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On Poincaré conjecture for M^5

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1. Introduction

Let M be a compact, connected differentiable n-manifold of class C^∞, with $n>1$. According to M. Morse [2] there is a non-degenerate function f on M such that f has just one critical point of index 0 and just one critical point of index n, which is termed the *polar function*. The similar way to the proof of the existence of canonical functions proves that we can modify a polar function f so that it satisfies the following properties (see [3]):

a) f is a polar function,

b) if P_j^i $(j=1,\cdots,n_i)$ are all critical points of index i, then

$$\eta_i = f(P_1^i) = \cdots = f(P_{n_i}^i)\,, \qquad i=0,\cdots,n\,,$$

c)

then

$$\eta_0<\eta_1<\cdots<\eta_n\,.$$

For an arbitrary real number c let $V_c=\{P\in M|f(P)\leqq c\}$, $V'_c=\{P\in M|f(P)\geqq c\}$. Introduce a Riemannian metric on M, ane denote by L_j^i the set of all ortho-f-arcs which are stretched to P_j^i, and denote by $L_j^i(c)$ the set of all points P on $L_j^i\cup P_j^i$ at which $c\leqq f(P)\leqq f(P_j^i)$. Then $L_j^i(c)$ is diffeomorphic with an i-dimensional ball in a euclidean space R^i.

By a *regular cell* in M we mean the image of the open unit ball under a regular imbedding map of its closure into M.

Suppose that U is an arbitrary regular cell such that $\partial L_j^i(c)\subset U$ and $L_j^i(c)$ is transversal to ∂U. Let τ_U be a C^∞-map of M into itself such that τ_U is regular 1-1 in $M-U$ and $\tau_U(\bar{U})=$a point in M. If for every $L_j^i(c)$ $(j=1,\cdots,n;\ i=1,\cdots,[n/2])$ $\tau_U L_j^i(c)$ is contained in a regular cell, we can show that V_c is contained in a regular cell. Similarly we can show that V'_c is also contained in a regular cell. Hence M is a sum of two cells and thus M is a sphere. (Theorem 1). We see that if $\pi_1(M^n)=0$ $(n>4)$ then $\tau_U L_j^1(c)$ is contained in a regular cell. Furthermore, when $n=5$, we can show that if $\pi_2(M^5)=0$ for every $L_j^2(c)$ and U then there is a cell containing $\tau_U L_j^2(c)$, thus we conclude that if $\pi_1(M^5)=0$ and $\pi_2(M^5)=0$, then M^5 is homeomorphic to a sphere S^5 (THEOREM 3).

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2. The sets L_j^i and $L_j^i(c)$

Let f be a canonical polar function as in §1. We choose in a neighborhood $U(P_j^i)$ of P_j^i a system of coordinates $x_1, \cdots, x_n$ so that f is represented in $U(P_j^i)$ as

$$f = a - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 .$$

By the *level manifold* of f we mean the set of all points P at which $f(P)=c$, where c is an arbitrary real number. Riemann metric into M which is represented in every $U(P_j^i)$ as $ds^2=\sum_{k=1}^{n} dx_k^2$. Then the trajectories orthogonal to the level manifolds of f are well defined in $M-\sum P_j^i$. These trajectories are called *ortho-f-arcs* on M. The differential equations defining them in $U(P_j^i)$ are

$$\frac{dx_k}{dt} = \varepsilon_k x_k , \quad 1 \leqq k \leqq n ,$$

where $\varepsilon_k=-1$ for $1\leqq k\leqq i$ and $\varepsilon_k=1$ for $i<k\leqq n$. Hence the solution of these equations is

$$x_k = c_k \exp \varepsilon_k t .$$

We suppose that the direction of every ortho-f-arc coincides with that of increasing of f.

If we put $c_{i+1}= \cdots =c_n=0$ and make $t\to+\infty$, we have $x\to 0$. Therefore, by putting $c_{i+1}= \cdots =c_n=0$ and by considering that $c_1, \cdots, c_i$ are variables, we get all ortho-f-arcs stretched into the critical point P_j^i. Therefore the set L_j^i of all points which are in $U(P_j^i)$ and on the ortho-f-arcs stretched into the critical point P_j^i is an i-dinensional space:

$$L_j^i = \{x \,|\, x_{i+1} = \cdots = x_n = 0,\ x \neq 0\} ,$$

and the set of points P on $L_j^i \cup P_j^i$ at which $c\leqq f(P) \leqq f(P_j^i)$ is written as

$$L_j^i(c) = \{x \,|\, x_{i+1} = \cdots = x_n = 0 ,\ c \leqq f(x) \leqq f(P_j^i)\} .$$

3. Uhe transformation $\tau_{U,\delta}$ and the varieties $\tau_U L(c)$.

By a *regular cell* in M we mean the image of the open unit ball B_1^n under a regular imbedding of $B_{1+\delta}^n$ into $M(\delta>0)$. Denote by $V_{c,c'}$ the subset of points P at which $c\leqq f(P)\leqq c'$. Then we have

LEMMA 1. *Assume that all critical points in $V_{c,c'}$ are of index i and that $\sum_j L_j^i(c) \cup V_c$ is contained in a regular cell. Then $V_{c'}$ is also contained in a regular cell.*

Proof. For simplicity we assume that $f(P_j^i)=0$, $c=-\varepsilon$ and $c'=\varepsilon$. Then, in terms of a certain coordinates x in a neighborhood $U(P_j^i)$ of P_j^i, f is written as

$$f = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 ,$$

Hence in $U(P_j^i)$ we have

$$V_{\pm\varepsilon} = \{x \mid -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 \leqq \pm\varepsilon\} ,$$
$$L_j^i(-\varepsilon) = \{x \mid x_{i+1} = \cdots = x_n = 0, \ -\varepsilon \leqq f(x) \leqq 0\} .$$

Let $\varphi(r)$ be a function of one variable r such that

$$\varphi(r) = \varepsilon + \frac{\delta}{2} \quad \text{for} \quad 0 \leqq r \leqq \varepsilon ,$$
$$= r \quad \text{for} \quad r \geqq \varepsilon + \delta ,$$
$$0 \leqq \frac{d\varphi(r)}{dr} \leqq 1 ,$$

where $\delta > 0$ is sufficiently small in comparison with $\varepsilon > 0$. Consider the set $\tilde{V}_{-\varepsilon,\delta}$ defined as follows:

$$\tilde{V}_{-\varepsilon,\delta} \cap (M - \sum U(P_j^i)) = V_{-\varepsilon} \cap (M - \sum U(P_j^i)) ,$$
$$\tilde{V}_{-\varepsilon,\delta} \cap U(P_j^i) = \{(x_1, \cdots, x_n) \mid -\varphi(x_1^2 + \cdots + x_i^2) + x_{i+1}^2 + \cdots + x_n^2 \leqq -\varepsilon\} .$$

Then $\partial \tilde{V}_{-\varepsilon,\delta} \cap U(P_j^i)$ is represented as

(1) $$-\varphi(x_1^2 + \cdots + x_i^2) + x_{i+1}^2 + \cdots + x_n^2 = -\varepsilon ,$$

and we have $\tilde{V}_{-\varepsilon,\delta} \subset V$. As we have already seen, the ortho-f-arc passing through a given point Q on $\partial V_\varepsilon \cap U(P_j^i)$ is given by

(2) $$x_k = a_k e^{\varepsilon_k t} , \qquad k = 1, \cdots, n ,$$

where $x(Q) = a$. Hence the intersection of the ortho-f-arc with $\partial \tilde{V}_{-\varepsilon,\delta} \cap U(P_j^i)$ is obtained by solving the equations (1) and (2). From (1) and (2) we have

(3) $$-\varphi((a_1^2 + \cdots + a_i^2)e^{-2t}) + (a_{i+1}^2 + \cdots + a_n^2)e^{2t} = -\varepsilon .$$

Put

$$g(t) = -\varphi((a_1^2 + \cdots + a_i^2)e^{-2t}) + (a_{i+1}^2 + \cdots + a_n^2)e^{2t} .$$

