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Note on Cohomological Operations

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H. Cartan gave an axiomatic theory of the Steenrod squares, and proved the product formulas for these operations, in [1]. There he used the theory of 'carapace' due to J. Leray. We shall now try to establish similar theorems in terms of the combinatorial theory in finite complexes. This is accomplished as a translation of a lemma in Steenrod [4], and is stated in Theorem I. Furthermore we give product formulas for the Pontrjagin and Postnikov squares in Theorem III. In Theorem II, we remark that the Shimada-Uehara operation $\mathscr{Q}_i$ [6] is a composition of two well-known operations.

§1. A system of cup-*i* products

We shall deal with finite simple complexes K [2]. That is to say, K is a finite abstract complex such that

(i) the closure $\mathrm{Cl}\,\sigma$ of each cell σ of K is acyclic,

(ii) a cochain γ^0 which takes $+1$ on each 0-cell is an integral cocycle,

(iii) there exists no cell with negative dimension.

γ^0 is called usually the fundamental cocycle of K. Let L be a subcomplex of K, then we denote by $C^p(K, L)$ the p-cochain group of K mod L with integer coefficients.

Let $\{\underset{i}{\smile}\}$ $(i=0, \pm 1, \pm 2, \cdots\cdots)$ be a sequence of *bilinear maps* $\underset{i}{\smile}: C^p(K) \times C^q(K) \to C^{p+q-i}(K)$ (p, q : arbitrary integers) such that

(1) *if* $u \in C^p(K, L_1)$, $v \in C^q(K, L_2)$, *then* $u \underset{i}{\smile} v \in C^{p+q-i}(K, L_1 \cup L_2)$, *where* L_1, L_2 *are arbitrary subcomplexes of* K,

(2) $\gamma^0 \underset{0}{\smile} u = u$ *for any* $u \in C^0(K)$,

(3) *if* $i < 0$, $u \underset{i}{\smile} v = 0$ *for arbitrary* u *and* v,

(4) *coboundary formula*: $\delta(u \underset{i}{\smile} v) = (-1)^{p+q-i} u \underset{i-1}{\smile} v + (-1)^{p+q+pq} v \underset{i-1}{\smile} u + \delta u \underset{i}{\smile} v + (-1)^p u \underset{i}{\smile} \delta v$, *where* $u \in C^p(K)$, $v \in C^q(K)$ *and* δ *is the coboundary operator.*

We refer to such a sequence $\{\underset{i}{\smile}\}$ as *a system of cup-i products.*

When any system of cup-i products is given, we can define in the usual way (for instance, see [3], [7], [8]) *Steenrod squares* Sq_i, *Pontrjagin squares* $\mathfrak{P}$, *Postnikov squares* $\bar{\mathfrak{P}}$ *and cup products* $\smile$ with respect to the natural coefficient groups:

$$\begin{aligned} \mathrm{Sq}_i &: H^p(K, L; I_2) \longrightarrow H^{2p-i}(K, L; I_2), \\ \mathfrak{P} &: H^p(K, L; I_{2t}) \longrightarrow H^{2p}(K, L; I_{4t}), \\ \bar{\mathfrak{P}} &: H^p(K, L; I_{2t}) \longrightarrow H^{2p+1}(K L; I_{4t}), \\ \smile &: H^p(K, L_1; I_s) \times H^q(K, L_2; I_t) \longrightarrow H^{p+q}(K, L_1 \cup L_2; I_{d(s,t)}), \end{aligned}$$

where I_m is the integer mod m, and $d(s,t)$ denotes the greatest common divisor of s and t. Explicitely, these are given by maps $\{u\} \longrightarrow \{u \underset{i}{\smile} u\}$, $\{u\} \longrightarrow \{u \underset{0}{\smile} u + u \underset{1}{\smile} \delta u\}$, $\{u\} \longrightarrow \{u \underset{0}{\smile} \delta u\}$ and $\{u\} \times \{v\} \longrightarrow \{u \underset{0}{\smile} v\}$ respectively, where $\{u\}$ denotes the cohomology class of a cocycle u.

