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Characteristic classes and cohomological operations

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1. Introduction

Throughout this paper, spaces are simply connected topological spaces with base points and have the homotopy types of CW-complexes: n-ad $(X; X_1, \cdots, X_{n-2}, x_0)$ is homotopy equivalent to a CW-n-ad $(W; W_1, \cdots, W_{n-2}, w_0)$ where x_0 and w_0 are base points. The fiber space is the one in the sence of Serre.

We shall say that the space X satisfies the condition $(A_{p,q})$ if the homotopy groups $\pi_i(X)$ of X vanish for $i<p$ and $i>q$.

Let X be a space and A, B be subspaces of X. Denote by $\Omega(X; A, B)$ the space of all paths in X starting in A and ending in B, and by π_0, π_1 the natural projections $\Omega(X; A, B)\to A$, $\Omega(X; A, B)\to B$ respectively. We write $LX=\Omega(X; x_0, X)$ and $\Omega X=\Omega(X; x_0, x_0)$; LX is the path space of X and ΩX is the loop space of X.

Let X, Y be two spaces. Denote by $\pi(X, Y)$ the set of homotopy classes of continuous maps $f: (X, x_0)\to(Y, y_0)$. It is known that:

(1.1) $h: (X; A, x_0)\to(Y; B, y_0)$ is a homotopy equivalence if $h_*: \pi_i(X)\to \pi_i(Y)$, $(h|A)_*: \pi_i(A)\to\pi_i(B)$ are isomorphism for each $i\geqq 0$ [4].

(1.2) $(LX; \Omega X, *)$ has the homotopy type of a CW-triad where $*$ is the constant loop. Also $(\Omega(E: e_0, F); LF, \Omega F, *)$ has the homotopy type of a CW-tetrad [5].

(1.3) If X satisfies the condition $(A_{p,2p-2})$ for some p, there exists a space X_0 such that X has the homotopy type of ΩX_0 [5], [9]: Such a space X_0 will be denoted by $\Omega^{-1}(X)$.

(1.4) If X, Y satisfy the conditions $(A_{p,q})$, $(A_{r,2p-2})$ respectively for some integers (p, q, r), $\pi(X, Y)$ forms an abelian group, natural with respect to maps $X\to X'$, $Y\to Y'$. Also there is the natural isomorphism $\Omega: \pi(X, Y)\to\pi(\Omega X, \Omega Y)$ [5], [9]: We shall denote its inverse isomorphism by Ω^{-1}.

Under the basic references of (1.1)$\sim$(1.4) we shall show in this paper the following:

(1) Let $\mathfrak{E}=(E, p, B, F)$ be a fiber space such that all of E, B, F satisfy the condition $(A_{p,2p-3})$ for an integer p. Then there is a class $\alpha\in\pi(B, \Omega^{-1}F)$ such that the equivalence class of the fiber space $\mathfrak{E}$ is uniquely determined by the triple (B, F, α). α is called the *characteristic class* of $\mathfrak{E}$.

(2) Under the same assumptions above, the following sequence is exact

$$\pi(\Omega F, X) \xleftarrow{\Omega(i)^*} \pi(\Omega E, X) \xleftarrow{\Omega(p)^*} \pi(\Omega B, X) \xleftarrow{\Omega(\alpha)^*} \pi(F, X)$$
$$\xleftarrow{i^*} \pi(E, X) \xleftarrow{p^*} \pi(B, X) \xleftarrow{\alpha^*} \pi(\Omega^{-1}(F), X)$$

where X is a space such that $\pi_i(X)=0$ for $i>2p-3$.

(3) Let (E, p, B, F), (F'', p'', F, F') be two fiber spaces such that all of E, B, F, F', F'' satisfy the condition $(A_{p,2p-3})$ for an integer p. Let α_1, α_2 be their characteristic classes. Then there exists a diagram

$$\begin{tikzcd} F' \arrow[r] & \tilde{F}'' \arrow[r] \arrow[d, "\tilde{p}''"] & E' \arrow[d, "p'"] \\ & F \arrow[r] & \tilde{E} \arrow[d, "\tilde{p}"] \\ & & B \end{tikzcd}$$

if and only if $\alpha_2 \cdot \Omega(\alpha_1)=0$, where $(\tilde{E}, \tilde{p}, B, F)$, $(\tilde{F}'', \tilde{p}'', F, F')$ are equivalent with (E, p, B, F), (F'', p'', F, F') respectively and $(E', \tilde{p}p', B, \tilde{F}'')$, $(E', p', \tilde{E}, F')$ are also fiber spaces: Such a system is called the *poly-fiber space*.

(4) We shall construct some higher operations as the obstructions to lift maps $f: X\to B$ (base space of poly-fiber space) to $\bar{f}: X\to E$ (total space of poly-fiber space): Some relations among operations induce the higher operations by constructing the appropriate poly-fiber spaces.

2. Preliminary

Let (E_1, p_1, B_1, F_1), (E_2, p_2, B_2, F_2) be two fiber spaces such that there exist homotopy equivalences $h_B: B_1\to B_2$, $h'_B: B_2\to B_1$, $h_F: F_1\to F_2$, $h'_F: F_2\to F_1$. We have:

LEMMA (2.1). *If there exists a map h_E: $(E_1, F_1)\to(E_2, F_2)$ sucn that $h_E|F_1 \simeq h_F: F_1\to F_2$, $h_B p_1 \simeq p_2 h_E$ rel. F_1, then h_E is a homotopy equivalence.*

Proof. Consider the diagram

$$\begin{tikzcd} \arrow[r, "(p_1 j_1)_*"] & \pi_{i+1}(B_1) \arrow[r, "\partial_1(p_{1*})^{-1}"] \arrow[d, "h_{B*}"] & \pi_i(F_1) \arrow[r, "i_{1*}"] \arrow[d, "(h_E|F_1)_*"] & \pi_i(E_1) \arrow[r, "(p_1 j_1)_*"] \arrow[d, "h_{E*}"] & \pi_i(B_1) \arrow[r, "\partial_1(p_{1*})^{-1}"] \arrow[d, "h_{B*}"] & \pi_{i-1}(F_1) \arrow[r] \arrow[d, "(h_E|F_1)_*"] & {} \\ \arrow[r, "(p_2 j_2)_*"] & \pi_{i+1}(B_2) \arrow[r, "\partial_2(p_{2*})^{-1}"] & \pi_i(F_2) \arrow[r, "i_{2*}"] & \pi_i(E_2) \arrow[r, "(p_2 j_2)_*"] & \pi_i(B_2) \arrow[r, "\partial_2(p_{2*})^{-1}"] & \pi_{i-1}(F_2) \arrow[r] & {} \end{tikzcd}$$

where $i_k: F_k\to E_k$ and $j_k: (E_k, e_{0k})\to(E_k, F_k)$ are the injections and $\partial_k: \pi_i(E_k, F_k)\to \pi_{i-1}(F_k)$ is the boundary homomorphisms $(k=1, 2)$. Since $h_E i_1 = i_2 h_E|F_1$, $h_B p_1 j_1 \simeq p_2 j_2 h_E$, $\partial_2 h_{E*} = (h_E|F_1)_* \partial_1$, the above diagram is commutative. Since h_{B*}, $(h_E|F_1)_*$ are isomorphisms the five lemma shows that h_{E*} is an isomorphism. This together with (1.1) implies $h_E: (E_1, F_1)\to(E_2, F_2)$ is a homotopy equivalence. q.e.d.

We shall say that fiber space $\mathfrak{F}_1=(E_1, p_1, B_1, F_1)$ is *equivalent* to $\mathfrak{F}_2=(E_2, p_2, B_2, F_2)$ if there exists a triple (h_E, h_B, h_F) as above, and denote by (h_E, h_B, h_F):

$\mathfrak{E}_1 \equiv \mathfrak{E}_2$. Clearly, this is an equivalence relation.

Let $\pi(A, B; LX, \Omega X)$ be the set of homotopy classes of continuous maps

$$f: (A, B, *) \to (LX, \Omega X, *)$$

where $*$ are the base points. Define a map η by $\eta(f) = f|B$. Then we have:

LEMMA (2.2). *$\eta_*: \pi(A, B; LX, \Omega X) \to \pi(B, \Omega X)$ is 1-1 and onto.*

Proof. Let $g_i: B \to \Omega X$ $(i=0, 1)$ denote the restrictions $f_i|B$ of maps $f_i: (A, B) \to (LX, \Omega X)$ and $G: B \times I \to \Omega X$ be a homotopy between g_i.

Define a map $F_0: A \times \dot{I} \cup B \times I \to LX$ by

$$F_0|A \times (i) = f_i \quad (i = 0, 1), \qquad F_0|B \times I = G.$$

Since LX is contractible, we can extend F_0 to a map $F: A \times I \to LX$, which shows $f_0 \sim f_1: (A, B) \to (LX, \Omega X)$. Hence η_* is a monomorphism.

Let $g: B \to \Omega X$ be a map. Since LX is contractible there exists a homotopy $G': B \times I \to LX$ between the map g and the constant map. Define a map $H_0: A \times (1) \cup B \times I \to LX$ by

$$H_0|A \times (1) = *, \quad H_0|B \times I = G';$$

we can extend H_0 to a map $H: A \times I \to LX$, and we have a map

$$f = F|A \times (0): (A, B) \to (LX, \Omega X)$$

such that $f|B = g$. Hence η_* is an epimorphism. q.e.d.

Let (E, p, B, F) be a fiber space such that all of E, B, F satisfy the condition $(A_{p, 2p-3})$ for an integer $p > 2$. According to (1.3) there exist spaces E_1, B_1, F_1 such that $E \simeq \Omega E_1$, $B \simeq \Omega B_1$, $F \simeq \Omega F_1$.

LEMMA (2.3). *Under the above condition there exists an appropriate fiber space $\mathfrak{E}_0 = (E_0, p_0, B_0, F_0)$ such that $\mathfrak{E} = (E, p, B, F)$ is equivalent to $\Omega\mathfrak{E}_0 = (\Omega E_0, \Omega p_0, \Omega B_0, \Omega F_0)$.*

Proof. Since $\pi(E, B) \simeq \pi(\Omega E_1, \Omega B_1) \simeq \pi(E_1, B_1)$, there exists a map $p_1: E_1 \to B_1$ such that $\Omega(p_1)h_E^1 \simeq h_B^1 p$, where $h_E^1: E \to \Omega E_1$, $h_B^1: B \to \Omega B_1$ are homotopy equivalences. Consider the mapping cylinder M_{p_1} of p_1, and construct the space $\Omega(M_{p_1}; E_1, M_{p_1})$ as usual. Denote by $p_0: \Omega(M_{p_1}; E_1, M_{p_1}) \to M_{p_1}$ the map which associates the end point to any path. We have the fiber space (E_0, p_0, B_0, F_0), where $E_0 = \Omega(M_{p_1}; E_1, M_{p_1})$, $F_0 = \Omega(M_{p_1}: E_1, *)$ and $B_0 = M_{p_1}$.

In view of the property of the mapping cylinder M_{p_1} there exist homotopy equivalences $h_E^2: E_1 \to E_0$ and $h_B^2: B_1 \to B_0$ such that $p_0 h_E^2 \simeq h_B^2 p_1$.
Thus $\Omega(p_0) \cdot \Omega(h_E^2) h_E^1 \simeq \Omega(h_B^2)\Omega(p_1)h_E^1 \simeq \Omega(h_B^2)h_B^1 p$. Denote $\Omega(h_B^2)h_B^1$ by $h_B: B \to \Omega B_0$. From the covering homotopy property there exists a map $h_E: E \to \Omega E_0$ such that $h_E \simeq \Omega(h_E^2)h_E^1$ and $\Omega(p_0)h_E = h_B p$. Hence $h_E(F) \subset \Omega F_0$, and by the five lemma we have that $h_E|F$ induces isomorphisms $\pi_i(F) \to \pi_i(\Omega F_0)$ for each i. q.e.d.

3. Characteristic class

Let (E, p, B, F) be a fiber space, and $(LB, \pi_1, B, \Omega B)$ be the fiber space of paths. By (1.2) there exists a CW-triad $(W; W_0, w_0)$ and there are homotopy equivalences $h: (W; W_0, w_0) \to (LB; \Omega B, *)$, $g: (LB; \Omega B, *) \to (W; W_0, w_0)$, such that $hg \simeq 1_{LB}$, $gh \simeq 1_w$, where $*$ is the constant loop in B.