The we have

$$g(0) = -\varphi(a_1^2 + \cdots + a_i^2) + a_{i+1}^2 + \cdots + a_n^2$$
$$\geqq -a_1^2 - \cdots - a_i^2 + a_{i+1}^2 + \cdots + a_n^2 = \varepsilon .$$

If $t < 0$ is sufficiently small we have

$$g(t) = -(a_1^2 + \cdots + a_i^2)e^{-2t} + (a_{i+1}^2 + \cdots + a_n^2)e^{2t}$$
$$\leqq (a_{i+1}^2 + \cdots + a_n^2)e^{2t} < \varepsilon .$$

Since $\frac{d}{dt} g(t) > 0$ the equation (3) has the unique solution $t = t(a) < 0$. Therefore, for a given $Q \in \partial V_\varepsilon \cap \sum U(P_j^i)$ the intersection of (1) and (2) is a unique point which we denote by $\tilde{Q}$. Furthermore, for $Q \in \partial V_\varepsilon - \sum U(P_j^i)$ the ortho-f-are passing through Q intersects with $\partial V_{-\varepsilon}$ at a unique point which we denote by $\tilde{Q}$. Now we have a correspondence $\partial V_\varepsilon \to \partial \tilde{V}$ given by $Q \to \tilde{Q}$. This is regular $1-1$.

Let κ_Q be the ortho-f-arc connecting two points Q and $\tilde{Q}$, and let $d(Q, Q')$ denote the length of κ_Q between Q and $Q'(Q' \in \kappa_Q)$.

Take a C^∞-function $\psi(r)$ such that

$$\begin{aligned} \psi(r) &= \frac{\delta'}{1-\delta'}r && \text{for } 0 \leqq r \leqq 1-\delta', \\ &= 1 && \text{for } r \geqq 1, \\ \frac{d\psi(r)}{dr} &\geqq 0. && \text{for } r \geqq 0. \end{aligned}$$

Define a C^∞-transformation $\sigma_{\delta'}: Q' \to Q''$ as follows:

$$\begin{aligned} \sigma_{\delta'} &= \text{identity} && \text{for } M-(V_\varepsilon - \tilde{V}_{-\varepsilon,\delta}), \\ \frac{d(Q'', \tilde{Q})}{d(Q, \tilde{Q})} &= \psi \frac{d(Q', \tilde{Q})}{d(Q, \tilde{Q})} && \text{for } V_\varepsilon - \tilde{V}_{-\varepsilon,\delta}. \end{aligned}$$

From the definition of φ and (1) we see that the point $x \in \tilde{V}_{-\varepsilon,\delta} \cap U(P^i_j)$ always satisfy

$$x^2_{i+1} + \cdots + x^2_n \leqq \delta$$

or

$$x^2_{i+1} + \cdots + x^2_n = -x - \cdots - x^2_i + \varepsilon .$$

On the other hand, since $\sum_j L^i_j(-\varepsilon) \cup V_{-\varepsilon}$ is contained in a regular cell U, we have

$$\tilde{V}_{-\varepsilon,\delta} \subset U \qquad \text{for sufficiently small } \delta > 0 .$$

Furthermore, since $\lim_{\delta' \to 0} \sigma_{\delta'} V_{\varepsilon/2} = V_{-\varepsilon,\delta}$, if $\delta' > 0$ is sufficiently small we have $\sigma_{\delta'} V_{\varepsilon/2} \subset U$ and hence $V_{\varepsilon/2} \subset \sigma^{-1}_{\delta'} U$. Thus the lemma is proved.

LEMMA 2. *Suppose that $\gamma_{i-1} < c < \gamma_i$ and V_c is contained in a regular cell. Then we can choose a regular cell U such that $V_c \subset U$ and $\sum_j L^i_j(c)$ is transversal to ∂U.*

Proof. Let f be a regular imbedding map of $B^n_{1+\delta}$ ($\delta > 0$) into M such that $fB^n_1 = U$. Let $\{W_\mu\}$, $\{W'_\mu\}$ be two sufficiently fine open coverings of ∂B^n_1 such that $W_\mu \supset \overline{W}'_\mu$. Consider all sets W_μ such that $\sum L^i_j(c)$ is transversal to ∂U in $f\overline{W}'_\mu$; we arrange these in a sequence $\overline{W}'_1, \overline{W}'_2, \cdots, \overline{W}'_\nu$. We shall say that a map $f': W_\mu \to M$ is an approximation of (f, W_μ, ζ) if the following is satisfied:
For any $u \in W_\mu$

$$|f_p(u) - f'_p(u)| < \zeta \text{ and } \left| \frac{\partial f_p(u)}{\partial u_k} - \frac{\partial f'_p(u)}{\partial u_k} \right| < \zeta$$
$$p, k = 1, \cdots, n ,$$

where $(f_1, \cdots, f_n)$, $(f'_1, \cdots, f'_n)$ are representations of f, f' in terms of local coordinates $x_1, \cdots, x_n$ in M. Let f' be an approximation of $(f, W_{\nu+1}, \zeta)$. Put $g(x_1, \cdots, x_n) = |f'^{-1}(x)|^2$. Then, from $|u|^2 = u^2_1 + \cdots + u^2_n = 1$ and $x = f'(u)$, we have

$$(4) \qquad g(x_1, \cdots, x_n) = 1 .$$

The tangent space of (4) at $x = a$ is

$$\sum_k \left(\frac{\partial g}{\partial x_k}\right)_{x=a}(x-a)=0\,.$$

Now we choose a system of local coordinates x so that $L_j^i(c)$ is represented as $x_{i+1}=\cdots=x_n=0$. If $L_j^i(c)$ is not transversal in $f'(W_{\nu+1})$ to the submanifold defined by (4), there is a point $x=a=(a_1,\cdots,a_i,\,0,\cdots,0)$ such that $g(a)=0$ and the tangent space of $L_j^i(c)$ at a is contained in that of (4) at a. Sinc the tangent space of $L_j^i(c)$ at a contains i vectors $(\alpha,0,\cdots,0)$, $(0,\alpha,0,\cdots,0)$, $\cdots$, $(0,\cdots,0,\alpha,\cdots,0)$ for an arbitrary real number α, we have

$$g(a)=1\,,\quad a=(a_1,\cdots,a_i,\,0,\cdots,0)\,,$$
$$\left(\frac{\partial g}{\partial x_p}\right)_{x=a}(\alpha-a_j)-\sum_{p\neq q}\left(\frac{\partial g}{\partial x_p}\right)_{x=a}a_p=0\,,\qquad j=1,\cdots,i\,.$$

Since α is a variable, it follows that

$$g(a_1,\cdots,a_i,\,0,\cdots,0)=1\,,$$
$$\frac{\partial g}{\partial x_j}(a_1,\cdots,a_i,\,0,\cdots,0)=0\,,\qquad j=1,\cdots,i\,. \tag{5}$$

Obviously for a given $\zeta>0$ there is an approximation f' of $(f,\,W_{\nu+1},\,\zeta)$ for which the equations (5) have not solution. Let $\lambda(u)$ be a C^∞-function which takes 1 on $W'_{\nu+1}$ and 0 on $B^n_{1+\delta}-W_{\nu+1}$. Put

$$f''_p=f_p+\lambda(f'_p-f_p)\,,$$

and take a map F such that

$$F=f'' \text{ on } \overline{W}'_{\nu+1}\,,$$
$$=f \quad\text{on } B^n_{1+\delta}-W_{\nu+1}\,.$$

For a given $\zeta'>0$ we choose δ so that F approximates $(f,\,B^n_{1+\delta},\,\zeta')$. If ζ' is sufficiently small it holds that F is regular imbedding of $B^n_{1+\delta}$ into M and $\sum_j L_j^i(c)$ is transversal to $F(\partial B^n_1)$ in $F(\bigcup_{\mu=1}^{\nu}\overline{W}'_\mu)$ and further $F(B^1_n)\supset V_c$. Hence $\sum_j L_j^i(c)$ is transversal to $F(\partial B^n_1\cap(\bigcup_{\mu=1}^{\nu+1}W'_\mu))$. By repreating this process we have the required result.

