1])

Now, let (A, q, B) be a cofibre space whose cofibre $C = B/qA$ is homotopy equivalent to the suspension ΣZ of a space Z.

We shall consider the following diagram:

$$\begin{array}{ccc} \Sigma Z & \xrightarrow{\phi_Z} & \Sigma Z \vee \Sigma Z \\ \kappa \downarrow & & \downarrow \kappa_0 \vee \kappa \\ C & \xrightarrow{\phi} & \Sigma A \vee C \end{array} \tag{2.4}$$

where κ is a homotopy equivalence, $\kappa_0 = \delta\kappa$, ϕ, δ and ϕ_Z are maps in (2.2) and (2.3).

PROPOSITION (2.5) Let (A, q, B) be a cofibre space such that q is an inclusion map. Assume that the cofibre C of (A, q, B) is homotopy equivalent to the suspension ΣZ of a space Z, and the diagram (2.4) is commutative. Then, there is a cofibre map $h: (Z, i_Z, TZ) \to (A, q, B)$.

Proof. Let $\bar{f}: TZ \to TA \cup B = \bar{B}$ be the map defined by

$$\bar{f} = \varepsilon\kappa\gamma .$$

Then, the commutativity of the diagram (2.4) shows that

$$\bar{f}(Z\times[0, t_0]) \subset TA \quad \text{and} \quad \bar{f}(Z\times[t_0, 1]) \subset B .$$

Define the homotopy $h_s: TZ \to TZ$ by

$$\begin{aligned} h_s(z, t) &= (z, (1-s)t+t_0 s), \qquad 0\leqq t\leqq t_0, \\ &= (z, t), \qquad t_0\leqq t\leqq 1 . \end{aligned}$$

Then, $h_0=id$, $h_1(Z\times[0, t_0])\subset Z\times t_0$, $h_s|Z\times[t_0, 1]=id$, $s\in I$ and $h_s(i_Z Z)\subset Z\times[0, t_0]$, $s\in I$.

Let $\bar{f}_s: TZ\to\bar{B}$ be the homotopy defined by

$$\bar{f}_s = \bar{f} h_s .$$

Then, $\bar{f}_0=\bar{f}$, $\bar{f}_1(Z\times[0, t_0])\subset TA\cap B$, $\bar{f}_s|Z\times[t_0, 1]=\bar{f}|Z\times[t_0, 1]$, $s\in I$, and $\bar{f}_s(i_Z Z) \subset TA$, $s\in I$. Hence, $\bar{f}_1$ defines the maps

$$\bar{h}: TZ\to B \quad \text{and} \quad h=\bar{h}|Z: Z\to A .$$

The map $\tilde{h}: \Sigma Z\to C$, defined by

$$\tilde{h}\gamma = \gamma'\bar{h},$$

is well-defined, and we have

$$\begin{aligned} \tilde{h}\gamma(z, t) = \gamma'\bar{h}(z, t) &= \begin{cases} y_0, & 0\leqq t\leqq t_0, \\ \gamma'\varepsilon\kappa\gamma(z, t), & t_0\leqq t\leqq 1, \end{cases} \\ &= p_C\bar{\gamma}\varepsilon\kappa\gamma(z, t) \\ &= p_C\phi\kappa\gamma(z, t), \end{aligned}$$

where $p_C: \Sigma A\vee C\to C$ is the projection onto C. Therefore, we have $\tilde{h}=p_C\phi\kappa$, because γ is the identification map.

On the other hand, by the commutativity of (2.4) and (2.3),

$$p_C\phi\kappa = p_C(\kappa_0\vee\kappa)\phi_Z = \kappa p_2\phi_Z \sim \kappa .$$

Hence, h is a homotopy equivalence, and therefore the triple $(h, \bar{h}, \tilde{h})$ is a cofibre map of (Z, i_Z, TZ) into (A, q, B). q.e.d.

REMARK. The maps ϕ_Z and ϕ define the cooperators in (Z, i_Z, TZ) and (A, q, B), respectively. Hence, the commutativity of (2.4) means that the homotopy equivalence κ commutes with the cooperators.

