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# *Arithmetical ideal theory in semigroups*

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The arithmetical ideal theory in rings may be regarded as that in semigroups, which can be treated as a generalization of the former. The arithmetical ideal theory in commutative semigroups was investigated for the firat time by Clifford ([4]) and then by Lorenzen ([8]). Some of the result of Clifford was extended to the noncommutative case by Kawada and Kondo ([7]). In the present paper we shall develop the arithmetical ideal theory in (noncommutative) semigroups, which is a generalization of that in noncommutative rings (cf. [1], [2] and [6]).

As preliminaries we deal in §1 with the factorization of integral elements in a lattice-ordered group and in §2 we give an abstract foundation of Artin-Hencke's ideal theory ([5]). Let $S$ be a semigroup with unity quantity. The concepts of orders, maximal orders, ideals etc. in $S$ are defined similarly as in rings. By using the results of §1, 2 we discus in §4 the theory of two-sided ideals with respect to a maximal order of $S$. We consider closed ideals (Lorenzen's $r$-ideals), i. e. ideals closed with respect to a given closure operation, by which a mapping of the set of all two-sided ideals in itself is defined. In order that the set of all closed two-sided $\mathfrak{o}$-ideals, $\mathfrak{o}$ a given regular order, forms an abelian group, which is a direct product of infinite cyclic groups, it is necessary and sufficient that Noether's axioms hold for $\mathfrak{o}$. Let $\mathfrak{o}$ be a regular order of $S$, for which Noether's axioms hold. The closure operation defined over two-sided $\mathfrak{o}$-ideals can be extended over $\mathfrak{o}$-sets containing regular elements. ($A$ subset $A$ of $S$ is called a $\mathfrak{o}$-set if $\mathfrak{o}A = A\mathfrak{o} = A$.) $A$ closed sub-semigroup of $S$ containing $\mathfrak{o}$ is called a $\mathfrak{o}$-semigroup. We determine in §5 all $\mathfrak{o}$-semigroups. They form a Boolian algebra with respect to inclusion relation. In §7 we shall consider the Brandt's gruppoid of normal ideals. The factorization of integral normal ideals may be regarded as the factorization of integral elements in a lattice-ordered gruppoid, which will be treated in §6.

## §1. Factorization of integral elements in a lattice-ordered group.

Let $G$ be a lattice-ordered group ($l$-group) with unity quantity $e$. Elements of $G$ will be denoted by small letters with or without suffices. We do not assume the multiplication to be commutative, except when we mention it particularly.

*Definition.* An element $a$ of $G$ is called *integral* if $a \leq e$.

If $a \leq b$ then there exist two integral elements $c$ and $d$ such that $a = bc = db$. Putting $c = b^{-1}a$ we get $c \leq b^{-1}b = e$ and $bc = a$. Similarly we obtain $a = bd$, $d \leq e$.

Let $x$ be any element of $G$. Then $a = x_\cap e$ is integral, hence there exist two integral elements $c$ and $d$ such that $a = xc = dx$. $x$ is, therefore, represented by a form of a right and a left quotients of two integral elements of $G$: $x = ac^{-1} = d^{-1}a$.

Let $a$ and $b$ be coprime. Then $a^\cup x = a^\cup bx = a^\cup xb$ for any integral element $x$ in $G$. Because, $a^\cup x = (a^\cup b)(a^\cup x) = a^2{}^\cup ax^\cup ba^\cup bx \leq a^\cup a^\cup a^\cup bx = a^\cup bx \leq a^\cup x$. If $a_i$ and $b_k$ are coprime ($i = 1, \cdots, m$; $k = 1, \cdots, n$), then so are $\Pi_{i=1}^m a_i$ and $\Pi_{k=1}^n b_k$.

If we assume that there exists $\sup X$ for a non-void subset $X$ of $G$, then there exist $\sup(Xa)$, $\sup(aX)$ for any $a$ in $G$, and $\sup(Xa) = (\sup X)a$, $\sup(aX) = a(\sup X)$. If there exists $\sup A$ for any non-void set $A$ of integral elements of $G$, then $G$ forms a conditionally complete lattice-ordered group ($cl$-group). By the well-known theorem $G$ forms a commutative group under multiplication. (*cf.* [3]).

In the following, we assume that $G$ forms a commutative group.

*Lemma* 1.1. *If two lattice-quotient $a/a'$ and $b/b'$ are projective, then* $a^{-1}a' = b^{-1}b'$.

Proof. Suppose that $a/a'$ is transposable to $b/b'$ and $b < a$. Put $c = a^{-1}a'$, $d = b^{-1}b'$. Then they are both integral and $b^\cup ac = a$, $b_\cap ac = bd$. Take an integral element $t$ such that $b = at$, then $bd = atd$. Hence $t^\cup c = e$, $t_\cap c = td$ and $e/c$ is transposable to $t/td$. Being $t$ and $c$ are coprime, we get $td = t_\cap c = tc$, $c = d$. If $a/a'$ is projective to $b/b'$, then by induction we complete our proof.

*Definition.* Let $a$ be an integral element of $G$, and $a = a_1 \cdots a_r$ a factorization with integral elements $a_i$ in $G$. A factorization of $a$ of the form $a = \Pi_{i=1}^r \Pi_{j=1}^{t_i} a_{ij}$, $a_i = \Pi_{j=1}^{t_i} a_{ij}$ ($i = 1, \cdots, r$) is called a *refinement* of the above factorization.

*Theorem* 1.1. (*Refinement theorem*) *Any two factorizations of an integral element a in G have the same refinement.*

Proof. Let $a = a_1 \cdots a_r = b_1 \cdots b_s$ be two factorizations of $a$. Putting $A_i = a_1 \cdots a_i$, $B_k = b_1 \cdots b_k$, we get two chains such that $e = A_0 > A_1 > \cdots > A_r = a$, $e = B_0 > B_1 > \cdots > B_s = a$. By Jordan-Hölder-Schreier's theorem in a modular lattice, we get two refinements of the same length: $e = A_0' > A_1' > \cdots > A_n' = a$, $e = B_0' > B_1' > \cdots > B_n' = a$ such that $A_{i-1}'/A_i'$ is projective to $B_{k-1}'/B_k'$

in pairs. Hence by lemma 1.1. $a_i' = A_{i-1}'^{-1}A_i' = B_{k-1}'^{-1}B_k' = b_k'$ and $a = a_1' \cdots a_n'$ $= b_1' \cdots b_n'$.

*Definition.* A *prime element* is an integral element $p$ such that $p$ is $\neq e$ and $ab \leq p$ implies $a \leq p$ or $b \leq p$ for integral elements $a, b$ of $G$. A *maximal element* is an element covered by the unity quantity $e$. An *irreducible element* is an integral element $p$ such that $p$ is $\neq e$ and not decomposed as a product of two integral elements other than $e$.

In a commutative $l$-group the following conditions are equivalent to one another.

(1) $p$ is an irreducible element.
(2) $p$ is a maximal element.
(3) $p$ is a prime element.

Proof. (1)$\rightarrow$(2): If $p$ is not maximal, then there exists an element $a$ such that $p < a < e$. Hence $p = ab$, $b < e$. (2)$\rightarrow$(3): If $ab \leq p$ for integral elements $a, b$ and $a \nleq p$, then $b = (a \cup p) b = ab \cup pb \leqq p \cup p = p$. (3)$\rightarrow$(1): If $p = ab$ for integral elements $a, b$ and $a \leq p$, then $p = ab \leq a$, $p = a$, $b = e$.

In the following we assume the ascending chain condition for integral elements of $G$.

*Corollary. Any integral element of a commutative l-group may be uniquely represented as a product of finite prime elements.*

A non-integral elements in $G$ can be represented as a quotient of two integral elements of $G$. Hence we have

*Theorem* 1.2. *G is a direct product of infinite cyclic groups with prime elements as its generators.*

## §3. Abstract foundation of Artin-Hencke's ideal theory in a maximal order of a ring.

Let $L$ be a complete lattice-ordered semigroup ($cl$-semigroup), (*cf.* [3]) such that there exists a mapping of $L$ into itself $a \rightarrow a^{-1}$ with the following properties:

(1) $aa^{-1}a \leq a$,
(2) $axa \leq a$ implies $x \leq a^{-1}$.

*Definition.* An element $a$ of $L$ is called *integral* if $a^2 \leq a$.

For example, an element $a$ satisfying $a \leq e$ is integral, where $e$ is a unity quantity of $L$.

*Lemma* 2.1. *Let $e$ be maximally integral, i.e. if $e\leq c$ and $c^2\leq c$ then $c=e$. Then the following conditions are equivalent*:

1. $ax\leq e$,
2. $axa\leq a$,
3. $xa\leq e$.