Consider the map $h': W\times I \to B$ such that $h'(w, t) = h(w)(t)$, $w \in W$, $0 \leqq t \leqq 1$. In view of the covering homotopy theorem, there exists a map $\bar{h}': W\times I \to E$ such that $p\bar{h}' = h'$ and $\bar{h}'(W\times(0) \cup w_0 \times I) = e_0$. Then we have maps

$$\begin{aligned} q' &: (LB, *) \longrightarrow (LE, *), \\ q &= q' | \Omega B : \Omega B \longrightarrow \Omega(E; e_0, F) \end{aligned}$$

such as $q'(\rho_B)(t) = \bar{h}'(g(\rho_B), t)$, $\rho_B \in LB$, $0 \leqq t \leqq 1$.

Also let $L(p): LE \to LB$ be the map such that $L(p)(\rho_E)(t) = p(\rho_E(t))$, $\rho_E \in LE$, $0 \leqq t \leqq 1$, and

$$p': \Omega(E; e_0, F) \to \Omega B, \quad {}^1p: \Omega E \to \Omega B$$

be its restrictions. Then we have:

LEMMA (3.1). *p', q are homotopy equivalences* (rel. $*$), *and the one of them is a homotopy inverse of the other.*

Proof. Since $(p'q)(\rho_B)(t) = p\bar{h}'(g(\rho_B), t) = h'(g(\rho_B), t) = (hg)(\rho_B)(t)$, we have $p'q = hg | \Omega B \simeq 1_{\Omega B}$ (rel. $*$). We shall next show that there exists a homotopy

$$qp' \simeq 1_{\Omega(E; e_0, F)}: (\Omega(E; e_0, F)\times I, LF\times I, *\times I) \to (\Omega(E; e_0, F), LF, *).$$

By (1.2), there exist a CW-triad $(V; V_0, v_0)$, a homotopy equivalence $h_0: (V; V_0, v_0) \to (\Omega(E; e_0, F), LF, *)$, and its homotopy inverse $g_0: (\Omega(E; e_0, F), LF, *) \to (V; V_0, v_0)$.

The above homotopy $p'q \simeq 1_{\Omega B}$ induces a map $Q: V\times I\times I \to B$ such that

$$Q(v, t, s) = \begin{cases} p\cdot h_0(v)(t) & \text{if } s = 0, \\ p(qp'h_0)(v)(t) & \text{if } s = 1, \\ b_0 & \text{if } t = 0, 1, \text{ or } v \in V_0. \end{cases}$$

In view of the covering homotopy theorem, we have a map $\bar{Q}: V\times I\times I \to E$ such that $p\bar{Q} = Q$ and

$$\bar{Q}(v, t, s) = \begin{cases} h_0(v)(t) & \text{if } s = 0, \\ (qp'h_0)(v)(t) & \text{if } s = 1, \\ e_0 & \text{if } t = 0, \\ h_0(v)(t) & \text{if } v \in V_0, \ t \leqq 1-s, \\ h_0(v)(1-s) & \text{if } v \in V_0, \ t > 1-s. \end{cases}$$

Hence we have a homotopy $H: h_0g_0 \simeq qp'h_0g_0: \Omega(E; e_0, F)\times I \to \Omega(E; e_0, F)$ such as

$$H(\rho, s)(t) = \bar{Q}(g_0(\rho), t, s), \quad \rho \in \Omega(E; e_0, F).$$

Since $h_0 g_0 \simeq 1_{\Omega(E;\, e_0,\, F)}$, we have the desired homotopy. q.e.d.

If we denote $\pi_1 q$ by f, in the diagram

$$\begin{tikzcd} & F \arrow[r, "i"] & E \arrow[d, "p"] \\ \Omega B \arrow[ur, "\pi_1 q = f"] \arrow[r, "i_1"] & LB \arrow[ur, "\pi_1 q'"] \arrow[r, "\pi_1"] & B \end{tikzcd}$$

we have $if = \pi_1 q' i_1$, and $p\pi_1 q' \simeq \pi_1$, since $hg \simeq 1_{LB}$. Also we have:

LEMMA (3.2). *The homotopy class* ${}^1\alpha \in \pi(\Omega B, F)$ *of* f *is uniquely determined by the given fiber space* $\mathfrak{E} = (E, p, B, F)$, *and* ${}^1p^{*1}\alpha = 0$.

Proof. Consider two CW-triads $(W^1;\, W^1_0,\, w^1_0)$, $(W^2;\, W^2_0,\, w^2_0)$ and two maps q_1, q_2 induced by W^1, W^2 respectively. From Lemma (3.1) we have $q_1 \simeq q_1 p' q_2 \simeq q_2$. Hence the induced maps $f_1 = \pi_1 q_1$ and $f_2 = \pi_1 q_2$ are homotopic. From $f\,{}^1p = \pi_1 q_1\,{}^1p \simeq \pi_1$ and $\pi_1(\Omega E) = e_0$ we have ${}^1p^{*1}\alpha = 0$. q.e.d.

LEMMA (3.3). *The fiber space* $\Omega\mathfrak{E} = (\Omega E,\, {}^1p,\, \Omega B,\, \Omega F)$ *is equivalent to the principal fiber space* $(\tilde{E}, \tilde{p}, \Omega B, \Omega F)$ *which is induced from the principal path fibering* $\pi_1 : LF \to F$ *by the map* f *above.*

Proof. Since ${}^1p = p' | \Omega E$, in view of Lemma (3.1) there exists a homotopy

$$H_1 : (\Omega E \times I,\ \Omega F \times I) \to (\Omega(E;\, e_0,\, F),\ LF)$$

between the inclusion map $\Omega E \subset \Omega(E;\, e_0,\, F)$ and the composition map $q\,{}^1p$. Define a map $\eta_F;\ \Omega E \to LF$ such as

$$\eta_F(\omega_E)(t) = \pi_1 H_1(\omega_E,\, t) \qquad \omega_E \in \Omega E,\ 0 \leqq t \leqq 1\,.$$

From the construction of H_1 (Lemma 3.1), it is easily verified that $\eta_F | \Omega F : \Omega F \to \Omega F$ is homotopic to the map $\omega_F \to \omega_F^{-1}$, $\omega_F \in \Omega F$ and $f\,{}^1p = \pi_1 \eta_F$. Hence we have a map $\eta : \Omega E \to \tilde{E} = \{(\omega_B,\, \rho_F) \mid \omega_B \in \Omega B,\ \rho_F \in LF,\ f\omega_B = \pi_1 \rho_F\}$

such as $\qquad \eta(\omega_E) = ({}^1p(\omega_E),\ \eta_F(\omega_E)) \qquad$ for any $\omega_E \in \Omega E$.

According to Lemma (2.1) we have

$$(\eta,\ \eta_F,\ i) : \Omega\mathfrak{E} \equiv (\tilde{E},\ \tilde{p},\ \Omega B,\ \Omega F)$$

where $i : \Omega B \to \Omega B$ is the identity map. q.e.d.

LEMMA (3.4). *Let* $\mathfrak{E}_1 = (E_1, p_1, B_1, F_1)$, $\mathfrak{E}_2 = (E_2, p_2, B_2, F_2)$ *be two fiber spaces each of which induces the homotopy class* ${}^1\alpha_i$ *of* $f_i : \Omega B_i \to F_i$ *respectively* $(i = 1, 2)$. *We make an assumption that there are homotopy equivalences* $h_B : B_1 \to B_2$, $h_F : F_1 \to F_2$. *If there exists a map* $h_E : E_1 \to E_2$ *such that* $(h_E, h_B, h_F) : \mathfrak{E}_1 \equiv \mathfrak{E}_2$ *then* $\Omega(h_B)^{*1}\alpha_2 = h_{F*}{}^1\alpha_1$. *Conversely, if* $\Omega(h_B)^{*1}\alpha_2 = h_{F*}{}^1\alpha_1$ *then* $\Omega\mathfrak{E}_1 \equiv \Omega\mathfrak{E}_2$.

Proof. Define a map $h : \Omega(E_1;\, e_{01},\, F_1) \to \Omega(E_2;\, e_{02},\, F_2)$ as $h(\rho)(t) = h_E[\rho(t)]$

then it is obvious that

$$p_2' h \simeq \Omega(h_B) p_1', \quad h_F \pi_1 q_1 \simeq h \pi_1 q_1$$

where $p_i' : \Omega(E_i;\, e_{0i},\, F_i) \to \Omega B_i$ and $q_i : \Omega B_i \to \Omega(E_i,\, e_{0i},\, F_i)$ are the homotopy equivalences (Lemma 3.1) and $\pi_1 q_i = f_i$. Thus

$$h_F f_1 = h_F \pi_1 q_1 \simeq h \pi_1 q_1 = \pi_1 h q_1 \simeq \pi_1 q_1 p_2' h q_1 = f_2 p_2' h q_1 \simeq f_2 \Omega(h_B) p_1' q_1 \simeq f_2 \Omega(h_B)\,.$$

Coversely, if $f_2 \Omega(h_B) \simeq h_F f_1$ there exists a map $H_1 : \Omega B_1 \times I \to F_2$ such as

$$H_1(\omega, t) = \begin{cases} f_2 \Omega(h_B)\omega & \text{if } t = 1\,, \\ h_F f_1 \omega & \text{if } t = 0\,. \end{cases}$$

Consider the two principal fiber spaces $\tilde{E}_i = \{(\omega_i, \rho_i) \mid \omega_i \in \Omega B_i,\, \rho_i \in LF_i,\, f_i \omega_i = \pi_1 \rho_i\}$ $i = 1.2.$ in (3.3). Define a map $\eta : \tilde{E}_1 \to \tilde{E}_2$ by $\eta(\omega_1, \rho_1) = (\omega_2, \rho_2)$ where $\omega_2 = \Omega(h_B)\omega_1$ and

$$\rho_2(t) = \begin{cases} h_F \rho_1(2t) & \text{if } 0 \leqq t \leqq \frac{1}{2}\,, \\ H_1(\omega_1,\, 2t-1) & \text{if } \frac{1}{2} \leqq t \leqq 1\,. \end{cases}$$

Now the proof is due to Lemmas (2.1), (3.3). q.e.d.

If all of E, B, F satisfy the condition $(A_{p,2p-3})$ for some integer $p > 2$, we have the natural isomorphism

$$\Omega^{-1} : \pi(\Omega B,\, F) \longrightarrow \pi(B,\, F_0)$$

where $F_0 = \Omega^{-1}(F)$, (see (1.3)). We shall denote the image $\Omega^{-1}({}^1\alpha)$ of ${}^1\alpha$ by α, and call it the *characteristic class* of the fiber space $\mathfrak{E} = (E, p, B, F)$. The following theorem justifies the terminology.