THEOREM 1. *Let U be a regular cell such that $\sum_j \partial L_j^i(c)\subset U$ and $\sum_j L_j^i(c)$ is is transversal to ∂U. Let τ_U be a C^∞-map of M into itself such that τ_U is a diffeomorphism on $M-\bar{U}$ and $\tau_U(\bar{U})$ is a point. Assume that every $\tau_U L_j^i(c)$, $i\leq[n/2]$, is contained in a regular cell. Then M is a sphere.*

Proof. Suppose that $\eta_{i-1}<c_{i-1}<\eta_i<c_i<\eta_{i+1}$, and that $V_{c_{i-1}}$ is contained in a regular cell U. Then by lemma 2 we may suppose that $\sum_k L_k^i(c_{i-1})$ is transversal to ∂U. Let f be a regular imbedding of B^n_2 into M such as $fB^n_i=U$. Take a C^∞-function $\varphi(r)$ such that

$$\varphi(r)=\delta\qquad\text{for } 0\leq r\leq 1\,,$$
$$=1\qquad\text{for } r\geq 3/2\,,$$

$$\frac{d\varphi(r)}{dr} \geqq 0 \quad \text{for } 0 \leqq r \leqq 3/2\,,$$

and define a C^∞-transformation τ'_δ on B^n_2 by

$$\tau'_\delta u = \varphi(|u|)u\,.$$

Define a map τ_δ of M into itself as follows:

$$\begin{aligned} \tau_\delta &= f\tau'_\delta f^{-1} && \text{on } f(B^n{}_{3/2})\,, \\ &= 1 && \text{on } M - f(B^n{}_{3/2})\,. \end{aligned}$$

Then we see that τ_δ ($\delta > 0$) are C^∞-transformations on M. Put $\tau_0 = \tau_U$, then τ_U is a diffeomorphism on $M - \bar{U}$ and $\tau_U \bar{U}$ is a point. By the hypothesis there is a regular cell U' which contains $\tau_U(\sum_k L^i_k(c_{i-1}) \cup V_{c_{i-1}}) \subset \tau_U(\sum_k L^i_k(c_{i-1}) \cup U) = \tau_U(\sum_k L^i_k(c_{i-1}))$. Since $\lim_{\delta \to 0} \tau_\delta = \tau_U$, if $\delta > 0$ is sufficiently small we have

$$\tau_\delta(\sum_k L^i_k(c_{i-1}) \cup V_{c_{i-1}}) \subset U'\,,$$

and so

$$\sum_k L^i_k(c_{i-1}) \cup V_{c_{i-1}} \subset \tau_\delta^{-1} U'\,.$$

Hence, by lemma 1, V_{c_i} is contained in a regular cell. Obviously V_{c_0} is a regular cell and so induction proves that $V_{c_{[n/2]}}$ is contained in a regular cell.

Now consider the function $-f(P)$ instead of $f(P)$. Then it is obvious that P^{n-i}_j are the critical points of $-f(P)$ with index i. Hence, putting $P^{n-i}_j = P'^i_j$ and $-\eta_{n-i} = \eta'_i$, we see that $-f(P'^{n-i}_j) = \eta'_j$ and $\eta'_0 < \eta'_1 < \cdots < \eta'_n$. Hence $-f(P)$ is also a canonical polar function on M. Furthermore, putting $-c_{n-i-1} = c'_{i-1}$, we have

$$\begin{aligned} V'_{c_{[n/2]}} &= \{P | f(P) \geqq c_{[n/2]}\} = \{P | -f(P) \leqq -c_{[n/2]}\}\,. \\ &= \{P | -f(P) \leqq c'_{n-[n/2]-1}\} \subset \{P | -f(P) \leqq c'_{[n/2]}\}\,. \end{aligned}$$

Therefore $V'_{c_{[n/2]}}$ is also contained in a regular cell. Thus M is the sum of two cells. Thus M is the sum of two cells, and according to [1] it is homeomorphic with a sphere.

By a singular k-sphere in M we shall mean the image under a C^0-map of S^k into M, where S^k is the unit k-sphere in R^{k+1}.

THEOREM 2. *If any singular k-sphere* ($k \leqq [n/2]$) *in M is contained in a cell then M is a sphere.*

Proof. We can modify the canonical polar function f so that $f(P^i_j) < f(P^{i+1}_k)$ for $j = 1, \cdots, n_i$, $k = 1, \cdots, n_{i+1}$, and $f(P^i_j) < f(P^i_{j+1})$ for $j = 1, \cdots, n_i - 1$. We arrange $f(P^i_j)$ in the sequence $\zeta_1 < \zeta_2 < \cdots$, and we put $P^i_j = P_\mu$, $L^i_j(c) = L_\mu(c)$ if $f(P^i_j) = \zeta_\mu$. Suppose that $\zeta_{\mu-1} < c_\mu < \zeta_\mu < c_{\mu+1} < \zeta_{\mu+1}$ and V_{c_μ} is contained in a cell. Then there are two cells U', U'' and real number $c(\eta_{\mu-1} < c < c_\mu)$ such that $V_{c_\mu} \subset U'$, $\bar{U}' \subset U''$ and $\partial L_\mu(c) \subset U'$. Let φ be a homeomorphism of U'' onto B^n_1. Then we have

$$\varphi\partial L_\mu(c) \subset \varphi U' \subset \varphi\bar{U}' \subset B_1^n,$$

so that there is B_r^n $(r<1)$ which contains $\varphi U'$. Denote by $(0, \varphi\partial L_\mu(c))$ the set of all points which are on segments connecting the origin 0 to points of $\varphi\partial L_\mu(c)$. This is a cone in B, and is a continuous image of a ball whose dimension is equal to $\dim L_\mu(c)$. Hence $L_\mu(c)\cup\varphi^{-1}(0, \varphi\partial L_\mu(c))$ is a simgular sphere in M. Define a transformation α_δ on B_1^n as follows:

$$(6) \qquad \begin{aligned} \alpha_\delta(u) &= \left(\frac{\delta-1}{r-1}(|u|-1)+1\right)u \qquad &&\text{for } r\leqq|u|\leqq 1, \\ &= \delta u &&\text{for } 0\leqq|u|\leqq r. \end{aligned}$$

α_0 is not a transformation in B_1^n, but it is a continuous map of B_1^n onto itself. From $L_\mu(c_\mu)\subset L_\mu(c)$ and $V_{c_\mu}\subset U'$, we have

$$L_\mu(c_\mu) \cup V_{c_\mu} \subset L_\mu(c) \cup U'.$$

Since $\alpha_\delta=$identity on ∂B_1, it follows that $\varphi^{-1}\alpha_\delta\varphi=$identity on $\partial U''$. Hence we may consider that $\varphi^{-1}\alpha_\delta\varphi$ $(\delta\neq 0)$ is a transformation on M and that $\varphi^{-1}\alpha_0\varphi$ is a continuous map of M into itself. Now we see

$$\begin{aligned} \varphi^{-1}\alpha_0\varphi(L_\mu(c_\mu) \cup V_{c_\mu}) &\subset \varphi^{-1}\alpha_0\varphi(L_\mu(c) \cup U') \\ &= \varphi^{-1}\alpha_0\varphi L_\mu(c) \cup \varphi^{-1}\alpha_0\varphi U' \\ &\subset \varphi^{-1}\alpha_0\varphi L_\mu(c) \cup \varphi^{-1}\alpha_0 B_r \\ &= \varphi^{-1}\alpha_0\varphi L_\mu(c) \cup \varphi^{-1}(0), \end{aligned}$$

and

$$\begin{aligned} &\varphi^{-1}\alpha_0\varphi(L_\mu(c) \cup \varphi^{-1}(0, \varphi\partial L_\mu(c))) \\ &\quad = \varphi^{-1}\alpha_0\varphi L_\mu(c) \cup \varphi^{-1}\alpha_0(0, \partial L_\mu(c)) \\ &\quad = \varphi^{-1}\alpha_0\varphi L_\mu(c) \cup \varphi^{-1}(0), \end{aligned}$$

so that

$$(7) \qquad \varphi^{-1}\alpha_0\varphi(L_\mu(c_\mu) \cup V_{c_\mu}) \subset \varphi^{-1}\alpha_0\varphi(L_\mu(c) \cup \varphi^{-1}(0, \varphi\partial L_\mu(c))).$$