§2. Equivalent conditions

In this section, we shall prove

Lemma 1. *The conditions* (1) *and* (2) *are equivalent with* (5) *and* (6):

(5) *$\bar{\sigma}^p \underset{i}{\smile} \bar{\tau}^q$ is a cochain in $St\,\sigma^p \cap St\,\tau^q$, where $\bar{\sigma}^p$ is an integral p-cochain defined by $\bar{\sigma}^p(\sigma^p) = +1$ and $\bar{\sigma}^p(\sigma'^p) = 0$ $(\sigma'^p \neq \sigma^p)$, and $St\,\sigma^p$ denotes the star of σ^p in K.*

(6) $$\bar{\sigma}^0 \underset{0}{\smile} \bar{\sigma}^0 = \bar{\sigma}^0 .$$

Proof. i) (1), (2) $\longrightarrow$ (5), (6).

Since $\bar{\sigma}^p \in C^p(K, K - \mathrm{St}\,\sigma^p)$ and $\bar{\tau}^q \in C^p(K, K - \mathrm{St}\,\tau^q)$, it follows from (1) that $\bar{\sigma}^p \underset{i}{\smile} \bar{\tau}^q \in C^{p+q-i}(K, K - \mathrm{St}\,\sigma^p \cup K - \mathrm{St}\,\tau^q) = C^{p+q-i}(K, K - \mathrm{St}\,\sigma^p \cap \mathrm{St}\,\tau^q)$. Thus $\bar{\sigma}^p \underset{i}{\smile} \bar{\tau}^q$ is a cochain in $\mathrm{St}\,\sigma^p \cap \mathrm{St}\,\tau^q$, and so we have (5). Especially, $\bar{\sigma}^0 \underset{0}{\smile} \bar{\tau}^0$ is a 0-cochain in $\sigma^0 \cap \tau^0$, so that $\bar{\sigma}^0 \underset{0}{\smile} \bar{\tau}^0 = 0$ if $\sigma^0 \neq \tau^0$. On the other hand, we have $\gamma^0 \underset{0}{\smile} \bar{\sigma}^0 = \bar{\sigma}^0$ from (2). Thus we see from the bilinearity of $\underset{0}{\smile}$ that the condition (6): $\bar{\sigma}^0 \underset{0}{\smile} \bar{\sigma}^0 = \bar{\sigma}^0$ holds.

ii) (5), (6) $\longrightarrow$ (1), (2).

Let $u = \sum_k a_j \bar{\sigma}_j^p (\sigma_j^p \in K - L_1)$ and $v = \sum_k b_k \bar{\tau}_k^r (\tau_k^q \in K - L_2)$ be elements of $C^p(K, L_1)$ and $C^q(K, L_2)$ respectively. Then it follows from (5) that $u \underset{i}{\smile} v$ is a cochain in $\bigcup_{j,k} (\mathrm{St}\,\sigma_j^p \cap \mathrm{St}\,\tau_k^q)$. Thus $u \underset{i}{\smile} v\,(\sigma^{p+q-i}) = 0$ if $\sigma^{p+q-i} \in L_1$ or $\in L_2$. Because, if we assume that $u \underset{i}{\smile} v\,(\sigma^{p+q-i}) \neq 0$, there exist j and k such that $\sigma^{p+q-i} \in \mathrm{St}\,\sigma_j^p$ and $\in \mathrm{St}\,\tau_k^q$. Therefore it follows that σ_j^p and σ_k^q belong to $\mathrm{Cl}\,\sigma^{p+q-i}$. If $\sigma^{p+q-i} \in L_1$, then $\mathrm{Cl}\,\sigma^{p+q-i} \subset L_1$, and so σ_j^p belongs to L_1. This is a contradiction. In case $\sigma^{p+q-i} \in L_2$, we have a contradiction in the similar way. Thus we have $u \underset{i}{\smile} v \in C^{p+q-i}(K, L_1 \cup L_2)$. This is (1). $\bar{\sigma}^0 \underset{0}{\smile} \bar{\tau}^0 = 0$ $(\sigma^0 \neq \tau^0)$ is a consequence of (5), and so $\gamma^0 \underset{0}{\smile} \bar{\sigma}^0 = \bar{\sigma}^0$ is obvious from (6). Thus we obtain (2) from the bilinearity of $\underset{0}{\smile}$. Q. E. D.