THEOREM (3.5). *The equivalent class of the fiber space $\mathfrak{E} = (E, p, B, F)$ is uniquely determined by the characteristic class α;* i.e. a) *$\mathfrak{E}$ is equivalent to the principal fiber space which is induced from the principal path fibering $\pi_1 : LF_0 \to F_0$ by a representative map $f_0 : B \to F_0$ of the characteristic class α;* b) *if α_i $(i=1,2)$ are the characteristic classes of the fiber spaces $\mathfrak{E}_i = (E_i, p_i, B_i, F_i)$ $(i = 1, 2)$ and if there exist homotopy equivalences $h_B : B_1 \to B_2$, $h_{F_0} : F_{01} \to F_{02}$ with $F_{0i} = \Omega^{-1}(F_i)$, then $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are equivalent if and only if $h_B^* \alpha_2 = h_{F_0 *} \alpha_1$.*

Proof. From the above assumption that all of E, B, F satisfy the condition $(A_{p,2p-3})$, there exists a fiber space (E_0, p_0, B_0, F_0) as in Lemma (2.3), there exist homotopy equivalences h_E, h_E', h_B, h_B' such that the following diagram is commutative up to homotopy

$$\begin{tikzcd} \Omega E_0 \arrow[r, shift left, "h_E"] \arrow[d, "\Omega(p_0)"'] & E \arrow[l, shift left, "h_E'"] \arrow[d, "p"] \\ \Omega B_0 \arrow[r, shift left, "h_B"] & B \arrow[l, shift left, "h_B'"] \end{tikzcd}$$

We denote by f_0 a map $\pi_1 q_0 : \Omega B_0 \to \Omega(E_0;\ e_{00},\ F_0) \to F_0$ associated to the given fiber space (E_0, p_0, B_0, F_0) in the sence of (3.2). The map $\Omega(q_0) : \Omega^2 B \to \Omega(\Omega(E_0;\ e_{00},\ F_0))$ induces a map $q' : \Omega B \to \Omega(E, e_0, F)$ such that $q'(\omega)(t) = h_E(\rho_{Et})$, where $\omega \in \Omega B$, $0 \leqq t \leqq 1$ and $\rho_{Et} \in \Omega E_0$ is determined by $\rho_{Et}(s) = [(\Omega(q_0)\Omega(h'_B)\omega)(s)](t)$ $0 \leqq s \leqq 1$. Therefore $q' = \bar{h}_E \Omega(q_0)\Omega(h'_B)$, where $\bar{h}_E : \Omega(\Omega(E_0;\ e_{00},\ F_0)) \to \Omega(E, e_0, F)$ is induced by the map h_E as above. By Lemma (3.1), $p'_0 q_0 \simeq 1$ and so $\Omega(p'_0)\Omega(q_0) \simeq 1$. Hence $\Omega(h'_B)p'\bar{h}_E\Omega(q_0) \simeq 1$. This implies that

$$p'q' \simeq \Omega(h_B)\Omega(h'_B)p'\bar{h}_E\Omega(q_0)\Omega(h'_B) \simeq 1\,.$$

By Lemma (3.1) $qp' \simeq 1$. Thus we have $q \simeq qp'q' \simeq q'$. This implies $f \simeq \Omega(h_E)\Omega(f_0)\Omega(h'_B)$.

Since $\Omega : \pi(B, F_0) \to \pi(\Omega B, F)$ is an isomorphism, we conclude that $f_0 h'_B$ belongs to the image $\Omega^{-1}({}^1\alpha)$. Now the proof is due to Lemmas (2.3), (3.3), (3.4), since $\Omega(h_B)^{*1}\alpha_2 = h_{F*}{}^1\alpha_1$ implies $h_B^*\alpha_2 = h_{F_0 *}\alpha_1$. q.e.d.

Owing to this theorem, we shall hereafter denote by $\mathcal{P}(B, F, \alpha)$ the equivalent class of the fiber spaces (E, p, B, F) each of which is associated to the characteristic class α.

COROLLARY (3.6). *If B and F satisfy the condition $(A_{p,2p-3})$ for an integer p, then*

$$\Omega\mathcal{P}(B, F, \alpha) = \mathcal{P}(\Omega B, \Omega F, {}^1\alpha) \qquad \text{for} \quad {}^1\alpha = \Omega(\alpha)\,.$$

COROLLARY (3.7). *If B, B', F, F' satisfy the condition $(A_{p,2p-3})$ for an integer p, and if $h_B : B' \to B$, $h_F : F_0 \to F'_0$ are homotopy equivalences with $F_0 = \Omega^{-1}(F)$ and $F'_0 = \Omega^{-1}(F')$, then we have*

$$\mathcal{P}(B, F, \alpha) = \mathcal{P}(B', F', h_B^* h_{F*}\alpha)\,.$$

4. Exact sequences

Let (E, p, B, F) be a fiber space such that B is q-connected and F is r-connected. It is known [10] that if X is a space such that $\pi_j(X) = 0$ for $j > q+r+1$ then the sequence of the sets of homotopy classes

$$\pi(F, X) \xleftarrow{i^*} \pi(E, X) \xleftarrow{p^*} \pi(B, X) \tag{4.1}$$

is exact.

On the other hand, let (E, p, B, F) be a fiber space such that all of E, B, F satisfy the condition $(A_{r,2r-3})$ for an integer $r > 2$. If we denote by α the characteristic class of the fiber space, then we have:

LEMMA (4.2). *The following sequence is exact for any space X*

$$\longrightarrow \pi(X, \Omega^s F) \xrightarrow{{}^s i_*} \pi(X, \Omega^s E) \xrightarrow{{}^s p_*} \pi(X, \Omega^s B) \xrightarrow{{}^s \alpha_*} \pi(X, \Omega^{s-1} F) \longrightarrow$$

$$\cdots \longrightarrow \pi(X, F) \xrightarrow{i_*} \pi(X, E) \xrightarrow{p_*} \pi(X, B) \xrightarrow{\alpha_*} \pi(X, F_0)$$

where ${}^{s}k=\Omega^{s}(k)$, $\Omega^{s}(\ )=\Omega(\Omega^{s-1}(\ ))$ *and* $F_0=\Omega^{-1}(F)$.

Proof. Since (E, p, B, F) is equivalent to a principal fiber space $(\tilde{E}, \tilde{p}, B, \Omega F_0)$ which is induced from the principal path fibering $\pi_1: LF_0 \to F_0$ by a representative map $f_0: B \to F_0$ of the characteristic class α, the proof of this lemma is due to the results of [8, pp. 282~3]. q.e.d.

LEMMA (4.3). *Let (E, p, B, F) be a fiber space such that B is q-connected, F is r-connected and the characteristic class of the fiber space $(\Omega E, {}^{1}p, \Omega B, \Omega F)$ is ${}^{1}\alpha$. Then the following sequence is exact for any space X such that $\pi_j(X)=0$ for $j>\min(q+r-1, 2q, 2r)$;*

$$\pi(\Omega F, X) \xleftarrow{{}^{1}i^{*}} \pi(\Omega E, X) \xleftarrow{{}^{1}p^{*}} \pi(\Omega B, X)$$
$$\xleftarrow{{}^{1}\alpha^{*}} \pi(F, X) \xleftarrow{i^{*}} \pi(E, X) \xleftarrow{p^{*}} \pi(B, X)\,.$$

Proof. Consider the fiber space $(\Omega(E; e_0, F), \pi_1, F, \Omega E)$ and the commutative diagram

$$\begin{tikzcd}
\pi(\Omega E, X) & \pi(\Omega B, X) \arrow[l, "{}^{1}p^{*}"'] \arrow[d, shift right, "p'^{*}"'] & \pi(F, X) \arrow[l, "{}^{1}\alpha^{*}"'] \arrow[dl, "\pi_1^{*}"] \\
 & \pi(\Omega(E; e_0, F), X) \arrow[ul, "j_1^{*}"] \arrow[u, shift right, "q^{*}"'] &
\end{tikzcd}$$

where p'^{*}, q^{*} are isomorphisms by Lemma (3.1). According to (4.1) we have that the upper row of this diagram is exact.

Consider the fiber space $(\Omega(E; E, F), \pi_0, E, \Omega(E; e_0, F))$ and the commutative diagram

$$\begin{tikzcd}
\pi(\Omega B, X) \arrow[d, shift right, "p'^{*}"'] & \pi(F, X) \arrow[l, "{}^{1}\alpha^{*}"'] \arrow[d, "\pi_1^{*}"] & \pi(E, X) \arrow[l, "i^{*}"'] \arrow[dl, "\pi_0^{*}"] \\
\pi(\Omega(E; e_0, F), X) \arrow[u, shift right, "q^{*}"'] & \pi(\Omega(E; E, F), X) \arrow[l, "j_2^{*}"'] &
\end{tikzcd}$$

where p'^{*}, q^{*}, π_1^{*} are isomorphisms. According to (4.1) we have that the upper row of this diagram is exact.

Combine the exact sequences

$$\pi(\Omega^{s}F, X) \xleftarrow{{}^{s}i^{*}} \pi(\Omega^{s}E, X) \xleftarrow{{}^{s}p^{*}} \pi(\Omega^{s}B, X) \qquad s=0, 1,$$
$$\pi(\Omega E, X) \xleftarrow{{}^{1}p^{*}} \pi(\Omega B, X) \xleftarrow{{}^{1}\alpha^{*}} \pi(F, X)\,,$$
$$\pi(\Omega B, X) \xleftarrow{{}^{1}\alpha^{*}} \pi(F, X) \xleftarrow{i^{*}} \pi(E, X)\,,$$

then we have Lemma (4.3). q.e.d.

Especially, if all of E, B, F satisfy the condition $(A_{r, 2r-3})$ for an integer $r>2$ and if X is a space whose homotopy groups $\pi_j(X)$ vanish for $j>2r-3$, then from Lemma (4.3) we have:

COROLLARY (4.4). *The following sequence is exact:*

$$\pi(\Omega F, X) \xleftarrow{{}^1i^*} \pi(\Omega E, X) \xleftarrow{{}^1p^*} \pi(\Omega B, X)$$
$$\xleftarrow{{}^1\alpha^*} \pi(F, X) \xleftarrow{i^*} \pi(E, X) \xleftarrow{p^*} \pi(B, X)$$
$$\xleftarrow{\alpha^*} \pi(F_0, X) \xleftarrow{i_0^*} \pi(E_0, X) \xleftarrow{p_0^*} \pi(B_0, X)$$

where (E_0, p_0, B_0, F_0) *is a fiber space as in Lemma* (2.3), (3.5).

Assume that (E, p, B, F) is a fiber space such that B, F are q-, r-connected respectively, and that X is a space whose homotopy groups $\pi_j(X)$ vanish for $j>2q-2$. Under these assumptions, we have a map

$$\tau({}^1\alpha) = \Omega^{-1}\cdot{}^1\alpha^*: \pi(F, \Omega X) \to \pi(\Omega B, \Omega X) \to \pi(B, X) .$$

This map will be called the *generalized transgression homomorphism.*

LEMMA (4.5). *If* X *is a space whose homotopy type is* $K(\Pi, n+1)$ *with* $1<n<\min(2q-3,\ q+r)$ *then the following diagram is commutative*

$$\begin{tikzcd}
H^n(F;\Pi) \arrow[r, "\delta"] \arrow[d, leftrightarrow, "\approx"] & H^{n+1}(E,F;\Pi) & H^{n+1}(B;\Pi) \arrow[l, "p^*"'] \arrow[d, leftrightarrow, "\approx"] \\
\pi(F,\Omega X) \arrow[rr, "\tau({}^1\alpha)"] & & \pi(B,X)
\end{tikzcd}$$

where $\approx$ *are the natural isomorphism*

Proof. Consider the diagram

$$\begin{tikzcd}
\pi(F,\Omega X) \arrow[r, "\pi_1^*"] \arrow[rr, bend left, "{}^1\alpha^*"] \arrow[d, leftrightarrow, "\eta_{1*}"] & \pi(\Omega(E;e_0,F),\Omega X) \arrow[r, "p'^*"] \arrow[d, leftrightarrow, "\eta_{2*}"] & \pi(\Omega B,\Omega X) \arrow[d, leftrightarrow, "\eta_{3*}"] & \\
\pi(E,F;LX,\Omega X) \arrow[r, "\pi_1^*"] \arrow[dr, "\pi_{1*}"'] & \pi(LE,\Omega(E;e_0,F);LX,\Omega X) & \pi(LB,\Omega B;LX,\Omega X) \arrow[l, "L_p^*"'] & \pi(B,X) \arrow[l, "L_B"'] \arrow[ul, "\Omega_B"'] \arrow[dll, "p^*"] \\
& \pi(E,F;X,x_0) \arrow[u, "L_E"'] & &
\end{tikzcd}$$

Here η_{i*}, $i=1,2,3$, are 1-1 onto by Lemma (2.2), and L_B, L_E are defined naturally. At first we shall prove that

$$L_E\cdot\pi_{1*} = \pi_1^* .$$

Let $f: (E;\ F,\ e_0) \to (LX;\ \Omega X,\ *)$ be a map representing a class $[f]$ of $\pi(E, F;\ LX, \Omega X)$. Then $L_E\cdot\pi_{1*}[f]$, $\pi_1^*[f]$ are represented by g, h respectively, where

$$\begin{aligned}
&g, h: (LE,\ \Omega(E;\ e_0,\ F),\ *) \to (LX,\ \Omega X,\ *) ,\\
&\left.\begin{aligned} g(\rho_E)(t) &= f(\rho_E(t))(1) ,\\ h(\rho_E)(t) &= f(\rho_E(1))(t) ,\end{aligned}\right\} \quad \text{for } \rho_E \in LE,\ 0\leq t\leq 1 .
\end{aligned}$$

Define a map $H_s: (LE, \Omega(E; e_0, F), *) \to (LX, \Omega X, *)$ $0 \leqq s \leqq 1$ by

$$H_s(\rho_E)(t) = \begin{cases} f(\rho_E(s))\left(\frac{t}{s}\right) & 0 \leqq t \leqq s, \\ f(\rho_E(t))(1) & s \leqq t \leqq 1. \end{cases}$$

Then we have $H_0 = g$, $H_1 = h$; i.e. $L_E \cdot \pi_{1*} = \pi_1^*$.

