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Cohomology modulo 2 of the compact exceptional groups E_6 and E_7

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Introduction

In the present work we determine the cohomology algebras modulo 2 of the compact *simply connected* exceptional Lie groups $\boldsymbol{E}_6$ and $\boldsymbol{E}_7$. The results are stated in Theorems 2.7 and 3.13.

For this purpose we compute the cohomology of loop spaces of the compact symmetric spaces $\boldsymbol{E}_6/\boldsymbol{F}_4$ and $\boldsymbol{E}_7/\boldsymbol{E}_6\times\boldsymbol{T}^1$ in lower degrees, using the method of Bott-Samelson [11]. On the course of this argument we see that all K-cycles [11] which describe the corresponding homology basis are orientable. This fact will be used elsewhere.

As another tool we use the spectral sequence arguments.

§1. Compact symmetric spaces.

1.1. Let G be a compact connected Lie group and T be a maximal torus in G. Denote by $\mathfrak{g}$ and $\mathfrak{t}$ the Lie algebras of G and T respectively. Using a positive definite invariant metric $(\ ,\ )$ in $\mathfrak{g}$ we have the Cartan orthogonal decomposition

$$\mathfrak{g} = t + \sum \mathfrak{e}_\alpha \tag{1.1}$$

where the summation runs over all positive roots α of $\mathfrak{g}$ relative to $\mathfrak{t}$ with respect to a linear order in $\mathfrak{t}^*$ (dual space of $\mathfrak{t}$). Since we are concerned only with compact symmetric spaces we mean by "roots" the angular parameters in the sence of E. Cartan. The space $\mathfrak{e}_\alpha$ is of dimension 2 and invariant under the adjoint action of T (or $\mathfrak{t}$). The adjoint action of $t\in T$ on $\mathfrak{e}_\alpha$ is a rotation through the angle $2\pi\alpha(t)$. We put $\mathfrak{e}_{-\alpha}=\mathfrak{e}_\alpha$ by convention.

We can choose an ortho-normal basis $\{U_\alpha,\ V_\alpha\}$ in $\mathfrak{e}_\alpha$ to satisfy the following structure equations:

$$\begin{aligned} &[H,\ U_\alpha] = 2\pi\alpha(H)V_\alpha \\ &[H,\ U_\alpha] = -2\pi\alpha(H)U_\alpha \qquad \text{for any}\quad H\in\mathfrak{t}, \\ &[U_\alpha,\ V_\alpha] = 2\pi H_\alpha \end{aligned} \tag{1.2}$$

where $H_\alpha\in\mathfrak{t}$ is defined by $(H_\alpha,\ H)=\alpha(H)$ for all $H\in\mathfrak{t}$. The frame $\{U_\alpha,\ V_\alpha\}$ is determined not uniquely, but up to rotations in $\mathfrak{e}_\alpha$. Hence, in particular the orientation of $\mathfrak{e}_\alpha$ is determined uniquely by any choice of the ortho-normal frame

$\{U_\alpha, V_\alpha\}$. Any frame $\{U_\alpha, V_\alpha\}$ satisfying (1.2) is called a *basic frame* of the oriented plane $\mathfrak{e}_\alpha$. If we put $U_{-\alpha}=U_\alpha$ and $V_{-\alpha}=-V_\alpha$, then the pair $\{U_{-\alpha}, V_{-\alpha}\}$ satisfies the equations (1.2) for the root $-\alpha$. Hence the orientation of $\mathfrak{e}_{-\alpha}$ given by a basic frame $\{U_{-\alpha}, V_{-\alpha}\}$ is opposite to that of $\mathfrak{e}_\alpha$ given by $\{U_\alpha, V_\alpha\}$. In this sense we can distinguish $\mathfrak{e}_\alpha$ from $\mathfrak{e}_{-\alpha}$ as oriented 2-planes.

If we choose a basic frame $\{U_\alpha, V_\alpha\}$ for each positive root α, we obtain an additive basis of $\mathfrak{t}^\perp$ (orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$). The so obtained basis is called a *canonical* basis of $(\mathfrak{g}, \mathfrak{t})$.

1.2. Let G be a compact connected Lie group and K be a closed connected subgroup of G. If there exists an involutive automorphism σ of G such that K is the e-component of the group $\hat{K}$ consisting of all fixed elements under σ, then the pair (G, K) is called a symmetric pair and the homogeneous space $G/\tilde{K}$ $(K\subseteq \tilde{K}\subseteq \hat{K})$ is a compact Riemannian symmetric space ([11]).

The symmetric space $G/\hat{K}$ can be canonically identified with a totally geodesic subspace M of G endowed with an invariant Riemannian metric. M is the diffeomorphic image of $G/\hat{K}$ under the map $\eta_*: G/K\to G$ induced by a map $\eta: G\to G$ defined by $\eta(g)=g\cdot\sigma(g^{-1})$. (Cf., [10]).

In this work we are concerned only with the case G is simply connected, then $K=\hat{K}$ by [11, 12] so that G/K is identified with M by η_*.

For a subset L of M its centralizer in K is denoted by $Z_K(L)$. An element a of M such that $Z_K(a)=K$ is called a *central* element of M. Such an element is contained in every maximal torus of M.

By $\mathfrak{k}$ we denote the Lie algebra of K and let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then $M=\exp\mathfrak{m}$ as is well known.

The involution σ of G induces an involutive automorphism of $\mathfrak{g}$ denoted by the same letter σ. $\mathfrak{k}$ is the eigenspace of σ with eigenvalue 1 and $\mathfrak{m}$ is the eigenspace with eigenvalue -1.

1.3. Let $\mathfrak{t}_-$ be a Cartan subalgebra of $\mathfrak{m}$, i.e., $T_-=\exp(\mathfrak{t}_-)$ is a maximal torus of M. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}_-$. Put $\mathfrak{t}_0=\mathfrak{t}\cap\mathfrak{k}$, then

$$\mathfrak{t}=\mathfrak{t}_0+\mathfrak{t}_- \quad \text{(direct sum)}. \tag{1.3}$$

$T=\exp(\mathfrak{t})$ is a maximal torus of G. Further we put $T_0=\exp(\mathfrak{t}_0)$, then $T=T_0T_-$ and $T_0\cap T_-$ is a finite group.

By (1.3) we see that $\mathfrak{t}$ is invariant by σ, whence the transposed linear map $\sigma^t: \mathfrak{t}^*\to\mathfrak{t}^*$ is induced. We put $\sigma^*=-\sigma^t$. σ^*-images of roots are also roots.

Let $\mathfrak{r}$ denote the system of non-zero roots of $\mathfrak{g}$ relative to $\mathfrak{t}$. By easy computations we see that

$$\sigma H_\alpha=-H_{\sigma^*\alpha} \quad \text{for all} \quad \alpha\in\mathfrak{r}, \tag{1.4}$$

and

(1.5) $$\sigma\mathfrak{e}_\alpha = \mathfrak{e}_{\sigma^*\alpha}.$$

Further, we see immediately,

(1.6) *if $\{U_\alpha, V_\alpha\}$ is a basic frame of $\mathfrak{e}_\alpha$, then $\{\sigma V_\alpha, \sigma U_\alpha\}$ is a basic frame of $\mathfrak{e}_{\sigma^*\alpha}$ for any $\alpha\in\mathfrak{r}$.*

For each pair $(\alpha, \sigma^*\alpha)$ of roots we say that a basic frame $\{U_\alpha, V_\alpha\}$ of $\mathfrak{e}_\alpha$ is *σ-related* to a basic frame $\{U_{\sigma^*\alpha}, V_{\sigma^*\alpha}\}$ of $\mathfrak{e}_{\sigma^*\alpha}$ when the relations

(1.7) $$\sigma U_\alpha = U_{\sigma^*\alpha}, \quad \sigma V_\alpha = -V_{\sigma^*\alpha}$$

are satisfied. When $\{U_\alpha, V_\alpha\}$ is σ-related to $\{U_{\sigma^*\alpha}, V_{\sigma^*\alpha}\}$, then $\{U_{\sigma^*\alpha}, V_{\sigma^*\alpha}\}$ is σ-related to $\{U_\alpha, V_\alpha\}$ and $\{U_{-\alpha}, V_{-\alpha}\}$ is σ-related to $\{U_{-\sigma^*\alpha}, V_{-\sigma^*\alpha}\}$, where $U_{-\beta}=U_\beta$ and $V_{-\beta}=-V_\beta$ for $\beta=\alpha$ or $\sigma^*\alpha$.

A canonical basis of $(\mathfrak{g}, \mathfrak{t})$ determines a basic frame $\{U_\alpha, V_\alpha\}$ of $\mathfrak{e}_\alpha$ for each positive root α. Let us use the convention that $U_{-\alpha}=U_\alpha$ and $V_{-\alpha}=-V_\alpha$, then a canonical basis of $(\mathfrak{g}, \mathfrak{t})$ determines a basic frame $\{U_\alpha, V_\alpha\}$ of $\mathfrak{e}_\alpha$ for each $\alpha\in\mathfrak{r}$. These basic frames are called the associated frames to the canonical basis. A canonical basis of $(\mathfrak{g}, \mathfrak{t})$ is said to be *σ-normalized* when the associated frame $\{U_\alpha, V_\alpha\}$ is σ-related to the associated frame $\{U_{\sigma^*\alpha}, V_{\sigma^*\alpha}\}$ for any $\alpha\in\mathfrak{r}$.

PROPOSITION 1.1. *Under the above choice of the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$* (i.e., if $\mathfrak{t}$ contains a Cartan subalgebra $\mathfrak{t}_-$ of $\mathfrak{m}$), *there exists a σ-normalized basis of* $(\mathfrak{g}, \mathfrak{t})$.

(Proof) Two transformations, $\alpha\to-\alpha$ and $\alpha\to\sigma^*\alpha$, of $\mathfrak{r}$ generates a transformation group Γ on $\mathfrak{r}$. Choose a positive root as a representative for each Γ-orbit. For each representative positive root α we discuss three cases separatedly.

Case i). $\sigma^*\alpha\neq\pm\alpha$. In this case the orbit $\Gamma\alpha$ consists of 4 roots $\pm\alpha$, $\pm\sigma^*\alpha$. Choose a basic frame $\{U_\alpha, V_\alpha\}$ of $\mathfrak{e}_\alpha$ arbitrarily. By (1.6) $\{\sigma V_\alpha, \sigma U_\alpha\}$ is a basic frame of $\mathfrak{e}_{\sigma^*\alpha}$. Rotate this frame by angle $\pi/2$, then we obtain a basic frame $\{\sigma U_\alpha, -\sigma V_\alpha\}$ of $\mathfrak{e}_{\sigma^*\alpha}$. We put $U_{\sigma^*\alpha}=\sigma U_\alpha$ and $V_{\sigma^*\alpha}=-\sigma V_\alpha$, then $\{U_{\sigma^*\alpha}, V_{\sigma^*\alpha}\}$ is a basic frame of $\mathfrak{e}_{\sigma^*\alpha}$ and $\{U_\alpha, V_\alpha\}$ is σ-related to $\{U_{\sigma^*\alpha}, V_{\sigma^*\alpha}\}$. When $\sigma^*\alpha$ is a negative root, we put $U_{-\sigma^*\alpha}=U_{\sigma^*\alpha}$, $V_{\sigma^*\alpha}=-V_{-\sigma^*\alpha}$ by our convention.