§3. Uniqueness and existence theorems

Corresponding to Theorem 2 in [1], we have

Theorem I. *There exists in K, at least, a system of cup-i products. If $\{\underset{i}{\smile}\}$, $\{\underset{i}{\smile}'\}$ are two such systems, and if Sq_i, $\mathfrak{P}$, $\bar{\mathfrak{P}}$, $\smile$ and Sq_i', $\mathfrak{P}'$, $\bar{\mathfrak{P}}'$, $\smile'$ are the operations induced by $\underset{i}{\smile}$ and $\underset{i}{\smile}'$ respectively, then we have*

$$\mathrm{Sq}_i = \mathrm{Sq}_i',\ \mathfrak{P} = \mathfrak{P}',\ \bar{\mathfrak{P}} = \bar{\mathfrak{P}}',\ \smile = \smile' .$$

Proof. Let us define a homomorphism $D^i: C^r(K\times K) \rightarrow C^{r-i}(K)$ $(i = 0, \pm 1, \pm 2, \cdots\cdots)$ by

$$D^i(\bar{\sigma}_j^p \times \bar{\sigma}_k^q) = (-1)^{i(p+q)+\frac{1}{2}i(i-1)}\, \bar{\sigma}_j^p \underset{i}{\smile} \bar{\sigma}_k^q , \quad (p+q = r) .$$

Then it is obvious from the bilinearity of $\underset{i}{\smile}$ that

$$(7) \qquad D^i(u\times v) = (-1)^{i(p+q)+\frac{1}{2}i(i-1)}\, u \underset{i}{\smile} v$$

for any $u \in C_p(K)$ and $v \in C^q(K)$.
Moreover, we have, by straightforward calculations, from (4) that

$$(8) \qquad \begin{aligned} &D^i\delta(u\times v) + (-1)^{i+1}\, \delta D^i(u\times v) \\ &= (-1)^{pq}\, D^{i-1}(v\times u) + (-1)^i D^{i-1}(u\times v) . \end{aligned}$$

Let $g_{\#}$: $C_r(K\times K) \rightarrow C_r(K\times K)$ be a chain transformation defined by

$$g_{\#}(\sigma_j^p \times \sigma_k^q) = (-1)^{pq} \sigma_k^q \times \sigma_j^p ,$$

and let $g^{\#}$: $C^r(K\times K) \rightarrow C^r(K\times K)$ be its dual. Then it is obvious that

$$g^{\#}(u\times v) = (-1)^{pq}\, v\times u ,\ u \in C^p(K) ,\ v \in C^q(K) ,$$

and that we can take as a carrier of $g_{\#}$ a map g: $K\times K \rightarrow K\times K$ defined by $g(\sigma_j \times \sigma_k) = \sigma_k \times \sigma_j$. If we use $g^{\#}$, (8) is written as follows:

$$(9) \qquad \begin{aligned} &D^i\delta(u\times v) + (-1)^{i+1}\, \delta D^i(u\times v) \\ &= D^{i-1}\, g^{\#}(u\times v) + (-1)^i D^{i-1}(u\times v) . \end{aligned}$$

Let D_i: $C_r(K) \rightarrow C_{r+i}(K\times K)$ be the dual of D^i. Then D_i is explicitely given by

$$(10) \qquad D_i c^r = \sum_{j,k} (D^i(\bar{\sigma}_j^p \times \bar{\sigma}_k^q)\cdot c^r)\, \sigma_j^p \times \sigma_k^q ,$$

where $p+q = r+i$, and $\cdot$ denotes the Kronecker index between a cochain and a chain.

We shall now prove three properties (12), (13), (14) of D_i.

Firstly, as the dual relation of (9), we obtain easily

$$(11) \qquad \partial D_i c^r + (-1)^{i+1}\, D_i \partial c^r = g_{\#} D_{i-1} c^r + (-1)^i D_{i-1} c^r ,$$

where $c^r \in C_r(K)$.
If we use the notations

$$\begin{aligned} \omega D_i &= \partial D_i + (-1)^{i+1} D_i \partial , \\ \alpha_i &= g_{\#} + (-1)^i e_{\#} \end{aligned}$$

($e_{\#}$: identical chain transformation) with Steenrod [4], then (11) becomes

(12) $$\omega D_i = \alpha_i D_{i-1}$$

Next, we shall prove

(13) $$\mathrm{In} D_0 c^0 = \mathrm{In} c^0, \qquad c^0 \in C_0(K),$$

where $\mathrm{In} c^0$ denotes the Kronecker index of a 0-chain.