The commutativities of the other parts in the above diagram are proved easily from the definition; i.e.

$$\tau(^1\alpha) = \Omega^{-1} \cdot {}^1\alpha^* = p^{*-1} \cdot \pi_{1*} \cdot \eta_{1*}^{-1}.$$

Take X a space whose homotopy type is $K(\Pi, n+1)$, and the lemma follows from the following commutative diagram

$$\begin{array}{ccccc} H^n(F, \Pi) & \xrightarrow{\delta} & H^{n+1}(E, F; \Pi) & \xleftarrow{p^*} & H^{n+1}(B; \Pi) \\ \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx \\ \pi(F, \Omega X) & \xrightarrow{\pi_{1*}\eta_{1*}^{-1}} & \pi(E, F; X, x_0) & \xleftarrow{p^*} & \pi(B, X) \end{array}$$

q.e.d.

This lemma implies that $\tau(^1\alpha)$ is just the same as the usual transgression homomorphism of the cohomology groups in this case.

We note that the verification of the Lemma (4.3) implies directly;

COROLLARY (4.6). *Let (E, p, B, F) be a fiber space such that B, F are q-, r-connected respectively, and let $^1\alpha$ be the characteristic class of the fiber space $(\Omega E, {}^1p, \Omega B, \Omega F)$. Then the following sequence is exact for any space X whose homotopy groups $\pi_j(X)$ vanish for $j >$* min $(2q-2, q+r-1, 2r)$

$$\pi(\Omega F, X) \xleftarrow{^1i^*} \pi(\Omega E, X) \xleftarrow{^1p^*} \pi(\Omega B, X) \xleftarrow{^1\alpha^*} \pi(F, X) \xleftarrow{i^*}$$
$$\pi(E, X) \xleftarrow{p^*} \pi(B, X) \xleftarrow{\tau(^1\alpha)} \pi(F, \Omega X) \xleftarrow{i^*} \pi(E, \Omega X) \xleftarrow{p^*}$$
$$\pi(B, \Omega X) \xleftarrow{\tau(^1\alpha)} \pi(F, \Omega^2 X) \xleftarrow{i^*} \cdots.$$

5. Poly-fiber spaces

Let (E, p, B, F), (E', p', E, F') be two fiber spaces such that all of E, B, F, E', F', satisfy the condition $(A_{r, 2r-3})$ for an integer $r > 2$. We shall denote by α, γ the characteristic classes of the above fiber spaces respectively.

(5.1)
$$\begin{array}{ccccc} F' & \xrightarrow{i''} & F'' & \xrightarrow{i'''} & E' \\ & & \downarrow p'' & & \downarrow p' \\ & & F & \xrightarrow{i} & E \\ & & & & \downarrow p \\ & & & & B \end{array}$$

In the above diagram, $pp': E' \to B$ is also a fiber mapping. We denote $(pp')^{-1}(b_0)$ by F''. Since $F' = p'^{-1}(e_0) \subset (pp)'^{-1}(b_0)$ there is the inclusion mappings i'', i''' such that $i''' \cdot i'' = i'$. Denote $p'|F''$ by p''. Then we have:

LEMMA (5.2). *(F'', p'', F, F') is also a fiber space whose characteristic class is $i^*\gamma$.*

Proof. From the facts $p''(F'') \subset p'(E') = E$ and $pp''(F'') = pp'(i'''F'') = \{b_0\}$, we have $p''(F'') \subset F$. Conversely let $e' \in E'$ be such an element that $p'(e') \subset F$, then we have $e' \in F''$ since $p(p'(e')) = b_0$ and $p''(F'') = F$. Hence (F'', p'', F, F') is a sub-fibering of (E', p', E, F'). From the definition of ${}^1\gamma \in \pi(\Omega E, F')$ it is easily verified that the characteristic class of the fiber space $(\Omega F'', \Omega(p''), \Omega F, \Omega F')$ is $\Omega(i)^* {}^1\gamma$. This implies that the characteristic class of the fiber space (F'', p'', F, F') is $i^*\gamma$. q.e.d.

If we denote by β the characteristic class of the fiber space (E', pp', B, F'') then we have:

LEMMA (5.3). $\Omega^{-1}(p'')_* \beta = \alpha$, $p^*\beta = \Omega^{-1}(i'')_* \gamma$.
Namely the following diagram is commutative up to homotopy

$$\begin{tikzcd}
E \arrow[r, "\gamma"] \arrow[d, "p"] & F'_0 \arrow[r, "\Omega^{-1}(i'')"] & F''_0 \arrow[d, "\Omega^{-1}(i'')"] \\
B \arrow[urr, "\beta"] \arrow[rr, "\alpha"] & & F_0
\end{tikzcd}$$

where $F_0^{(i)} = \Omega^{-1}(F^{(i)})$, $i = 0, 1, 2$.

Proof. It is sufficient to prove that $p''_* {}^1\beta = {}^1\alpha$, ${}^1p^* {}^1\beta = i''_* {}^1\gamma$ where ${}^1\alpha$, ${}^1\beta$, ${}^1\gamma$ are $\Omega(\alpha)$, $\Omega(\beta)$, $\Omega(\gamma)$ respectively.

Let $(W; W_0, w_0)$ be a CW-triad and the one of $h: (W; W_0, w_0) \to (LB; \Omega B, *)$, $g: (LB; \Omega B, *) \to (W; W_0, w_0)$ be a homotopy inverse of the other. Consider the map $h': W \times I \to B$ such that $h'(w, t) = h(w)(t)$. In view of the covering homotopy property there exist mappings $\bar{h}': W \times I \to E$, $\bar{\bar{h}}': W \times I \to E'$ such that $p\bar{h}' = h'$, $p'\bar{\bar{h}}' = \bar{h}'$ and $\bar{h}'(W \times (0) \cup w_0 \times I) = e_0$, $\bar{\bar{h}}'(W \times (0) \cup w_0 \times I) = e'_0$. Then we have two mappings $q: (LB, \Omega B, *) \to (LE, \Omega(E; e_0, F), *)$, $\bar{q}: (LB, \Omega B, *) \to (LE', \Omega(E'; e'_0, F''), *)$ such as $q(\rho_B)(t) = \bar{h}'(g(\rho_B), t)$, $\bar{q}(\rho_B)(t) = \bar{\bar{h}}'(g(\rho_B), t)$ for $\rho_B \in LB$, $0 \leq t \leq 1$. From Lemma (3.2) we have ${}^1\alpha \ni \pi_1 q | \Omega B$, ${}^1\beta \ni \pi_1 \bar{q} | \Omega B$, and hence $p''_* {}^1\beta = {}^1\alpha$ since $q = L(p')\bar{q}$.

Let $(V; V_1, V_0, v_0)$ be a CW-tetrad and the one of H, G;

$$(V; V_1, V_0, v_0) \underset{G}{\overset{H}{\rightleftarrows}} (LE; \Omega(E; e_0, F), \Omega E, *),$$

be a homotopy inverse of the other whose existence are due to the Theorem 3 of [5]. Consider the map $H': V \times I \to E$ such that $H'(v, t) = H(v)(t)$. By the covering homotopy property there exists a mapping $\bar{H}': V\times I\to E'$ such that $p'\bar{H}'=H'$ and $\bar{H}'(V\times(0)\cup v_0\times I)=e_0'$. Then we have a mapping $Q:(LE, \Omega(E; e_0, F), \Omega E, *)\to(LE', \Omega(E'; e_0', F''), \Omega(E': e_0', F'), *)$ such as $Q(\rho_E)(t)=\bar{H}'(G(\rho_E), t)$ for $\rho_E\in LE$, $0\leqq t\leqq 1$. From the Lemma (3.2) we have ${}^1\gamma\ni\pi_1 Q|\Omega E$. And since $L(p')Q(\rho_E)(t)=p'\bar{H}'(G(\rho_E), t)=H'(G(\rho_E), t)=HG(\rho_E)(t)$ we have $L(p')Q\simeq$ identity map. Hence there exists a homotopy $qL(p)\simeq 1\simeq L(p')Q$, namely there exists a map $H_0: \Omega(E; e_0, F)\times I\times I\to E$ such that

$$H_0(\rho_E, t, s) = \begin{cases} (qL(p)\rho_E)(s) & \text{if } t=0\,, \\ (L(p')Q\rho_E)(s) & \text{if } t=1\,, \\ e_0 & \text{if } s=0\,. \end{cases}$$

In view of the covering homotopy property we have a map $\bar{H}_0: V_1\times I\times I\to E'$ lifting the composition map $(H|V_1)\times 1\times 1$ such that

$$\bar{H}_0(v, t, s) = \begin{cases} (qL(p)(H(v)))(s) & \text{if } t=0\,, \\ (Q(H(v)))(s) & \text{if } t=1\,, \\ e_0' & \text{if } s=0\,. \end{cases}$$

Hence, we have a map $H_1: \Omega E\times I\to F''$ such that

$$H_1(\rho_E, t) = \bar{H}_0(G(\rho_E), t, 1) \qquad \rho_E\in\Omega E,\ 0\leqq t\leqq 1\,.$$

This implies that $\pi_1\bar{q}\,{}^1p\simeq i''\pi_1 Q$ since $HG\simeq 1$; i.e. ${}^1p^*\,{}^1\beta=i''_*\,{}^1\gamma$. q.e.d.

We shall call the *poly-fiber space* such a system as in (5.1).

LEMMA (5.4). *Let $\mathfrak{E}=(E, p, B, F)$, $\mathfrak{F}''=(F'', p'', F, F')$ be two fiber spaces such that all of E, B, F, F', F'' satisfy the condition $(A_{r,2r-3})$ for an integer $r>2$ and having the characteristic classes α_1, α_2 respectively. Then there exists a poly-fiber space* (5.1) *up to equivalence, if and only if*

$$\alpha_2\cdot{}^1\alpha_1=0 \qquad \text{where} \quad {}^1\alpha_1=\Omega(\alpha_1)\,.$$

Proof. If there exists a poly-fiber space (5.1), we denote by β the characteristic class of (E', pp', B, F''). Then $\alpha_2\cdot{}^1\alpha_1=\alpha_2(p''^*\,{}^1\beta)=(p''^*\alpha_2)\,{}^1\beta=0$ by Lemmas (3.2) and (5.3).

Conversely, if $\alpha_2\cdot{}^1\alpha_1={}^1\alpha_1^*(\alpha_2)=0$, from the Corollary (4.4) there exists a class $\gamma\in\pi(E, F_0')$ such that $i^*(\gamma)=\alpha_2$ where $F_0'=\Omega^{-1}(F')$. Hence we have a space $E'(\gamma)\in\mathcal{P}(E, F', \gamma)$ and a poly-fiber space

$$\begin{tikzcd} F' \arrow[r] & F''(\gamma) \arrow[r] \arrow[d, "p''(\gamma)"] & E'(\gamma) \arrow[d, "p'(\gamma)"] \\ & F \arrow[r, "i"] & E \arrow[d] \\ & & B \end{tikzcd}$$

where $F''(\gamma)=p'(\gamma)^{-1}F\in\mathcal{P}(F, F', \alpha_2)$; i.e. $(F''(\gamma), p''(\gamma), F, F'')\equiv\mathfrak{F}''$. q.e.d.