As a sufficient condition that the diagram (2.4) is commutative, we have the following lemma.

LEMMA (2.6) If Y is the space obtained from X by attaching cells, independently to each other, (i.e., $Y=X\cup\bigcup_i e_i$), or, more generally, if Y is the space obtained from X by attaching a space TZ by a map $u: Z\to X$, then, the commutativity of (2.4) holds for the cofibre space (X, i_f, Y_f) associated with the inclusion map $f: X\to Y$.

Proof. If is sufficient to prove the latter case. Since C_f is homotopy equivalent to Y/fX, it is also to ΣZ by the map $\kappa: \Sigma Z\to C_f$ such that

$$\begin{aligned}\kappa(z, t) &= (u(z), 1-2t), && 0\leqq t\leqq 1/2,\\ &= (z, 2t-1), && 1/2\leqq t\leqq 1.\end{aligned}$$

Let $\phi_Z: \Sigma Z\to\Sigma Z\vee\Sigma Z$ and $\phi: C_f\to\Sigma X\vee C_f$ be the maps defined by

$$\begin{aligned}\phi_Z(z, t) &= ((z, 4t), z_0), && 0\leqq t\leqq 1/4,\\ &= (z_0, (z, (4t-1)/2)), && 1/4\leqq t\leqq 1/2,\\ &= (z_0, (z, t)), && 1/2\leqq t\leqq 1,\end{aligned}$$

and

$$\begin{aligned}\phi(y) &= (y_0, y), && y\in Y,\\ \phi(x, t) &= (y_0, (x, 2t)), && x\in X,\ 0\leqq t\leqq 1/2,\\ &= (y_0, y_0), && x\in X,\ 1/2\leqq t\leqq 3/4,\\ &= ((x, 4t-3), y_0), && x\in X,\ 3/4\leqq t\leqq 1.\end{aligned}$$

Then, the commutativity of (2.4) is easily verified. q.e.d.

3. An exact sequence

For given spaces X and Y, the set *of* homotopy classes of maps $X\to Y$ is denoted by $\pi(X, Y)$, and the constant map and the class containing it by the same letter 0.

THEOREM (3.1) Let X and Y be simply connected spaces and $f: X\to Y$ be a map. Assume that the cofibre C_f of the cofibre space (X, i_f, Y_f) associated with f is homotopy equivalent to the suspension ΣZ of a spaces Z such that the diagram (2.4) is commutative. Then, there exists a cofibre map $h: (Z, i_Z, TZ)\to(X, i_f, Y_f)$ and the following sequence of sets of homotopy classes is exact for any space V:

$$\pi(C_f, V)\xrightarrow{\gamma^*}\pi(Y, V)\xrightarrow{f^*}\pi(X, V)\xrightarrow{h^*}\pi(Z, V).$$

Proof. The exactness of the first three terms is well-known. (See Puppe [7], p. 305)

The existence of a cofibre map $(h, \bar{h}, \hat{h}): (Z, i_Z, TZ)\to(X, i_f, Y_f)$ such that

$i_f h = \bar{h} i_Z$ is proved in (2.5). Since Y is a deformation retract of Y_f, there is a retraction $r: Y_f \to Y$ such that $r i_f = f$. Since TZ is contractible, $i_Z \sim 0$ and hence $fh = r i_f h = r h i_Z \sim 0$. Therefore, we have $h^* f^* = 0$, i.e., $\operatorname{Ker} h^* \supset \operatorname{Im} f^*$.

Conversely, let $g: X \to V$ be a map such that $gh \sim 0$. Then, there is a map $G: TZ \to V$ defined by a null-homotopy of gh. We define the map $G': h_{\#}(TZ) \to V$ by

$$\begin{aligned} G'(i_{TZ}(z, t)) &= G(z, t), & z \in Z,\ t \in I, \\ G'(i_X(x)) &= g(x), & x \in X, \end{aligned}$$

where $h_{\#}(TZ)$ is the space defined in (1.7). Since X, Y are simply connected, Y_f is homotopy equivalent to $h_{\#}(TZ)$, by (1.10). Hence, there is a map G'': $Y_f \to V$ such that $G'' i_f \sim g$. Therefore, the map $g' = G'' j: Y \to V$ satisfies $g' f \sim g$, where $j: Y \to Y_f$ is the inclusion map. This shows that $\operatorname{Ker} h^* \subset \operatorname{Im} f^*$, and we have the exactness of the last three terms. q.e.d.