Proof. If $b$ is integral, then $b\leq e$, for putting $c=b\cup e$ we obtain $e\leq c$ and $c^2=b^2\cup b\cup e\leq b\cup e=c$, hence $e=c$ and $b\leq e$. Let $axa\leq a$. Since $axax\leq ax$ we obtain $ax\leq e$. The converse is evident. Hence we get (1) $\rightleftarrows$ (2), similarly (2) $\rightleftarrows$ (3).

In particular we have $aa^{-1}\leq e$ and $a^{-1}a\leq e$.

*Theorem* 2.1. *The following conditions are equivalent to one another.*

1) *$e$ is maximally integral.*
2) *If $a$ is integral, then $a\leq e$.*
3) *If $a^n\leq c$ $(n=1,2,\cdots)$, then $a\leq e$.*
4) *If $ax\leq a$ then $x\leq e$.*
5) *If $xa\leq a$ then $x\leq e$.*

Proof. (1) $\rightarrow$ (2) has already been shown in the proof of Lemma 2.1. Since (2) $\rightarrow$ (1) is evident we have (1) $\rightleftarrows$ (2). (2) $\rightarrow$ (3): Putting $b=\bigcup_{n=1}^{\infty}a^n$, we get $a\leq b\leq c$ and $b^2=\bigcup_{n=2}^{\infty}a^n\leq\bigcup_{n=1}^{\infty}a^n=b$. Hence $a\leq b\leq e$. (3) $\rightarrow$ (4): If $ax\leq a$, then $ax^n\leq a$ $(n=1,2,\cdots)$ and $a^{-1}ax^na^{-1}a\leq a^{-1}aa^{-1}a\leq a^{-1}a$. Hence $x^n\leq(a^{-1}a)^{-1}$ $(n=1,2,\cdots)$, $x\leq e$. (4) $\rightarrow$ (2) is evident. Similarly we obtain (3) $\rightarrow$ (5) and (5) $\rightarrow$ (2).

In the following we assume that $e$ is maximally integral. Every integral element is therefore $\leq e$.

*Theorem* 2.2. *$L$ forms a residuated lattice.*

Proof. If $ax\leq b$, then $b^{-1}ax\leq b^{-1}b\leq e$, $b^{-1}axb^{-1}a\leq b^{-1}a$; $x\leq(b^{-1}a)^{-1}$, i.e. $X=\{x\mid x\in L,\ ax\leq b\}$ is bounded. Hence there exists $c=\sup X$ and $ac\doteq\sup(aX)\leq b$. $c$ is a left-residual $(b:a)_l$ of $b$ by $a$. Similarly there exists a right-residual $(b:a)_r=\sup Y$, where $Y=\{y\mid y\in L,\ ya\leq b\}$.

We have the following:

1) $e=(a:a)_r=(a:a)_l$.
2) $a^{-1}=(e:a)_r=(e:a)_l$. If $a=e$ then $e^{-1}=e$.
3) $((c:a)_r:b)_l=((c:b)_l:a)_r$. If $c=e$ then $(a^{-1}:b)_l=(b^{-1}:a)_r$.
4) If $a\leq b$ then $(c:a)_r\geq(c:b)_r$ and $(c:a)_l\geq(c:b)_l$. If $c=e$ then $a\leq b$ implies $a^{-1}\geq b^{-1}$.
5) If $b\leq c$ then $(b:a)_r\leq(c:a)_r$ and $(b:a)_l\leq(c:a)_l$.

6) $(a:bc)_r = ((a:c)_r:b)_r$, $(a:bc)_l = ((a:b)_l:c)_l$.

7) $(a:b\cup c)_r = (a:b)_r \cap (a:b)_r$, $(a:b\cup c)_l = (a:b)_l \cap (a:c)_l$. If $a = e$ then $(b\cup c)^{-1} = b^{-1} \cap c^{-1}$.

8) $(a\cap b:c)_r = (a:c)_r \cap (b:c)_r$, $(a\cap b:c)_l = (a:c)_l \cap (b:c)_l$.

9) $(ab:c)_r \geq a(b:c)_r$, $(ab:c)_l \geq (a:c)_l b$.

Since $aa^{-1} \leq e$, $a^{-1}aa^{-1} \leq a^{-1}$, we get $a \leq (a^{-1})^{-1}$. If we define $a^* = (a^{-1})^{-1}$, then $a \leq a^*$. If $a \leq b$ then $a^{-1} \geq b^{-1}$. Hence $a^* \leq b^*$. From $a^{-1} \geq (a^*)^{-1} = (a^{-1})^*$ and $a^{-1} \leq (a^{-1})^*$ we get $a^{-1} = (a^*)^{-1} = (a^{-1})^* = ((a^{-1})^{-1})^{-1}$.

We obtain the following:

(1) $a \leq a^*$

(2) $a^{**} = a^*$

(3) $a \leq b$ implies $a^* \leq b^*$.

(4) $a^*b^* \leq (ab)^*$

Being $ab(ab)^{-1} \leq e$, we have $b(ab)^{-1} \leq a^{-1} = (a^*)^{-1}$, $b(ab)^{-1}a^* \leq (a^*)^{-1}a^* \leq e$. Hence $(ab)^{-1}a^* \leq b^{-1} = (b^*)^{-1}$ and $(ab)^{-1}a^*b^* \leq e$. Hence $a^*b^* \leq (ab)^*$.

We have also the following:

$$(ab)^* = (a^*b)^* = (ab^*)^* = (a^*b^*)^*,$$
$$(a\cup b)^* = (a^*\cup b)^* = (a\cup b^*) = (a^*\cup b^*)^*,$$
$$(a^*\cap b^*)^* = a^*\cap b^*.$$

*Definition.* Two elements $a$ and $b$ of $L$ are called *quasi-equal* if $a^* = b^*$. Symbol: $a \sim b$.

Since $a^{-1} = (a^*)^{-1}$, $a \sim b$ is equivalent to $a^{-1} = b^{-1}$. It is evident that this relation fulfils the equivalence relation.

$a \sim a^*$ is evident by (2). Since $(aa^{-1})^{-1} = (e:aa^{-1})_l = ((e:a)_l:a^{-1})_l = (a^{-1}:a^{-1})_l = e = e^{-1}$, we have $aa^{-1} \sim e$. Similarly $a^{-1}a \sim e$. If $a$ is integral, $a^*$ is so. Because, $(a^*)^2 \leq (a^2)^* \leq a^*$. $a \sim b$ implies $b \leq a^*$. If $a \sim b$ and $a$ is integral, then $b$ is so. If $a \leq c \leq b$ and $a \sim b$, then $a^* \leq c^* \leq b^*$, $a^* = b^*$, hence $a \sim c$. If $a \leq b^*$ then there exist two integral elements $c$ and $d$ such that $a \sim cb \sim bd$. For, putting $c = ab^{-1}$, we have $c \leq b^*b^{-1} = b^*(b^*)^{-1} \leq e$ and $cb = ab^{-1}b \sim ae = a$. (cf. the following.) Similarly we obtain $a \sim bd$. If $a \sim b$ then there exist two elements $u$ and $v$ such that $au = vb$ and $u \sim v \sim e$. For, since $a^* = b^*$, i. e. $a^{-1} = b^{-1}$, we get $ab^{-1}b = aa^{-1}b$, $u = b^{-1}b \sim e$ and $v = aa^{-1} \sim e$.

From $a \sim b$ and $c \sim d$, it follows that

$$ac \sim bd\,,\quad a\cup c \sim b\cup d\,,\quad a\cap c = b\cap d$$

Proof. $(ac)^{-1} = (e:ac)_r = ((e:c)_r:a)_r = (c^{-1}:a)_r = (d^{-1}:a)_r = (a^{-1}:d)_l = (b^{-1}:d)_l = ((e:b)_l:d)_l = (e:bd)_l = (bd)^{-1}$, i. e. $ac \sim bd$. $(a\cup c)^{-1} = a^{-1}\cap c^{-1} = b^{-1}\cap d^{-1} = (b\cup d)^{-1}$, i. e. $a\cup c \sim b\cup d$. $bb^{-1}(a\cap c) \leq bb^{-1}a \cap bb^{-1}c \leq ba^{-1}a \cap ec$

$\leq be_\cap c = b_\cap c$. Hence $aa^{-1}bb^{-1}(a_\cap c) \leq aa^{-1}(b_\cap c) \leq a_\cap c$. Since $aa^{-1} \sim bb^{-1} \sim e$, we get $a_\cap c \sim b_\cap c$. Similarly $b_\cap c \sim b_\cap d$.