Case ii). $\sigma^*\alpha=\alpha$. In this case the orbit $\Gamma\alpha$ consists of 2 roots $\pm\alpha$. If we take a basic frame $\{U'_\alpha, V'_\alpha\}$ of $\mathfrak{e}_\alpha$, then $\{\sigma V'_\alpha, \sigma U'_\alpha\}$ is also a basic frame of $\mathfrak{e}_\alpha$ by (1.6). Hence the frames $\{U'_\alpha, V'_\alpha\}$ and $\{\sigma U'_\alpha, \sigma V'_\alpha\}$ determine opposite orientation of $\mathfrak{e}_\alpha$, i.e., the restriction $\sigma|\mathfrak{e}_\alpha$ is a reflection in the plane $\mathfrak{e}_\alpha$. We have a unit vector U_α of $\mathfrak{e}_\alpha$, which is invariant under σ. Choose another vector V_α of $\mathfrak{e}_\alpha$ so as to make the pair $\{U_\alpha, V_\alpha\}$ a basic frame of $\mathfrak{e}_\alpha$. Then $\sigma V_\alpha=-V_\alpha=-V_{\sigma^*\alpha}$. Hence $\{U_\alpha, V_\alpha\}$ is σ-related to itself.

Case iii). $\sigma^*\alpha=-\alpha$. The orbit $\Gamma\alpha$ consists of 2 roots $\pm\alpha$. Let $\mathfrak{g}'$ be the semi-simple part of $\mathfrak{g}$. $\mathfrak{g}'$ is invariant under σ, and $\mathfrak{t}'=\mathfrak{t}\cap\mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}'$ satisfying the assumption of our proposition for $\mathfrak{g}'$ and $\sigma'=\sigma|\mathfrak{g}'$. Let $\mathfrak{g}'_c$ and $\mathfrak{t}'_c$ be the complexification of $\mathfrak{g}'$ and $\mathfrak{t}'$ respectively. Let $\{E_\beta; \beta\in\mathfrak{r}\}$ be a Weyl base of $\mathfrak{g}'_c$ relative to $\mathfrak{t}'_c$. The involution σ' of $\mathfrak{g}'$ is extended uniquely to an anti-involution of $\mathfrak{g}'$. Under this situation the Lemma 5 of [17, Exp. 11] can be applied and we conclude that

$$(*) \qquad \sigma' E_\alpha = E_{-\alpha}\,.$$

If we put $U_\alpha=c(E_\alpha+E_{-\alpha})$ and $V_\alpha=\sqrt{-1}\,c(E_\alpha-E_{-\alpha})$, where c is a positive constant to normalize $(U_\alpha, U_\alpha)=(V_\alpha, V_\alpha)=1$, then the pair $\{U_\alpha, V_\alpha\}$ is a basic frame of $\mathfrak{e}_\alpha$ as is easily seen. Then $\sigma U_\alpha=U_\alpha$ and $\sigma V_\alpha=V_\alpha$ by (*). Hence $\{U_\alpha, V_\alpha\}$ is σ-related to the basic frame $\{U_{-\alpha}, V_{-\alpha}\}$ of $\mathfrak{e}_{-\alpha}$, where $U_{-\alpha}=U_\alpha$ and $V_{-\alpha}=-V_\alpha$ by our convention.

The cases i)-iii) exhaust all possible cases. Therefore we have chosen a basic frame $\{U_\alpha, V_\alpha\}$ for each positive root. The totality of these basic frames constitute a canonical basis of $(\mathfrak{g}, \mathfrak{t})$. By our construction it is clear that this canonical basis is σ-normalized. Q.E.D.

1.4. Put

$$\mathfrak{r}_0=\{\alpha\in\mathfrak{r}\,;\,\sigma^*\alpha=-\alpha\}\,.$$

This is a closed subsystem of roots of $\mathfrak{r}$. The restriction of any root $\alpha\in\mathfrak{r}-\mathfrak{r}_0$ to $\mathfrak{t}_-$ is a non-zero linear form on $\mathfrak{t}_-$, called a root of $\mathfrak{m}$ relative to $\mathfrak{t}_-$ [10, 11]. By $\mathfrak{r}_-$ we denote the totality of roots of $\mathfrak{m}$ relative $\mathfrak{t}_-$. About the basic properties of roots and Weyl groups of symmetric spaces we refer to [10, 11, 12, 14].

We remark that the root system $\mathfrak{r}_-$ is slightly different from root system of reductive Lie algebras in the point that 2 times of a root of $\mathfrak{r}_-$ may be a root of $\mathfrak{r}_-$. However fundamental systems of roots of $\mathfrak{r}_-$ are isomorphic to some fundamental systems of roots of some reductive Lie algebras (Cf., Satake [14]).

When G is simply-connected and $\mathfrak{r}_-$ is isomorphic to a root system of some semi-simple Lie algebra, then basic translations in $\mathfrak{r}_-$, corresponding to simple roots of a fundamental system of roots of $\mathfrak{r}_-$, generate the unit lattice $\exp^{-1}(e)\cap\mathfrak{t}_-$ of M, as is easily seen from (6.8) of Bott [10].

1.5. Let $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-)$ be the centralizer of $\mathfrak{t}_-$ in $\mathfrak{k}$. This is the Lie algebra of $Z_K(T_-)$, and we see easily that

$$(1.8) \qquad \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-)=\mathfrak{t}_0+\sum_{\mathfrak{r}_0\ni\alpha>0}\mathfrak{e}_\alpha\,.$$

In particular we have the

PROPOSITION 1.2. *Let l, l_0 and λ be respectively the ranks of G, $Z_K(T_-)$ and G/K. Then*

i) *$l_0 = l - \lambda$ and T_0 is a maximal torus of $Z_K(T_-)$,*

ii) *$\mathfrak{r}_0$, as the set of linear forms on $\mathfrak{t}_0$, is the root system of $Z_K(T_-)$.*

1.6. A linear order in $\mathfrak{t}^*$ satisfying the property: "$\alpha \in \mathfrak{r} - \mathfrak{r}_0$, $\alpha > 0$" implies "$\sigma^*\alpha > 0$", is called a σ-order. The usage of σ-order is convenient to describe the relations between fundamental systems of $\mathfrak{r}_0$, $\mathfrak{r}$ and $\mathfrak{r}_-$. Cf. Satake [14]. Let Δ be a fundamental system of $\mathfrak{r}$ defined by a σ-order, then $\Delta_0 = \Delta \cap \mathfrak{r}_0$ is a fundamental system of roots $\mathfrak{r}_0$, and the subset Δ_- of $\mathfrak{r}_-$, which is obtained by restricting $\Delta - \Delta_0$ to $\mathfrak{t}_-$, is a fundamental system of roots of $\mathfrak{r}_-$ [14].

Choose a σ-order in $\mathfrak{t}^*$ and a σ-normalized canonical basis of $(\mathfrak{g}, \mathfrak{t})$. For each positive root of $\mathfrak{r} - \mathfrak{r}_0$ such that $\sigma^*\alpha \neq \alpha$, we put

$$
\begin{aligned}
A_\alpha &= A_{\sigma^*\alpha} = \frac{1}{\sqrt{2}}(U_\alpha + U_{\sigma^*\alpha}), \quad B_\alpha = B_{\sigma^*\alpha} = \frac{1}{\sqrt{2}}(V_\alpha + V_{\sigma^*\alpha}), \\
A'_\alpha &= -A'_{\sigma^*\alpha} = \frac{1}{\sqrt{2}}(V_\alpha - V_{\sigma^*\alpha}), \quad B'_\alpha = -B'_{\sigma^*\alpha} = \frac{1}{\sqrt{2}}(-U_\alpha + U_{\sigma^*\alpha}).
\end{aligned} \tag{1.9}
$$

Then A_α, $A'_\alpha \in \mathfrak{k}$ and B_α, $B'_\alpha \in \mathfrak{m}$. For each $\alpha \in \mathfrak{r}$ we put $\tilde{\alpha} = \alpha | \mathfrak{t}_-$ and $\hat{\alpha} = \alpha | \mathfrak{t}_0$. Then we have that

$$
\begin{aligned}
[H, A_\alpha] &= 2\pi\tilde{\alpha}(H) B_\alpha, \quad [H, B_\alpha] = -2\pi\tilde{\alpha}(H) A_\alpha, \\
[H, A'_\alpha] &= 2\pi\tilde{\alpha}(H) B'_\alpha, \quad [H, B'_\alpha] = -2\pi\tilde{\alpha}(H) A'_\alpha
\end{aligned} \tag{1.10}
$$

for all $H \in \mathfrak{t}_-$. Further

$$
[H, A_\alpha] = 2\pi\hat{\alpha}(H) A'_\alpha, \quad [H, A'_\alpha] = -2\pi\hat{\alpha}(H) A_\alpha \tag{1.11}
$$

for all $H \in \mathfrak{t}_0$. Here we put

$$
\mathfrak{e}_{\hat{\alpha}} = \text{2-plane spanned by } A_\alpha \text{ and } A'_\alpha. \tag{1.12}
$$

When $\sigma^*\alpha = \alpha$ we see that $U_\alpha \in \mathfrak{k}$ and $V_\alpha \in \mathfrak{m}$, which satisfy

$$
[H, U_\alpha] = 2\pi\tilde{\alpha}(H) V_\alpha, \quad [H, V_\alpha] = -2\pi\tilde{\alpha}(H) U_\alpha \tag{1.10'}
$$

for all $H \in \mathfrak{t}_-$.

Let $\gamma \in \mathfrak{r}_-$ and $\mathfrak{t}_-^\gamma$ denote the singular plane in $\mathfrak{t}_-$ defined by $\gamma(H) = 0$ for $H \in \mathfrak{t}_-$. Further we put $T_-^\gamma = \exp \mathfrak{t}_-^\gamma$. $\mathfrak{z}_\mathfrak{k}(\mathfrak{t}_-^\gamma)$ is the Lie algebra of $Z_K(T_-^\gamma)$.

Hereafter we shall make the following assumption:

(1.*) *$\mathfrak{r}_-$ is isomorphic to a root system of some semi-simple Lie algebra.*

If a positive root γ of $\mathfrak{r}_-$ have even multiplicity, then the set of roots $\mathfrak{r}_\gamma = \{\alpha \in \mathfrak{r}; \tilde{\alpha} = \gamma\}$, is divided to pairs $(\alpha, \sigma^*\alpha)$ such that $\sigma^*\alpha \neq \alpha$. (1.13) proves that

$$
\mathfrak{z}_\mathfrak{k}(t_-^\gamma) = \mathfrak{z}_\mathfrak{k}(t_-) + \sum \mathfrak{e}_{\hat{\alpha}} \tag{1.13}
$$

where the summation runs over all pairs $(\alpha, \sigma^*\alpha)$ such that $\tilde{\alpha} = \gamma$.