4. Dual situation for fibre spaces

DEFINITION (4.1) A triple (E, p, B) of two spaces E, B and a map $p: E \to B$ is called a (strong) *fibre space*, if the homotopy lifting property holds for any space U (which is not necessarily homotopy equivalent to a CW-complex), i.e., for any homotopy $g_t: U \to B$ of $g_0 = p f_0$, there is a homotopy $f_t: U \to E$ of f_0 such that $g_t = p f_t$.

For a fibre space (E, p, B), the space $F = p^{-1}(b_0)$ is called the *fibre*; E and B are called the total space and the base, respectively. We shall denote the injection $F \to E$ by i.

DEFINITION (4.2) Let (E, p, B) and (E', p', B') be two fibre spaces whose fibres F and F' are homotopy equivalent. A triple of maps $(\tilde{f}, \bar{f}, f)$ is called a *fibre map* if the following diagram is commutative:

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ \tilde{f}\downarrow & & \bar{f}\downarrow & & \downarrow f \\ F' & \xrightarrow{i'} & E' & \xrightarrow{p'} & B' \end{array}$$

and $\tilde{f}$ is a homotopy equivalence.

The following proposition is well-known. (See Serre [8], p. 479)

PROPOSITION (4.3) Let X, Y be spaces, $f: X \to Y$ be a map and Y_f be its mapping cylinder. Then, the triple $(\Omega(Y_f; X, Y_f), p, Y_f)$ is a fibre space where $\Omega(Y_f; X, Y_f)$ is the set of all maps $l: ([0, r]; 0, r) \to (Y_f; X, Y_f)$, $0 < r < +\infty$, with the compact open topology and $p: \Omega(Y_f; X, Y_f) \to Y_f$ is the map defined by $p(l) = l(r)$. (We shall call such a triple the *fibre space associated with the map f*.)

Now, assume that the fibre $F_f = \Omega(Y_f; X, \bar{y}_0)$ of $(\Omega(Y_f; X, Y_f), p, Y_f)$, $(\bar{y}_0$

is the base point of Y_f), is homotopy equivalent to the loop space ΩZ of a space Z, and we shall consider the following diagram:

$$(4.4) \qquad \begin{array}{ccc} \Omega(Y_f;\, X, \bar{y}_0)\times \Omega Y_f & \xrightarrow{\ \phi\ } & \Omega(Y_f;\, X, \bar{y}_0) \\ {\scriptstyle \kappa\times\kappa_0}\downarrow & & \downarrow{\scriptstyle \kappa} \\ \Omega Z\times \Omega Z & \xrightarrow{\ \phi_Z\ } & \Omega Z \end{array}$$

where $\kappa: \Omega(Y_f;\, X, \bar{y}_0)\to \Omega Z$ is a homotopy equivalence, κ_0 is its restriction to ΩY_f and $\phi: \Omega(Y_f;\, X, \bar{y}_0)\times \Omega Y_f\to \Omega(Y_f;\, X, \bar{y}_0)$ and $\phi_Z: \Omega Z\times\Omega Z\to\Omega Z$ are the maps defined by the path addition $^\vee$ in the sense of Moore.

PROPOSITION (4.5) Let X, Y be two simplicial complexes and $f: X\to Y$ be a simplicial map. Assume that the fibre $\Omega(Y_f: X, \bar{y}_0)$ of the fibre space $(\Omega(Y_f;\, X, Y_f), p, Y_f)$ is homotopy equivalent to the loop space ΩZ of a space Z, and the diagram (4.4) is commutative. Then, there exists a fibre map $(\tilde{h}, \bar{h}, h)$: $(\Omega(Y_f;\, X, Y_f), p, Y_f)\to (LZ, p_Z, Z)$ such that $\tilde{h}: \Omega(Y_f;\, X, \bar{y}_0)\to\Omega Z$ is the given homotopy equivalence κ, where LZ is the path space over Z.