If we classify $L$ by the quasi-equal relation, then the set $G$ of all classes $E, A, B, \cdots$ forms a partly ordered set when we define $A \leq B$ by $a^* \leq b^*$ $(a \in A,\ b \in B)$ in $L$. Moreover $G$ forms a lattice with respect to this order. $A^\cup B$ and $A_\cap B$ are the classes containing $(a^{*\cup} b^*)^*$ and $(a^*{}_\cap b^*)^* = a^*{}_\cap b^*$ $(a \in A,\ b \in B)$ respectively. And moreover $G$ forms a $l$-semigroup when we define $AB$ by the class containing $(a^*b^*)^*$ $(a \in A,\ b \in B)$. Since the inverse of $A$ is the class containing $a^{-1}(a \in A)$, $G$ forms a group. As above mentioned $a^\cup b$, $a_\cap b$ and $ab$ $(a \in A, b \in B)$ are contained in $A^\cup B$, $A_\cap B$ and $AB$ respectively

If a subset $\{B_\alpha\}$ of $G$ is bounded, then the subset $\{b_\alpha^*\}$ $(b_\alpha \in B_\alpha)$ of $L$ is bounded, and it is easily verified that $\cup_\alpha B_\alpha$ is the class containing $(\cup_\alpha b_\alpha^*)^*$ and $(a(\cup_\alpha b_\alpha^*))^* = (\cup_\alpha ab_\alpha^*)^* = (\cup_\alpha (ab_\alpha^*)^*)^* = (\cup_\alpha (ab_\alpha)^*)^*$. Hence $G$ forms a $cl$-group. We state this in

*Theorem* 2.3. *If we classify $L$ by the quasi-equal relation, then the set $G$ of all classes forms a cl-group. $G$ is, therefore, commutative as a group and distributive as a lattice.*

*Corollary. Multiplication of $L$ is commutative in the sence of quasi-equality, i.e. $ab \sim ba$ for any $a$ and $b$ in $L$.*

Let $a$ be an integral element of $L$. We call $a \sim \Pi_{i=1}^r a_i$ a *factorization of $a$ in the sense of quasi-equality*, where $a_i$ is integral $(i = 1, \cdots, r)$.

By theorem 2.3 and Theorem 1.1 we have

*Theorem* 2.4. *Two factorizations of an integral element of $L$ in the sense of quasi-equality have the same refinement.*

*Definition. A prime element* is an integral element $p$ of $L$ such that $p$ is $\neq e$ and $ab \leq p$ implies $a \leq p$ or $b \leq p$ for integral elements $a$ and $b$ in $L$.

If $p$ is prime and not quasi-equal to $e$, then $p^* = p$. For, since $p \sim p^*$, there exist $u$ and $v$ such that $up = p^*v$, $u \sim v \sim e$. Hence $p^*v \leq p$. Assume $v \leq p$, then $p \sim e$, a contradiction. Hence $p^* \leq p$, $p^* = p$.

*Theorem* 2.5. *If we assume the ascending chain condition for integral elements of $L$ in the sense of quasi-equality, then any integral element in $L$ is quasi-equal to a product of finite prime elements of $L$, and this factorization is uniquely determined apart from its quasi-equality.*

*Theorem* 2.6. *Assume the following conditions for integral elements of $L$.*

1) *Ascending chain condition holds for integral elements of $L$.*

2) *Any prime element is maximal.*

3) *Any prime element contains such an element* $a^* = a$.

*Then quasi-equality implies equality. Hence* $L$ *forms a commutative cl-group, which is a direct product of infinite cyclic groups generated by prime elements.*

Proof. Any prime element $p$ in $L$ is not quasi-equal to $e$. For, if we take an element $a$ such that $a = a^* \leq p$, and decompose it as $a \sim p_1 \cdots p_n$, where $p_i$ is prime element not quasi-equal to $e$ $(i = 1, \cdots, n)$, then $p_1 \cdots p_n \leq a^* = a \leq p$. Hence there exists some $p_i$ such that $p_i \leq p$. Hence $p = p_i$. If $u \sim e$ then $u = e$. For, if we assume $u \neq e$ and take a maximal element $p$ such that $u \leq p < e$, then $p$ is a prime element and we get $p \sim e$. This is a contradiction. Finally, if $a \sim b$ then there exist $u$ and $v$ such that $au = vb$, $u \sim v \sim e$, Since $u = v = e$, we obtain $a = b$. Q.E.D.

Let $L$ be a *cl*-semigroup such that there exists a mapping of $L$ into itself $a \rightarrow \bar{a}$ with the following properties:

(1) $a \leq \bar{a}$,
(2) $\bar{\bar{a}} = \bar{a}$,
(3) $a \leq b$ implies $\bar{a} \leq \bar{b}$,
(4) $\bar{a}\bar{b} \leq \overline{ab}$.

We have the following: $\overline{\bar{a} \cap \bar{b}} = \bar{a} \cap \bar{b}$, $\overline{ab} = \overline{\bar{a} b} = \overline{a \bar{b}} = \overline{\bar{a} \bar{b}}$, $\overline{a \cup b} = \overline{\bar{a} \cup b} = \overline{a \cup \bar{b}} = \overline{\bar{a} \cup \bar{b}}$ and, more generally, if there exists $\cup_\alpha a_\alpha$ then $\overline{\cup_\alpha a_\alpha} = \overline{\cup_\alpha \bar{a}_\alpha}$.

*Definition.* An elements $a$ of $L$ is called *equivalent* to $b$ if $\bar{a} = \bar{b}$. Symbol: $a \sim b$.

It is evidedt that this relation fulfils the equivalence relation.

If $a \sim b$ and $c \sim d$, then $\overline{ac} = \overline{\bar{a} \bar{c}} = \overline{\bar{b} \bar{d}} = \overline{bd}$ and $\overline{a \cup c} = \overline{\bar{a} \cup \bar{c}} = \overline{\bar{b} \cup \bar{d}} = \overline{b \cup d}$. Hence $ac \sim bd$ and $a \cup c \sim b \cup d$. But $a \cap c \sim b \cap d$ does not hold in general.

If we classify $L$ by the equivalence relation, then the set $H$ of all classes $A, B, \cdots$ forms a partly ordered set when we define $A \leq B$ if $\bar{a} \leq \bar{b}$ $(a \in A,\ b \in B)$. Moreover $H$ forms a *cl*-semigroup with respect to this order.

We can easily verify that the classification mentioned above is characterized by the following properties:

1) Every class contains the greatest element.

2) Let $A, B$ be two classes. Then the class containing $ab$ and the class containing $a \cup b (a \in A,\ b \in B)$ are determined by $A$ and $B$ only not depending upon the choice of $a$ and $b$.

In the following we suppose that the unity quantity $e$ is maximally integral in $L$. Then $\bar{e} = e$, since $e \leq \bar{e}$, $(\bar{e})^2 \leq \overline{ee} = \bar{e}$.

*Theorem* 2.7. *If in* $L$ *the mapping* $a \rightarrow a^{-1}$ *is defined, then* $a \leq \bar{a} \leq a^*$

$=(a^{-1})^{-1}$, $(\bar{a})^{*}=a^{*}$. $\bar{a}=\bar{b}$ *implies* $a^{*}=b^{*}$. *If* $H$ *forms a group then* $\bar{a}=a^{*}$.

Proof. From $a\overline{a^{-1}}\leq\bar{a}\overline{a^{-1}}\leq\overline{a\,a^{-1}}\leq\bar{e}=e$, we have $\overline{a^{-1}}\leq a^{-1}$. Hence $\overline{a^{-1}}=a^{-1}$. Since $\bar{a}\overline{a^{-1}}\leq e$ we get $a\leq\bar{a}\leq(\overline{a^{-1}})^{-1}=a^{*}$, $(\bar{a})^{*}=a^{*}$. Now let $H$ form a group. Let $E$ be the unity quantity of $H$ and $A^{-1}$ the inverse of $A$ in $H: AA^{-1}=A^{-1}A=E$. Then the unity quantity $e$ of $L$ is contained in $E$. For if $e\in A$ then $A^{2}=A$, $A=E$. Let $a$ be an element of $A$ and $A'$ be $a$ class containing $a^{-1}$. If we take $x$ in $A^{-1}$, then $ax\in AA^{-1}=E$, $ax\leq e$. Hence $x\leq a^{-1}$, $ax\leq aa^{-1}\leq e$, $aa^{-1}\in E$, $AA'=E$, $A'=A^{-1}$. Therefore if $a\in A$ then $a^{-1}\in A^{-1}$, $a^{*}=(a^{-1})^{-1}\in A$, $a^{*}\leq\bar{a}$, hence $a^{*}=\bar{a}$.