By (1.11), (1.13) and Prop. 1.2 we can see easily the

PROPOSITION 1.3. *When a root γ or $\mathfrak{r}_-$ has an even multiplicity, then, under the assumption* (1.*),

i) *rank of* $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\gamma})$ *=rank of* $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-)=l-\lambda$*, and* $\mathfrak{t}_0$ *is a Cartan subalgebra of* $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\gamma})$,

ii) $\mathfrak{r}_0\cup\hat{\mathfrak{r}}_\gamma$ *is a root system of* $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\gamma})$ *relative to* $\mathfrak{t}_0$*, where* $\hat{\mathfrak{r}}_\gamma$ *is the set of linear forms on* $\mathfrak{t}_0$ *obtained by restricting the roots of* $\mathfrak{r}_\gamma$ *to* $\mathfrak{t}_0$.

If a positive root γ of $\mathfrak{r}_-$ have odd multiplicity, than γ itself, extended as a linear form on $\mathfrak{t}$ by $\gamma|\mathfrak{t}_0=0$, contained in $\mathfrak{r}_\gamma$, and $\mathfrak{r}_\gamma-\{\gamma\}$ is divided to pairs $(\alpha, \sigma^*\alpha)$ such that $\sigma^*\alpha\neq\alpha$. (1.10) and (1.10′) proves that

$$(1.14) \qquad \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}^{\gamma}) = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-)+\Re\{U_\gamma\}+\sum \mathfrak{e}_{\hat{\alpha}}$$

where $\Re\{U_\gamma\}$ denotes the 1-dimensional space generated by U_γ and the summation runs over all pairs $(\alpha, \sigma^*\alpha)$ such that $\sigma^*\alpha\neq\alpha$ and $\tilde{\alpha}=\gamma$.

Since $\gamma|\mathfrak{t}_0=0$, U_γ commutes with $\mathfrak{t}_0$. Further, if multiplicity of γ is 1, then U_γ commutes with $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-)$ because $\gamma\pm\alpha$ is not a root for all $\alpha\in\mathfrak{r}_0$. Hence we obtain the

PROPOSITION 1.4. *When a root γ of $\mathfrak{r}_-$ has an odd multiplicity, then, under the assumption* (1.*),

$$\text{rank of } \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\gamma}) - \text{rank of } \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-) = 1\,.$$

Further, if the multiplicity of $\gamma(\in\mathfrak{r}_-)$ *is* 1, *then* $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\gamma})$ *is the direct sum of Lie algebras* $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-)$ *and* $\Re\{U_\gamma\}$.

1.7. The groups $Z_K(T_-)$ and $Z_K(T_-^{\gamma})$, $\gamma\in\mathfrak{r}_-$, are generally non-connected.

PROPOSITION 1.5. *Let $Z'_K(T_-)$ denote the e-component of $Z_K(T_-)$. If G is simply connected, then we have the isomorphism*

$$Z_K(T_-)/Z'_K(T_-) \cong K\cap T_-/T_0\cap T_-\,.$$

Under the assumption of this proposition $K=\hat{K}$ [11, 12]. Hence

$$(1.15) \qquad K\cap T_- = \{t\in T_-\,;\ t^2=e\}\,.$$

The two groups $K\cap T_-$ and $T_0\cap T_-$ are computable by diagrams of G, G/K and $Z_K(T_-)$.

(Proof) First we remark that the centralizer of T_- in G, $Z(T_-)$ is σ-closed and connected. Hence $(Z(T_-), Z'_K(T_-))$ is a symmetric pair with the induced involution. We see easily that T_- is a maximal torus of this symmetric pair. Each connected component of $Z_K(T_-)$ is $Z'_K(T_-)\times Z'_K(T_-)$ orbit and intersects witb T_- [11, p. 1024]. Therefore

$$Z_K(T_-)/Z'_K(T_-) \cong Z_K(T_-)\cap T_-/Z'_K(T_-)\cap T_-\,.$$

It is clear that $Z_K(T_-)\cap T_-=K\cap T_-$. On the other hand: for each element $a\in Z'_K(T_-)\cap T_-$ choose a maximal torus T'_0 of $Z'_K(T_-)$ such that $a\in T'_0$. Since

T_0 is a maximal torus of $Z'_K(T_-)$ by Prop. 1.2, there exists $b \in Z'_K(T_-)$ such that $bT'_0b^{-1}=T_0$. Since $a \in T_-$ and $b \in Z'_K(T_-)$, $bab^{-1}=a$. Hence $a \in T_0 \cap T_-$. Therefore $Z'_K(T_-) \cap T_- = T_0 \cap T_-$. Q.E.D.

Similarly we have the

PROPOSITION 1.6. *Let L be a subset of T_- and $Z'_K(L)$ be the e-component of $Z_K(L)$. If G is simply connected, then we have the isomorphism*

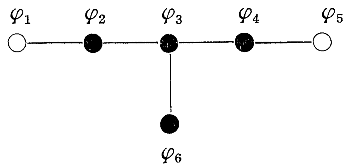
$$Z_K(L)/Z'_K(L) \cong K \cap T_-/Z'_K(L) \cap T.$$

§ 2. $\boldsymbol{H}^*(\boldsymbol{E}_6, \boldsymbol{Z}_2)$.

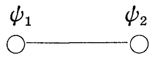
2.1. First we discuss the symmetric pair (G, K) with $G=\boldsymbol{E}_6$ and $K=\boldsymbol{F}_4$. The space $\boldsymbol{E}_6/\boldsymbol{F}_4$ is a compact, simply connected symmetric space of type EIV in E. Cartan's notation. This space is of rank 2 and has the root system isomorphic to that of the Lie algebra A_2. The multiplicity of every root is 8 [12, p. 422].

$Z_K(T_-)$ is of rank 4 by Prop. 1.2 and of dimension $52-26+2=28$ by [11], p. 1019, (2.2) or [12], p. 353. Furthermore, every root of $Z_K(T_-)$ has the same length since the root system of $Z_K(T_-)$ is a subsystem of the root system of $\boldsymbol{E}_6$. Hence $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-)=D_4$.

The σ-fundamental system of roots of $\boldsymbol{E}_6$ is described by the rule of Satake [14] as follows:

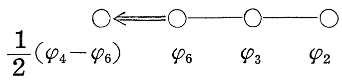


where $\sigma^*\varphi_1=\varphi_1+2\varphi_2+2\varphi_3+\varphi_4+\varphi_6$ and $\sigma^*\varphi_5=\varphi_5+\varphi_2+2\varphi_3+2\varphi_4+\varphi_6$. Put $\tilde{\varphi}_1=\psi_1$ and $\tilde{\varphi}_5=\psi_2$. Then the Schläfli figure of $\boldsymbol{E}_6/\boldsymbol{F}_4$ is



and the roots of $\mathfrak{r}_-$ are $\pm\psi_1$, $\pm\psi_2$, $\pm(\psi_1+\psi_2)$.

The groups $Z_K(T_-^{\psi_i})$ ($i=1, 2, 3$, $\psi_3=\psi_1+\psi_2$) are isomorphic, of rank 4 by Prop. 1.3 and of dimension $28+8=36$. The root system of these groups can be discussed by Prop. 1.3. $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\psi_i})=B_4$, e.g., the Schläfli figure of $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\psi_1})$ is



where the arrow $\Leftarrow$ directs from longer root to shorter one.

The unit lattice $\exp^{-1}(e)\cap\mathfrak{t}$ of $\boldsymbol{E}_6$ in $\mathfrak{t}$ is generated by basic translations τ_i ($1 \leqq i \leqq 6$), where τ_i is a vector perpendicular to the plane $\varphi_i=0$ such that $\varphi_i(\tau_i)=2$ [17]. Since τ_2, τ_3, τ_4 and τ_6 generate $\exp^{-1}(e)\cap\mathfrak{t}_0$, $Z'_K(T_-)$ is simply connected. Basic translations of $\mathfrak{t}_-$, corresponding to the roots ψ_i, are denoted by $\bar{\tau}_i$, $i=1,2$. By (6.8) of [10] and the fact that $\mathfrak{r}_-$ is exactly isomorphic to the root system of Lie algebra A_2, we see that $\bar{\tau}_1$ and $\bar{\tau}_2$ generate $\exp^{-1}(e)\cap\mathfrak{t}_-$. $\bar{\tau}_1$ and $\bar{\tau}_2$ are expressed by basic translations of $\boldsymbol{E}_6$ in $\mathfrak{t}$ as follows:

$$\bar{\tau}_1 = 2\tau_1+2\tau_2+2\tau_3+\tau_4+\tau_6\,,$$
$$\bar{\tau}_2 = \tau_2+2\tau_3+2\tau_4+2\tau_5+\tau_6\,.$$

By (1.15), $K\cap T_-$ is generated by $\exp\frac{1}{2}\tau_i$, $i=1,2$. Now

$$\frac{1}{2}\bar{\tau}_1\equiv\frac{1}{2}(\tau_4+\tau_6)\,,\quad \frac{1}{2}\bar{\tau}_2\equiv\frac{1}{2}(\tau_2+\tau_6)$$

modulo the unit lattice of $\boldsymbol{E}_6$ in $\mathfrak{t}$. Hence

$$\exp\frac{1}{2}\bar{\tau}_i\in T_0 \qquad \text{for} \quad i=1 \text{ and } 2\,.$$

Therefore $K\cap T_-=T_0\cap T_-$. Then by Prop. 1.5 we have the

PROPOSITION 2.1. *In the symmetric pair* $(\boldsymbol{E}_6, \boldsymbol{F}_4)$, $Z_K(T_-)$ *is connected and simply-connected.* $Z_K(T_-)\cong \boldsymbol{Spin}\ (8)$.

Further, $K\cap T_-=Z_K(T_-^{\psi_i})\cap T_-$ for $i=1,2,3$, whence by Prop. 1.6 we have the

PROPOSITION 2.1′. *In the symmetric pair* $(\boldsymbol{E}_6, \boldsymbol{F}_4)$, $Z_K(T_-^{\psi_i})$ *is connected and*

$$Z_K(T_-^{\psi_i})\cong \boldsymbol{Spin}\ (9)$$

for $i=1,2,3$.