The proof of this proposition will be given in the next section.

REMARK. The maps ϕ and ϕ_Z define the operators in $(\Omega(Y_f;\, X, Y_f), p, Y_f)$ and (LZ, p_Z, Z), respectively, in the sense of Eckmann-Hilton [2]. Hence, the commutativity of (4.4) means that the homotopy equivalence κ commutes with the operators.

THEOREM (4.6) Let X, Y be two spaces and $f: X\to Y$ be a map. Assume that the fibre F_f of the fibre space $(\Omega(Y_f;\, X, Y_f), p, Y_f)$ associated with f is homotopy equivalent to the loop space ΩZ of a space Z such that the diagram (4.4) is commutative. Then, the following sequence of sets of homotopy classes is exact for any space U:

$$\pi(U, F_f) \xrightarrow{\ i_*\ } \pi(U, X) \xrightarrow{\ f_*\ } \pi(U, Y) \xrightarrow{\ h_*\ } \pi(U, Z)\,.$$

Proof. The exactness of the first three terms is well-known. (For example, see Nomura [5], p. 118)

Since X, Y have the homotopy type of a CW-complex, and any CW-complex has the homotopy type of a simplicial complex (see Milnor [4], Theorem 2), we may replace X, Y by simplicial complexes K_X, K_Y, respectively, and f by a simplicial map φ. Hence, it is easily seen that $\Omega(K_\varphi;\, K_X, K_\varphi)$ is homotopy equivalent to $\Omega(Y_f;\, X, Y_f)$ where K_φ is the mapping cylinder of $\varphi: K_X\to K_Y$. Since the commutativity of (4.4) for $(\Omega(Y_f;\, X, Y_f), p, Y_f)$ implies that for $(\Omega(K_\varphi;\, K_X, K_\varphi), p, K_\varphi)$, (4.5) shows the existence of a fibre map h: $(\Omega(K_\varphi;\, K_X, K_\varphi), p, K_\varphi)\to (LZ, p_Z, Z)$ such that $p_Z\bar{h}=hp$. Since LZ is contractible, we have $hp\sim 0$. On the other hand, the injection map i_X: $K_X\to\Omega(K_\varphi;\, K_X, K_\varphi)$ is a homotopy equivalence and $pi_X=\varphi$. Hence we have $h_*\varphi_*=0$, i.e., $\operatorname{Ker} h_*\supset \operatorname{Im}\varphi_*$.

Conversely, let $g: U \to Y$ be a map such that $hg \sim 0$. Then, there is a map $G: U \to LZ$ defined by a null-homotopy of hg. As is well-known, $\Omega(K_\varphi; K_X, K_\varphi)$ is homotopy equivalent to the space which consists of all pairs (y, l), $y \in K_\varphi$, $l \in LZ$ such that $h(y) = p_Z(l)$. Therefore, there is a map $g': U \to \Omega(K_\varphi; K_X, K_\varphi)$ defined by g and G, and satisfies $pg' \sim g$. This shows that there is a map $g'': U \to K_X$ such that $\varphi g'' \sim g$, and we have $\operatorname{Ker} h_* \subset \operatorname{Im} \varphi_*$.

Since X, Y are homotopy equivalent to K_X, K_Y, and f is equivalent to φ, the exactness of the last three terms is proved. q.e.d.

THEOREM (4.7) Let X and Y be spaces and $f: X \to Y$ be a map. Let F_f be the fibre $\Omega(Y_f; X, \bar{y}_0)$ of the fibre space associated with f, and C_f be the cofibre $Y_f/i_f X$ of the cofibre space associated with f.