*Theorem* 2.8. *Suppose that* $H$ *forms a group under multiplication. Then there exists a mapping of* $L$ *into itself* $a\rightarrow a^{-1}$ *with the following properties*:

(1) $aa^{-1}a\leq a$,

(2) $axa\leq a$ *implies* $x\leq a^{-1}$.

Proof. Let $a$ be any element in $A\in H$ and $t$ in $A^{-1}$, then $\bar{t}\in A^{-1}$, $a\bar{t}\leq e$. If we define $a^{-1}=\bar{t}$ then $aa^{-1}\leq e$. Hence we get $aa^{-1}a\leq a$. If $axa\leq a$ then $ax\leq e$ by maximality of $e$. And $x\leq\bar{x}=e\bar{x}=\overline{a^{-1}a}\,\bar{x}\leq\overline{a^{-1}ax}\leq\overline{a^{-1}e}=\overline{a^{-1}}$ $=\bar{\bar{t}}=\bar{t}=a^{-1}$; *i.e.* $axa\leq a$ implies $x\leq a^{-1}$.

*Theorem* 2.9. *Let in* $L$ *the mapping* $a\rightarrow a^{-1}$ *be defined. The classification of* $L$ *by quasi-equality is the only one in order that a set* $G'$ *of all classes by some partitions of* $L$, *such that the class* $E'$ *containing* $e$ *consists of integral elements of* $L$ *and* $a\leq x\leq e$ $(a\in E')$ *implies* $x\in E'$, *forms a group under multiplication.*

Proof. $E'$ is the unity quantity of the group $G'$. As in the proof of theorem 2.7, $aa^{-1}\in E'$, $a^{-1}a\in E'$ for any element $a$ of $L$. If $a^{*}=b^{*}$, i.e. $a^{-1}=b^{-1}$ then $A'=B'$, $a\in A'$, $b\in B'(A', B'\in G')$, because $c=ab^{-1}b=aa^{-1}b$, $c\in A'E'=A'$, $c\in E'B'=B'$. $G'$ is, therefore, obtained by some classification of $G$ modulo a subgroup $H$, and $G'$ is isomorphic to $G/H$. $E'$ is the set-sum of all classes $E, A, B, \cdots$ in $H$. If $E'\neq E$ then there exists an element $A\neq E$ of $H$, hence if $a\in A$ then $a<e$, $a^{-1}\neq e$, $a^{-1}>e$, but $a^{-1}\in A^{-1}\in H$, $a^{-1}<e$. This is a contradiction. We have $E'=E$.

## §3. Orders, ideals in a semigroup.

In this section we shall consider orders and ideals in a semigroup. Since the theory is similar to that of K. Asano, ([2]) we shall only state the main results omitting the proof.

Let $\mathfrak{o}$ be a semigroup, and $M$ the semigroup consisting of all $\lambda$ in $\mathfrak{o}$ such that $a\lambda = b\lambda$ implies $a = b$ and $\lambda a = \lambda b$ implies $a = b$. Let further $M'$ be a subsemigroup of $M$. A semigroup $S$ is called left quotient semigroup of $\mathfrak{o}$ by $M'$ when 1) $S$ contains $\mathfrak{o}$ and has a unity quantity 1, 2) any element $\alpha$ in $M'$ has an inverse $\alpha^{-1}$ in $S$: $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$, and 3) for any $x$ in $S$ there exists $\alpha$ in $M'$ such that $\alpha x$ is contained in $\mathfrak{o}$. Any element in $S$ is, therefore, expressible in the form $\alpha^{-1}a$ where $\alpha \in M'$ and $a \in \mathfrak{o}$.

In order that there exists a left quotient semigroup of $\mathfrak{o}$ by $M'$, it is necessary and sufficient that for any $a$ in $\mathfrak{o}$ and any $\alpha$ in $M'$ there exist $a'$ in $\mathfrak{o}$ and $\alpha'$ in $M'$ satisfying $a'\alpha = \alpha' a$ ([9]). And this quotient semigroup is uniquely determined by $\mathfrak{o}$ and $M'$ apart from its isomorphism. If $S$ is a left quotient semigroup of $\mathfrak{o}$ by $M'$, then for any $\alpha_i$ $(i = 1, \cdots, n)$ in $M'$ there exists $c_i$ $(i = 1, \cdots, n)$ in $\mathfrak{o}$ such that $\gamma = c_1\alpha_1 = \cdots = c_n\alpha_n \in M'$, therefore for any finite set of elements $x_1, \cdots, x_n$ in $S$ we can take an element $\gamma$ in $M'$ such that $\gamma x_i \in \mathfrak{o}$ $(i = 1, \cdots, n)$. If $M' = M$ we call $S$ a left quotient semigroup of $\mathfrak{o}$. We can analogously consider a right quotient semigroup of $\mathfrak{o}$ by $M'$. If $S$ is a left and a right quotient semigroup of $\mathfrak{o}$, then $S$ is called a quotient semigroup of $\mathfrak{o}$.

Let $S$ be a given semigroup with unity quantity 1. An element of $S$ is called *regular* if it has a left and a right inverse. The subset $S^*$ consisting of all regular elements of $S$ forms a group under multiplication.

*Definition.* A subset $\mathfrak{o}$ of $S$ is called an *order* of $S$ when

1) $\mathfrak{o}$ forms a subsemigroup with 1,
2) $S$ is a quotient semigroup of $\mathfrak{o}$ by $S^* \cap \mathfrak{o}$.

Let $\mathfrak{o}$ be an order of $S$ and $\mathfrak{o}'$ a subsemigroup of $S$ with 1. If there exist two regular elements $\lambda, \mu$ such that $\lambda\mathfrak{o}\mu \subseteq \mathfrak{o}'$, then $\mathfrak{o}'$ is also an order of $S$.

*Definition.* Let $\mathfrak{o}$ be an order of S. A subset $A$ of $S$ is called a *left* (*right*) $\mathfrak{o}$*-set* when $\mathfrak{o}A \subseteq A (A\mathfrak{o} \subseteq A)$. A left and a right $\mathfrak{o}$-set is called a *two-sided* $\mathfrak{o}$*-set* or in short $\mathfrak{o}$*-set* of $S$. A left (right) $\mathfrak{o}$-set $\mathfrak{a}$ is called a *left* (*right*) $\mathfrak{o}$*-ideal* of S, if $\mathfrak{a}$ contains a regular Element of S and there exists a regular element $\lambda$ such that $\mathfrak{a}\lambda \subseteq \mathfrak{o}$ $(\lambda\mathfrak{a} \subseteq \mathfrak{o})$. $\mathfrak{a}$ is called an $\mathfrak{o}$-$\mathfrak{o}'$*-ideal* if it forms a left $\mathfrak{o}$-ideal and a right $\mathfrak{o}'$-ideal. An $\mathfrak{o}$-$\mathfrak{o}$-ideal is called a *two-sided* $\mathfrak{o}$*-ideal* or in short $\mathfrak{o}$*-ideal.*

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two left (right) $\mathfrak{o}$-ideals of $S$. Then $\mathfrak{a} \cup \mathfrak{b}$ (set-sum) and $\mathfrak{a} \cap \mathfrak{b}$ (intersection) are also left (right) $\mathfrak{o}$-ideals.

Let $\mathfrak{a}$ be an $\mathfrak{o}$-$\mathfrak{o}'$-ideal and $\mathfrak{b}$ an $\mathfrak{o}'$-$\mathfrak{o}''$-ideal, then $\mathfrak{a}\mathfrak{b} = \{ab \mid a \in \mathfrak{a},\ b \in \mathfrak{b}\}$ is an $\mathfrak{o}$-$\mathfrak{o}''$-ideal. Particularly a product of two $\mathfrak{o}$-ideals is also an $\mathfrak{o}$-ideal.

*Definition.* Two sub-set $M$ and $N$ of $S$ are called *equivalent* if there exist regular elements $\lambda, \mu, \lambda', \mu'$ snch that $\lambda N \mu \subseteq M$ and $\lambda' M \mu' \subseteq N$.

Two orders $\mathfrak{o}$ and $\mathfrak{o}'$ are called *equivalent*, when they are equivalent as subsets of $S$. In this case, $\lambda, \mu$ satisfying $\lambda\mathfrak{o}'\mu \subseteq \mathfrak{o}$ may be taken as elements of $\mathfrak{o}$, and similarly $\lambda', \mu'$ satisfying $\lambda'\mathfrak{o}\mu' \subseteq \mathfrak{o}'$ as elements of $\mathfrak{o}'$. In order that two orders $\mathfrak{o}$ and $\mathfrak{o}'$ are equivalent it is necessary and sufficient that there exists an $\mathfrak{o}$-$\mathfrak{o}'$-ideal of $S$.