For the future convenience we shall state another proof of sufficiency: If $\alpha_2 \cdot {}^1\alpha_1 = \alpha_{2*}({}^1\alpha_1) = 0$, then there exists a class ${}^1\beta \in \pi(\Omega B, F'')$ such that $p''_* {}^1\beta = {}^1\alpha_1$ since the following sequence is exact by Lemma (4.2).

$$\pi(\Omega B, F') \xrightarrow{i''_*} \pi(\Omega B, F'') \xrightarrow{p''_*} \pi(\Omega B, F) \xrightarrow{\alpha_{2*}} \pi(\Omega B, F'_0) .$$

Hence we have a space $E'(\beta) \in \mathcal{P}(B, F'', \beta)$ where $\beta = \Omega^{-1}({}^1\beta)$. Namely if f is a representative map of β, then we have

$$E'(\beta) = \{(b, \rho'') \mid b \in B, \rho'' \in LF''_0, f(b) = \pi_1(\rho'')\} ,$$

where $(F''_0, p''_0, F_0, F'_0)$ is a fiber space associated to the given fiber space (F'', p'', F, F') in the sence of Theorem (3.5). We define the principal fiber space $E(\beta)$ by

$$E(\beta) = \{(b, \rho) \mid b \in B, \ \rho \in LF_0, \ p''_0 f(b) = \pi_1(\rho)\}$$

and the maps $p'(\beta) : E'(\beta) \to E(\beta)$, $p(\beta) : E(\beta) \to B$ by

$$p'(\beta)(b, \rho'') = (b, L(p''_0)(\rho'')), \ p(b, \rho) = b .$$

Since $p''_{0*}(\beta) = \alpha_1$, $(E(\beta), p(\beta), B, \Omega F_0) \equiv \mathfrak{E}$ by Theorem (3.5). It is easily seen that $p'(\beta) : E'(\beta) \to E(\beta)$ is a fiber map with fiber $\{(b_0, \rho'') \mid \rho'' \in \Omega F'_0\}$ which is homotopy equivalent to F'. Namely we have the desired poly-fiber space

$$\begin{array}{ccccc} \Omega F'_0 & \longrightarrow & \Omega F''_0 & \longrightarrow & E'(\beta) \\ & & \Big\downarrow {\scriptstyle \Omega(p''_0)} & & \Big\downarrow {\scriptstyle p'(\beta)} \\ F & \xleftarrow{h''} & \Omega F_0 & \xrightarrow{i(\beta)} & E(\beta) \\ & & & & \Big\downarrow \\ & & & & B \end{array}$$

LEMMA (5.5). *Under the same conditions above, there exists one to one correspondence among* (1) *the strongly homotopy types of* E', (2) *the classes* β *or* $\pi(B, F''_0)$, *and* (3) *the classes* γ *or* $\pi(E, F'_0)$.

Proof. (1)$\leftrightarrow$(2) and (1)$\leftrightarrow$(3) are obvious from the Theorem (3.5). We shall give here the direct correspondence of (2)$\leftrightarrow$(3).

Denote by $h: F''(\gamma) \to F'' \to \Omega F''_0$ the homotopy equivalence and by β' the characteristic class of the fiber space $(E'(\gamma), pp'(\gamma), B, F''(\gamma))$. Then according to Corollary (3.7) we have $\mathcal{P}(B, F''(\gamma), \beta') = \mathcal{P}(B, \Omega F''_0, h_* \beta')$. Namely, if we denote $h^* \beta'$ by $\beta(\gamma)$ we have $E'(\gamma) \in \mathcal{P}(B, F'', \beta(\gamma))$.

Similary, if we denote by $h' : E \to E(\beta)$ the homotopy equivalence and by γ' the characteristic class of the fiber space $(E'(\beta), p'(\beta), E(\beta), \Omega F'_0)$, then we have $\mathcal{P}(E(\beta), \Omega F'_0, \gamma') = \mathcal{P}(E, F', h'^* \gamma')$. Namely $E'(\beta) \in \mathcal{P}(E, F' \gamma(\beta))$ if we denote $h'^* \gamma'$ by $\gamma(\beta)$. In the above two diagram we have $p''(\gamma) = p'' \cdot h$, $i(\beta) = h' \cdot i \cdot h''$

$$p''_*({}^1\beta(\gamma)) = p''_*(h_*{}^1\beta') = p''(\gamma)_*{}^1\beta' = {}^1\alpha_1\,,$$
$$i^*(\gamma(\beta)) = i^*(h'^*\gamma') = i(\beta)^*h''^{*-1}\gamma' = \alpha_2\,.$$

Therefore the desired correspondence is given by $\gamma\to\beta(\gamma)$, $\beta\to\gamma(\beta)$. q.e.d.

6. Lifting problems—Secondary operations

Let X be a space. We shall consider the lifting problem of the map $f_1: X\to B$ where B is the base space of a poly-fiber space

$$\begin{array}{ccccc} F' & \longrightarrow & F'' & \longrightarrow & E' \\ & & \downarrow{\scriptstyle p''} & & \downarrow{\scriptstyle p'} \\ & & F & \longrightarrow & E \\ & & & & \downarrow{\scriptstyle p} \\ & & & & B \end{array}$$

By $\alpha_1, \alpha_2, \beta, \gamma$ we mean the characteristic classes of the fiber spaces (E, p, B, F), (F'', p'', F, F') (E', pp', B, F''), (E', p', E, F') respectively. In view of Lemma (4.2), there exists a map $f_2: X\to E$ with $p_*[f_2]=[f_1]$ if and only if $\alpha_{1*}[f_1]=0$; $\alpha_{1*}[f_1]$ is called the first obstructions.

If $[f_1]\in\pi(X, B)$ satisfies $\alpha_{1*}[f_1]=0$, $[f_1]$ is $p_*[f_2]$ for a class $[f_2]\in\pi(X, E)$. $[f'_2]\in\pi(X, E)$ is $p'_*[f_3]$ for a class $[f_3]\in\pi(X, E')$ if and only if $\gamma_*[f'_2]=0$, by Lemma (4.2). The condition for existency of such $f'_2\in p_*^{-1}[f_1]$ for a given map f_1 is that there exists f_3 such that $[f_1]=p_*p'_*[f_3]$, namely

(6.1.1) if $f_2\in p_*^{-1}[f_1]$, then $\gamma_*[f_2]$ belongs to the $\mathrm{Im}\,(i^*\delta)_*=\mathrm{Im}\,\alpha_{2*}$,

(6.1.2) $\beta_*[f_1]=0$.

From the condition (6.1.1) we can define a secondary operation

$$\Phi: \pi(X, B)\cap\mathrm{Ker}\,\alpha_{1*} \longrightarrow \pi(X, F'_0)/\alpha_{2*}\pi(X, F)$$

such that the coset $\Phi[f_1]$ contains $\gamma_*[f_2]$. Namely, $\Phi[f_1]=0$ if and only if $[f_1]$ is representable as $p_*p'_*[f_3]$; $\Phi[f_1]$ is called the secondary obstruction.

Consider the diagram

$$(6.2)\qquad \begin{array}{ccccc} \pi(X, E) & \xrightarrow{\gamma_*} & \pi(X, F'_0) & \xrightarrow{\Omega^{-1}(i'')_*} & \pi(X, F''_0) \\ \downarrow{\scriptstyle p_*} & & {\scriptstyle\beta_*}\nearrow & & \downarrow{\scriptstyle\Omega^{-1}(p'')_*} \\ \pi(X, B) & & \xrightarrow{\alpha_{1*}} & & \pi(X, F_0)\,, \end{array}$$

then by Lemma (5.3) this diagram is commutative. Since $\alpha_{1*}[f_1]=0$, $\beta_*[f_1]$ belongs to the $\mathrm{Ker}\,\Omega^{-1}(p'')_*$. By the exactness of the sequence

$$\pi(X, F) \xrightarrow{\alpha_{2*}} \pi(X, F'_0) \xrightarrow{\Omega^{-1}(i'')_*} \pi(X, F''_0) \xrightarrow{\Omega^{-1}(p'')_*} \pi(X, F_0)$$

there exists an isomorphism

$$\bar{i}: \operatorname{Coker} \alpha_{2*} \approx \operatorname{Ker} \Omega^{-1}(p'')_* .$$

From the commutativity of the diagram (6.2) we can define $\Phi[f_1]$ as $\bar{i}^{-1}\beta_*[f_1]$. It is obvious from our definitions that

(6.3.1) $\Phi[f_1]=0$ if and only if $\beta_*[f_1]=0$,

(6.3.2) $\Phi[p]$ is represented by γ,

(6.3.3) Φ is natural: i.e. if $g: Y \to X$ is a map and $[f] \in \pi(X, B) \cap \operatorname{Ker} \alpha_{1*}$, then $\Phi[fg]=g^*\Phi[f]$, where

$$g^*: \pi(X, F'_0)/\alpha_{2*}\pi(X, F) \to \pi(Y, F'_0)/\alpha_{2*}\pi(Y, F) .$$

Lemma (6.4). *Let $\mathfrak{E}=(E, p, B, F)$ be a fiber space. If $[f_1], [f_2] \in \pi(X, B) \cap \operatorname{Ker} \alpha_{1*}$ then*

$$\Phi([f_1]\circ[f_2]) = \Phi[f_1]\circ\Phi[f_2] .$$

Here $\circ$ denote the group multiplications of $\pi(\ ,\ )$, which are same to the multiplications induced by the loop structures of $\mathfrak{E} \equiv \Omega\mathfrak{E}_0$. (Lemma 2.3).

Proof. If $[f] \in \pi(X, B)$ satisfies the condition $\alpha_{1*}[f]=0$, there exists a mapping $g: X \to E$ such that $pg=f$. Hence $\Phi[f]=\Phi[pg]=g^*\Phi[p]=g^*\{\gamma\}$ by (6.3.2) and (6.3.3). If $pg_1=f=pg_2$ then there exists a mapping $h: X \to F$ such that $[g_2]=i_*[h]\circ[g_1]$. We have $g_2^*\gamma=g_1^*\gamma\circ h^*i^*\gamma=g_1^*\gamma\circ h^*\alpha_2$, and so $g^*\gamma$ is uniquely determined as the coset of α_{2*}-Image. If $[f_1], [f_2] \in \pi(X, B) \cap \operatorname{Ker} \alpha_{1*}$, there exist mappings $g_1, g_2: X \to E$ such that $pg_i=f_i$, $i=1,2$. Then

$$\begin{aligned} \Phi([f_1]\circ[f_2]) &= \Phi((g_1\circ g_2)^*[p]) = (g_1\circ g_2)^*\Phi[p] \\ &= (g_1\circ g_2)^*\{\gamma\} = g_1^*\{\gamma\}\circ g_2^*\{\gamma\} = \Phi[f_1]\circ\Phi[f_2] . \end{aligned}$$

q.e.d.

Summarizing the results of (5.4), (5.5), (6.3), (6.4) we have:

Theorem (6.5). *The relation $\alpha_2 \cdot {}^1\alpha_1=0$ induces a secondary operation*

$$\Phi: \pi(X, B) \cap \operatorname{Ker} \alpha_{1*} \longrightarrow \pi(X, F'_0)/\alpha_{2*}\pi(X, F) ,$$

and it is determined uniquely mod primary operations associated to $\pi(B, F'_0)$.

Proof. It is obvious that the secondary operation Φ is uniquely determined by the strongly homotopy type of E', namely by the class β or by the class γ. Here γ is determined for the class α_2 such as $i^*\gamma=\alpha_2$, namely γ is uniquely determined mod $p^*\pi(B, F'_0)$. Fix an element γ and construct a poly-fiber space (5.1), then we have a secondary operation $\Phi(\gamma)$ as in Lemma (6.4). Let $\alpha \in \pi(B, F'_0)$ be a class and γ' be a class $\gamma\circ p^*\alpha$. Then we have

$$\Phi(\gamma')[f] = g^*\{\gamma'\} = g^*\{\gamma\circ p^*\alpha\} = g^*\{\gamma\}\circ g^*\{p^*\alpha\} = \Phi(\gamma)[f]\circ\alpha_*[f]$$

where f, g are the same as in the proof of Lemma (6.4). This show that $\{\Phi(\gamma')\}$ are uniquely determined mod primary operations $\{\alpha_*\}$. q.e.d.