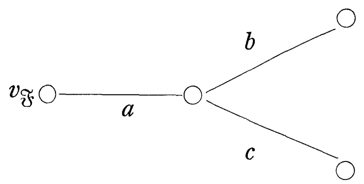
2.2. Since $Z_K(T_-^{\psi_i})/Z_K(T_-)\approx\boldsymbol{Spin}\ (9)/\boldsymbol{Spin}\ (8)\approx\boldsymbol{S}^8$ and $\boldsymbol{Spin}\ (8)$ operates on $\boldsymbol{S}^8$ through $\boldsymbol{SO}(8)$, the K-cycles [11] to describe the homology basis mod 2 of $\Omega(\boldsymbol{E}_6/\boldsymbol{F}_4)$ are all orientable iterated 8-sphere bundles over 8-spheres with cross-sections. In particular, these K-cycles are orientable as is easily seen. Hence, by [11],

PROPOSITION 2.2. $H^*(\Omega(\boldsymbol{E}_6/\boldsymbol{F}_4)\,;Z)$ *has no torsion.*

Let

$$\mathfrak{F} = \{X\in\mathfrak{t}_-;\ \psi_i(X)>0,\ i=1,2\}$$

be a Weyl chamber in $\mathfrak{t}_-$. $\mathfrak{F}$ is subdivided in cells by singular planes in it. A subgraph of the 1-skeleton of the dual subdivision of $\mathfrak{F}$ in the sense of Bott-Samelson [11], is described as follows:



where $v_{\mathfrak{F}}$ is the dual vertex to the fundamental cell in $\mathfrak{F}$ and the singular planes dual to edges a, b, c are respectively $(\psi_1+\psi_2, 1)$, $(\psi_1, 1)$, $(\psi_2, 1)$. By this we see immediately that

$$(2.1)\qquad \begin{aligned} H_i(\Omega(\boldsymbol{E}_6/\boldsymbol{F}_4)\,;\,Z) &\cong 0 && \text{for } i \not\equiv 0 \pmod 8,\\ &\cong Z && \text{for } i = 0, 8,\\ &\cong Z+Z && \text{for } i = 16, \end{aligned}$$

by discussing along the scheme of Bott-Samelson [11].

The set of central elements of the pair $(\boldsymbol{E}_6, \boldsymbol{F}_4)$ form a group of order 3 which coincides with the center of $\boldsymbol{E}_6$. Hence we have a central element a of the pair $(\boldsymbol{E}_6, \boldsymbol{F}_4)$ which is not identical with the neutral element e. Since a minimal geodesic segment from e to a in T_- is contained exactly in a singular plane, the set of minimal geodesic segments in $\Omega_{e,a}(M)$, $M=\eta_*(\boldsymbol{E}_6/\boldsymbol{F}_4)$, is a homeomorphic image of the space $\boldsymbol{F}_4/\boldsymbol{Spin}(9)=\boldsymbol{W}$, the projective plane of Cayley-Graves octanions, by Prop. 2.2 and [10]. Using the method of Bott [10], we have a cellular space decomposition:

$$\Omega_{e,\,a}(\boldsymbol{E}_6/\boldsymbol{F}_4) = \boldsymbol{W}\cup e_{16}\cup\cdots$$

where $\cdots$ denotes higher dimensional cells. The cohomology of $\boldsymbol{W}$ implies the

LEMMA 2.3. *Let u_8 denote the generator of $H^8(\Omega(\boldsymbol{E}_6/\boldsymbol{F}_4)\,;\,Z)$; then $(u_8)^2$ is not divisible by any integer larger than* 1.

Discussing the spectral sequence associated with the standard fibre space of loop spaces of $\boldsymbol{E}_6/\boldsymbol{F}_4$, we see easily the

LEMMA 2.4. $H^i(\boldsymbol{E}_6/\boldsymbol{F}_4\,;\,Z)\cong 0$ *for* $0<i<9$ *and* $9<i<17$,
$\cong Z$ *for* $i=0$, 9 *and* 17.
Further, let z_9 and z_{17} denote the generators of H^9 and H^{17}, then

$$z_{17} = Sq^8 z_9 \ mod\ 2,$$

where Sq^8 is the Steenrod squaring operation.

The later half of this lemma is concluded from Lemma 2.3 and $(u_8)^2=Sq^8u_8$. Since $\dim \boldsymbol{E}_6/\boldsymbol{F}_4=26$ we can now apply the Poincaré duality, then we have the

PROPOSITION 2.5. *$H^*(\boldsymbol{E}_6/\boldsymbol{F}_4\,;\,Z)$ has no torsion and*

$$H^*(\boldsymbol{E}_6/\boldsymbol{F}_4\,;\,Z) = \wedge_Z(z_9,\, z_{17}),$$

where $\wedge_Z$ denotes an exterior algebra over Z with generators described in parentheses.

2.3. Let us discuss the cohomology spectral sequence $E_r^{p,q}$ mod 2 associated with the fibration $(\boldsymbol{E}_6, \boldsymbol{E}_6/\boldsymbol{F}_4, \boldsymbol{F}_4, q)$.

$$(2.2)\qquad H^*(\boldsymbol{F}_4\,;\,Z_2) = Z_2[x_3]/(x_3^4)\otimes\wedge_2(x_5,\, x_{15},\, x_{23})$$

by Borel [5], p. 330, Théo. 19.2 (c), where $\deg x_i=i$, $x_5=Sq^2x_3$. These generators are universally transgressive. Hence they are transgressive in the spectral sequence considered here. By Prop. 2.6, $E_2^{i+1,0}=0$ for $i=3,5,15$ and 23. Hence all generators of $E_2^{0,*}$ must be *parmanent*, i.e., remains up to E_∞-terms unkilled, and the spectral sequence considered here is collapsed, whence we obtain the

LEMMA 2.6. *F_4 is totally non homologous zero* mod 2 *in E_6*.

By this Lemma we have the additive isomorphism

$$H^*(E_6\,;\,Z_2)\cong H^*(F_4\,;\,Z_2)\otimes\wedge_2(q^*z_9,\,q^*z_{17})\,. \tag{2.3}$$

Since $H^{10}(E_6\,;\,Z_2)\cong H^{12}(E_6\,;\,Z_2)\cong 0$ by (2.3), we see that

$$(i^{*-1}x_3)^4=0 \quad\text{and}\quad (i^{*-1}x_5)^2=0$$

in $H^*(E_6\,;Z_2)$, where $i:F_4\to E_6$ is the inclusion, which in tern concludes that the isomorphism is a *multiplicative* one.

By Bott-Samelson [11], p. 995, Theo. V,

$$\begin{aligned}\pi_i(E_6)&\cong 0 &&\text{for}\quad 4\leqq i\leqq 8\,,\\ &\cong Z &&\text{for}\quad i=3\text{ and }9\,.\end{aligned}$$

We have a map $f:E_6\to K(Z,3)$ such that $f^*u_3=x_3$, a generator of $H^3(E_6\,;\,Z)$, where u_3 is the fundamental class of $H^3(Z,3\,;\,Z)$. f^* mod p is bijective in degrees$\leqq 8$ and injective in degree 9 for any prime p. The cohomology mod 2 of $K(Z,3)$ is computed in Serre [16]. From these we see that $f^*H^9(Z,3\,;\,Z_2)$ is 2-dimensional with generators $(x_3)^3$ and $Sq^4Sq^2x_3$. On the other hand $H^9(E_6\,;\,Z_2)$ is 2-dimensional with generators $(x_3)^3$ and q^*z_9. Hence we can use $Sq^4Sq^2x_3$ as the generator of deg 9 instead of q^*z_9, and $Sq^8Sq^4Sq^2x_3$ as the generator of deg 17 instead of $q^*z_{17}=Sq^8q^*z_9$. (Note that every element of deg 9 and deg 17 of $H^*(E_6\,;\,Z_2)$, has square zero by (2.3)).

Hence we obtain the

THEOREM 2.7. $H^*(E_6;Z_2)=Z_2[x_3]/(x_3^4)\otimes\wedge_2(x_5,x_9,x_{15},x_{17},x_{23})$ *where* $\deg x_i=i$, *and $\wedge_2$ denotes an exterior algebra over Z_2 with generators described in parentheses. Further we have the following relations of generators with Steenrod squarings*:

$$x_5=Sq^2x_3\,,\quad x_9=Sq^4x_5\quad\textit{and}\quad x_{17}=Sq^8x_9\,.$$

2.4. Consider the cohomology spectral sequence $E_r^{p,q}$ mod p associated with the fibration $(E_6, E_6/F_4, F_4, q)$ for any odd prime. Cohomology mod p of E_6 and F_4 is determined for every odd prime in Borel [7] Cor. to Théo. 1, [6] Théo. 2.2 and 2.3, [5] Théo. 19.2 (a), (b). These results of Borel, combined with Prop. 2.5, proves that

$$\text{Poincaré polynomial of } E_2^{p,q}=\text{Poincaré polynomial of } E_\infty^{p,q}$$

for every odd prime. Hence the considered spectral sequences are collapsed. Then, combined with Lemma 2.6, we have the

Proposition 2.8. *$\boldsymbol{F}_4$ is totally non-homologous zero* mod p *in $\boldsymbol{E}_6$ for any prime* $p\geqq 2$.

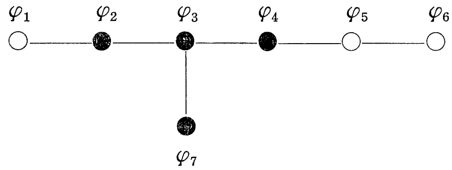
This proposition completes a partial result of [1], p. 256, Prop. 1.

About the loop spaces, $\Omega\boldsymbol{F}_4$ is totally non-homologous zero in $\Omega\boldsymbol{E}_6$ for any coefficients. The proof is easy. Cf., also the final proposition of [2].

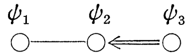
§3. $H^*(\boldsymbol{E}_7; \boldsymbol{Z}_2)$.

3.1. Let us consider the symmetric pair (G, K) with $G=\boldsymbol{E}_7$ and $K=\boldsymbol{E}_6\cdot\boldsymbol{T}^1$, where K is not the direct of direct of $\boldsymbol{E}_6$ and $\boldsymbol{T}^1$ in the strict sense but a 3-fold covering group of K is the direct product, or equivalently $\boldsymbol{E}_6\cap\boldsymbol{T}^1=\boldsymbol{Z}_3$. However we shall write $K=\boldsymbol{E}_6\times\boldsymbol{T}^1$ according to the usual description. The space $\boldsymbol{E}_7/\boldsymbol{E}_6\times\boldsymbol{T}^1$ is a compact, simply connected symmetric space of type EVII in E. Cartan's notation, of rank 3 and has the root system isomorphic to that of Lie algebra C_3. The multiplicities of long roots are 1 and those of short roots are 8 [12, p. 423].