By the hypothesis, if $\dim L_\mu(c_\mu)\leqq\left[\frac{n}{2}\right]$, the singular sphere $\varphi^{-1}\alpha_0\varphi(L_\mu(c) \cup \varphi^{-1}(0, \varphi\partial L_\mu(c)))$ is contained in a cell. Hence it follows from (6) and (7) that if $\delta>0$ is sufficiently small $\varphi^{-1}\alpha_\delta\varphi(L_\mu(c_\mu)\cup V_{c_\mu})$ is contained in in a cell and so $L_\mu(c_\mu)\cup V_{c_\mu}$, is in a cell. Therefore, by Lemma 1, $V_{c_{\mu+1}}$ is in a cell. Clearly V_{c_1} is in a cell, and hence induction proves that V_{c_ν} is in a cell, where P_ν is the critical point of index $\left[\frac{n}{2}\right]$ and $P_{\nu+1}$ is that of index $\left[\frac{n}{2}\right]+1$. If we consider $-f$ instead of f, the same way as in the proof of Theorem 1 proves that $V'_{c_\nu}=\{P|f(P)\geqq c_\nu\}$ is also in a cell. Thus M is the sum of two cells, and consequently M is a sphere.

4. Diffeomorphical deformations

Let N_0 be a closed C^∞-submanifold of M, and let $N'=N_0\times I$ be the product of

N_0 and I, where I denote the closed interval $[0,1]$. N' is a C^∞-manifold. Let f be a C^∞-map of N' into M such that each ϕ_t is a regular 1-1 map of N_0 into M and $N_0=\phi_0(N_0)$, where we put $\phi_t(P)=f(P, t)$ for $P\in N_0$ and $t\in I$. Then $N_1=\phi_1(N_0)$ is a C^∞-submanifold in M. In this section we say that the set of maps ϕ_t, $t\in I$, forms a diffeomorphical deformation of N_0 onto N_1. We say also that ϕ_t is a diffeomorphical deformation of N_0 to the points P_μ $(\mu=1,\cdots,k)$ if each ϕ_t $(0\leqq t<1)$ is a regular 1-1 and $\sum P_\mu=\phi_1(N_0)$ is a finite sum of points.

LEMMA 3. *If ϕ_t is a diffeomorphical deformation of N_0 to N_1, then there is a C^∞-transformation on M which maps N_0 onto N_1.*

Proof. To prove this lemma it is sufficient to show that there is a C^∞-transformation which maps N_a onto N_b when $b-a>0$ is sufficiently small.

Choose two coverings $\{W_i\}$ and $\{U_i\}$ of $N_a\cup N_b$ and a system of local coordinates x^i in a neighborhood of $\bar{U}_i$ so that $\overline{W}_i\subset U_i$ and N_a is represented as $x^i_{p+1}=\cdots=x^i_n=0$. Then N_b is written as

$$x^i_\mu=f_\mu(x^i_j,\cdots,x^i_p,b),\quad \mu=p+1,\cdots,n\,,$$
$$f_\mu(x^i_1,\cdots,x^i_p,b)\to 0\quad(b\to a)\,.$$

We suppose that

$$\overline{W}_i=\{x^i\,|\,|x^i|\leqq 1\}\ \text{and}\ \bar{U}_i=\{x^i\,|\,|x^i|\leqq 2\}\,.$$

Take a C^∞-function $\psi(x^i)$ such that $\psi(x^i)=0$ for $|x^i|\leqq 1$, and $=1$ for $|x^i|\geqq 2$.

Put

$$y^1_\mu=x^1_\mu+f_\mu(x^1_1\cdots x^1_p,b)\psi(x^1),\ \mu=1,\cdots,n\,. \tag{8}$$

Then we see that if $|f_\mu|$ are sufficiently small then the map defined by (8) is a C^∞-transformation on U_1 which is the identity on ∂U_1. Denote it by τ_1, and extend it over M by setting $\tau_1=$identity for $M-U_1$. Then we have

$$\tau_1 N_b=N_a\ \text{in}\ W_1\,.$$

Clearly $\tau_1 N_a$ is represented as

$$x^i_\mu=f^1_\mu\ (x^i_1,\cdots,x^i_p;\ b),\qquad \text{for}\ \mu=p+1,\cdots,n$$

where

$$f^1_\mu(x^i_1,\cdots,x^i_p;\ b)\to 0\ \ (b\to a)$$

and

$$f^1_\mu(x^i_1,\cdots,x^i_p;\ b)=0\qquad \text{for}\ (x^i_1,\cdots,x^i_p)\in W_1\,.$$

Let τ_2 be a map represented by

$$\begin{aligned} y^2_\mu&=x^2_\mu\,,\qquad \text{for}\ \mu=1,\cdots,p\\ &=x^2_\mu+f^1_\mu(x^2_1,\cdots,x^2_p,b)\psi(x),\qquad \text{for}\ \mu=p+1,\cdots,n\,.\end{aligned}$$

Then τ_2 is a C^∞-transformation such as

$$\tau_2 N_b=N_a\ \text{in}\ W_1\cup W_2\,.$$

By applying the above process for $W_1,W_2,\cdots,W_m$, we have

$$\tau_m N_b = N_a .$$

Thus the lemma is proved.

5. Regular spheres S^k in M.

LEMMA 4. *Suppose that $\pi_k(M)=0$ and $2k+1<n$. Then for given regular spheres S^k there is a diffeomorphical deformation ϕ_t of S^k to the sum of points.*

Proof. We shall divide the proof into the following three parts.

a) Let φ be the regular 1-1 map of ∂B_1^{k+1} onto S^k, and π be the projection of $B_1^{k+1}-0$ onto ∂B_1^{k+1} by the radii of B_1^{k+1}. For an arbitrary point P in $B^{k+1}{}_{1/4,1}$ we define φP to be $\varphi\pi P$. Then it follows from the hypothesis that there is a continuous extension of φ over B_1^{k+1}. We denote it by the same notation φ. We choose two coverings $\{U_i\}$, $\{U_i'\}$ $(i=1, \cdots, \mu)$ of $B^{k+1}{}_{1/4}$ such that $U_i \supset \bar{U}_i'$, $\cup U_i \subset B^{k+1}{}_{1/2}$ and each φU_i is contained in neighborhood for which a system of local coordinates is defined. Let $u_1^2+\cdots+u_{k+1}^2=1$ and $t(P)$ be the distance between P and the origin. Then we can take $u_1(\pi P), \cdots, u_k(\pi P)$, $t(P)$ as coordinates of P in U_1. Let $x_1 \cdots x_n$ be a system of coordinates in a neighborhood of φU_1. Then φ is written in U_1 as

$$x_j = f_j(u, t), \quad u = (u_1, \cdots, u_k) \, ; \, j = 1, \cdots, n .$$

Let $f_j'(u, t)$ be a C^∞-functions which approximates $(f; U_1, \zeta)$ where $\zeta>0$ is sufficiently small. Let $\lambda(u, t)$ be a C^∞-function which takes 1 on $\bar{U}_1'$ and 0 on U_1-U_1'', where $U_1 \supset U_1''$, $U_1'' \supset \bar{U}_1'$. Put

$$f_j'' = f_j+\lambda(f_j'-f_j)$$

and define φ' as

$$\begin{aligned} \varphi^1 &= f'' && \text{on } U_1 \\ &= \varphi && \text{on } B_1^{k+1}-U_1 . \end{aligned}$$