Let $c_0 = \sum_j a_j \sigma_j^0$, then it follows from (7) and (10) that

$$D_0 c_0 = \sum_{j\,k} ((\bar{\sigma}_j^0 \underset{0}{\smile} \bar{\sigma}_k^0)\cdot c^0) \sigma_j^0 \times \sigma_k^0 .$$

Since

$$\begin{aligned} \bar{\sigma}_j^0 \underset{0}{\smile} \bar{\sigma}_k^0 &= \bar{\sigma}_j^0, \quad \text{if } j = k, \\ &= 0, \quad \text{otherwise}, \end{aligned}$$

we have

$$\begin{aligned} (\bar{\sigma}_j^0 \underset{0}{\smile} \bar{\sigma}_k^0)\cdot c^0 &= a_j, \quad \text{if } j = k, \\ &= 0, \quad \text{otherwise}. \end{aligned}$$

Thus it holds that

$$D_0 c^0 = \sum_j a_j (\sigma_j^0 \times \sigma_j^0),$$

and so

$$\mathrm{In} D_0 c^0 = \sum_j a_j = \mathrm{In} c^0,$$

which is (13).

Let $C: K \to K \times K$ be a carrier defined by $C(\tau) = \mathrm{Cl}\,\tau \times \mathrm{Cl}\,\tau$. Then we have

(14) C is an acyclic carrier such that $gC(\tau) = C(\tau)$ for any $\tau \in K$, and D_i is carried by C.

Since $\mathrm{Cl}\,\tau$ is acyclic, $\mathrm{Cl}\,\tau \times \mathrm{Cl}\,\tau$ is also acyclic. Thus C is acyclic. $gC(\tau) = C(\tau)$ is obvious. From (10) we have

$$D_i(\tau^r) = \sum_{j,k} (D^i(\bar{\sigma}_j^p \times \bar{\sigma}_k^q)\cdot \tau^r) \sigma_j^p \times \sigma_k^q .$$

Since $D^i(\bar{\sigma}_j^p \times \bar{\sigma}_k^q) = (-1)^{i(p+q)+\frac{1}{2}i(i-1)} \bar{\sigma}_j^p \underset{i}{\smile} \bar{\sigma}_k^q$ is a cochain in $\mathrm{St}\,\sigma_j^p \cap \mathrm{St}\,\sigma_k^q$ from (5), $D^i(\bar{\sigma}_j^p \times \bar{\sigma}_k^q)\cdot \tau^r \neq 0$ implies that $\tau^r \in \mathrm{St}\,\sigma_j^p$ and $\in \mathrm{St}\,\sigma_k^q$. Thus both σ_j^p and σ_k^q belong to $\mathrm{Cl}\,\tau^r$, and so $D_i(\tau^r)$ is a chain in $\mathrm{Cl}\,\tau^r \times \mathrm{Cl}\,\tau^r$. Therefore we can take C as a carrier of D_i, and (14) is proved.

If we carry out the above arguments as to $\underset{i}{\smile}'$, we find that the dual D_i' of a homomorphism D'^i defined by

(15) $$D'^i(\bar{\sigma}_j^p \times \bar{\sigma}_k^q) = (-1)^{i(p+q)+\frac{1}{2}i(i-1)} \bar{\sigma}_j^p \underset{i}{\smile}' \bar{\sigma}_k^q$$

has the properties (12), (13), (14).

Thus both D_i and D_i' satisfy the all conditions of Lemma 5.5 in [4, p. 56], and so it follows from this lemma that

(16) there exists a sequence of homomorphisms $E_i: C_r(K) \to C_{r+i}(K \times K)$

($i = 0, 1, 2, \cdots$) such that

i) $E_0 = 0$ ii) $\omega E_{i+1} = D_i' - D_i - \alpha_i E_i$

iii) E_i is carried by C.