By $\Phi(\alpha_2, \alpha_1)$ we mean a class of secondary operations which are determined in the sence of Theorem (6.5) by the given relation $\alpha_2 \cdot {}^1\alpha_1 = 0$. Also, we shall denote by ${}^1\Phi$, ${}^{-1}\Phi$ the secondary operations determined by the class $\{{}^1\gamma\}$, $\{{}^{-1}\gamma\}$ respectively if the conditions for dimension are satisfied, where ${}^1\gamma = \Omega(\gamma)$ and ${}^{-1}\gamma = \Omega^{-1}(\gamma)$.

For example let $B = K(Z_2, n)$, $F = F' = K(Z_2, n)$, $\alpha_1 = \alpha_2 = Sq^1$ (*) $= \frac{1}{2}\delta$, then $\Phi(Sq^1, Sq^1)$ is the Bockstein operation $\frac{1}{4}\delta$, where modulous is zero. We shall denote $\Phi(Sq^1, Sq^1)$ by Φ_{00} for any integer n.

Let $B = K(Z_2, n)$, $F = K(Z_2, n) \times K(Z_2, n+1)$, $F' = K(Z_2, n+2)$, $\alpha_1 = Sq^1 \times Sq^2$, $\alpha_2 = Sq^3 \circ Sq^2$, then $\Phi(Sq^3 \circ Sq^2, Sq^1 \times Sq^2)$ is the Adem operation [2], where modulous is zero. We shall denote this operation by Φ_{11}.

LEMMA (6.6). *$\Phi(\alpha_2, \alpha_1)\alpha_* = \Phi(\alpha_2, \alpha_1\alpha)$, $\bar{\alpha}_*\Phi(\alpha_2, \alpha_1) = \Phi(\bar{\alpha}\alpha_2, \alpha_1)$, where $\alpha \in \pi(\bar{B}, B)$ and $\bar{\alpha} \in \pi(F'_0, \bar{F}'_0)$ satisfy the conditions for dimension* [3].

Proof. Let $\beta \in \pi(B, F''_0)$ be a class such that $p''_*{}^1\beta = {}^1\alpha_1$. Then we have $p''_*({}^1\beta\,{}^1\alpha) = {}^1\alpha_1{}^1\alpha$. This implies that we may chose ${}^1\beta\,{}^1\alpha$ as the characteristic class of $(\bar{E}', \bar{p}\bar{p}', \bar{B}, F'')$ where $\bar{E}'$ is the total space of poly-fiber space associated with the relation $\alpha_2 \cdot {}^1\alpha_1{}^1\alpha = 0$. Thus the first part of the lemma is trivial by Lemmas (5.4), (5.5), (6.5).

Let $F'' \in \mathcal{P}(F, F', \alpha_2)$ be a principal fiber space induced from the principal path-fiber space $(LF'_0, \pi_1, F'_0, \Omega F'_0)$. Let $\bar{F}'' \in \mathcal{P}(F, \bar{F}', \alpha\alpha_2)$ be a principal fiber space induced from the principal path-fiber space $(L\bar{F}'_0, \pi_1, \bar{F}'_0, \Omega\bar{F}'_0)$. Then there exists natural mapping $\bar{\bar{\alpha}}: F'' \to \bar{F}''$ such that $\bar{\bar{\alpha}}(\omega_F, \rho_{F'}) = (\omega_F, L(\bar{\alpha})\rho_{F'})$ where $\omega_F \in F$, $\rho_{F'} \in LF'_0$ and $L(\bar{\alpha}): LF'_0 \to L\bar{F}'_0$ is induced by $\bar{\alpha}$. This implies that we may choose $\bar{\bar{\alpha}}{}^1\beta$ as the characteristic class of $(\bar{E}', \bar{p}\bar{p}', B, \bar{F}'')$ where $\bar{E}'$ is the total space of poly-fiber space associated with the relation $\bar{\alpha}\alpha_2 \cdot {}^1\alpha_1 = 0$. Now the second part of the lemma is obvious since the following diagram is commutative

$$\begin{array}{ccc}
\pi(X, F'_0) & \xrightarrow{i_*} & \pi(X, F''_0) \\
\Big\downarrow \bar{\alpha}_* & & \Big\downarrow \Omega^{-1}(\bar{\bar{\alpha}})_* \\
\pi(X, \bar{F}'_0) & \xrightarrow{\bar{i}_*} & \pi(X, \bar{F}''_0)\,.
\end{array}$$

q.e.d.

LEMMA (6.7). *If $\alpha_3 \cdot {}^1\alpha_2 \cdot {}^2\alpha_1 = 0$ then $\Phi(\alpha_3{}^1\alpha_2, {}^1\alpha_1) = \Phi(\alpha_3, \alpha_2{}^1\alpha_1): \mathrm{Ker}\,{}^1\alpha_1 \to \mathrm{Coker}\,\alpha_3$, where ${}^2\alpha_1 = \Omega({}^1\alpha_1)$* [3].

(*) Here Sq^I is the homotopy class of the map $f: K(Z_2, n) \to K(Z_2, n+i)$ $i = \deg I$ such that $f^*\iota_{n+i} = Sq^I\iota_n$.

Proof. Let (E_1, p_1, B, F_1), (E_2, p_2, F_1, F_2), (E_3, p_3, F_2, F_3), $(E'_1, p'_1, \Omega B, F_2)$ and $(E'_2, p'_2, \Omega F_1, F_3)$ be fiber-spaces associated with given classes $\alpha_1, \alpha_2, \alpha_3, \alpha_2{}^1\alpha_1$ and $\alpha_3{}^1\alpha_2$ respectively.

$$\begin{tikzcd} F_3 \arrow[r] & E_3 \arrow[r] \arrow[d] & \tilde{E}_1 \arrow[d] \\ & F_2 \arrow[r] & E'_1 \arrow[d] \\ & & \Omega B \end{tikzcd} \qquad \begin{tikzcd} F_3 \arrow[r] & E'_2 \arrow[r] \arrow[d] & \tilde{E}_2 \arrow[d] \\ & \Omega F_1 \arrow[r] & \Omega E_1 \arrow[d] \\ & & \Omega B \end{tikzcd}$$

There exists a natural mapping $\bar{\alpha}_2: \Omega E_1 \to E'_1$ such that $\bar{\alpha}_2 | \Omega F_1 = {}^1\alpha_2$. If $[f] \in \pi(X, \Omega B)$ satisfies the condition ${}^1\alpha_{1*}[f]=0$, then there exists a class $[g_1] \in \pi(X, \Omega E_1)$ such that $\Omega(p_1)_*[g_1]=[f]$. Let γ_2 be the characteristic class associated to the poly-fiber space $\tilde{E}_1$: which induces the operation $\Phi(\alpha_3, \alpha_2{}^1\alpha_1)$; i.e. $i_{21}{}^*\gamma_2=\alpha_3$. Since $\bar{\alpha}_2 i_1 = i_{21}{}^1\alpha_2$ we may choose $\bar{\alpha}_2{}^*\gamma_2$ as the characteristic class associated to the poly-fiber space $\tilde{E}_2$ which induces the operation $\Phi(\alpha_3{}^1\alpha_2, {}^1\alpha_1)$.

$$\begin{tikzcd} \Omega F_1 \arrow[r, "i_1"] \arrow[d, "{}^1\alpha_2"] & \Omega E_1 \arrow[d, "\bar{\alpha}_2"] & \\ F_2 \arrow[r, "i_{21}"] & E'_1 \arrow[r, "\gamma_2"] & F_3 \end{tikzcd} \qquad \begin{tikzcd} & \Omega E_1 \arrow[r, "\bar{\alpha}_2"] \arrow[d, "\Omega p_1"] & E'_1 \arrow[dl, "p'_1"] \\ X \arrow[ur, "g_1"] \arrow[urr, "g_2"'] \arrow[r, "f"'] & \Omega B & \end{tikzcd}$$

Since $p'_{1*}(\bar{\alpha}_{2*}[g_1])=[f]$, $\Phi(\alpha_3, \alpha_2{}^1\alpha_1)$ can be represented by $(\bar{\alpha}_{2*}g_1)^*\gamma_2$. Now the proof is due to the fact that $(\bar{\alpha}_{2*}g_1)^*\gamma_2 = g_1^*(\bar{\alpha}_2^*\gamma_2) = g_1^*\gamma_1$. q.e.d.

LEMMA (6.8). *If* $\alpha_{2i}\cdot{}^1\alpha_{1i}=0 \quad i=1,2$, *then*

$$\Phi(\alpha_{21}\circ\alpha_{22}, \alpha_{11}\times\alpha_{12}) = \Phi(\alpha_{21}, \alpha_{11})\circ\Phi(\alpha_{22}, \alpha_{12}):$$
$$\operatorname{Ker}\alpha_{11}\cap\operatorname{Ker}\alpha_{12} \longrightarrow \pi(\ , F'_0) \ / \ \operatorname{Im}\alpha_{21}\cup\operatorname{Im}\alpha_{22} \ [3].$$

Proof. Let (E_i, p_i, B, F_i), (F''_i, p''_i, F_i, F') be fiber spaces associated with the given classes α_{1i}, α_{2i} $i=1,2$ respectively.

$$\begin{tikzcd} F' \arrow[r] & F''_1\times F''_2 \arrow[r] \arrow[d] & E'_1\times E'_2 \arrow[d] \\ & F'_1\times F'_2 \arrow[r] & E_1\times E_2 \arrow[d] \\ & & B \end{tikzcd}$$

If $[f] \in \pi(X, B)$ satisfies the conditions $\alpha_{11*}[f]=0$, $\alpha_{12*}[f]=0$, then there exists a class $[g_i] \in \pi(X, E_i)$ such that $p_{i*}[g_i]=[f]$. Since $\alpha_{2i}\cdot{}^1\alpha_{1i}=0$, there exists poly-fiber spaces E'_i each of which induces the operation $\Phi(\alpha_{2i}, \alpha_{1i})$ respectively $i=1,2$; i.e. if we denote by γ_i the characteristic classes of (E'_i, p'_i, E_i, F') $i=1,2$, $\Phi(\alpha_{2i}, \alpha_{1i})[f]$ is represented by $g_i^*\gamma_i$ $i=1,2$ respectively.

On the other hand, $\Phi(\alpha_{21}\circ\alpha_{22}, \alpha_{11}\times\alpha_{12})[f]$ is represented by $(g_1\times g_2)^*(\gamma_1\circ\gamma_2)$.

Then the proof follows from that $(g_1\times g_2)^*(\gamma_1\circ\gamma_2)=g_1^*\gamma_1\circ g_2^*\gamma_2$. q.e.d.

For examples, $Sq^3\Phi_{11}=\Phi(Sq^3Sq^2\circ Sq^3Sq^3,\ Sq^2\times Sq^1)=\Phi(q^5Sq^1,\ Sq^1)=Sq^5\Phi_{00}$, and if we denote $\Phi(Sq^1\circ Sq^2Sq^1\circ Sq^4,\ Sq^4\times Sq^2\times Sq^1)$ by Φ_{02} then $Sq^1\Phi_{02}=\Phi(Sq^1Sq^1\circ Sq^1Sq^2Sq^1\circ Sq^1Sq^4,\ Sq^4\times Sq^2\times Sq^1)=\Phi(Sq^2Sq^2\circ(Sq^2Sq^1Sq^2+Sq^4Sq^1),\ Sq^2\times Sq^1)=\Phi(Sq^2Sq^2\circ Sq^2Sq^3,\ Sq^2\times Sq^1)+\Phi(Sq^4Sq^1,\ Sq^1)=Sq^2\Phi_{11}+Sq^4\Phi_{00}$.

Thus we have:

(6.9.1) $\quad Sq^3\Phi_{11}=Sq^5\Phi_{00}$ mod primary operations [1],

(6.9.2) $\quad Sq^1\Phi_{02}=Sq^2\Phi_{11}+Sq_4\Phi_{00}$.