If $\mathfrak{a}$ is a left (right) $\mathfrak{o}$-ideal of $S$, then the set $\mathfrak{o}_l$ consisting of all $x$ such that $x\mathfrak{a} \subseteq \mathfrak{a}$, $x \in S$, forms an order of $S$ and is equivalent to $\mathfrak{o}$. The set $\mathfrak{o}_r$ consisting of all $y$ such that $\mathfrak{a}y \subseteq \mathfrak{a}$, $y \in S$, forms also an order equivalent to $\mathfrak{o}$. $\mathfrak{o}_l(\mathfrak{o}_r)$ is, moreover, a left (right) $\mathfrak{o}$-ideal containing $\mathfrak{o}$. And $\mathfrak{a}$ is an $\mathfrak{o}_l$-$\mathfrak{o}_r$-ideal of $S$.

*Definition.* $\mathfrak{o}_l$ and $\mathfrak{o}_r$ are called a *left order* and a *right order* of $\mathfrak{a}$ respectively.

*Definition.* An one-sided $\mathfrak{o}$-ideal $\mathfrak{a}$ is called *integral* if it forms a semigroup, i. e. $\mathfrak{a}^2 \subseteq \mathfrak{a}$.

It is easily verified that the following conditions are equivalent:

(1) $\mathfrak{a} \subseteq \mathfrak{o}_l$, (2) $\mathfrak{a} \subseteq \mathfrak{o}_r$, (3) $\mathfrak{a}^2 \subseteq \mathfrak{a}$.

*Definition.* An order $\mathfrak{o}$ of $S$ is called *maximal* when there exists no order which is equivalent to $\mathfrak{o}$ and contains $\mathfrak{o}$ properly.

Let $\mathfrak{o}$ be an order of $S$. Then the following conditions on $\mathfrak{o}$ are equivalent.

1) $\mathfrak{o}$ is a maximal order (in the set of all orders equivalent to $\mathfrak{o}$).

2) There exists no integral left and no integral right $\mathfrak{o}$-ideal containing $\mathfrak{o}$.

3) $\mathfrak{o}$ is a left order of any left $\mathfrak{o}$-ideal, and also a right order of any right $\mathfrak{o}$-ideal.

4) $\mathfrak{o}$ is a left and a right order of any two-sided $\mathfrak{o}$-ideal.

Let $\mathfrak{o}$ be a maximal order of $S$. A left or a right $\mathfrak{o}$-ideal is integral if and only if it is contained in $\mathfrak{o}$. A left (right) $\mathfrak{o}$-set equivalent to $\mathfrak{o}$ is a left (right) $\mathfrak{o}$-ideal of $S$.

If $\mathfrak{o}$ is an order of $S$, and if $\mathfrak{a}$ is a left or a right $\mathfrak{o}$-ideal and $\mathfrak{o}_l$, $\mathfrak{o}_r$ the left, right orders of $\mathfrak{a}$ respectively, then we have $\mathfrak{a}^{-1} = \}c\,|\,\mathfrak{a}c \subseteq \mathfrak{o}_l,\ c \in S\}$ $= \{c\,|\,\mathfrak{a}c\mathfrak{a} \subseteq \mathfrak{a},\ c \in S\} = \{c\,|\,c\mathfrak{a} \subseteq \mathfrak{o}_r,\ c \in S\}$. $\mathfrak{a}^{-1}$ forms an $\mathfrak{o}_r$-$\mathfrak{o}_l$-ideal of $S$.

*Definition.* $\mathfrak{a}^{-1}$ is called the *inverse ideal* of $\mathfrak{a}$.

If $\mathfrak{a}$ and $\mathfrak{b}$ have the same left or right order, then $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\mathfrak{a}^{-1} \supseteq \mathfrak{b}^{-1}$.

Let $\mathfrak{o}$ be a maximal order and $\mathfrak{a}$ a left $\mathfrak{o}$-ideal. Then the left order of $\mathfrak{a}^{-1}$ is maximal. If $\mathfrak{m}$ is a subsemigroup of $S$ such that $\lambda\mathfrak{m}\mu \subseteq \mathfrak{o}$ $(\lambda, \mu \in S^* \frown \mathfrak{o})$, then there exists a maximal order which contains $\mathfrak{m}$ and is equivalent to $\mathfrak{o}$, namely the left order $\mathfrak{o}'$ of the inverse ideal of a left $\mathfrak{o}$-ideal $\mathfrak{a} = \mathfrak{o}\lambda \smile \mathfrak{o}\lambda\mathfrak{m}$. For any order $\mathfrak{o}'$ equivalent to $\mathfrak{o}$, there exists therefore a maximal order which contains $\mathfrak{o}'$ and is equivalent to $\mathfrak{o}$.

*Defintiton.* An order $\mathfrak{o}$ of $S$ is called *regular* when for any $x$ in $S$ there exist two regular elements $\alpha$ and $\beta$ in $\mathfrak{o}$ such that $x\mathfrak{o}\alpha \subseteq \mathfrak{o}$ and $\beta\mathfrak{o}x \subseteq \mathfrak{o}$.

Let $\mathfrak{o}$ be an order of $S$. Then the following conditions are equivalent.

1. $\mathfrak{o}$ is regular.
2. For any $x$ in $S$ there exists a two-sided $\mathfrak{o}$-ideal which contains $x$.
3. For any $\mu$ in $S^*$, $\mathfrak{o}\mu\mathfrak{o}$ forms a two-sided $\mathfrak{o}$-ideal.
4. If $M$ is a subset of $S$ such that $\lambda M\mu \subseteq \mathfrak{o}$ $(\lambda, \mu \in S^*)$, then there exist regular elements $\alpha, \beta$ in $\mathfrak{o}$ such that $\alpha M \subseteq \mathfrak{o}$, $M\beta \subseteq \mathfrak{o}$.
5. For any regular element $\alpha$ in $\mathfrak{o}$ there exist regular elements $\alpha', \alpha''$ in $\mathfrak{o}$ such that $\mathfrak{o}\alpha \supseteq \alpha'\mathfrak{o}$, $\alpha\mathfrak{o} \supseteq \mathfrak{o}\alpha''$.
6. Any one-sided $\mathfrak{o}$-ideal contains a two-sided $\mathfrak{o}$-ideal.

If a subsemigroup $\mathfrak{o}'$ containing unity quantity 1 is equivalent to a regular order $\mathfrak{o}$ of $S$, then $\mathfrak{o}'$ is a regular order of $S$.

The intersection of two equivalent regular orders $\mathfrak{o}, \mathfrak{o}'$ is also a regular order equivalent to $\mathfrak{o}$ and $\mathfrak{o}'$.

A regular order of $S$ is a maximal order, if and only if there exists no integral two-sided $\mathfrak{o}$-ideal containing $\mathfrak{o}$ properly.

## §4. Two-sided $\mathfrak{o}$-ideals.

In this section we shall denote by $\mathfrak{o}$ a fixed maximal order of a semigroup $S$. A two-sided $\mathfrak{o}$-ideal will be called briefly an "ideal".

Let $L$ be a set of all ideals $\mathfrak{o}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots$ in $S$. If a subset $X$ of $L$ be bounded, i.e. $\mathfrak{a}_\nu \subseteq \mathfrak{c}$ for any $\mathfrak{a}_\nu$ in $X$, then it is easily verified that the set-union $\cup_\nu \mathfrak{a}_\nu$ of all $\mathfrak{a}_\nu$ in $X$ forms an ideal and $\sup X = \cup_\nu \mathfrak{a}_\nu$, $\mathfrak{a}(\sup X) = \sup(\mathfrak{a}X)$, $(\sup X)\mathfrak{a} = \sup(X\mathfrak{a})$. Hence $L$ forma a $cl$-semigroup under multiplication and inclusion relation. A mapping of $L$ into itself $\mathfrak{a} \to \mathfrak{a}^{-1}$ has the following properties 1. $\mathfrak{a}\mathfrak{a}^{-1}\mathfrak{a} \subseteq \mathfrak{a}$ and 2. $\mathfrak{a}\mathfrak{x}\mathfrak{a} \subseteq \mathfrak{a}$ implies $\mathfrak{x} \subseteq \mathfrak{a}^{-1}$. We may, therefore, apply the results of §2.

*Theorem* 4.1. *L forms a residuated lattice.*

If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of $S$, then $(\mathfrak{b}:\mathfrak{a})_l$ $((\mathfrak{b}:\mathfrak{a})_r)$ is nothing but the ideal consisting of all $c \in S$ such that $\mathfrak{a}c \subseteq \mathfrak{b}$ $(c\mathfrak{a} \subseteq \mathfrak{b})$.