If we denote by E^i the dual of E_i, (ii) is written in terms of cohomology as follows:

$$E^{i+1}\delta + (-1)^{i+2}\delta E^{i+1} = D'^i - D^i - E^i g^{\#} - (-1)^i E^i . \tag{17}$$

Thus, if u is an element of the p-cocycle group $Z^p(K, L; I_2)$, it follows from (7), (15) and (17) that

$$u \smile_i' u - u \smile_i u \equiv \delta E^{i+1}(u \times u) \quad \text{mod } 2. \tag{18}$$

Since $E_{i+1}\tau$ is a chain in $\text{Cl}\,\tau \times \text{Cl}\,\tau$ from (iii) of (16), $E_{i+1}\tau$ is a chain in $L \times L$ if $\tau \in L$. Since $u \times u \in C^{2p}(K \times K, K \times L \cup L \times K)$, it follows that

$$E^{i+1}(u \times u)(\tau) = (u \times u)(E_{i+1}\tau) = 0$$

if $\tau \in L$. Thus $E^{i+1}(u \times u) \in C^{2p-i-1}(K, L)$. Therefore we have from (18)

$$\text{Sq}_i'\{u\} = \text{Sq}_i\{u\} .$$

As for the Pontrjagin square, it holds for $u \in Z^p(K, L; I_{2t})$ that

$$\begin{aligned}(u \smile_0' u + u \smile_1' \delta u) - (u \smile_0 u + u \smile_1 \delta u)\\ \equiv \delta(E^1(u \times u) + E^2(u \times \delta u)) \quad \text{mod } 4t.\end{aligned}$$

Since $E^1(u \times u) + E^2(u \times \delta u) \in C^{2p-1}(K, L)$, we have

$$\mathfrak{P}'\{u\} = \mathfrak{P}\{u\} .$$

Similarly, we have $\ddot{\mathfrak{P}}' = \ddot{\mathfrak{P}}$, $\smile = \smile'$. Thus the second part of Theorem I is proved.

Steenrod showed in [4] that there exists in K, at least, a sequence $\{D_i\}$ satisfying the condition (12), (13), (14). Precisely, K in [4] is a geometrical cell complex. However, as is shown easily, the arguments in there still hold in a finite simple complex. If we now define $u \smile_i v$ by (7), we can easily prove that $\{\smile_i\}$ is a system of cup-i products. Thus the first part of Theorem is proved, and this completes the proof.

§4. On the Shimada-Uehara operation

Shimada-Uehara defined in [6] the homomorphism $\mathcal{Q}_i : H^p(K, L; I_{2t}) \to H^{2p-i}(K, L; I)$ for odd $p-i$. This was originally defined by the map of $u \in Z^p(K, L; I_{2t})$ to $q_i u = u \smile_i u + u \smile_{i+1} \delta u + (-1)^p \frac{1}{2} \delta u \smile_{i+2} \delta u \in Z^{2p-i}(K, L; I)$. However, we can prove that this operation is nothing but a composition of two well-known operations, as is shown in the following.

Let

$$\mathrm{Sq}_{i+1}: H^p(K, L: I_{2t}) \longrightarrow H^{2p-i-1}(K, L; I_2)$$

be the Steenrod square corresponding to the natural coefficients homomorphism $I_{2t} \to I_2$, and let

$$\Delta: H^{2p-i-1}(K, L; I_2) \longrightarrow H^{2p-i}(K, L; I)$$

be the coboundary operator associated with the exact coefficient sequence

$$0 \longrightarrow I \overset{\xi}{\longrightarrow} I \overset{\eta}{\longrightarrow} I_2 \longrightarrow 0,$$

where $\xi(n) = 2n$ and η is the natural factorization $I/\xi I$. Then we have

Theorem II

$$\mathscr{Q}_i = \Delta \mathrm{Sq}_{i+1}.$$

Proof. It follows from the definition of Δ that $\Delta\mathrm{Sq}_{i+1}\{u\}$ is a cohomology class containing a cocycle v such that $2v = \delta(u \underset{i+1}{\smile} u)$. Since

$$\delta(u \underset{i+1}{\smile} u) = (-1)^{i+1} 2u \underset{i}{\smile} u + \delta u \underset{i+1}{\smile} u + (-1)^{i+1} u \underset{i+1}{\smile} \delta u,$$
$$\delta(\delta u \underset{i+2}{\smile} u) = (-1)^{i+1} \delta u \underset{i+1}{\smile} u - u \underset{i+1}{\smile} \delta u + (-1)^{p+1} \delta u \underset{i+2}{\smile} \delta u,$$

we have

$$\delta(u \underset{i+1}{\smile} u) = (-1)^{i+1} 2(u \underset{i}{\smile} u + u \underset{i+1}{\smile} \delta u + (-1)^p \frac{1}{2} \delta u \underset{i+2}{\smile} \delta u)$$
$$+ (-1)^{i+1} \delta(\delta u \underset{i+2}{\smile} u).$$