LEMMA (6.10). *Let $g: Y\to X$ be a map. If $[f]\in\pi(X, B)$ satisfies the conditions $\alpha_{1*}[f]=0$, $g^*[f]=0$, then $g^*\Phi(\alpha_2, \alpha_1)[f]=\alpha_{2*}\alpha_{1_g}[f]$ where α_{1_g} is the functional operation associated with the following commutative diagram*

$$\begin{array}{ccccccccc}
 & & \pi(X,F) & \xrightarrow{i_*} & \pi(X,E) & \xrightarrow{p_*} & \pi(X,B) & \xrightarrow{\alpha_{1*}} & \pi(X,F_0)\\
 & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} & & \\
\pi(Y,\Omega B) & \xrightarrow{{}^1\alpha_{1*}} & \pi(Y,F) & \xrightarrow{i_*} & \pi(Y,E) & \xrightarrow{p_*} & \pi(Y,B) & &
\end{array}$$

Proof. Let $[f]$ be a class of $\pi(X, B)\cap\operatorname{Ker}\alpha_{1*}\cap\operatorname{Ker} g^*$. Then from the commutativity and naturality of the diagram (6.2) we have

$$g^*\Phi(\alpha_2, \alpha_1)[f] = g^*\{\gamma_*[\bar f]\} = \{\gamma_* g^*[\bar f]\} = \{\alpha_{2*}i_*^{-1}g^*[\bar f]\}$$

where $[\bar f]\in\pi(X, E)$ is a class such that $p_*[\bar f]=[f]$ and the existency of $i_*^{-1}g^*[\bar f]\in\pi(Y, F)$ is due to that $p_*(g^*[\bar f])=g^*(p_*[\bar f])=g^*[f]=0$. Since $i_*^{-1}g^*[\bar f]$ represents the coset $\alpha_{1_g}[f]$ and $\alpha_2\cdot{}^1\alpha_1=0$, we have $g^*\Phi[f]=\alpha_{2*}\alpha_{1_g}[f]$ mod $\operatorname{Im}\alpha_{2*}g^*$. q.e.d.

7. Generalizations

Consider the diagram

$$(7.1)\qquad \begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
Y_3^3 & \longrightarrow & Y_2^3 & \longrightarrow & Y_1^3 & \longrightarrow & Y^3\\
 & & \downarrow{\scriptstyle p_2^3} & & \downarrow{\scriptstyle p_1^3} & & \downarrow{\scriptstyle p^3}\\
 & & Y_2^2 & \longrightarrow & Y_1^2 & \longrightarrow & Y^2\\
 & & & & \downarrow{\scriptstyle p_1^2} & & \downarrow{\scriptstyle p^2}\\
 & & & & Y_1^1 & \longrightarrow & Y^1\\
 & & & & & & \downarrow{\scriptstyle p^1}\\
 & & & & & & B
\end{array}$$

where all spaces satisfy the condition $(A_{r,2r-3})$ for an integer $r>2$, (Y^1, p^1, B, Y_1^1), (Y^2, p^2, Y^1, Y_2^2), (Y^3, p^3, Y^2, Y_3^3), $\cdots$ are fiber spaces. We denote inductively $(p_{n-1}^n p_{n-1}^{n+1}\cdots p_{n-1}^m)^{-1}(*_{n-1}^{n-1})$ by Y_n^m, $0<n\leqq m$, where $p_0^n=p^n$, $Y_0^n=Y^n$, $Y^0=B$ and $*_{n-1}^{n-1}$ is the base point of Y_{n-1}^{n-1}. It is easily seen similarly as in §5 that

(7.2.1) $\quad (Y^m_n, p^{k+1}_n \cdots p^m_n, Y^k_n, Y^m_{k+1})$

is a fiber space and, we denote its characteristic class by $\alpha(m, k, n)$, $m > k \geqq n$.

(7.2.2). The following diagram is commutative up to homotopy

$$\begin{tikzcd}
Y^j_n \arrow[r, "\alpha(m,j,n)"] \arrow[d, "p"'] & Y^m_{j+1,0} \arrow[r, "i"] & Y^m_{k+1,0} \arrow[d, "p'_0"] \\
Y^k_n \arrow[urr, "\alpha(m,k,n)"] \arrow[rr, "\alpha(j,k,n)"'] & & Y^j_{k+1,0}
\end{tikzcd}$$

$m > j > k \geqq n$, where $p = p^{k+1}_n \cdots p^j_n$, $p'_0 = \Omega^{-1}(p^{j+1}_{k+1} \cdots p^m_{k+1})$ and $Y_0 = \Omega^{-1}(Y)$.

(7.2.3). Let the following systems (1), (2) be as above. Then there exists a system (3) as above, if and only if $\alpha(m, k, k) \cdot {}^1\alpha(k, k-1, i) = 0$ where ${}^1\alpha = \Omega(\alpha)$, $m > k > i \geqq 0$.

$$(1)\quad \begin{array}{ccc} Y^k_k \to \cdots \to & Y^k_i \\ & \downarrow \\ & \vdots \\ & \downarrow \\ & Y^i_i \end{array} \qquad (2)\quad \begin{array}{ccc} Y^m_m \to \cdots \to & Y^m_k \\ & \downarrow \\ & \vdots \\ & \downarrow \\ & Y^k_k \end{array} \qquad (3)\quad \begin{array}{ccc} Y^m_m \to \cdots \to & Y^m_i \\ & \downarrow \\ & \vdots \\ & \downarrow \\ & Y^i_i . \end{array}$$

We shall refer such a system (3) as also a poly-fiber space.

Now, we shall consider the lifting problems of the map $f_1 : X \to B$ where B is the base space of a poly-fiber space (7.1).

If $[f_1] \in \pi(X, B)$ satisfies the conditions $\alpha(1, 0, 0)_*[f_1] = 0$, $\alpha(2, 0, 0)_*[f_1] = 0$ (i.e. $\Phi(\alpha(2, 1, 1), \alpha(1, 0, 0))[f_1] = 0$), $[f_1]$ is representable by $(p^1p^2)_*[f_3]$ for a class $[f_3] \in \pi(X, Y^2)$. The third obstruction for that $[f_1]$ can be represented by $(p^1p^2p^3)_*[f_4]$ for a class $[f_4] \in \pi(X, Y^3)$ is the class $\alpha(3, 0, 0)_*[f_1]$. Consider the diagram

$$\begin{tikzcd}
\pi(X, Y^2_2) \arrow[r] & \pi(X, Y^2_1) \arrow[r] \arrow[d, "p^2_{1*}"] & \pi(X, Y^2) \arrow[r, "\alpha(3,2,0)^*"] \arrow[d, "p^2_*"] & \pi(X, Y^3_{30}) \arrow[r] & \pi(X, Y^3_{20}) \arrow[r] \arrow[d, "p^3_{20*}"] & \pi(X, Y^3_{10}) \arrow[d, "p^3_{10*}"] \\
& \pi(X, Y^1_1) \arrow[r] & \pi(X, Y^1) \arrow[urr, "\alpha(3,1,0)_*"] \arrow[rr, "\alpha(2,1,0)_*"] \arrow[d, "p^1_*"] & & \pi(X, Y^2_{20}) \arrow[r] & \pi(X, Y^2_{10}) \arrow[d, "p^2_{10*}"] \\
& & \pi(X, B) \arrow[uurrr, "\alpha(3,0,0)_*"] \arrow[urrr, "\alpha(2,0,0)^*"] \arrow[rrr, "\alpha(1,0,0)_*"] & & & \pi(X, Y^1_{10}).
\end{tikzcd}$$

By (7.2.2) this diagram is commutative. Since $\alpha(2, 0, 0)_*[f_1] = 0$ $\alpha(3, 0, 0)_*[f_1]$ belongs to the $\operatorname{Ker} p^3_{10*}$. From the exactness of the sequence

$$\pi(X, Y^2_1) \xrightarrow{\alpha(3,2,1)_*} \pi(X, Y^3_{30}) \xrightarrow{i_{0*}} \pi(X, Y^3_{10}) \xrightarrow{p^3_{10*}} \pi(X, Y^2_{10})$$

we have an isomorphism

$$\bar{i} : (\pi(X, Y^3_{30}) / \operatorname{Im} \alpha(3, 2, 2)_*) \ / \ \operatorname{Im} \Phi \ \approx \ \operatorname{Ker} p^3_{10*}$$

as follows, where $\Phi=\Phi(\alpha(3,2,2),\ \alpha(2,1,1))$; Let $[g]$ be a class of $\pi(X, Y^2_1)$. Then $\alpha(3,2,1)_*[g]$ represents a class $\Phi(p^2_{1*}[g])\in \operatorname{Coker}\alpha(3,2,2)_*$. Thus we have $\bar{i}$ as the composition of the isomorphisms.

$$\operatorname{Coker}\alpha(3,2,2)_*/\operatorname{Im}\Phi \approx \operatorname{Coker}\alpha(3,2,1)_* \approx \operatorname{Ker} p^3_{10*}.$$

We denote by $\operatorname{Coker}\Phi$ the quotient group $\operatorname{Coker}\alpha(3,2,2)_*/\operatorname{Im}\Phi$. Define the third operation

$$\Psi: \operatorname{Ker}\Phi(\alpha(2,1,1),\ \alpha(1,0,0)) \longrightarrow \operatorname{Coker}\Phi(\alpha(3,2,2),\ \alpha(2,1,1))$$

by $\Psi[f_1]=\bar{i}^{-1}\alpha(3,0,0)_*[f_1]$.

It is easily seen similarly as in §6 that:

(7.3.1) $\Psi[f]=0$ if and only if $\alpha(3,0,0)_*[f]=0$,

(7.3.2) $\Psi[p^1p^2]$ is represented by $\alpha(3,2,0)$,

(7.3.3) Ψ is natural: i.e. if $g: Y\to X$ is a map and $[f]\in\pi(X, B)\cap \operatorname{Ker}\Phi(\alpha(2,1,1),\ \alpha(1,0,0))$, then $\Psi[fg]=g^*\Psi[f]$, where

$$g^*: (\pi(X, Y^3_{30})/\operatorname{Im}\alpha(3,2,2)_*)\ /\ \operatorname{Im}\Phi \longrightarrow (\pi(Y, Y^3_{30})/\operatorname{Im}\alpha(3,2,2)_*)\ /\ \operatorname{Im}\Phi,$$
$$\Phi=\Phi(\alpha(3,2,2),\ \alpha(2,1,1)).$$

(7.3.4) $\Psi([f_1]\circ[f_2])=\Psi[f_1]\circ\Psi[f_2]$ (see Lemma (6.4)).

THEOREM (7.4). *If either the relations* $\alpha(3,2,2)_*\,{}^1\Phi(\alpha(2,1,1),\ \alpha(1,0,0))=0$ *or* $\Phi(\alpha(3,2,2),\ \alpha(2,1,1))\cdot{}^1\alpha(1,0,0)_*=0$ *is satisfied, there exists a third operation* Ψ *as above, and it is determined uniquely mod secondary operations.*

Proof. It is obvious that the third operation Ψ is uniquely determined by the equivalent class of $(Y^3,\ p^1p^2p^3,\ B,\ Y^3_1)$ namely by the class $\alpha(3,0,0)$ or by the class $\alpha(3,2,0)$.

We assume that $\alpha(3,2,2)\cdot{}^1\Phi(\alpha(2,1,1),\ \alpha(1,0,0))=0$. This relation implies that $\alpha(3,2,2)\ {}^1\alpha(2,1,0)=0$, and so by Lemma (5.4) there exists a poly-fiber space:

$$\begin{array}{ccccc}
Y^3_3 & \longrightarrow & Y^3_2 & \longrightarrow & \bar{Y}^3 \\
 & & \downarrow & & \downarrow p' \\
 & & Y^2_2 & \xrightarrow{i^2} & Y^2 \\
 & & & & \downarrow p^2 \\
 & & Y^1_1 & \xrightarrow{i^1} & Y^1.
\end{array}$$

The characteristic class $\bar{\alpha}(3,2,0)$ of the fiber space $(\bar{Y}^3,\ p',\ Y^2,\ Y^3_3)$ is uniquely determined mod $p^{2*}\pi(Y^1, Y^3_{30})$ since $i^{2*}\bar{\alpha}(3,3,0)=\alpha(3,2,2)$. Fix an element $\bar{\alpha}(3,2,0)$ and construct a poly-fiber space as above.