1) Assume that X and Y are simply connected, and C_f is homotopy equivalent to the suspension of a space Z such that the diagram (2.4) is commutative. Then, the following sequence is exact for any space V:

$$\pi(C_f, V) \xrightarrow{\gamma^*} \pi(Y, V) \xrightarrow{f^*} \pi(X, V) \xrightarrow{i^*} \pi(F_f, V) .$$

2) Assume that F_f is homotopy equivalent to the loop space of a space Z' such that the diagram (4.4) is commutative. Then, the following sequence is exact for any space U:

$$\pi(U, F_f) \xrightarrow{i_*} \pi(U, X) \xrightarrow{f_*} \pi(U, V) \xrightarrow{\gamma_*} \pi(U, C_f) .$$

Proof. 1) By (4.6), the sequence

$$\pi(Z, F_f) \xrightarrow{i_*} \pi(Z, X) \xrightarrow{f_*} \pi(Z, Y)$$

is exact. On the other hand, by (2.5), there is a map $h: Z \to X$ such that $fh \sim 0$. Hence, there is a map $u: Z \to F_f$ such that $iu \sim h$.

Consider the following diagram

$$\begin{array}{ccccccc} \pi(C_f, V) & \xrightarrow{\gamma^*} & \pi(Y, V) & \xrightarrow{f^*} & \pi(X, V) & \xrightarrow{h^*} & \pi(Z, V) \\ & & & & & {\scriptstyle i^*}\searrow & \uparrow{\scriptstyle u^*} \\ & & & & & & \pi(F_f, V) \end{array}$$

where the horizontal sequence is exact, by (3.1). If $\alpha \in \pi(X, V)$ is an element such that $i^*(\alpha) = 0$, then we have $h^*(\alpha) = u^* i^*(\alpha) = 0$. Therefore, there is an element $\beta \in \pi(Y, V)$ such that $f^*(\beta) = \alpha$. Hence, $\operatorname{Ker} i^* \subset \operatorname{Im} f^*$. $\operatorname{Ker} i^* \supset \operatorname{Im} f^*$ is obviuos.

2) A similar argument is valid for this case. q.e.d.

The following corollary is an immediate consequence of (4.7).

COROLLARY (4.8) 1) Under the same assumptions of (4.7), 1), $f^*: \pi(Y, V) \to \pi(X, V)$ is onto for any space V if, and only if, F_f is contractible in X.

2) Under the same assumptions of (4.7), 2), $f_*: \pi(U, X) \to \pi(U, Y)$ is onto for any space U if, and only if, $\gamma \sim 0$.

5. Proof of (4.5)

For a convenience, we shall denote Y_f by Y, $\Omega(Y; X, Y)$ by E, $\Omega(Y; X, y_0)$ by F and the portion of E over a subset Y' of Y by $E|Y'$. Also, denote the k-skeleton of Y by Y^k.

Define the maps $\bar{h}_0: E|Y^0 \to LZ$ and $h_0: Y^0 \to Z$ by

$$\bar{h}_0(l) = \kappa(l \vee \tilde{l}_v^{-1}), \qquad \text{for } v \in Y,\ l \in E|v,$$

and

$$h_0(v) = z_0 \qquad (\text{base point of } Z),$$

where $\tilde{l}_v$ is a fixed path starting at y_0 and ending at v. Then obviuosly we have $pz\bar{h}_0 = h_0 p$, and $\bar{h}_0|F = \kappa$.

We shall prove the proposition by the induction on the dimension of skeleton of Y under the following assumption:

Assumption $(5.1)_k$: For $k \geqq 0$, maps $\bar{h}_k: E|Y^k \to LZ$ and $h_k: Y^k \to Z$ are defined such that

$$pz\bar{h}_k = h_k p,$$

and the following diagram is commutative:

$$(5.2) \qquad \begin{array}{ccc} \Omega(Y; X, y_0) \times \Omega(Y; y_0, Y^k) & \longrightarrow & \Omega(Y; X, Y^k) \\ \downarrow{\scriptstyle \kappa \times \bar{h}_k^0} & & \downarrow{\scriptstyle \bar{h}_k} \\ \Omega Z \times LZ & \longrightarrow & LZ \end{array}$$

where $\bar{h}_k^0: \Omega(Y; y_0, Y^k) \to LZ$ is the restriction of $\bar{h}_k$ to $\Omega(Y; y_0, Y^k) \subset \Omega(Y; X, Y^k)$ and the horizontal maps are defined by the path additions.