If we define $\mathfrak{a}^* = (\mathfrak{a}^{-1})^{-1}$, then

$$\mathfrak{a} \subseteq \mathfrak{a}^*$$
$$\mathfrak{a}^{**} = \mathfrak{a}^*$$
$$\mathfrak{a} \subseteq \mathfrak{b} \text{ implies } \mathfrak{a}^* \subseteq \mathfrak{b}^*,$$
$$\mathfrak{a}^*\mathfrak{b}^* \subseteq (\mathfrak{a}\mathfrak{b})^*.$$

*Definition.* Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are called *quasi-equal* if they are quasi-equal as elements of *cl*-semigroup $L$, i. e. $\mathfrak{a}^* = \mathfrak{b}^*$ or $\mathfrak{a}^{-1} = \mathfrak{b}^{-1}$.

*Definition.* An ideal $\mathfrak{a}$ is called *v-ideal* if $\mathfrak{a} = \mathfrak{a}^*$.

For example $\mathfrak{a}^{-1}$ and $\mathfrak{a}^*$ are $v$-ideal for any ideal $\mathfrak{a}$. $\mathfrak{o}$ is evidently $v$-ideal. Also a prime ideal $\mathfrak{p}$ not equivalent to $\mathfrak{o}$ is a $v$-ideal.

We have immediately the following: $\mathfrak{a} \sim \mathfrak{a}^*$; $\mathfrak{a}\mathfrak{a}^{-1} \sim \mathfrak{a}^{-1}\mathfrak{a} \sim \mathfrak{o}$; if $\mathfrak{a} \sim \mathfrak{b}$ and $\mathfrak{a}$ is integral, then $\mathfrak{b}$ is also integral; if $\mathfrak{a} \sim \mathfrak{b}$ and $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \mathfrak{b}$, then $\mathfrak{a} \sim \mathfrak{c}$; if $\mathfrak{a} \subseteq \mathfrak{b}^*$ then $\mathfrak{a} \sim \mathfrak{c}\mathfrak{b} \sim \mathfrak{b}\mathfrak{d}$, where $\mathfrak{c}$ and $\mathfrak{d}$ are integral; if $\mathfrak{a} \sim \mathfrak{b}$ then $\mathfrak{a}\mathfrak{u} = \mathfrak{v}\mathfrak{b}$, $\mathfrak{u} \sim \mathfrak{v} \sim \mathfrak{o}$; if $\mathfrak{a} \sim \mathfrak{b}$ and $\mathfrak{c} \sim \mathfrak{d}$ then $\mathfrak{a}\mathfrak{c} \sim \mathfrak{b}\mathfrak{d}$, $\mathfrak{a} \cup \mathfrak{c} \sim \mathfrak{b} \cup \mathfrak{d}$ and $\mathfrak{a} \cap \mathfrak{c} \sim \mathfrak{b} \cap \mathfrak{d}$.

If we classify the set of all two-sided $\mathfrak{o}$-ideals, then the set $G$ of all classes forms a *cl*-group. This group is, therefore, abelian as a group and distributive as a lattice.

*Theorem* 4. 2. *The set $G'$ of all v-ideals $\mathfrak{a}, \mathfrak{b}, \cdots$ forms a cl-group under multiplication: $(\mathfrak{a}\mathfrak{b})^*$, join: $(\mathfrak{a} \cup \mathfrak{b})^*$ and meet: $\mathfrak{a} \cap \mathfrak{b}$. And $G'$ is isomorphic to $G$ as a cl-group. Hence $G'$ is abelian as a group and distributive as a lattice. Moreover if we assume the ascending chain condition for integral v-ideals, then $G'$ is a direct product of infinite cyclic groups with prime v-ideals as their generators.*

The multiplication of ideals is therefore commutative in the sense of quasi-equality and we get the Artin's Refinement theorem.

*Theorem* 4. 3. *If $L$ forms a group under multiplication, then it is a cyclic group generated by a maximal integral ideal $\mathfrak{p}$.*

Proof. Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$ be any ascending chain of integral ideals $\mathfrak{a}_i$, $i = 1, 2, \cdots$. And let further $\mathfrak{a}$ be an integral ideal of set-sum of all $\mathfrak{a}_i$. Then we get $\mathfrak{a}_1\mathfrak{a}^{-1} \subseteq \mathfrak{a}_2\mathfrak{a}^{-1} \subseteq \cdots \subseteq \mathfrak{a}\mathfrak{a}^{-1} = \mathfrak{o}$. Since $\mathfrak{a}\mathfrak{a}^{-1}$ is the set-sum of all $\mathfrak{a}_i\mathfrak{a}^{-1}$, $i = 1, 2, \cdots$, there exists a finite number $n$ such that $\mathfrak{a}_n\mathfrak{a}^{-1} \ni 1$, hence $\mathfrak{a}_n\mathfrak{a}^{-1} = \mathfrak{o}$, $\mathfrak{a} = \mathfrak{a}_n$, $\mathfrak{a} = \mathfrak{a}_{n+i}$, $i = 1, 2, \cdots$. Let $\mathfrak{p} \neq \mathfrak{o}$ be a maximal integral ideal. Then $\mathfrak{p}$ is not quasi-equal to $\mathfrak{o}$. For if not, then $\mathfrak{p} \sim \mathfrak{o}$, $\mathfrak{p}^{-1} = \mathfrak{o}^{-1} = \mathfrak{o}$, $\mathfrak{o} = \mathfrak{p}\mathfrak{p}^{-1} = \mathfrak{p}\mathfrak{o} = \mathfrak{p}$, a contradiction. Let $\mathfrak{a}$ be any integral ideal in $L$. Then $\mathfrak{a} \cup \mathfrak{p} = \mathfrak{o}$ or $= \mathfrak{p}$. If $\mathfrak{a} \cup \mathfrak{p} = \mathfrak{o}$ then $\mathfrak{a} \ni 1$, $\mathfrak{a} = \mathfrak{o}$. If $\mathfrak{a} \cup \mathfrak{p} = \mathfrak{p}$ then $\mathfrak{a} \subseteq \mathfrak{p}$, i. e. $\mathfrak{p}$ is a divisor of any integral ideal in $L$. Hence $\mathfrak{p}$ is one and only one maximal (of course prime) ideal in $L$. If $\mathfrak{a}$ is $\neq \mathfrak{o}$ and integral then $\mathfrak{a} = \mathfrak{p}\mathfrak{a}_1$. If $\mathfrak{a}_1$ is $\neq \mathfrak{o}$ then $\mathfrak{a}_1 = \mathfrak{p}\mathfrak{a}_2$. Thus we get a chain $\mathfrak{a} \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$. Hence there exists a positive integer such that $\mathfrak{a} = \mathfrak{p}^n$. Any ideal in $L$ is represented as a quotient of two integral ideals. Q. E. D.

Let us suppose that there exists a mapping of $L$ into itself $\mathfrak{a} \rightarrow \bar{\mathfrak{a}}$ with the following properties:

1) $\mathfrak{a} \subseteq \bar{\mathfrak{a}}$,

2) $\bar{\bar{\mathfrak{a}}} = \bar{\mathfrak{a}}$,

3) $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\bar{\mathfrak{a}} \subseteq \bar{\mathfrak{b}}$,

4) $\bar{\mathfrak{a}}\bar{\mathfrak{b}} \subseteq \overline{\mathfrak{a}\mathfrak{b}}$.

For example the mapping $\mathfrak{a} \rightarrow \mathfrak{a}^* = (\mathfrak{a}^{-1})^{-1}$ has such properties. By Theorem 2.7 $\bar{\mathfrak{a}} \subseteq \mathfrak{a}^*$.

We get the following: $\overline{\mathfrak{a}_\cap \bar{\mathfrak{b}}} = \bar{\mathfrak{a}}_\cap \bar{\mathfrak{b}}$, $\overline{\mathfrak{a}\mathfrak{b}} = \overline{\bar{\mathfrak{a}}\mathfrak{b}} = \overline{\mathfrak{a}\bar{\mathfrak{b}}} = \overline{\bar{\mathfrak{a}}\,\bar{\mathfrak{b}}}$, $\overline{\mathfrak{a}^\cup \mathfrak{b}} = \overline{\bar{\mathfrak{a}}^\cup \mathfrak{b}} = \overline{\mathfrak{a}^\cup \bar{\mathfrak{b}}}$ $= \overline{\bar{\mathfrak{a}}^\cup \bar{\mathfrak{b}}}$ and more generally if there exists $\bigcup_\alpha \mathfrak{a}_\alpha$ then $\overline{\bigcup_\alpha \mathfrak{a}_\alpha} = \overline{\bigcup_\alpha \bar{\mathfrak{a}}_\alpha}$.