Let $\alpha\in\pi(Y^1, Y^3_{30})$ be a class and $\bar{\alpha}'(3,2,0)$ be a class $\bar{\alpha}(3,2,0)\cdot p^{2*}\alpha$. Then we have $\Psi(\bar{\alpha}')[f]=\bar{f}^*\{\bar{\alpha}'\}=\bar{f}^*\{\bar{\alpha}\circ p^{2*}\alpha\}=\bar{f}^*\{\bar{\alpha}\}\circ\bar{f}^*\{p^{2*}\alpha\}=\Psi(\bar{\alpha})[f]\circ\Phi[f]$ where $[f]\in\pi(X,B)\cap\operatorname{Ker}\alpha(2,0,0)_*$, $\bar{f}: X\to Y^2$ is a map such that $(p^1p^2)_*[\bar{f}]=[\bar{f}]$

and Φ is a secondary operation $\Phi(i^{1*}\alpha,\ \alpha(1,0,0))$ associated to the poly-fiber space

$$\begin{tikzcd} Y^3_3 \arrow[r] & F \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ & Y^1_1 \arrow[r, "i^1"] & Y^1 \arrow[d, "p^1"] \\ & & B \end{tikzcd}$$

(The characteristic classes of (E, p, Y^1, Y^3_3), (Y^1, p^1, B, Y^1_1) are $\alpha, \alpha(1,0,0)$ respectively).

Hence Ψ is uniquely determined mod secondary operations whose type is $\Phi(i^{1*}\alpha,\ \alpha(1,0,0))$.

We assume that $\Phi(\alpha(3,2,2),\ \alpha(2,1,1))\cdot{}^1\alpha(1,0,0))_*=0$. This relation implies that $\alpha(3,1,1)\ {}^1\alpha(1,0,0)=0$, and so by Lemma (5.4) there exists a poly-fiber space:

$$\begin{tikzcd} Y^3_2 \arrow[r, "i^3"] \arrow[d, "p^3_2"] & Y^3_1 \arrow[r] \arrow[d, "p''"] & \bar{Y}^3 \arrow[d, "p'"] \\ Y^2_2 & Y^1_1 \arrow[r] & Y^1 \arrow[d, "p^1"] \\ & & B\,. \end{tikzcd}$$

The characteristic class $\bar{\alpha}(3,0,0)$ of the fiber space $(\bar{Y}^3, p^1p', B, Y^3_1)$ is uniquely determined mod $i^{3*}\pi(B, Y^3_{20})$ since $p''^*\bar{\alpha}(3,0,0)=\alpha(1,0,0)$. Fix an element $\bar{\alpha}(3,0,0)$ and construct a poly-fiber space as above.

Let $\alpha\in\pi(B, Y^3_{20})$ be a class and $\bar{\alpha}'(3,0,0)$ be a class $\bar{\alpha}(3,0,0)\circ i^{3*}\alpha$.
Then we have

$$\Psi(\bar{\alpha}')[f]=\bar{i}^{-1}\bar{\alpha}'_*[f]=\bar{i}^{-1}(\bar{\alpha}\circ i^{3*}\alpha)_*[f]=\bar{i}^{-1}\bar{\alpha}_*[f]\circ\bar{i}^{-1}i^{3*}\alpha[f]=\Psi(\bar{\alpha})[f]\circ\Phi[f]$$

where $[f]\in\pi(X,B)\cap\mathrm{Ker}\,\alpha(2,0,0)_*$ and Φ is a secondary operation $\Phi(\alpha(3,2,2),\ p^3_{20*}\alpha)$ associated to the poly-fiber space

$$\begin{tikzcd} Y^3_3 \arrow[r] & Y^3_2 \arrow[r] \arrow[d, "p^3_2"] & E' \arrow[d, "p'"] \\ & Y^2_2 \arrow[r] & E \arrow[d, "p"] \\ & & B \end{tikzcd}$$

(The characteristic classes of (E', pp', B, Y^3_2), $(Y^3_2, p^3_2, Y^2_2, Y^3_3)$ are $\alpha, \alpha(3,2,2)$ respectively).

Hence Ψ is uniquely determined mod secondary operations whose type is $\Phi(\alpha(3,2,2),\ p^3_{20*}\alpha)$. q.e.d.

THEOREM (7.5). *Let Ψ be a third operation associated with the relation $\alpha^1\Phi=0$.*

Let $g: Y \to X$ *be a map. If* $[f] \in \pi(X, B)$ *satisfies the conditions* $\Phi[f]=0$, $g^*[f]=0$, *then* $g^*\Psi[f]=\alpha_*\Phi_g[f]$ *mod* $\operatorname{Im}\alpha_* g^*$ *where* Φ_g *is the functional operations associated with the following commutative diagram*

$$\begin{array}{ccccccc} \pi(X, Y_2^2)/\operatorname{Im}{}^1\alpha_{2*} & \xrightarrow{i_*} & \pi(X, Y^2) & \xrightarrow{(p^1p^2)_*} & \pi(X, B)\cap\operatorname{Ker}\alpha_{1*} & \xrightarrow{\Phi} & \\ \downarrow g^* & & \downarrow g^* & & \downarrow g^* & & \\ \xrightarrow{{}^1\Phi} \pi(Y, Y_2^2)/\operatorname{Im}{}^1\alpha_{2*} & \xrightarrow{i_*} & \pi(Y, Y^2) & \xrightarrow{(p^1p^2)_*} & \pi(Y, B)\cap\operatorname{Ker}\alpha_{1*}. & & \end{array}$$

Here $\Phi=\Phi(\alpha_2, \alpha_1)$, $\alpha_1=\alpha(1, 0, 0)$, $\alpha_2=\alpha(2, 1, 1)$, $\alpha=\alpha(3, 2, 2)$, *and we use the same notations as above.*

Proof. Let $[f]$ be a class of $\pi(X, B)\cap\operatorname{Ker}\Phi\cap\operatorname{Ker} g^*$. Then we have

$$g^*\Psi[f] = g^*\{\alpha(3, 2, 0)_*[\bar f]\} = \{\alpha(3, 2, 0)_* g^*[\bar f]\} = \{\alpha(3, 2, 2)_* i_*^{-1} g^*[\bar f]\}$$

where $[\bar f]\in\pi(X, Y^2)$ is a class such that $(p^1p^2)_*[\bar f]=0$. Since $i_*^{-1}g^*[\bar f]$ represents the coset $\Phi_g[f]$ and $\alpha^1\Phi=0$, we have $g^*\Psi[f]=\alpha_*\Phi_g[f]$ mod $\operatorname{Im}\alpha_* g^*$. q.e.d.

We may continue these considerations about higher poly-fiber spaces; and we have some higher operations similarly as above.

8. Applications

We denote by η the essential map: $S^{m+1}\to S^m$ for any $m\geqq 2$. Let n be an integer $\geqq 7$. $X=S^n\cup e^{n+4}$ where e^{n+4} is attached to S^n by the composition map $\eta\cdot\eta\cdot\eta: S^{n+3}\to S^n$. Let Ψ be a third operation associated to the relation $Sq^1\Phi_{02}+Sq^2\Phi_{11}+Sq^4\Phi_{00}=0$ (6.9.2). We denote the generators of $H^n(X, Z_2)(\approx\pi(X, K(Z_2, n)))$, $H^{n+4}(X, Z_2)$ by s^n, e^{n+4} respectively. Then we have:

THEOREM (8.1). $\Psi(s^n)=e^{n+4}$.

Proof. Since $Sq^1(s^n)=0$, $Sq^2(s^n)=0$, $Sq^4(s^n)=0$, we can define $\Phi_{02}(s^n)$, $\Phi_{11}(s^n)$, $\Phi_{00}(s^n)$. By the conditions for dimension it is obvious that $\Phi_{11}(s^n)=0$, $\Phi_{00}(s^n)=0$. Also $\Phi_{02}(s^n)=0$ (see Lemma (8.2)). Thus we can define $\Psi(s^n)$.

Let $Y=S^{n+2}\cup\bar e^{n+4}$, where $\bar e^{n+4}$ is attached to S^{n+2} by the map η. Let $V=\bar S^n\cup e^{n+3}$, where e^{n+3} is attached to $\bar S^n$ by the composition map $\eta\cdot\eta: S^{n+2}\to\bar S^n$. We denote the generators of $H^{n+2}(Y)$, $H^{n+4}(Y)$, $H^n(V)$, $H^{n+3}(V)$ by s^{n+2}, $\bar e^{n+4}$, $\bar s^n$, e^{n+3} respectively, where $H^*(\ )=H^*(\ , Z_2)$. Then it is known [2] that $\Phi_{11}(\bar s^n)=e^{n+3}$.

Let $g: Y\to X$ be a map such that $g|S^{n+2}=\eta\cdot\eta$; we denote by f, i, j the map $g|S^{n+2}$ and the inclusion maps $S^n\to X$, $S^{n+2}\to Y$ respectively.

$$\begin{array}{ccc} S^n & \xleftarrow{f} & S^{n+2} \\ \downarrow i & & \downarrow \\ X & \xleftarrow{g} & Y \end{array}$$

If we denote by C the mapping cylinder of f, it is easily seen that $H^m(C, S^{n+2}) \approx H^m(V)$ for any $m>1$. Thus we have $\Phi_{11_f}(s_0^n)=s_0^{n+2}$ where s_0^n, s_0^{n+2} are the generators of $H^n(S^n)$, $H^{n+2}(S^{n+2})$ respectively.

From naturality of the functional operations we have $\Phi_{11_g}(s^n)=s^{n+2}$ since $j^*s^{n+2}=s_0^{n+2}=\Phi_{11_f}(s_0^n)=\Phi_{11_f}(i^*s^n)=j^*\Phi_{11_g}(s^n)$ and j^* is isomorphisms. By the conditions for dimension, it is obvious that $\Phi_{00_g}(s^n)=0$, $\Phi_{02_g}(s^n)=0$.

Hence by Theorem (7.5) we have $g^*\Psi(s^n)=Sq^2\Phi_{11_g}(s^n)=Sq^2(s^{n+2})=\bar{e}^{n+4}$. This implies $\Psi(s^n)=e^{n+4}$ since $g^*: H^{n+4}(X)\approx H^{n+4}(Y)$. q.e.d.

We denote by $\nu: S^{n+3}\to S^n$ the suspension of the Hopf map $S^7\to S^4$. $X_1=S_1^n\cup e_1^{n+4}$, where e_1^{n+4} is attached to S_1^n by ν. $X_2=S_2^n\cup e_2^{n+4}$, where e_2^{n+4} is attached to S_2^n by 6ν. We denote the generators of $H^n(X_i, Z_2)$, $H^{n+4}(X_i, Z_2)$ by s_i^n, e_i^{n+4}, $i=1,2$, respectively. Then we have:

Lemma (8.2) $\Phi_{02}(s_2^n)=e_2^{n+4}$, $\Phi_{02}(s^n)=0$.

Proof. Since $Sq^1(s_2^n)=0$, $Sq^2(s_2^n)=0$, $Sq^4(s_2^n)=0$, we can define $\Phi_{02}(s_2^n)$.

Let $g: X_1\to X_2$ be a map such that $g|S_1^n: S_1^n\to S_2^n$ (degree 6). Similarly as in the proof of Theorem (8.1) we have

$$\begin{aligned} g^*\Phi_{02}(s_2^n) &= Sq^4\cdot Sq^1{}_g(s_2^n)+Sq^2Sq^1\cdot Sq^2{}_g(s_2^n)+Sq^1\cdot Sq^4{}_g(s_2^n) \\ &= Sq^4\cdot Sq^1{}_g(s_2^n) = Sq^4(s_1^n) = e_1^{n+4}. \end{aligned}$$

This implies $\Phi_{02}(s_2^n)=e_2^{n+4}$.

Let $g': X_1\to X$ be a map such that $g'|S_1^n: S_1^n\to S^n$ (degree 12). It is obvious that $Sq^1{}_{g'}(s^n)=0$, then

$$\Phi_{02}(s^n) = 0 \quad \text{since} \quad 12\nu = \eta\cdot\eta\cdot\eta. \qquad \text{q.e.d.}$$

(Added in proof) See 579-36: M. Mahowald; On obstructions to extending a maps, Notices, Amer. Math. Soc., Vol. 8 (1961) p. 241.

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