where f'' is the map defined by

$$x_j = f_j''(u, t) .$$

Then φ^1 is a C^∞-map in $\bar{U}_1'$ and $\varphi^1=\varphi$ on $B_1^{k+1}-U_1$. Applying the above process to φ^1, U_2 and U_2' instead of φ, U_1 and U_1', we have φ^2 such that φ^2 is a C^∞-map in $U_1' \cup U_2'$ and $\varphi^2=\varphi$ on $B_1^{k+1}-(U_1 \cup U_2)$. By repeating these, we have $\varphi^1, \varphi^2, \cdots,$ φ^μ ; φ^μ is a C^∞-map of B_1^{k+1} into M such that $\varphi^\mu=\varphi$ in $B^{k+1}{}_{1/2,1}$. Thus the map φ of $B^{k+1}{}_{1/2,1}$ is extended to a C^∞-map of B_1^{k+1} into M. The extended map will be denoted by the same notation φ. It is written in U as

$$x_j = g_j(u, t)$$

where g_j are C^∞-functions of u, t. We remark that φ^μ is regular 1-1 on every ∂B_t^{k+1} $(1/2 \leq t \leq 1)$ (see the construction of φ^μ). We write simply φ instead of φ^μ.

b) Consider the equations

$$\frac{\partial(x_1, \cdots, x_{k-1}, x_p)}{\partial(u_1, \cdots, u_{k-1}, u_k)} = 0,$$
$$(p = 1, 2, \cdots, n-k+1).$$

Since $n-k+1>k+1$ it follows that the number of the the equations is greater than that of the unknowns. Hence, for an arbitrary $\zeta>0$, we can take $g'_j(u, t)$ so that $|g_j-g'_j|<\zeta$ and there is p_0 satisfying

$$\frac{\partial(g'_1, \cdots, g'_{k-1}, g'_{p_0})}{\partial(u_1, \cdots, u_{k-1}, u_k)} \neq 0.$$

Let $\lambda(u, t)$ be a C^∞-function in U_1 which takes 1 on $\bar{U}'_1$ and 0 on $U_1-U''_1$, where $U_1\supset\bar{U}''_1$, $U''_1\supset\bar{U}'_1$. Put

$$g''_j = g'_j+\lambda(g'_j-g_j);$$

and define φ^1 by

$$\begin{aligned} \varphi^1 &= g'' && \text{in } U_1, \\ &= \varphi && \text{in } B_1^{k+1}-U_1, \end{aligned}$$

where g'' is the map defined by

$$x_j = g''_j(u, t).$$

Then φ^1 is regular on every $\partial B_t^{k+1}\cap U_1$. Applying the above process to ζ^1, φ^1 and U_2, we have φ^2 such that φ^2 is regular on $\partial B_t^{k+1}\cap(U'_1\cup U'_2)$ and $\varphi^2=\varphi$ on $B_1^{k+1}-(U_1\cup U_2)$. By repeating this process we have $\varphi^1, \varphi^2, \cdots, \varphi^\mu$; φ^μ is regular on every ∂B_t^{k+1} and is equal to φ on $B^{k+1}{}_{1/2,1}$. For an arbitrary $\zeta^1>0$ we can choose $\zeta, \zeta^1, \zeta^2, \cdots$ so that φ^μ approximates $(\varphi, B_1^{k+1}, \zeta')$ and hence from the fact that φ is regular 1-1 on every ∂B_t^{k+1} $\left(\frac{1}{2}\leqq t\leqq 1\right)$, if ζ' is sufficiently smal it is hold that φ^μ is regular 1-1 on every ∂B_t^{k+1} $\left(\frac{1}{2}\leqq t\leqq 1\right)$. This is shown as follows. Suppose that for $\zeta'>\zeta''>\cdots\to 0$ there is a sequence φ', φ'', which approximate $(\varphi, B^{k+1}{}_{1/2,1}, \zeta')$, $(\varphi, B^{k+1}{}_{1/2,1}, \zeta'')$, $\cdots$ and are not regular 1-1 on some ∂B_t^{k+1} $\left(\frac{1}{2}\leqq t\leqq 1\right)$. Then there is a sequence $t_1, t_2, \cdots\to a$ $\left(\frac{1}{2}\leqq t_i\leqq 1,\ \frac{1}{2}\leqq a\leqq 1\right)$ such that $\varphi', \varphi'', \cdots$ are not regular 1-1 on $\partial B_{t_1}^{k+1}, \partial B_{t_2}^{k+1}, \cdots$, respectively. By π_m denote the projection of ∂B_a^{k+1} on $\partial B_{t_m}^{k+1}$ by radii of B_1^{k+1}. Then, in terms of the local coordinates $u_1, \cdots, u_k$, π_m is written as $u'_i=u_i$, $i=1, \cdots, k$. Hence $\varphi'\pi_1$, $\varphi''\pi_2, \cdots$ approximate $(\varphi, \partial B_1^{k+1}, \zeta')$, $(\varphi, \partial B_1^{k+1}, \zeta'')$, $\cdots$ respectively and they are not regular 1-1 on ∂B_a^{k+1}. On the other hand φ is regular 1-1 on ∂B_a^{k+1}, and hence it follows that $\varphi^{(m)}\pi_m$ is regular 1-1 on ∂B_a^{k+1} if $\zeta^{(m)}$ is sufficiently small. We write simply φ instead of φ^μ.

c) Let $\{U_i\}$, $\{U'_i\}$ be two open coverings of $B^{k+1}{}_{1/2}$; we choose them so that $U_i\supset\bar{U}'_i$, $\cup\bar{U}_i\subset B^{k+1}{}_{3/4}$ and if $\bar{U}_i$ and $\bar{U}_k$ have common points then φ is 1-1 on $\partial B_t^{k+1}\cap(U_i\cup U_k)$. Consider all sets $U_i\cup U_k$ with $\bar{U}_i\cap\bar{U}_k=0$, and arrange them in a sequence $W_1, W_2, \cdots$. Take ζ so small that any approximation map φ' of $(\varphi,$

$B_1^{k+1}, \zeta)$ is regular and 1-1 on each $\partial B_t^{k+1} \cap (\bar{U}_i \cup \bar{U}_k)$ for which $\bar{U}_i \cap \bar{U}_k \neq 0$. Let u, t and u', t be coordinates in U_i and in U_k respectively, and let $x_1 \cdots x_n$ be coordinates in a neighborhood containing $\varphi(\bar{U}_i \cup \bar{U}_k)$. Then $\varphi(\partial B_t^{k+1} \cap U_i)$ and $\varphi(\partial B_t^{k+1} \cap U_k)$ are written as

$$x_j = f_j(u, t), \quad x_j = g_j(u', t)$$

respectively. Consider the equations

$$(9) \qquad f_j(u, t) = g_j(u', t), \qquad j = 1, \cdots, n.$$

Since $n > 2k+1$, it follows that the number of the equations is greater than that of the unknowns. Hence, for a given ζ', $0 < \zeta' < \zeta$ there are C^∞-functions f'_j, g'_j such that f'_j approximates (f_j, W_1, ζ') and g'_j approximates (g_j, W_1, ζ'), and $f'_r(u, t) = g'_j(u', t)$ have no solution. Let $\lambda(u, t)$ be a C^∞-function which takes 1 on $\bar{U}'_i$ and 0 on $U_i - U''_i$, where $U_i \supset \bar{U}''_i$, $\bar{U}''_i \supset \bar{U}'_i$. Let $\lambda'(u', t)$ be a C^∞-function which takes 1 on $\bar{U}'_k$ and 0 on $U_k - U''_k$, where $U_k \supset \bar{U}''_k$, $U''_k \supset \bar{U}'_k$. Suppose $W_1 = U_i \cup U_k$, and put

$$\begin{aligned} \overline{W}'_1 &= \bar{U}'_i \cup \bar{U}'_k, \\ f''_j &= f_j + \lambda(f'_j - f_j), \\ g''_j &= g_j + \lambda'(g'_j - g_j). \end{aligned}$$