*Definition.* An ideal $\mathfrak{a}$ is called a *closed ideal* or briefly a *c-ideal* if $\bar{\mathfrak{a}} = \mathfrak{a}$.

*Definition.* An ideal $\mathfrak{a}$ is called *equivalent* to $\mathfrak{b}$ if $\bar{\mathfrak{a}} = \bar{\mathfrak{b}}$. Symbol: $\mathfrak{a} \sim \mathfrak{b}$.

If $\mathfrak{a} \sim \mathfrak{b}$ and $\mathfrak{c} \sim \mathfrak{d}$ then we get $\mathfrak{a}\mathfrak{c} \sim \mathfrak{b}\mathfrak{d}$ and $\mathfrak{a}^\cup \mathfrak{c} \sim \mathfrak{b}^\cup \mathfrak{d}$. But $\mathfrak{a}_\cap \mathfrak{c} \sim \mathfrak{b}_\cap \mathfrak{d}$ does not hold in general.

If we classify $L$ by the equivalence relation, then the set $H$ of all closses $A, B, \cdots$ forms a partly ordered set when we define $A \leq B$ by $\bar{\mathfrak{a}} \subseteq \bar{\mathfrak{b}}$ where $\mathfrak{a} \in A$ and $\mathfrak{b} \in B$. Moreover $H$ forms a $cl$-semigroup with respect to this order. The set $H'$ of all $c$-ideals $\mathfrak{a}, \mathfrak{b}, \cdots$ forms a $cl$-semigroup under multiplication $\overline{\mathfrak{a}\mathfrak{b}}$, join $\overline{\mathfrak{a}^\cup \mathfrak{b}}$ and meet $\mathfrak{a}_\cap \mathfrak{b}$. And $H'$ is isomorphic to $H$ as a $cl$-semigroup. If $H(H')$ forms a group then by Theorem 2.7 $\bar{\mathfrak{a}} = \mathfrak{a}^*$. If we classify $H'$ by quasi-equal relation, then the set $H''$ of all classes forms a $cl$-group isomorphic to $G$.

*Theorem* 4.4. *Let us assume the following conditions*:

1) *The ascending chain condition holds for integral c-ideals.*

2) *Any prime c-ideal is maximal.*

3) *Any prime c-ideal contains a v-ideal.*

*Then quasi-equality implies equality. Hence the c-ideals forms a cl-group, hence* $\bar{\mathfrak{a}} = \mathfrak{a}^*$. *Moreover if* $\mathfrak{o}$ *is regular, then the condition 3 is always satisfied.*

For any integral ideal $\mathfrak{a}$ and any regular element $a$ in $\mathfrak{a}$, there exists an ideal $\mathfrak{c}$ such that $\mathfrak{o}a \supseteq \mathfrak{c}$, hence $(\mathfrak{o}a)^{-1} = a^{-1}\mathfrak{o} \subseteq \mathfrak{c}^{-1}$, $\mathfrak{o}a = (a^{-1}\mathfrak{o})^{-1} \supseteq \mathfrak{c}^*$, and we get $\mathfrak{a} \supseteq \mathfrak{c}^*$.

We get easily:

*Theorem* 4.5. *Let* $\mathfrak{o}$ *be a regular order. In order that the set of all closed two-sided* $\mathfrak{o}$*-ideals forms an abelian gronp which is a direct product of infinite cyclic gronps, it is necessary and sufficient that the following conditions hold for* $\mathfrak{o}$.

$A_1$: $\mathfrak{o}$ *is maximal.*

$A_2$: *Ascending chain condition holds for integral c-ideals.*

$A_3$: *A prime c-ideal is a maximal two-sided c-ideal.*

## §5. Closed $\mathfrak{o}$--semigroups

Let S be a semigroup with unity quantity 1, and $\mathfrak{o}$ a maximal order of S.

An "ideal" means a "two-sided $\mathfrak{o}$-ideals" in $S$. In this section we assume that the set $L$ of all ideals has a mapping into itself $\mathfrak{a}\rightarrow\mathfrak{a}'$ with following conditions:

1) $\mathfrak{a}\subseteq\mathfrak{a}'$ 2) $\mathfrak{a}''=\mathfrak{a}'$ 3) $\mathfrak{a}\subseteq\mathfrak{b}$ implies $\mathfrak{a}'\subseteq\mathfrak{b}'$ 4) $\mathfrak{a}'\mathfrak{b}'\subseteq(\mathfrak{a}\mathfrak{b})'$.

An ideal $\mathfrak{a}$ is called a *closed ideal* or in short *c-ideal* if $\mathfrak{a}'=\mathfrak{a}$.

In the following we shall assume

$A_1$: $\mathfrak{o}$ is a regular maximal order.

$A_2$: Ascending chain condition holds for integral $c$-ideals.

$A_3$: A prime $c$-ideal is maximal.

Therefore the $c$-ideals form an abelian group under the multiplication $\mathfrak{a}\cdot\mathfrak{b}=(\mathfrak{a}\mathfrak{b})'$ and the $c$-ideals coincide with the $v$-ideals: $\mathfrak{a}'=\mathfrak{a}^*=(\mathfrak{a}^{-1})^{-1}$.

*Lemma* 5.1. *Ascending chain condition holds for c-ideals which are contained in a fixed c-iideals* $\mathfrak{c}$.

Proof. Let $N$ be any set of $c$-ideals contained in $\mathfrak{c}$. If $\mathfrak{a}\in N$ then $\mathfrak{c}^{-1}\cdot\mathfrak{a}\subseteq\mathfrak{o}$, hence of all $\mathfrak{c}^{-1}\cdot\mathfrak{a}$ there exists a maximal $c$-ideal $\mathfrak{b}$. Obviously $\mathfrak{a}_0=\mathfrak{c}\cdot\mathfrak{b}$ is a maximal ideal in $N$.

*Lemma* 5.2. *If* $\mathfrak{a}$ *is an ideal then there exist finite elements* $c_1,\cdots,c_n$ *in* $\mathfrak{a}$ *such that* $\mathfrak{a}'$ *is a c-ideal generated by* $c_1,\cdots,c_n$:

$$\mathfrak{a}'=(c_1,\cdots,c_n)=(\mathfrak{o}c_1\mathfrak{o}\cup\cdots\cup\mathfrak{o}c_n\mathfrak{o})^*.$$

Proof. Let $c_1,\cdots,c_m$ be $m$ elements in $\mathfrak{a}$ and $c_1$ be regular. If there exists an element $c_{m+1}\in\mathfrak{a}$ and $c_{m+1}\notin(c_1,\cdots,c_m)$, then we make $(c_1,\cdots,c_m,c_{m+1})$. Since the chain $(c_1)\subset(c_1,c_2)\subset\cdots\subseteq\mathfrak{a}'$ does not continue infinitely, there exists a number $n$ such that $(c_1,\cdots,c_n)\supseteq\mathfrak{a}$. Hence $(c_1,\cdots,c_n)\supseteq\mathfrak{a}'$ and $\mathfrak{a}'=(c_1,\ldots,c_n)$.

*Definition.* Let $A$ be an $\mathfrak{o}$-set of $S$ containg regular elements of $S$. The set-sum $\bar{A}$ of all $c$-ideals genesated by finite elements in $A$ is called the *closure* of $A$, and $A$ is called *closed* if $\bar{A}=A$.

If $\mathfrak{a}$ is a $c$-ideal, then by lemma 5.2 we get $\bar{\mathfrak{a}}=\mathfrak{a}'$. A $c$-ideal is closed in the sense of this definition.

The closure of an $\mathfrak{o}$-set has the following properties:

1. $A\subseteq\bar{A}$,
2. $\bar{\bar{A}}=\bar{A}$,
3. $A\subseteq B$ implies $\bar{A}\subseteq\bar{B}$.
4. $\bar{A}\,\bar{B}\subseteq\overline{AB}$

By Definition, 1 and 3 are evident. Let $a$ be an element in $\bar{\bar{A}}$, then there exists a $c$-ideal $(a_1,\cdots,a_r)$ containing $a$, generated by $a_i$ in $\bar{A}$, $i=1,\cdots,r$. There exists a $c$-ideal $(b_{i1},\cdots,b_{is_i})$ containing $a_i$, where $b_{ij}\in A$, $j=1,\cdots,s_i$;

$i=1,\cdots,r$. Let $b_1,\cdots,\cdots,b_n$ be the set of all $b_{ij}$. Then $a_i\in(b_1,\cdots,b_n)$, $i=1,\cdots,r$, and $(b_1,\cdots,b_n)\subseteq\bar{A}$, $a\in\bar{A}$. Hence we get $\bar{\bar{A}}\subseteq\bar{A}$, $\bar{\bar{A}}=\bar{A}$.