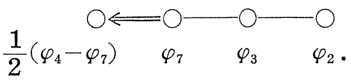
By the same reason as in 2.1, $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-)=D_4$. The σ-fundamental system of roots of $\boldsymbol{E}_7$ is described by the rule of [14] as follows:



where $\sigma^*\varphi_1=\varphi_1+2\varphi_2+2\varphi_3+\varphi_4+\varphi_7$, $\sigma^*\varphi_5=\varphi_5+\varphi_2+2\varphi_3+2\varphi_4+\varphi_7$ and $\sigma^*\varphi_6=\varphi_6$. Put $\tilde{\varphi}_1=\psi_1$, $\tilde{\varphi}_5=\psi_2$ and $\tilde{\varphi}_6=\psi_3$. Then the Schläfli figure of $\boldsymbol{E}_7/\boldsymbol{E}_6\times\boldsymbol{T}^1$ is



For all short roots ψ of $\mathfrak{r}_-$ the groups $Z_K(T^{\psi}_-)$ are isomorphic, of rank 4 by Prop. 1.3 and of dimension 36. The root system of these groups can discussed by Prop. 1.3, and we obtain that $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}^{\psi}_-)=B_4$ for every short root ψ of $\mathfrak{t}_-$, e.g., the Schläfli figure of $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}^{\psi_1}_-)$ is



For all long roots ψ' of $\mathfrak{r}_-$ the groups $Z_K(T^{\psi'}_-)$ are isomorphic, of rank 5, and $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}^{\psi'}_-)=D_4\times T^1$ by Prop. 1.4.

3.2. The unit lattice $\exp^{-1}(e)\cap\mathfrak{t}$ of $\boldsymbol{E}_7$ in $\mathfrak{t}$ is generated by basic translations τ_i corresponding to simple roots φ_i for $1\leqq i\leqq 7$. Since τ_2, τ_3, τ_4 and τ_7 generate $\exp^{-1}(e)\cap\mathfrak{t}_0$, $Z'_K(T_-)$ is simply connected. Basic translations of $\mathfrak{t}_-$, corresponding to the simple roots ψ_i, are denoted by $\bar{\tau}_i$, $1\leqq i\leqq 3$. $\bar{\tau}_i$, $1\leqq i\leqq 3$, generate $\exp^{-1}(e)$

$\cap \mathfrak{t}_-$ by the same reason as in No. 2.1, which are expressed by τ_i, $1 \leqq i \leqq 7$ as follows:

$$\bar{\tau}_1 = 2\tau_1 + 2\tau_2 + 2\tau_3 + \tau_4 + \tau_7,$$
$$\bar{\tau}_2 = \tau_2 + 2\tau_3 + 2\tau_4 + 2\tau_5 + \tau_7,$$
$$\bar{\tau}_3 = \tau_6.$$

By (1.15), $K \cap T_-$ is generated by $\exp \frac{1}{2} \bar{\tau}_i$, $1 \leqq i \leqq 3$. Now

$$\frac{1}{2}\bar{\tau}_1 \equiv \frac{1}{2}(\tau_4 + \tau_7), \quad \frac{1}{2}\bar{\tau}_2 \equiv \frac{1}{2}(\tau_2 + \tau_7)$$

modulo the unit lattice of $\boldsymbol{E}_7$ in $\mathfrak{t}$. Hence

$$\exp \frac{1}{2} \bar{\tau}_i \in T_0 \quad \text{for} \quad i = 1 \text{ and } 2, \tag{3.1}$$

whereas $\exp \frac{1}{2} \bar{\tau}_3 = \exp \frac{1}{2} \tau_6$ is not contained in T_0. Therefore

$$K \cap T_- / T_0 \cap T_- \cong Z_2$$

generated by $\exp \frac{1}{2} \tau_6$. Hence $Z_K(T_-)$ has two components by Prop. 1.5.

The center of $\boldsymbol{E}_7$ is a group of order 2. Let c' be an element of the central lattice [17] of $\boldsymbol{E}_7$ defined by

$$\begin{aligned} \varphi_i(c') &= 0 \quad \text{for} \quad i \neq 6, \\ &= 1 \quad \text{for} \quad i = 6. \end{aligned}$$

Then $c = \exp c'$ is the generator of the center of $\boldsymbol{E}_7$, i.e., $c \neq e$ and $c^2 = e$. c' is expressed as a linear combination of basic translations τ_i, $1 \leqq i \leqq 7$, as follows:

$$c' = \tau_1 + 2\tau_2 + 3\tau_3 + \frac{5}{2}\tau_4 + 2\tau_5 + \frac{3}{2}\tau_6 + \frac{3}{2}\tau_7.$$

Then

$$c' \equiv \frac{1}{2}(\tau_4 + \tau_6 + \tau_7)$$

modulo the unit lattice in $\mathfrak{t}$. Hence

$$c \cdot \left(\exp \frac{1}{2} \tau_6\right) = \exp \frac{1}{2}(\tau_4 + \tau_7) \in T_0,$$

which implies that c belongs to the component of $Z_K(T_-)$ not containing e.

Therefore we obtain the

PROPOSITION 3.1. *In the symmetric pair* $(\boldsymbol{E}_7, \boldsymbol{E}_6 \times \boldsymbol{T}^1)$,

$$Z_K(T_-) = Z'_K(T_-) + c \cdot Z'_K(T_-) \cong \boldsymbol{Spin}\,(8) \times \boldsymbol{Z}_2,$$

where c is the central element of $\boldsymbol{E}_7$, not identical with the neutral element.

3.3. Let ψ be a short root of $\mathfrak{r}_-$. $Z'_K(T_-) \cong \boldsymbol{Spin}\,(9)$ as is easily seen from the above discussions. In the inclusion $\boldsymbol{Spin}\,(8) \subset \boldsymbol{Spin}\,(9)$, the center of the latter is contained in the center of the former. Hence the center of $Z'_K(T^{\psi}_-)$ is contained in the center of $Z'_K(T_-)$, which does not contain c by Prop. 3.1. Therefore

(3.2) $$Z'_K(T^{\psi}_-) \not\ni c, \quad Z'_K(T^{\psi}_-) \cap Z_K(T_-) = Z'_K(T_-).$$

(3.2), Prop. 1.6 and Prop. 3.1 conclude easily that

$$Z_K(T^{\psi}_-)/Z'_K(T^{\psi}_-) \cong Z_2,$$
$$Z_K(T^{\psi}) = Z'_K(T^{\psi}_-) + c \cdot Z'_K(T^{\psi}_-).$$

Hence we obtain the

PROPOSITION 3.2. *For any short root ψ of $\mathfrak{r}_-$,*

$$Z_K(T^{\psi}_-) = Z'_K(T^{\psi}_-) + c \cdot Z'_K(T^{\psi}_-) \cong \boldsymbol{Spin}\,(9) \times \boldsymbol{Z}_2.$$

Let ψ' be a positive long root of $\mathfrak{r}_-$. Let $S^3_{\psi'}$ be the 3-sphere in $\boldsymbol{E}_7$ tangential to $\mathfrak{e}_{\psi'} + \Re\{\bar{\tau}_{\psi'}\}$, where $\Re\{\bar{\tau}_{\psi'}\}$ is 1-dimensional subspace of $\mathfrak{g}$, generated by the basic translation $\bar{\tau}_{\psi'}$ corresponding to ψ', and ψ' is also considered as a root of $\mathfrak{r}$ such that $\psi'|\mathfrak{t}_- = \psi'$. $\exp \frac{1}{2}\bar{\tau}_{\psi'}$ is the antipode of e in the 3-sphere $S^3_{\psi'}$. Let $(U_{\psi'}, V_{\psi'})$ be a σ-related basic frame of $\mathfrak{e}_{\psi'}$. $U_{\psi'}$ is in $\mathfrak{k}$ and generates a Cartan subalgebra of $S^3_{\psi'}$, whose exponential (a circle) passes through the antipode of e in $S^3_{\psi'}$. Hence there is a real number a such that $\exp aU_{\psi'} = \exp \frac{1}{2}\bar{\tau}_{\psi'}$, which means in particular, that

$$\exp \frac{1}{2}\bar{\tau}_{\psi'} \in Z'_K(T^{\psi'}_-).$$

ψ' is one of the following three roots of $\mathfrak{r}_-$:

$$\psi_3, \; 2\psi_2 + \psi_3, \; 2\psi_1 + 2\psi_2 + \psi_3.$$

The unique root of $\mathfrak{r}$, which gives ψ' restricted to $\mathfrak{r}_-$, is:

$$\varphi_6, \; \varphi_2 + 2\varphi_3 + 2\varphi_4 + 2\varphi_5 + \varphi_6 + \varphi_7 \quad \text{or} \quad 2\varphi_1 + 3\varphi_2 + 4\varphi_3 + 3\varphi_4 + 2\varphi_5 + \varphi_6 + 2\varphi_7.$$

Hence $\bar{\tau}_{\psi'}$ is the one of the following three vectors:

$$\tau_6, \; \tau_2 + 2\tau_3 + 2\tau_4 + 2\tau_5 + \tau_6 + \tau_7, \; 2\tau_1 + 3\tau_2 + 4\tau_3 + 3\tau_4 + 2\tau_5 + \tau_6 + 2\tau_7.$$

Then we see easily that

$$\left(\exp \frac{1}{2}\bar{\tau}_{\psi'}\right)\left(\exp \frac{1}{2}\tau_6\right) \in T_0 \subset Z'_K(T^{\psi'}_-).$$

Therefore

(3.3) $$\exp \frac{1}{2}\tau_6 \in Z'_K(T^{\psi'}_-)$$

for any long root ψ' of $\mathfrak{r}_-$. By (3.1), (3.3) and Prop. 1.6 we obtain the

PROPOSITION 3.3. *For any long root ψ' of $\mathfrak{r}_-$, $Z_K(T^{\psi'}_-)$ is connected, and*

$$Z_K(T^{\psi'}_-) \cong \boldsymbol{Spin}\,(8) \times \boldsymbol{T}^1$$

in the sense that the semi-simple part is **Spin** (8) *and the connected center is* $\boldsymbol{T}^1$.

3.4. Let us discuss K-cycles [11] to describe the additive homology basis mod 2 of $\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1)$. They are iterated fibre bundle with cross sections with base spaces and successive fibres $Z_K(T_-^{\psi})/Z_K(T_-)\approx Z'_K(T_-^{\psi})/Z'_K(T_-)\approx \boldsymbol{S}^8$ (8-sphere) by Props. 3.1 and 3.2 when ψ is a short root of $\mathfrak{r}_-$, and $Z_K(T_-^{\psi'})/Z_K(T_-)\approx \boldsymbol{S}^1$ (a circle) by Props. 3.1 and 3.3 when ψ' is a long root of $\mathfrak{r}_-$.

When ψ is a short root, the non-trivial central element $c(\in Z_K(T_-)-Z'_K(T_-))$ of $\boldsymbol{E}_7$ operates as the identity transformation on $Z_K(T_-^{\psi})/Z_K(T_-)\approx \boldsymbol{S}^8$ from the left, whence $Z_K(T_-)$ operates through $\boldsymbol{SO}(8)\subset \boldsymbol{SO}(9)$ on the 8-sphere.

When ψ' is a long root, $Z_K(T_-)$ is a normal subgroup of $Z_K(T_-^{\psi'})$, whence $Z_K(T_-)$ operates trivially on $Z_K(T_-^{\psi'})/Z_K(T_-)\approx \boldsymbol{S}^1$.