Define φ^1 by

$$\begin{aligned} &= f'' && \text{in } U_i, \\ &= g'' && \text{in } U_k, \\ &= \varphi && \text{in } B_1^{k+1} - U_i - U_k, \end{aligned}$$

where f'' and g'' are the maps defined by $x_j = f_j(u, t)$ and $x_j = g_j(u, t)$ respectively. Then φ^1 is regular 1-1 on every $\partial B_t^{k+1} \cap \overline{W}_1$ and

$$\varphi^1 = \varphi \qquad \text{on } B_1^{k+1} - W_1.$$

Applying this process to φ^1, W_2 instead of φ, W_1, we have φ^2 which is regular 1-1 on every $\partial B_t^{k+1} \cap (\overline{W}'_1 \cup \overline{W}'_2)$ and satisfies

$$\varphi^2 = \varphi \qquad \text{in } B_1^{k+1} - W_1 - W_2.$$

By repeating such process, we have an approximation φ^ν of $(\varphi, B_1^{k+1}, \zeta')$ which is regular 1-1 on every ∂B_t^{k+1}, $0 \leqq t \leqq \frac{2}{3}$, and $= \varphi$ on ${B^{k+1}}_{3/4,1}$. Since φ is regular 1-1 on every ∂B_t^{k+1}, $\frac{2}{3} \leqq t \leqq 1$, and φ^ν approximates $(\varphi, B_t^{k+1}, \zeta')$, it follows that φ^ν is regular 1-1 on every ∂B_t^{k+1}, $0 \leqq t \leqq 1$.

Put $\phi_t = \varphi^\nu \pi_t \partial B_t^{k+1}$, where π_t is the projection of ∂B_1^{k+1} onto ∂B_t^{k+1} by the radii of the ball B_1^{k+1}. Then ϕ_t is the required deformation.

LEMMA 5. *If $\pi_k(M^n) = 0$ $(2k+1 < n)$, then for given disjoint regular spheres S_μ^k $(\mu = 1, \cdots, \nu)$ there is a diffeomorphical deformation of $\sum S_\mu^k$ to the sum of points $\sum P_\mu$.*

Proof. In virtue of lemma 4, for every S_μ^k there is a diffeomorphical deformation $\phi_{\mu,t}$ of S_μ^k to P_μ. Let $S^k=\{u \mid u_1^2+\cdots+u_{k+1}^2=1\}$ be the unit sphere in R^{k+1}, and let ψ_μ be a diffeomorphism of S^k onto S_μ^k. Let $\{U_i\}$ be a sufficiently fine open coverings of S^k. If $\phi_{\mu,t}S_\mu^k \cap \phi_{\mu' t}S_{\mu'}^k \neq 0$ the equation

$$\phi_{\mu,t}\psi_\mu(u) = \phi_{\mu',t}\psi_{\mu'}(u'), \quad u \in U_i, \quad u' \in U_k$$

has a solution for some i and k. In therms of local coordinates x in M, the above equation is written as

$$\begin{aligned} f_j(u,t) &= g_j(u',t), \quad j=1,\cdots,n \\ |u| &= |u'| = 1. \end{aligned} \tag{10}$$

It follows from the hypothesis that the equations (10) have no solution when t ($t \geqq 0$) is sufficiently small. Since the number of these equations is greater than that of unknowns, by the similar way to (c) in the proof of lemma 4 we can modify $\phi_{\mu,t}$ so that $\phi_{\mu,t}S_\mu^k \cap \phi_{\mu',t}S_{\mu'}^k=0$ if $\mu \neq \mu'$. Put $\phi_t=\phi(tx)$, $x \in \partial B_1^{k+1}$. Then ϕ_t is a desired diffeomorphical deformation.

By making use of lemma 3 we have immediately

LEMMA 6. *If $\pi_k(M^n)=0$ ($n \geqq 2k+1$), then arbitrary disjoint regular spheres $S_\mu^k(\mu=1,\cdots,\nu)$ in M are contained in a regular cell in M.*

LEMMA 7. *Suppose that $\pi_1(M^n)=0$ and that $n \geqq 5$. Then, for a given regular cell U and disjoint arcs α_μ ($\mu=1,\cdots,\nu$) such that every α_μ is transversal to ∂U and $\partial\alpha_\mu \subset U$, there is a regular cell containing $U \cup (\bigcup_{\mu=1}^{\nu} \alpha_\mu)$.*

Proof. Put $\sum \alpha_\mu - U = \sum \beta_i$, every β_i being a connected component of $\sum \alpha_\mu - U$. Take a point P in U, and connect the two points $\partial\beta_i$ by a certain regular arc γ_i such that $\gamma_i \subset U$, $P \in \gamma_i$. We can take γ_i so that $\beta_i \cup \gamma_i$ is a regular circle in M and $(\beta_i \cup \gamma_i) \quad (\beta_k \cup \gamma_k)=P$ if $i \neq k$. Moreover we see that for a given neighborhood W of $\cup \gamma_i$ there is a transformation σ in M such that $\sigma U \subset W$ and $\sigma \sum_i (\beta_i \cup \gamma_i) = \sum_i (\beta_i \cup \gamma_i)$. Hence if there is a cell U' which contains $\sum_i (\beta_i \cup \gamma_i)$, then we have

$$\sigma(\sum_i \alpha_i \cup U) = \sigma(\sum_i \beta_i \cup U) \subset \sum_i \beta_i \quad W \subset U'$$

and hence

$$\sum \alpha_i \cup U \subset \sigma^{-1} U'$$

The existence of the cell containing $\sum_i (\beta_i \cup \gamma_i)$ is shown as follows. According to lemma 6, every $\beta_i \cup \gamma_i$ is in a regular cell. Hence there is a regular disk D such as $\partial D = \beta_i \cup \gamma_i$. Since $n \geqq 5$, it is easy to see that D_i has no self-intersection and that $D_i \quad D_k = P$ ($i \neq k$) if $\sum_i D_i$ is in a general position. Then for a given cell U' containing P there is a transformation τ such that $\tau \sum_i D_i \subset U'$. Hence $\sum_i D \subset \tau^{-1} U'$ and it follows that $\sum_i (\beta_i \cup \gamma_i) \subset \tau^{-1} U'$. Thus the lemma is proved.

LEMMA 8. *If* $\dim M \geqq 5$ *and if* $c < \eta_2$, *then* V_c *is contained in a regular cell.*

Proof. Consider $L_j^1(c_0) \cup V_{c_0}$, $\eta_0 < c_0 < \eta_1$. Then, since V_{c_0} is in a regular cell, it follows from lemma 6 that $\sum L(c_0) \cup V_{c_0}$ is in a regular cell. Hence by lemma 1 we get the required result.

LEMMA 9. *Suppose that* $n = 2k+1$ *and that* $\pi_k(M^n) = 0$. *Then for given disjoint regular spheres* S_μ^k $(\mu = 1, \cdots, \nu)$ *there is a deformation* ϕ_t *of* $\sum_\mu S_\mu^k$ *to the sum of points* $\sum P_\mu$ *such that*

1) $\phi_t(\sum_\mu S_\mu^k)$ *has no self-intersection if* $t \neq t_i$ $(i = 1, \cdots, m)$,

2) *for every* t_i, $\phi_{t_i}(\sum_\mu S_\mu^k)$ *has only one self-intersection point.*

PROOF. This is proved in the same way as in the proof of lemmas 4 and 5. Clearly (a) and (b) hold in this case. Now the equations (9) and (10) in (c) becomes

$$f_j(u, t) = g_j(u', t), \quad j = 1, \cdots, 2k+1 \qquad \qquad u = (u_1, \cdots, u_k), \quad u' = (u, \cdots, u) . \tag{11}$$

Since in (11) the number of the equations is equal to that of the unknowns, it follows that for a given $\zeta > 0$ there are approximations f_j' and g_j' of (f_j, W_1, ζ) and (g_j, W_1, ζ) such that the number of the solutions of the equations $f_j'(u, t) = g(u', t)$ are finite in W_1. Now the same way as in (c) proves the required result.