Thus $\frac{1}{2}\delta(u \underset{i+1}{\smile} u) = (-1)^{i+1} q_i u + (-1)^{i+1} \delta\left(\frac{1}{2}\delta u \underset{i+1}{\smile} u\right)$, and so we have

$$\{v\} = (-1)^{i+1}\{q_i u\}.$$

Therefore
$$\Delta\mathrm{Sq}_{i+1}\{u\} = \mathscr{Q}_i\{u\}.$$

§5. Product formulas

Let

$$\mu: H^q(K, L; I) \longrightarrow H^q(K, L; I_{4t})$$

be the natural homomorphism corresponding to the natural factorization $I \to I_{4t}$. Then we have

Theorem III (*product formulas*). *Defining in the natural manner, namely by the tensor products, the group pairings between coefficient groups in the cross or cup products, we have the following formulas:*

A) i) *If* $x \in H^p(K_1, L_1; I_2)$ *and* $y \in H^q(K_2, L_2; I_2)$, *we have*

$$\mathrm{Sq}_i(x \times y) = \sum_{j+k=i} \mathrm{Sq}_j x \times \mathrm{Sq}_k y \tag{19}$$

in $H^{2(p+q)-i}(K_1 \times K_2, K_1 \times L_2 \cup L_1 \times K_2; I_2)$ (Cartan).*

ii) *Let p and q are even. If* $x \in H^p(K_1, L_1; I_{2s})$ *and* $y \in H^q(K_2, L_2; I_{2t})$, *we have*

$$\mathfrak{P}(x \times y) = \mathfrak{P}x \times \mathfrak{P}y + \bar{\mathfrak{P}}x \times \mu\varDelta \mathrm{Sq}_2\, y + \mu\varDelta \mathrm{Sq}_2\, x \times \bar{\mathfrak{P}}y . \tag{20}$$

in $H^{2(p+q)}(K_1 \times K_2, K_1 \times L_2 \cup L_1 \times K_2; I_{4d(s \cdot t)})$.

iii) *If* $x \in H^p(K_1, L_1; I_{2s})$ *and* $y \in H^q(K_2, L_2; I_{2t})$, *we have*

$$\bar{\bar{\mathfrak{P}}}(x \times y) = \mathfrak{P}x \times \bar{\bar{\mathfrak{P}}}y + \bar{\bar{\mathfrak{P}}}x \times \mathfrak{P}y . \tag{21}$$

in $H^{2(p+q)+1}(K_1 \times K_2, K_1 \times L_2 \cup L_1 \times K_2; I_{4d(s,t)})$.

B) *Let us assume that* $K_1 = K_2 = K$ *in* i), ii), iii). *Then* (19), (20), (21) *hold with the cup product* $\smile$ *in place of the cross product* $\times$, *and with* $K, L_1 \cup L_2$ *in places of* $K_1 \times K_2$, $K_1 \times L_2 \cup L_1 \times K_2$ *respectively.*

We shall first prepare two lemmas, before we proceed to prove the theorem.

Let K_1 and K_2 be finite simple complexes. Then $K_1 \times K_2$ is also finite and simple. Let $\{\underset{i}{\smile}^1\}$ and $\{\underset{i}{\smile}^2\}$ be arbitrary systems of cup-i products in K_1 and K_2 respectively. Following H. Cartan, we shall now define a bilinear map $\underset{i}{\smile}$ of $C^p(K_1 \times K_2) \times C^q(K_1 \times K_2)$ in $C^{p+q-i}(K_1 \times K_2)$ by

$$\begin{aligned}(u_1 \times u_2) \underset{i}{\smile} (v_1 \times v_2) &= (-1)^{p_2 q_1} \sum_j (u_1 \underset{2j}{\smile}^1 v_1) \times (u_2 \underset{i-2j}{\smile}^2 v_2) \\ &+ (-1)^{p_2(q_1+q_2)+p_2+q_2} \sum_j (u_1 \underset{2j+1}{\smile}^1 v_1) \times (v_2 \underset{i-2j-1}{\smile}^2 u_2) .\end{aligned} \tag{22}$$

where $u_l \in C^{p_l}(K_l)$, $v_l \in C^{q_l}(K_l)$ $(l = 1, 2)$.