Hence we obtain the

PROPOSITION 3.4. *Every K-cycle to describe the additive homology basis* mod 2 *of* $\Omega(\boldsymbol{E}_7/E_6\times \boldsymbol{T}^1)$ *is an iterated orientable sphere bundle with cross-sections whose base space and successive fibres are* 8-*spheres or* 1-*spheres. Every K-cycle is an orientable manifold.*

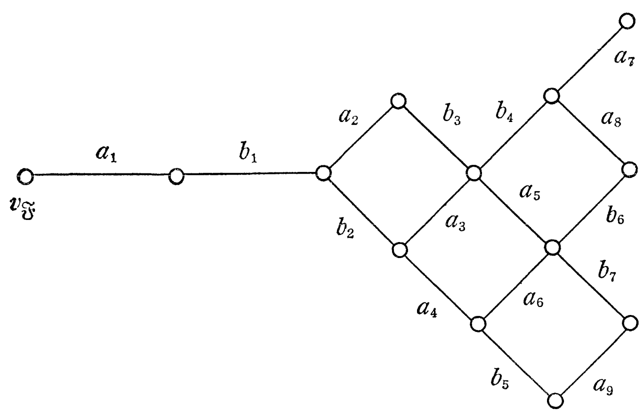
The later half of this proposition is proved easily by computing Gysin sequence of an orientable sphere bundle with an orientable manifold as the base space.

3.5. By Prop. 3.4, and [11] we see that $H_*(\Omega(\boldsymbol{E}_7/E_6\times \boldsymbol{T}^1); Z)$ has no torsion. An additive basis of $H_*(\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1); Z)$ in low degrees can be obtained following the scheme of Bott-Samelson [11].

Let $\mathfrak{F}$ be a Weyl chamber in $\mathfrak{t}_-$ defined by

$$\mathfrak{F} = \{X\in \mathfrak{t}_-;\ \psi_i(X)>0 \quad \text{for} \quad i=1,2,3\}.$$

$\mathfrak{F}$ is subdivided in cells by singular planes in it. A subgraph of the 1-skeleton of the dual subdivision of $\mathfrak{F}$, which is sufficient to describe an additive basis of $H_*(\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1); Z)$ in degrees $\leqq 27$ is described as follows:



The roots corresponding to the edges a_i (or b_i) are long roots (or short roots). If we express the singular planes dual to edges a_i and b_i by a_i^* and b_i^*, then we have

$$\begin{aligned}
&a_1^* = (2\psi_1+2\psi_2+\psi_3, 1), \quad b_1^* = (\psi_1+2\psi_2+\psi_3, 1)\,, \\
&a_2^* = a_4^* = a_6^* = a_9^* = (2\psi_2+\psi_3, 1)\,, \\
&b_2^* = b_3^* = (\psi_1+\psi_2+\psi_3, 1)\,, \quad b_4^* = b_6^* = (\psi_2+\psi_3, 1)\,, \\
&a_4^* = a_5^* = a_8^* = (2\psi_3+2\psi_2+\psi_3, 2)\,, \quad a_7^* = (\psi_3, 1) \\
&b_5^* = b_7^* = (\psi_1+\psi_2, 1)\,.
\end{aligned}$$

Since the multiplicities of a_i^* (or b_i^*) are 1 (or 8), we see by [11] that

$$\begin{aligned}
H_i(\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1)\,;\,Z) &\cong Z \quad \text{for} \quad i = 0, 1, 9, 10, 17, 19\,, \\
&\cong Z+Z \quad \text{for} \quad i = 18, 26\,, \\
&\cong Z+Z+Z \quad \text{for} \quad i = 27 \\
&\cong 0 \quad \text{otherwise for} \quad i < 27\,,
\end{aligned}$$

Since $H^*(\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1))$ is a Hopf algebra, the aboves and the Borel-Hopf structure theorem of Hopf algebras [4] conclude the

PROPOSITION 3.5. *$H^*(\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1)\,;\,Z)$ has no torsion.*

$$H^*(\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1)\,;\,Z) = \wedge_Z(u_1, u_9, u_{17}) \otimes Z[u_{18}, u_{26}]$$

in degrees ≤ 27, where $\deg u_i = i$ for each generators.

3.6. The central elements of the symmetric pair $(\boldsymbol{E}_7, \boldsymbol{E}_6\times \boldsymbol{T}^1)$ form a group of order 2. Let b be the generator of this group. Let b' be an element of the central lattice in $\mathfrak{t}_-$ defined by

$$\psi_1(b') = \psi_2(b') = 0\,, \quad \psi_3(b') = 1\,.$$

Then $b = \exp b'$.

A minimal geodesic segment in $\Omega_{e,b}(M)$, $M = \eta_*(\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1)$, is given by $\exp tb'$, $0 \leq t \leq 1$, whose index is zero. A geodesic segment in $\Omega_{e,b}(M)$, whose index has a positive minimal value, is given by

$$\exp t(b'+\tau_1)\,, \qquad 0 \leq t \leq 1\,,$$

of which the index is 18, calculated by the rule of Bott [10], §6.

Hence, by [10], we have a cellular space decomposition

$$\Omega_{e,b}(\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1) \cong \boldsymbol{N} \cup e_{18} \cup \cdots \tag{3.4}$$

where $\cdots$ denotes higher dimensional cells and

$$\boldsymbol{N} = Z_K(b)/Z_K(\exp tb', \ 0 \leq t \leq 1)$$

is homeomorphic to the set of minimal geodesics in $\Omega_{e,b}(M)$.

$Z_K(b) = \boldsymbol{E}_6\times \boldsymbol{T}^1$. Since $\{\exp tb', 0 \leq t \leq 1\} = T_-^{\psi_1} \cap T_-^{\psi_2}$, we see that

$$L = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\psi_1}) + \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\psi_2}) + \mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\psi_1+\psi_2})$$

where L is the Lie algebra of $Z_K(\exp tb',\ 0\leqq t\leqq 1)$. L is of rank 4 and of dimension 52. $\mathfrak{t}_0$ is a Cartan subalgebra of L and the root system of L relative to $\mathfrak{t}_0$ is the union of root systems of $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\psi_1})$, $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\psi_2})$ and $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{t}_-^{\psi_1+\psi_2})$. From these we see that $L=F_4$. Hence

$$Z'_K(\exp tb',\ 0\leqq t\leqq 1) = \boldsymbol{F}_4 .$$

Discussing similarly with Prop. 3.2, we see that

$$\begin{aligned}&Z_K(\exp tb',\ 0\leqq t\leqq 1)\\ &= Z'_K(\exp tb',\ 0\leqq t\leqq 1)+c\cdot Z'_K(tb',\ 0\leqq t\leqq 1) = \boldsymbol{F}_4\times \boldsymbol{Z}_2 .\end{aligned}$$

Hence

$$\boldsymbol{N}\approx \boldsymbol{E}_6\times \boldsymbol{T}^1/\boldsymbol{F}_4\times \boldsymbol{Z}_2 = \boldsymbol{E}_6\times(\boldsymbol{T}^1/\boldsymbol{Z}_2)/\boldsymbol{F}_4$$

where $\boldsymbol{Z}_2\subset \boldsymbol{T}^1$.

Putting $\boldsymbol{T}'=\boldsymbol{T}^1/\boldsymbol{Z}_2$, consider the principal bundle

$$(\boldsymbol{E}_6\times \boldsymbol{T}',\ \boldsymbol{E}_6\times \boldsymbol{T}'/\boldsymbol{E}_6,\ \boldsymbol{E}_6)$$

where $\boldsymbol{E}_6\times \boldsymbol{T}'/\boldsymbol{E}_6=\boldsymbol{S}^1$ a circle. Since fibre $\boldsymbol{E}_6$ is connected and the base space is a circle, this bundle has a cross-section. Hence the associated bundle

$$(\boldsymbol{N},\ \boldsymbol{S}^1,\ \boldsymbol{E}_6/\boldsymbol{F}_4)$$

is a product bundle, and $\boldsymbol{N}\approx \boldsymbol{S}^1\times \boldsymbol{E}_6/\boldsymbol{F}_4$ (the product space). Then by Prop. 2.5 and Lemma 2.4 we have the

LEMMA 3.6.

$$H^*(\boldsymbol{N};\ Z) = \wedge_Z(z_1,\ z_9,\ z_{17})$$

and $z_{17}=Sq^8z_9 \bmod 2$.

By (3.4) the cohomology maps induced by the inclusion $\boldsymbol{N}\subset \Omega_{e,b}(M)$ are isomorphic in degrees $\leqq 16$ and injective in degree 17. Then Lemma 3.6 prove the

LEMMA 3.7. *We have the relation*

$$u_{17} = Sq^8u_9 \bmod 2 .$$

in generators of Prop. 3.5.

3.7. Consider a fibre bundle $(\boldsymbol{E}_7/\boldsymbol{E}_6,\ \boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1,\ \boldsymbol{E}_6\times \boldsymbol{T}^1/\boldsymbol{E}_6,\ q)$, and we put $A=\boldsymbol{E}_7/\boldsymbol{E}_6$, $B=\boldsymbol{E}_7/\boldsymbol{E}_6\times \boldsymbol{T}^1$ and $F=\boldsymbol{E}_6\times \boldsymbol{T}^1/\boldsymbol{E}_6$, then $F=\boldsymbol{S}^1$ a circle and $\boldsymbol{T}^1$ is a 3 fold covering of $\boldsymbol{S}^1$.

Let $b\in F$ and $\Omega_{F,b}(A)$ denote the space of paths in A from F to b. Let

$$\Omega q:\ \Omega_{F,b}(A)\rightarrow \Omega_B$$

be the induced map by q. Then, as is well known,

$$\Omega q_*:\ \pi_i(\Omega_{F,b}(A))\cong \pi_i(\Omega B)$$

for all $i\geqq 0$, namely Ωq is a weak homotopy equivalence. Hence

$$\Omega q^*:\ H^*(\Omega B)\cong H^*(\Omega_{F,b}(A))$$

for any coefficients.