LEMMA 10. *If* $n = 2k+1$, $\pi_1(M^n) = 0$ *and* $\pi_k(M^n) = 0$, *then arbitrary disjoint regular spheres* S_μ^k $(\mu = 1, \cdots, \nu)$ *in* M *are contained in a regular cell.*

Proof. According to lemma 8, there is a deformation ϕ_t of $\sum_\mu S_\mu^k$ to $\sum_\mu P_\mu$ such that $\phi_t \sum_\mu S_\mu^k$ is regular except for $t = t_1, \cdots, t_l$, and $\phi_{t_i} \sum S_\mu^k$ has only one self-intersection point P_i. Hence, in virtue of lemma 1, in order to prove lemma 9 it is sufficient to verify that if $\phi_t \sum_\mu S_\mu^k$ is contained in a regular cell for every $t < t_i$, then $\phi_{t_i} \sum_\mu S_\mu^k$ is also contained in a regular cell. Choose local coordinates x such that $x(P_i) = 0$ in a neighborhood of P. Then we may consider that in this neighborhood $\phi_t \sum_\mu S_\mu^k$ is the sum of the following two k-planes:

$$\begin{aligned} x_j &= a_j(t - t_i) + \sum_{p=1}^k b_j^p x_p , \\ x_j &= c_j(t - t_i) + \sum_{p=1}^k d_j^p x_p , \end{aligned} \tag{12}$$

where $j = k+1, \cdots, n$ and the determinant of the matrix of the coefficients in the equations

$$\sum_{p=1}^k (b_j^p - d_j^p) x_p = (c_j - a_j)(t - t_i)$$
$$j = 1, 2, \cdots, n$$

is not 0. Choose coordinates axis $y_1, \cdots, y_n$ so that $y_1, \cdots, y_k$ are on the plane

$x_i=\sum_{p=1}^{k} b_j^p x_p$ $(j=k+1,\cdots,n)$ and $y_{k+1},\cdots,y_{2k}$ are on the plane $x_j=\sum_{p=1}^{k} d_j^p x_p$ $(j=k+1,\cdots,n)$. Then (12) becomes

$$y_n=a(t-t_i)\,,\quad y_{k+1}=\cdots=y_{2k}=0\,,$$
$$y_n=c(t-t_i)\,,\quad y_1=\cdots=y_k=0\,.$$

Suppose that $|a|<|c|$, and consider a segment defined by $y_j=0$ $(j=1,\cdots,n-1)$ and $|y_n|<\delta|c|$. By lemma 7 there is a regular cell U containing $\phi_{t_i-\delta}\sum S_\mu^k$ and the segment. Therefore if $\zeta>0$ is sufficiently small the set W of point y for which $y_1^2+\cdots+y_{n-1}^2\leqq\zeta$ and $|y_n|\leqq\delta|c|$ is contained in U. Now take a function $\psi(r)$ such that

$$\begin{aligned}\psi(r)&=-a+c \qquad \text{for } 0\leqq r\leqq\frac{\zeta}{2}\,,\\ &=0 \qquad\qquad \text{for } r\geqq\zeta\\ 0\leqq\psi(r)&\leqq-a+c \qquad (\text{or } 0\geqq\psi(r)\geqq-a+c)\,.\end{aligned}$$

Consider the set of varieties N, $t_i-\delta\leqq t\leqq t_i$, which coincides with $\phi_t\sum S_\mu^k$ in $M-W$ and is the sum of two submanifolds

$$y_n=a(t-t_i)+(t-t_i)\psi(y_1^2+\cdots+y_{n-1}^2)\,,$$
$$y_{k+1}=\cdots=y_{2k}=0$$

and

$$y_n=c(t-t_i)\,,\quad y_1=\cdots=y_k=0$$

in W. Obviously every N_t has only one self-intersection point $y=0$, and in the neighborhood $W'=\{y_1|y_1^2+\cdots+y_{n-1}^2<\frac{\zeta}{2},\ |y_n|<\delta|c|\}$ we have $N_t\cap W'=N_{t_i}\cap W'$. Hence by the same way as in the proof of lemma 3, we see that there is a transformation τ such as $\tau N_{t_i-\delta}=N_{t_i}$. On the other hand, since $\phi_{t_i}\sum_\mu S_\mu^k=N_{t_i}$ and $N_{t_i-\delta}\subset(\phi_{t_i-\delta}\sum_\mu S_\mu^k)\cup W\subset U$, we have $\phi_{t_i}\sum S_\mu^k\subset\tau U$.

6. Poincaré conjecture for M.

LEMMA 11. *For given circles α_i,β_i $(i=1,2,\cdots,p)$ on ∂B_1^5 which do not intersect each others, there is a transformation τ on $B_{1-\delta,1+\delta}^5$ such that $\tau\alpha_i=\beta_i$, $\tau\partial B_1^5=\partial B_1^5$ and $\tau=$identity on $\partial B_{1-\delta,1+\delta}^5$.*

Proof. Obviously there is a deformation ϕ_t of $\sum\alpha_i$ to $\sum\beta_i$ such as $\phi_t\sum\alpha_i\subset\partial B_1^5$. Hence by the same way as in the proof of lemma 5, we can modify ϕ_t so that ϕ_t is a diffeomorphical deformation of $\sum_i\alpha_i$ to $\sum\beta_i$ and $\phi_t\sum\alpha_i\subset\partial B_1^5$. To prove the lemma it is sufficient to show that there is a transformation τ such that $\tau\phi_b\sum\alpha_i=\phi_a\sum\alpha_i$ where $a,b\in I$ aud $b-a>0$ is sufficiently small. Choose a covering $\{W_k\}$ of $\phi_a\sum\alpha_i\cup\phi_b\sum\alpha_i$ in $B_{1-\delta,1+\delta}^5$ and coordinates x^k so that $W_k=\{x^k||x^k|<2\}$. Suppose that $\phi_a\sum\alpha_i$ and ∂B_1^5 are written in W_k as $\phi_a\sum\alpha_i=\{x^k|x_2^k=x_3^k=x_4^k=x_5^k=0\}$ and $\partial B_1^5=\{x^k|x_5^k=0\}$ respectively. Then $\phi_b\sum\alpha_i$ is written in W_1 as

$$x_j^1 = f_j^1(x_1^1, b), \qquad j = 2, 3, 4,$$
$$x_5^1 = 0\,,$$
$$f_j^1(x, b) \to 0 (b \to a)\,.$$

Put

$$y_j^1 = x_j^1 \qquad \text{for } j = 1, 5,$$
$$= x_j^1 + f_j^1(x_1^1, b)\psi(x^1)\,, \qquad \text{for } j = 2, 3, 4,$$

where $\psi(x^1)$ is a C^∞-function which takes 0 on W_1 and 1 on $W_1' = \{x^1 | |x^1| < 1\}$. Then it induces a C^∞-transformation in $B^5_{1-\delta,1+\delta}$ when $b-a>0$ is sufficiently small. Denoting it by τ_1 we see that τ_1=identity of $B^5_{1-\delta,1+\delta} - W_1$ and $\tau_1\phi_b \sum_i \alpha_i = \phi_a \sum_i \alpha_i$ on W_1'. We suppose that $\{W_k'\}$ is an open covering of $\phi_a \sum \alpha_i$ where $W_k' = \{x^k | |x^k| \leqq 1\}$. Similarly to the proof in lemma 3, repeating of the above process for W_2, W_2', W_3, W_3', $\cdots$, yield the required transformation.

THEOREM 3. *If we have $\pi_1(M^5)=\pi_2(M^5)=0$ for a compact C^1-manifold M, then M is homeomorphic with a sphere S^5.*

Proof. Since any C^r-manifold $(r \geqq 1)$ is C^r-homeomorphic with a analytic manifold, we may assume that M^5 is C^∞-manifold.