Lemma 2. $\{\underset{i}{\smile}\}$ *is a system of cup-i products in* $K_1 \times K_2$.

Proof. Straightforward calculations show that $\{\underset{i}{\smile}\}$ satisfies the condition (4). The conditions (5), (6) and (3) can be proved easily. Thus Lemma 2 follows from Lemma 1.

Let K, K' be finite simple complexes, and let L, L' be their subcomplexes. Let $\{\underset{i}{\smile}\}$ and $\{\underset{i}{\smile}'\}$ be arbitrary systems of cup-i products in K and K' respectively. Suppose that $f_{\#}: C^p(K, L) \to C^p(K', L')$ be a chain transformation which is carried by an acyclic carrier. Let us denote by f^* the homomorphism of cohomology groups induced by $f_{\#}$. Then we have

Lemma 3.

$$\mathrm{Sq}_i f^* = f^* \mathrm{Sq}_i' , \quad \mathfrak{P} f^* = f^* \mathfrak{P}' , \quad \bar{\bar{\mathfrak{P}}} f^* = f^* \bar{\bar{\mathfrak{P}}}' .$$

This lemma can be proved in the similar way as in the proof of Theorem 8.1 in [4]. Therefore we will not prove this lemma.

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* We find an elegant proof of this formula in [5].

Proof of Theorem III. (A) It follows from Theorem I and Lemma 2 that we may calculate by the rule (22) each square in $K_1 \times K_2$. Thus (19) is obvious.

We shall prove (20). Let $u \in Z^p(K_1, L_1 ; I_{2s})$, $v \in Z^q(K_2, L_2 ; I_{2t})$, and let p, q are even. Then it follows from (22) and Theorem II that

$$\begin{aligned}
\mathfrak{P}\{u \times v\} &= \{(u \times v) \underset{0}{\smile} (u \times v) + (u \times v) \underset{1}{\smile} \delta(u \times v)\} \\
&= \{(u \underset{0}{\smile} u) \times (v \underset{0}{\smile} v) + (u \underset{0}{\smile} \delta u) \times (v \underset{1}{\smile} v) \\
&\quad + (u \underset{1}{\smile} \delta u) \times (v \underset{0}{\smile} v) + (u \underset{0}{\smile} u) \times (v \underset{1}{\smile} \delta v) \\
&\quad - (u \underset{1}{\smile} u) \times (\delta v \underset{0}{\smile} v)\} \\
&= \{(u \underset{0}{\smile} u + u \underset{1}{\smile} \delta u) \times (v \underset{0}{\smile} v + v \underset{1}{\smile} \delta v) \\
&\quad - (u \underset{1}{\smile} \delta u) \times (v \underset{1}{\smile} \delta v) + (u \underset{0}{\smile} \delta u) \times (v \underset{1}{\smile} v) \\
&\quad - (u \underset{1}{\smile} u) \times (\delta v \underset{0}{\smile} v)\} \\
&= \mathfrak{P}\{u\} \times \mathfrak{P}\{v\} + \{(u \underset{0}{\smile} \delta u) \times (v \underset{1}{\smile} v + v \underset{2}{\smile} \delta v + \frac{1}{2} \delta v \underset{3}{\smile} \delta v) \\
&\quad + (u \underset{1}{\smile} u + u \underset{2}{\smile} \delta u + \frac{1}{2} \delta u \underset{3}{\smile} \delta u) \times (v \underset{0}{\smile} \delta v)\} \\
&= \mathfrak{P}\{u\} \times \mathfrak{P}\{v\} + \bar{\mathfrak{P}}\{u\} \times \mu\Delta \mathrm{Sq}_2\{v\} + \mu\Delta \mathrm{Sq}_2\{u\} \times \bar{\mathfrak{P}}\{v\} ,
\end{aligned}$$

Therefore we obtain (20).

The proof of (21) is similar. Thus (A) is proved.

(B) If we note that $D^0(u \times v) = u \underset{0}{\smile} v$ from (7), and that D^0 is the dual of a chain transformation D_0 with an acyclic carrier C, (B) is obvious from (A) and Lemma 3. This completes the proof.

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