Let

$$r: \Omega_{F,b}(A) \to F = \boldsymbol{S}^1$$

be a mapping defined so as to map each path of $\Omega_{F,b}(A)$ to its starting point. This is a fibration in the sense of Serre [15], with fibre ΩA. Since the base space of the fibration r is a circle, the associated spectral sequence collapses for any coefficient. In particular

$$E_\infty^{*,*}(Z) \cong H^*(\boldsymbol{S}^1, H^*(\Omega A; Z)).$$

Since $\pi_1(A) \cong 0$ as is easily seen from the homotopy sequence of the bundle $(\boldsymbol{E}_7, \boldsymbol{E}_7/\boldsymbol{E}_6, \boldsymbol{E}_6)$, $\pi_1(\boldsymbol{S}^1)$ operates trivially on the local coefficients $H^*(\Omega A)$ and we have the isomorphism

$$E_\infty^{*,*}(Z) \cong H^*(\boldsymbol{S}^1; Z) \otimes H^*(\Omega A; Z).$$

Now consider the exact sequence

$$0 \to E_\infty^{*,*}(Z) \to H^*(\Omega B; Z) \to E_\infty^{*,*}(Z) \to 0,$$

where $E_\infty^{1,*}(Z) = E_\infty^{0,*} \cong H^*(\Omega A; Z)$. $H^*(\Omega A; Z)$ is isomorphic to a subgroup of $H^*(\Omega B; Z)$, which has no torsion by Prop. 3.5, whence $H^*(\Omega A; Z)$ has no torsion.

$$H^*(\Omega A; Z) = H^*(\Omega B; Z) /\!/ H^*(\boldsymbol{S}^1; Z)$$

using a notation $/\!/$ of Milnor-Moore [13], where $H^*(\boldsymbol{S}^1; Z)$, identified with a subalgebra of $H^*(\Omega B; Z)$ by $(r \circ \Omega q)^*$, is a normal Hopf subalgebra of $H^*(\Omega A; Z)$ in the sense of [13].

Finally, by applying Prop. 3.5 and Lemma 3.7, we obtain the

PROPOSITION 3.8. *$H^*(\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6); Z)$ has no torsion, and*

$$H^*(\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6); Z) = \wedge_Z(u_9, u_{17}) \otimes Z[u_{18}, u_{26}]$$

in degrees $\leqq 26$. *Furthermore*

$$u_{17} = Sq^8 u_9 \bmod 2.$$

3.8. Let us consider the spectral sequence mod 2 associated with the standard fibre space of loop spaces on $\boldsymbol{E}_7/\boldsymbol{E}_6$. By Prop. 3.8 and the well known properties of suspension (e.g., use [15], p. 468, Prop. 5) we see immediately the

(3.5) *u_9, u_{17} is transgressive. $H^i(\boldsymbol{E}_7/\boldsymbol{E}_6; Z_2) \cong 0$ for $0 < i < 10$ and $10 < i < 18$. $H^{10} \cong H^{18} \cong Z_2$, whose generators z_{10} and z_{18} are transgression images of u_9 and u_{17} respectively. Further $z_{18} = Sq^8 z_{10}$.*

About the behavior of u_{18} of Prop. 3.8 in the considered spectral sequence there are two possibilities:

pos. (a) u_{18} is d_r-cocycles for $i \leqq 9$, and

$$d_{10}\kappa_{10}^2(u_{18}) = \kappa_{10}^2(z_{10} \otimes u_9)\ ,^{*)}$$

or

pos. (b) u_{18} is transgressive.

In No. 3.9 we will prove that *pos.* (b) is impossible.

We shall determine $H^i(\boldsymbol{E}_7/\boldsymbol{E}_6; Z_2)$ for $19 \leqq i \leqq 26$. Consider

$$E_r^{i,0} \overset{d_r}{\longleftarrow} E_r^{i-r,r-1} \tag{3.6}$$

for $2 \leqq r \leqq i$ and $19 \leqq i \leqq 26$. By Prop. 3.8 and (3.5) $E_r^{i-r,r-1}=0$ if $r \leqq 17$ and $r \neq 10$, or if $0 < i-r < 18$ and $i-r \neq 10$. Hence d_r of (3.6) may be non-trivial only if *i*) $r=10$ and $i=20$, or *ii*) $i=r=19$. In particular, regarding $E_{i+1}^{i,0}=0$, we have the

$$E_2^{i,0} = 0 \qquad \text{for} \quad 21 \leqq i \leqq 26\ . \tag{3.7}$$

About the remaining two cases, the two possibilities (a) and (b) are related.

When *pos.* (a) occurs: $E_{10}^{10,9}$ is generated by $\kappa_{10}^2(z_{10} \otimes u_9)$, which is a d_{10}-image. Hence d_{10} of (3.6) is trivial for the case *i*), and $E_2^{20,0} \cong 0$. On the other hand for the case *ii*), $E_{19}^{0,18}=0$, whence d_{19} of (3.6) is trivial, and $E_2^{19,0} \cong 0$.

When *pos.* (b) occurs: $E_{10}^{10,9}$ is generated by $\kappa_{10}^2(z_{10} \otimes u_9)$, which is not a d_{10}-image and $E_{11}^{10,9}=0$. Hence d_{10} of (3.6) is non-travial for the case *i*), and $E_2^{20,0} \cong Z_2$ generated by $(z_{10})^2$. For the case *ii*), $E_{19}^{0,18}$ is generated by $\kappa_{19}^2(u_{18})$ since u_{18} is transgressive. $E_{20}^{19,0} \cong 0$ implies that d_{19} is isomorphic, and $E_2^{19,0} \cong Z_2$ with generator z_{19} which is a transgression image of u_{18}.

Summarizing the above we have the

LEMMA 3.9. *If pos.* (a) *occur, then*

$$H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\ Z_2) = \wedge_2(z_{10},\ z_{18})$$

in degrees $\leqq 26$. *If pos.* (*b*) *occur, then*

$$H^*(\boldsymbol{E}_7/E_6;\ Z_2) = Z_2[z_{10},\ z_{18},\ z_{19}]$$

in degrees $\leqq 26$, *where* z_{19} *is a transgression image of* u_{18}. *In any case*

$$z_{18} = Sq^8 z_{10}\ .$$

3.9. Let us discuss the spectral sequence mod 2 associated with the fibration $(\boldsymbol{E}_7,\ \boldsymbol{E}_7/\boldsymbol{E}_6,\ p,\ \boldsymbol{E}_6)$.

By [11], p. 995, Theo. V, $\pi_i(\boldsymbol{E}_7) \cong 0$ for $4 \leqq i \leqq 10$. Hence

$$\begin{aligned} H^*(\boldsymbol{E}_7;\ Z_2) &= H^*(Z, 3;\ Z_2) \quad \textit{in degrees} \leqq 10\ , \\ &= Z_2[x_3,\ Sq^2x_3,\ Sq^4Sq^2x_3] \quad \textit{in degrees} \leqq 10 \end{aligned} \tag{3.8}$$

---

* $\kappa_s{}^r$ is the natural projection of a subgroup of $E_r{}^{a,b}$ consisting of d_t-cocyles for $r \leqq i \leqq s-1$, onto $E_s{}^{a,b}$. Cf. [4].

by [16].

By Theo. 2.7, we know that

$$H^*(\boldsymbol{E}_6\,;\,Z_2) = Z_2[y_3]/(y_3^4) \otimes \wedge_2(y_5, y_9, y_{15}, y_{17}, y_{23})$$

with relations: $y_5 = Sq^2y_3$, $y_9 = Sq^4Sq^2y_3$, $y_{17} = Sq^8Sq^4Sq^2y_3$.

In the spectral sequence considered here, $y_3 = i^*(x_3)$ is *parmanent*, i.e., remains up to the E_∞ term, where $i^*: H^*(\boldsymbol{E}_7\,;\,Z_2) \to H^*(\boldsymbol{E}_6\,;\,Z_2)$ is induced by the inclusion $\boldsymbol{E}_6 \subset \boldsymbol{E}_7$. Hence y_5, y_9 and y_{17} are also *parmanent*, and

(3.9) $\quad Sq^8Sq^4Sq^2x_3 \neq 0$ and *indecomposable*

in $H^*(\boldsymbol{E}_7\,;\,Z_2)$. Since the remaining generators of fibre cohomology are y_{15} and y_{23}, the elements z_{10} and z_{18} of base cohomology remain unkilled up to E_∞ by the reason of degrees. Therefore

$$p^*(z_{10}) \neq 0, \quad p^*(z_{18}) \neq 0.$$

Since $H^{10}(\boldsymbol{E}_7\,;\,Z_2) \cong Z_2$ with generator $(Sq^2x_3)^2$ by (3.8),

(3.10) $$p^*(z_{10}) = (Sq^2x_3)^2.$$

Then

(3.11) $$p^*(z_{18}) = p^*(Sq^8z_{10}) = (Sq^4Sq^2x_3)^2 \neq 0.$$

Now we state the

(3.12) *In any system of generators of type M* ([4]) *of* $H^*(\boldsymbol{E}_7\,;\,Z_2)$, *the generators of degrees* $\leqq 9$ *are* x_3, x_5 *and* x_9. x_3 *and* x_5 *are unique,* $x_5 = Sq^2x_3$. $x_9 \equiv Sq^4Sq^2x_3$ modulo *the decomposable elements. The heights of these generators are all* $\geqq 4$.

The first statement of (3.12) and the uniqueness of x_3 and x_5 follows from (3.8). (3.8) also proves that the heights of x_3 and x_5 are $\geqq 4$. Since $H^9(\boldsymbol{E}_7\,;\,Z_2) \cong Z_2 + Z_2$ with generators $Sq^4Sq^2x_3$ and $(x_3)^3$, $x_9 = Sq^4Sq^2x_3$ or $= Sq^4Sq^2x_3 + (x_3)^3$. $(Sq^4Sq^2x_3)^2 \neq 0$ by (3.11). And $(Sq^4Sq^2x_3 + (x_3)^3)^2 = (Sq^4Sq^2x_3)^2 + (x_3)^6$. If $(x_3)^6 \neq 0$, then this element is not primitive, whereas $(Sq^4Sq^2x_3)^2$ is primitive about the coproduct of $H^*(\boldsymbol{E}_7\,;\,Z_2)$, whence $(Sq^4Sq^2x_3)^2 \neq (x_3)^6$. Therefore $(x_9)^2 \neq 0$ in any case, and the height of x_9 is $\geqq 4$. Therefore (3.12) is proved.

We remark that, when we use the generator x_9 only to the exponent $\leqq 3$, we can use $Sq^4Sq^2x_3$ as the generator of degree 9. For example, we can conclude $(Sq^2x_3)^2(Sq^4Sq^2x_3)^3 \neq 0$ from (3.12) and the structure theorem of Hopf algebras [4].

We shall determine the behavior of the generator y_{15} in the present spectral sequence. y_{15} is d_r-cocycles for $r \leqq 9$. If $d_{10}\kappa_{10}^2(y_{15}) \neq 0$, then $d_{10}\kappa_{10}^2(y_{15}) = \kappa_{10}^2(z_{10} \otimes (y_3)^2)$, which implies that $p^*z_{10}(x_3)^2 = (x_3)^2(Sq^2x_3)^2$ represent zero in $E_\infty^{10,6}$. But $E_\infty^{10+i,6-i} = 0$ for $1 \leqq i \leqq 6$. Hence $(x_3)^2(Sq^2x_3)^2 = 0$, which contradicts to (3.12).

Therefore y_{15} is d_{10}-cocycle. Then y_{15} is *parmanent* by the reason of $E_2^{i,a}=0$ for $11 \leqq i \leqq 16$.

We have an element $x'_{15} \in H^{15}(\boldsymbol{E}_7; Z_2)$ such that $i^*(x'_{15})=y_{15}$. Since y_{15} is indecomposable in $H^*(\boldsymbol{E}_6; Z_2)$, x'_{15} is also *indecomposable* in $H^*(\boldsymbol{E}_7; Z_2)$.