In virtue of lemma 2 we can choose a regular cell U such that $V_c \subset U$ $(\eta_1 < c < \eta_2)$ and $\sum L_j^2(c)$ is transversal to ∂U. Then $\sum L_j^2(c) \cap \partial U$ is 1-dimensional submanifold, and it consists of finite regular circles which are denoted by α_μ $(\mu = 1, \cdots, \nu)$. Hence $\sum L_j^2(c) - U$ is the sum of 2-dimensional domains whose boundaries are circles α_μ.

Let σ_t and τ_t be maps of M into itself such that

1) σ_t and τ_t are C^∞-transformations if $t>0$,

2) $\lim_{t\to 0} \sigma_t = \sigma_0$ and $\lim_{t\to 0} \tau_t = \tau_0$ are regular 1-1 in $M-\bar{U}$,

3) $\tau_0\sigma_0\alpha_\mu = P_\mu$ and $\tau_0\sigma_0\bar{U} = \sum_\mu OP_\mu$, where P_μ $(\mu = 1, \cdots, \nu)$ are points on ∂U and 0 is a point in U.

Then the boundaries of every domain are mapped to points by $\tau_0\sigma_0$. Hence $\tau_0\sigma_0(\sum_j L_j^2 - U)$ is the sum of spheres which are regular except for points $\sum P_\mu$. We suppose that

4) $\tau_0\sigma_0(\sum_j L - U)$ is the sum of disjoint regular spheres.

For the moment we shall assume the existence of such maps σ_t and τ_t.

Then it follows from lemma 9 that there is a regular cell W containing these spheres and point 0. Apply lemma 7 for W and $\sum_\mu OP_\mu$. Then we get a cell W' containing $W \cup \sum_\mu OP_\mu$, and it follows

$$\tau_0\sigma_0(\sum L_j^2 \cup V_c) \subset \tau_0\sigma_0(\sum L_j^2 - U) \cup \tau_0\sigma_0 U \subset W'\,.$$

Hence if $t<0$ is sufficiently small we have

$$\tau_t\sigma_t(\sum L_j^2 \cup V_c) \subset W'\,,$$

namely

$$\sum L_j^2 \cup V_c \subset \sigma_t^{-1}\tau_t^{-1}W' .$$

Consequintly, according to theorem 1, M^5 is a sphere.

Now we shall show the existence of such σ_t and τ_t.

Let φ be a regular 1-1 map of the ball $\bar{B}_2^5$ into M such as $\varphi\bar{B}_1^5=\bar{U}$. We can consider that u is a coordinate system in $\varphi\bar{B}_2^5$, and that U is a ball of radius 1 and center 0. From now on, we put $U_\nu=\{u \mid |u|<r\}$, $U_{r,r'}=\{u \mid r\leqq |u|\leqq r'|$.

Take distinct points P_μ ($\mu=1, \cdots, \nu$) on ∂U and coordinates u^μ in φB_1 such as $u^\mu=T_\mu u$, where T_μ is an element of rotation group in R^5 such as ${}^t(0, 0, 0, 0, 1) = T_\mu {}^t u(P_\mu)$ where ${}^t(0, 0, 0, 0, 1)$ and ${}^t u(P_\mu)$ are transposed vectors of $(0, 0, 0, 0, 1)$ and $u(P_\mu)$ respectively. Put

$$A_\mu(\zeta) = \{u^\mu ; |u_\mu^\mu|/|u^\mu|\leqq 2\zeta ,\ 1 = 1, 2, 3, 4\}.$$

If $\zeta>0$ is sufficiently small we have

$$A_\mu(\zeta) \frown A_{\mu'}(\zeta) = 0 \qquad (\mu \neq \mu') .$$

By lemma 10 we may suppose $\alpha_\mu \subset A_\mu(\zeta/2)$. Furthermore if $\delta(\delta>0)$ is sufficiently small we have

$$\sum L_j^2(c) \frown \bar{U}_{1-\delta,1+\delta} \subset \sum A_\mu \frown \bar{U}_{1-\delta,1+\delta} .$$

Take a C^∞-function $\psi_\delta(y)$ of one variable auch thrt

$$\psi_\delta(y) = 1 \quad \text{for } y\geqq 1 ,$$
$$= \delta \quad \text{for } 0\leqq y\leqq 1-\delta ,$$
$$\frac{d}{dy}\psi_\delta(y) \geqq 0 ,$$

and define a C^∞-map σ_δ of U into itself by

$$\sigma_\delta u = \psi_\delta(|u|)u .$$

Clearly σ_δ is a C^∞-transformation in U for $\delta>0$. Furthermore, take three C^∞-functions $f(y)$, $g(y)$ and $h(y)$ satisfying following conditions:

$$f(y) = \begin{cases} -\dfrac{1}{\log y} & \text{if } 0<y<1 , \\ 0 & \text{if } y\leqq 0 , \end{cases}$$

$$g(y) = \begin{cases} 1 & \text{if } y\leqq 1+\varepsilon , \\ 0 & \text{if } y\geqq 1+2\varepsilon , \end{cases} \quad \frac{dg}{dy}\leqq 0 ,$$

$$h(y) = \begin{cases} 1 & \text{if } y\geqq \zeta , \\ 0 & \text{if } y\geqq 3/2\cdot\zeta , \end{cases} \quad \frac{dh}{dy}\leqq 0 ,$$

where ε is a sufficiently small positive number. For simplicity, we write u instead of u^μ. Now we use u_1, u_2, u_3, u_4, $|u|$ as coordinates in $U-0$. Consider a C^∞-map $\tau_t ; u \rightarrow (u_i')$, $t\geqq 0$, of A_μ into itself defined by

(13)
$$u'_i = (f(|u|-1+t)-1)g(|u|)h(\sum_{i=1}^{4} u/|u|^2)u_i+u_i\,, \qquad (i=1, 2, 3, 4)$$
$$|u'| = |u|\,.$$

Obviously if $t>0$ is sufficiently small then τ_t is C^∞-transformations. If $\sum_{i=1}^{4} u_i^2/|u|^2 \leq \zeta$ and $|u|<1+\varepsilon$, (13) is written as

(14)
$$u'_i = -\frac{u_i}{\log(|u|-1+t)}\,, \qquad (i = 1, 2, 3, 4)\,.$$
$$|u'| = |u|\,.$$

By lemma 11 we may assume that α_μ is written as $u_1^2+u_2^2=\zeta^2$, $u_3=u_4=0$, $|u|=1$. Then $\sum L_j^2 \cap A_\mu$ is represented as

(15)
$$u_3 = (|u|-1)f_3(u_1, u_2, |u|)\,,$$
$$u_4 = (|u|-1)f_4(u_1, u_2, |u|)\,,$$

where $u_1^2+u_2^2=\zeta^2$ and f_3, f_4 are C^∞-functions with respect to $u_1, u_2, |u|$. From $u_1^2+u_2^2=\zeta^2$ and (14) we have

(16)
$$|u|-1+t = \exp(-\zeta/\sqrt{u'^2_1+u'^2_2})\,.$$

Hence $\tau_0(\sum L_j^2 \cap A_\mu)$ is represented as

$$u'_k = \sqrt{u'^2_1+u'^2_2}/\zeta \exp(-\zeta/\sqrt{u'^2_1+u'^2_2})$$
$$\times f_k(\zeta u'_1/\sqrt{u'^2_1+u'^2_2},\ \zeta u'_2/\sqrt{u'^2_1+u'^2_2},\ 1+\exp(-\zeta/\sqrt{u'^2_1+u'^2_2}))$$
$$(k = 3, 4)\,.$$

Hence u'_4, u'_4 and $|u|$ are C^∞-functions of u'_1 and u'_2. Thus $\tau_0(\sum L_j^2(c)-U)$ is the sum of regular spheres S_p^2 which do not intersect each other. Hence σ_t and τ_t satisfy (1), (2), (3) and (4). This completes the proof of Theorem 3.

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