The assumption $(5.1)_k$ is satisfied when $k=0$. For, if $\bar{\lambda} \in \Omega(Y; X, y_0)$ and $\bar{\mu} \in \Omega(Y; y_0, v)$,

$$\begin{aligned} \kappa(\bar{\lambda}) \vee \bar{h}_v(\bar{\mu}) &= \kappa(\bar{\lambda}) \vee \kappa(\bar{\mu} \vee \tilde{l}_v^{-1}) \\ &= \kappa(\bar{\lambda} \vee \bar{\mu} \vee \tilde{l}_v^{-1}) \qquad (\text{by } (4.4)) \\ &= h_v(\bar{\lambda} \vee \bar{\mu}). \end{aligned}$$

Now, assume $(5.1)_{n-1}$: Let s^n be an n-simplex of Y. Each point y of s^n is represented by

$$y = (x, t), \qquad x \in \dot{s}^n,\ t \in I,$$

such that

$$(x, 0) = x \quad \text{and} \quad (x, 1) = v,$$

for a fixed point v in $s^n - \dot{s}^n$. Hence, there is a continuous set of paths μ_y, $y \in s^n$, definep by

$$\mu_y(s) = (x, t+s), \quad 0 \leqq s \leqq 1-t, \qquad \text{for } y = (x, t).$$

Define the maps $g_{s^n}: s^n \to LZ$ and $h_{s^n}: s^n \to Z$ by

$$\begin{aligned} g_{s^n}(x, t) &= \bar{h}_{n-1}(\tilde{l}_v \vee \mu_x^{-1}), & 0 \leqq t \leqq 1/2, \\ &= \bar{h}_{n-1}(\tilde{l}_v \vee \mu_x^{-1}) \,|\, (2-2t), & 1/2 \leqq t \leqq 1, \end{aligned}$$

and

$$h_{s^n}(x, t) = p_Z g_{s^n}(x, t) ,$$

where $\tilde{l}_v$ is a fixed path starting at y_0 and ending at v, and $l|t=l|[0, tr]$, for $l: [0, r] \to Z$.

Then, by $(5.1)_{n-1}$,

$$h_{s^n}(x) = p_Z\bar{h}_{n-1}(\tilde{l}_v \vee \mu_x^{-1}) = h_{n-1}p(\tilde{l}_v \vee \mu_x^{-1}) = h_{n-1}(x) , \qquad \text{for } x \in \dot{s}^n .$$

Therefore, $h_{s^n}|\dot{s}^n = h_{n-1}$.

Let μ'_y, $y=(x, t)$, be the path defined by

$$\begin{aligned} \mu'_y(s) &= (x, t-s) , \quad 0 \leqq s \leqq t , \qquad 0 \leqq t \leqq 1/2 , \\ &= (x, t-s) , \quad 0 \leqq s \leqq 1-t , \quad 1/2 \leqq t \leqq 1 . \end{aligned}$$

Since $\mu_x \vee \tilde{l}_v^{-1} \vee \tilde{l}_v \vee \mu_x^{-1}$ is homotopic in LY to the constant path, fixing its end points, let $F_u: \dot{s}^n \to LY$ be its homotopy such that

$$F_0(x) = \mu_x \vee \tilde{l}_v^{-1} \vee \tilde{l}_v \vee \mu_x^{-1} , \qquad F_1(x) = \text{constant path at } x.$$