The above argument, combined with (3.9), conclude the

(3.12′) *In any system of generators of type M of* $H^*(\boldsymbol{E}_7; Z_2)$, *the generators of degrees* $\geqq 10$ *and* $\leqq 17$ *are* x_{15} *and* x_{17}. $x_{15} \equiv x'_{15}$ *and* $x_{17} \equiv Sq^8 Sq^4 Sq^2 x_3$ modulo *the decomposable elements.*

LEMMA 3.10. *pos. (b) does not occur.*

(Proof) Assume that *pos.* (b) occurs. Then

$$p^*(z_{10})^2 \neq 0 \quad \text{and} \quad p^*(z_{19}) \neq 0$$

by the reason of their degrees. Then the generator x_5 of $H^*(\boldsymbol{E}_7; Z_2)$ has height $\geqq 8$ since $(x_5)^4 = p^*(z_{10})^2 \neq 0$. Remark that the spectral sequence considered in this No. collapses in total degrees $\leqq 22$ by the above discussions. Then, since $p^*(z_{19})$ can not be a square of some other element, it is indecomposable, and any system of generators of type M of $H^*(\boldsymbol{E}_7; Z_2)$ contains a generator x_{19} of degree 19 such that $x_{19} \equiv p^*(z_{19})$ modulo decomposable elements. Now

$$p^*(z_{10})^2 p^*(z_{18}) p^*(z_{19}) \equiv (x_5)^4 (x_9)^2 x_{19}$$

modulo a subalgebra of $H^*(\boldsymbol{E}_7; Z_2)$ generated by the elements of degrees $\leqq 18$. The later is non-zero by the above discussions, (3.12) and the structure theorem of Hopf algebras [4]. Hence

$$p^*((z_{10})^2 z_{18} z_{19}) \neq 0 \quad \text{and} \quad (z_{10})^2 z_{18} z_{19} \neq 0 .$$

But this is impossible because

$$\deg (z_{10})^2 z_{18} z_{19} = 57 > 55 = \dim \boldsymbol{E}_7/\boldsymbol{E}_6 .$$

Therefore *pos.* (b) does not occur.

3.10. By Lemmas 3.9 and 3.10 $H^*(\boldsymbol{E}_7/\boldsymbol{E}_6; Z_2)$ is determined in degrees $\leqq 26$. Now

$$p^*(z_{10}) p^*(z_{18}) = (Sq^2 z_3)^2 (Sq^4 Sq^2 x_3)^2 \neq 0$$

by (3.10)-(3.12) and the structure theorem of Hopf algebras [4]. Hence

(3.13) $$z_{10} z_{18} \neq 0$$

in $H^{28}(\boldsymbol{E}_7/\boldsymbol{E}_6; Z_2)$.

Let us discuss the behavior of the generator u_{26} of $H^{26}(\Omega(\boldsymbol{E}_7/\boldsymbol{E}_6); Z_2)$ in the spectral sequence considered in No. 3.8. u_{26} is d_r-cocycles for $r \leqq 9$. If $d_{10}\kappa_{10}^2(u_{26}) \neq 0$, then $d_{10}\kappa_{10}^2(u_{26}) = \kappa_{10}^2(z_{10} \otimes u_{17})$ which is also a d_{10}-image of $u_9 u_{17}$. Hence, by replacing u_{26} by $u_{26} + u_9 u_{17}$ if necessary, we can choose the generator u_{26} such that it is also d_{10}-cocycle. Then, u_{26} is d_r-cocycles for $r \leqq 17$. If $d_{18}\kappa_{18}^2(u_{26}) \neq 0$, then $d_{18}\kappa_{18}^2(u_{26}) = \kappa_{18}^2(z_{18} \otimes u_9)$, which implies that $z_{10}z_{18}=0$ in $H^*(\boldsymbol{E}_7/\boldsymbol{E}_6\,;\,Z_2)$. This contradicts to (3.13). Hence u_{26} is d_{18}-cocycle. Then u_{26} is *transgressive* since $E_r^{a,0}=0$ for $19 \leqq a \leqq 26$ by Lemmas 3.9 and 3.10. Therefore we proved the

(3.14) *u_{26} can be chosen to be transgressive. $H^{26}(\boldsymbol{E}_7/\boldsymbol{E}_6\,;\,Z_2) \cong Z_2$ with generator z_{27}, which is a transgression image of u_{26}.*

Lemmas 3.9, 3.10 and (3.14) determines $H^*(\boldsymbol{E}_7/\boldsymbol{E}_6\,;\,Z_2)$ in degrees $\leqq 27$. Then, by Poincaré duality and $\dim(\boldsymbol{E}_7/\boldsymbol{E}_6)=55$, we can determine $H^*(\boldsymbol{E}_7/\boldsymbol{E}_6\,;\,Z_2)$ additively as follows:

$$H^i(\boldsymbol{E}_7/\boldsymbol{E}_6\,;\,Z_2) \cong Z_2 \quad \text{for } i = 0, 10, 18, 27, 28, 37, 45, 55$$
$$\cong 0 \quad \text{for other degrees } i\,.$$

By (3.13), $z_{10}z_{18}$ generates $H^{28}(\boldsymbol{E}_7/\boldsymbol{E}_6\,;\,Z_2)$. Then $z_{10}z_{18}z_{27}$ generates $H^{55}(\boldsymbol{E}_7/\boldsymbol{E}_6\,;\,Z_2)$ by Poincaré duality. Hence $z_{10}z_{18}z_{27} \neq 0$, $z_{10}z_{27} \neq 0$ and $z_{18}z_{27} \neq 0$. $z_{10}z_{27}$ and $z_{18}z_{27}$ generates H^{37} and H^{45} respectively. Hence we obtained the

Proposition 3.11. *$H^*(\boldsymbol{E}_7/\boldsymbol{E}_6;\,Z_2) = \bigwedge_2(z_{10},\, z_{18},\, z_{27})$.*

3.11. We shall continue the discussion of the spectral sequence discussed in No. 3.9. The only thing to be discussed is the behavior of the generator y_{23} of fibre cohomology since all other generators of fibre cohomology are parmanent.

y_{23} is d_r-cocycles for $r \leqq 9$. Since $H^{14}(\boldsymbol{E}_6\,;\,Z_2) \cong Z_2 + Z_2$, generated by $(y_3)^3y_5$ and y_5y_9,

$$d_{10}\kappa_{10}^2(y_{23}) = \kappa_{10}^2(z_{10} \otimes (a(y_3)^3y_5 + by_5y_9))$$

with $a,\, b \in Z_2$, which implies that $z_{10} \otimes (a(y_3)^3y_5 + by_5y_9)$ represents zero in $E_\infty^{10,14}$, namely

$$a(x_3)^3(Sq^2x_3)^3 + b(Sq^2x_3)^3x_9 \equiv 0$$

modulo the elements of filtration >10. On the other hand, $E_\infty^{10+i,14-i} \cong 0$ for $1 \leqq i \leqq 14$ expect for $i=8$ by Prop. 3.11. And, if $d_{10}\kappa_{10}^2(y_{23}) \neq 0$, then $E_\infty^{18,6} \cong Z_2$ generated by $\kappa_\infty^2(z_{18} \otimes (y_3)^2)$. Hence

$$a(x_3)^3(Sq^2x_3)^3 + b(Sq^2x_3)^3x_9 + c(x_3)^2(Sq^4Sq^2x_3)^2 = 0$$

with $c \in Z_2$. Then (3.12) and the structure theorem of Hopf algebras [4] conclude that $a=b=c=0$. Consequently, y_{23} is d_{10}-cocycle.

Next y_{23} is d_r-cocycles for $11 \leqq r \leqq 17$ by Prop. 3.11. A discussion similar as above concludes that y_{23} is also d_{18}-cocycle. Then, by Prop. 3.11 $E_2^{i,0}=0$ for $19 \leqq i \leqq 24$, which implies that y_{23} is *parmanent*.

Consequently, the spectral sequence discussed here collapses. Therefore

Proposition 3.12. *$\boldsymbol{E}_6$ is totally non-homologous zero* mod 2 *in $\boldsymbol{E}_7$.*

And we have the additive isomorphism

(3. 15) $$H^*(\boldsymbol{E}_7; Z_2) \cong H^*(\boldsymbol{E}_7/\boldsymbol{E}_6; Z_2) \otimes H^*(\boldsymbol{E}_6; Z_2)$$
$$\cong \wedge_2(p^*z_{27}) \otimes Z_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \wedge_2(x_{15}, x_{17}, x_{23})$$

by (3. 10)-(3. 12), where $i^*(x_{23}) = y_{23}$. Since $H^{12}(\boldsymbol{E}_7; Z_1)$ is generated by an imprimitive element x_3x_9 about the coproduct of $H^*(\boldsymbol{E}_7; Z_2)$ and $(x_3)^4$ must be primitive, $(x_3)^4 = 0$. By Prop. 3. 11, $(z_{10})^2 = (z_{18})^2 = 0$. Hence, $(Sq^2x_3)^4 = (p^*z_{10})^2 = 0$ and $(Sq^4Sq^2x_3)^4 = 0$, i.e., Sq^2x_3 and $Sq^4Sq^2x_3$ have heights 4. Therefore we can choose the generator of degree 9 as $x_9 = Sq^4Sq^2x_3$. p^*z_{27} is *indecomposable* in $H^*(\boldsymbol{E}_7; Z_2)$ since it can not be a square of any other element and (3. 15) indicates this. Hence $H^*(\boldsymbol{E}_7; Z_2)$ contains a generator x_{27}. $(z_{27})^2 = 0$ (by Prop. 3. 12) implies that we can choose the generator of degree 27 as $x_{27} = p^*z_{27}$ with height 2. Finally x_{15}, x_{17} and x_{23} have height 2, and

$$(Sq^8Sq^4Sq^2x_3)^2 = Sq^{16}(Sq^4Sq^2x_3)^2$$
$$= p^*(Sq^{16}z_{18}) = 0\,,$$

for $H^{34}(\boldsymbol{E}_7/\boldsymbol{E}_6; Z_2) \cong 0$ by Prop. 3. 11. Hence, by (3. 9), we can choose the generator of degree 17 as $x_{17} = Sq^8Sq^4Sq^2x_3$.

In the above we determined a system of generators of type M of $H^*(\boldsymbol{E}_7; Z_2)$, and by the structure theorem of Hopf algebras [4] we obtain the

THEOREM 3. 13. *The cohomology ring* $H^*(\boldsymbol{E}_7; Z_2)$ *is described as*

$$H^*(\boldsymbol{E}_7; Z_2) = Z_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \wedge_2(x_{15}, x_{17}, x_{23}, x_{27})\,,$$

where $\deg x_i = i$, *with relations*: $x_5 = Sq^2x_3$, $x_9 = Sq^4x_5$ *and* $x_{17} = Sq^8x_9$.

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