Define the map $\bar{h}_{s^n}: E|s^n \to LZ$ by

$$\begin{aligned} \bar{h}_{s^n}(l) &= \bar{h}_{n-1}(l \vee \mu'_y \vee F_{1-2t}(x)) , & 0 \leqq t \leqq 1/2 , \\ &= \kappa(l \vee \mu'_y \vee \mu'^{-1}_y \vee \mu_y \vee \tilde{l}_v^{-1}) \vee g_{s^n}(y) , & 1/2 \leqq t \leqq 1 , \end{aligned}$$

for $y=(x, t)$ and $l \in E|y$. Then, if $y=(x, 1/2)$ and $l \in E|y$,

$$\begin{aligned} \bar{h}_{n-1}(l \vee \mu'_y \vee F_0) &= \bar{h}_{n-1}(l \vee \mu'_y \vee \mu_x \vee \tilde{l}_v^{-1} \vee \tilde{l}_v \vee \mu_x^{-1}) \\ &= \bar{h}_{n-1}(l \vee \mu'_y \vee \mu'^{-1}_y \vee \mu_y \vee \tilde{l}_v^{-1} \vee \tilde{l}_v \vee \mu_x^{-1}) \\ &= \kappa(l \vee \mu'_y \vee \mu'^{-1}_y \vee \mu_y \vee \tilde{l}_v^{-1}) \vee \bar{h}_{n-1}(\tilde{l}_v \vee \mu_x^{-1}) \qquad \text{(by (5.2))} \\ &= \kappa(l \vee \mu'_y \vee \mu'^{-1}_y \vee \mu_y \vee \tilde{l}_v^{-1}) \vee g_{s^n}(y) , \end{aligned}$$

since $\mu'^{-1}_y \vee \mu_y = \mu_x$, $0 \leqq t \leqq 1/2$. Therefore, $\bar{h}_{s^n}$ is well-defined.

If $l \in E|x$, $x=(x, 0) \in \dot{s}^n$,

$$\bar{h}_{s^n}(l) = \bar{h}_{n-1}(l \vee \mu'_x \vee F_1) = \bar{h}_{n-1}(l) ,$$

since μ'_x and F_1 are constant paths. Hence $\bar{h}_{s^n}|(E|\dot{s}^n) = \bar{h}_{n-1}$.

It is easily verified that $p_Z\bar{h}_{s^n} = h_{s^n}p$, and also

$$\kappa(\bar{\lambda}) \vee \bar{h}_{s^n}(\bar{\mu}) = \bar{h}_{s^n}(\bar{\lambda} \vee \bar{\mu})$$

for $\bar{\lambda} \in \Omega(Y; X, y_0)$, $\bar{\mu} \in \Omega(Y; y_0, s^n)$.

Thus, the maps $\bar{h}_n: E|Y^n \to LZ$ and $h_n: Y^n \to Z$, defined by

$$\bar{h}_n|(E|s^n) = \bar{h}_{s^n} , \quad \text{and} \quad h_n|s^n = h_{s^n} ,$$

for a simplex $s^n \in Y^n$, are well-defined and satisfy the inductive assumption $(5.1)_n$. This completes the proof of Proposition (4.5).

References

[1] B. Eckmann and P. J. Hilton, Groupes d'homotopie et dualité, I, II, C. R. Acad. Sci. Paris, 246 (1958), pp. 2444–2447 and pp. 2555–2558.

[2] B. Eckmann and P. J. Hilton, Operators and Cooperators in homotopy theory, Math. Ann., 141 (1960), pp. 1–21.

[3] P. J. Hilton, Homotopy theory and duality (mimeographed), Cornell University, 1959.

[4] J. Milnor, On spaces having the homotopy type of a CW-complex, Trans. Amer. Math. Soc., 90 (1959), pp. 272–280.

[5] Y. Nomura, On mapping sequences, Nagoya Math. J., 17 (1960), pp. 111–145.

[6] P. Olum, Non abelian cohomology and van Kampen's theorem, Ann. of Math., 68 (1959), pp. 658–668.

[7] D. Puppe, Homotopiemengen und ihre indizierte Abbildungen I, Math. Zeit., 68 (1958), pp. 299–344.

[8] J. P. Serre, Homologie singulaire des espaces fibrés, Ann. of Math., 54 (1951), pp. 425–505.