Let $A$ and $B$ be two $\mathfrak{o}$-sets of $S$. If $a\in\bar{A}$, $b\in\bar{B}$, then there exist $c$-ideals $(a_1,\cdots,a_r)$, $(b_1,\cdots,b_s)$ such that $a\in(a_1,\cdots,a_r)$, $b\in(b_1,\cdots,b_s)$, $a_i\in A$, $b_k\in B$. Hence $ab\in(a_1,\cdots,a_r)\cdot(b_1,\cdots,b_s)=\{(\bigcup_i\mathfrak{o}a_i\mathfrak{o})(\bigcup_k\mathfrak{o}b_k\mathfrak{o})\}^*=(c_1,\cdots,c_t)$ where $c_j\in(\bigcup_i\mathfrak{o}a_i\mathfrak{o})(\bigcup_k\mathfrak{o}b_k\mathfrak{o})\subseteq AB$, $j=1,\cdots,t$. Hence $ab\in\overline{AB}$, i.e. $\bar{A}\,\bar{B}\subseteq\overline{AB}$.

*Lemma* 5.3. *Let $A$ and $B$ be $\mathfrak{o}$-sets and $M$ be a subset of $S$. If $AM\subseteq B$ and $A\lambda\subseteq B$ for a regular element $\lambda$, then $\bar{A}M\subseteq\bar{B}$. Particularly if $\mathfrak{a},\mathfrak{b}$ are ideals then $\mathfrak{a}M\subseteq\mathfrak{b}$ implies $\bar{\mathfrak{a}}M\subseteq\bar{\mathfrak{b}}$.*

Proof. $\bar{A}M\subseteq\bar{A}(\overline{\mathfrak{o}M\mathfrak{o}\cup\mathfrak{o}\lambda\mathfrak{o}})\subseteq\overline{A(\mathfrak{o}M\mathfrak{o}\cup\mathfrak{o}\lambda\mathfrak{o})}\subseteq\bar{B}$.

*Lemma* 5.4. *Let $\mathfrak{a},\mathfrak{b}$ and $\mathfrak{c}$ be $c$-ideals. If $\mathfrak{a}M\subseteq\mathfrak{b}$ then $(\mathfrak{c}\cdot\mathfrak{a})M\subseteq\mathfrak{c}\cdot\mathfrak{b}$. If, particularly, $\mathfrak{a}c\subseteq\mathfrak{b}$ then $c\in\mathfrak{a}^{-1}\cdot\mathfrak{b}$.*

Proof. Since $\mathfrak{c}\mathfrak{a}M\subseteq\mathfrak{c}\mathfrak{b}$, we get $\overline{\mathfrak{c}\mathfrak{a}}M\subseteq\overline{\mathfrak{c}\mathfrak{b}}$, i.e. $(\mathfrak{c}\cdot\mathfrak{a})M\subseteq\mathfrak{c}\cdot\mathfrak{b}$.

Let $P$ be a set of prime $c$-ideals. For a $c$-ideal $\mathfrak{a}$, let $\mathfrak{a}_P$ be the set of all $S$-elements $c$ such that $\mathfrak{n}_c c\subseteq\mathfrak{a}$, where $\mathfrak{n}_c$ is a suitable integral $c$-ideal coprime to $P$, i.e. coprime to all $\mathfrak{p}$ in $P$. $\mathfrak{n}c\subseteq\mathfrak{a}$ implies $c\mathfrak{n}\subseteq\mathfrak{a}$ and conversely. In fact, from $\mathfrak{n}c\subseteq\mathfrak{a}$ we get $c\in\mathfrak{n}^{-1}\cdot\mathfrak{a}=\mathfrak{a}\cdot\mathfrak{n}^{-1}$, $c\mathfrak{n}\subseteq\mathfrak{a}$. It is easy to see that $\mathfrak{a}_P$ is the set-union of all $\mathfrak{n}^{-1}\cdot\mathfrak{a}$ with $\mathfrak{n}$ coprime to $P$.

*Definition.* $\mathfrak{a}_P$ is called the *$P$-component* of $\mathfrak{a}$. If $P$ consists of a single prime ideal $\mathfrak{p}$, then we denote $\mathfrak{a}_P$ by $\mathfrak{a}_\mathfrak{p}$.

*Lemma* 5.5. *For any $P$ $\mathfrak{o}_P$ forms an order containing $\mathfrak{o}$.*

Proof. If $c,c'\in\mathfrak{o}_P$, i.e. $\mathfrak{n}c\subseteq\mathfrak{o}$, $\mathfrak{n}'c'\subseteq\mathfrak{o}$, then $\mathfrak{n}'\mathfrak{n}cc'\subseteq\mathfrak{n}'\mathfrak{o}c'=\mathfrak{n}'c'\subseteq\mathfrak{o}$. Hence by Lemma 5.4 $(\mathfrak{n}'\cdot\mathfrak{n})(cc')\subseteq\mathfrak{o}$. Since $\mathfrak{n}'\cdot\mathfrak{n}$ is obviously coprime to $P, cc'\in\mathfrak{o}_P$.

*Lemma* 5.6. *If $\mathfrak{a}$ and $\mathfrak{b}$ are $c$-ideals such that $\mathfrak{a}x\subseteq\mathfrak{b}(x\in S)$ then $\mathfrak{a}_P x\subseteq\mathfrak{b}_P$..*

Proof. Since $\mathfrak{a}x\subseteq\mathfrak{b}$, $(\mathfrak{n}^{-1}\cdot\mathfrak{a})x\subseteq\mathfrak{n}^{-1}\cdot\mathfrak{b}$, $(\bigcup\mathfrak{n}^{-1}\cdot\mathfrak{a})x=\bigcup(\mathfrak{n}^{-1}\cdot\mathfrak{a})x\subseteq\bigcup\mathfrak{n}^{-1}\cdot\mathfrak{b}$, hence $\mathfrak{a}_P x\subseteq\mathfrak{b}_P$.

*Lemma* 5.7. *$\mathfrak{a}_P$ is an $\mathfrak{o}_P$-ideal.*

Proof. If $a\in\mathfrak{a}_P$ and $c\in\mathfrak{o}_P$, i.e. there exist $\mathfrak{n}$ and $\mathfrak{n}'$ such that $\mathfrak{n}a\subseteq\mathfrak{a}$ and $\mathfrak{n}'c\subseteq\mathfrak{o}$, then $\mathfrak{n}\mathfrak{n}'ca\subseteq\mathfrak{a}$; hence $(\mathfrak{n}\cdot\mathfrak{n}')(ca)\subseteq\mathfrak{a}$, $ca\in a_P$. Similarly $ac\in\mathfrak{a}_P$. Since there exists a regular element $\lambda$ such that $\mathfrak{a}\lambda\subseteq\mathfrak{o}$, we get $\mathfrak{a}_P\lambda\subseteq\mathfrak{o}_P$; and similarly there exists a regular element $\mu$ such that $\mu\mathfrak{a}_P\subseteq\mathfrak{o}_P$.

*Lemma* 5.8. *$\mathfrak{o}_P$ is a regular order of $S$.*

Proof. For any $x$ in $S$ there exists a regular element $\alpha$ in $\mathfrak{o}$ such that $\mathfrak{o}\alpha\mathfrak{o}x\subseteq\mathfrak{o}$, hence $\overline{\mathfrak{o}\alpha\mathfrak{o}}x\subseteq\mathfrak{o}$, $\alpha\mathfrak{o}_P x\subseteq(\overline{\mathfrak{o}\alpha\mathfrak{o}})_P\,x\subseteq\mathfrak{o}_P$. Similarly there exists a regular element $\beta$ in $\mathfrak{o}$ such that $x\mathfrak{o}_P\beta \subseteq \mathfrak{o}_P$.

*Lemma* 5.9. *If* $\mathfrak{a}$ *is a c-ideal, then* $\overline{\mathfrak{a}_P} = \mathfrak{a}_P$. *Particularly* $\overline{\mathfrak{o}_P} = \mathfrak{o}_P$.

Proof. Let $a$ be any element in $\overline{\mathfrak{a}_P}$. Then there exist finite elements $a_1, \cdots, a_r$ in $\mathfrak{a}_P$ such that $a$ is contained in $(a_1, \cdots, a_r)$ generated by $a_1, \cdots, a_r$. Since $a_i \in \mathfrak{n}_i^{-1}\cdot\mathfrak{a}$ $(i = 1, \cdots, r)$, there exists an $\mathfrak{n}$ satisfying $a_i \in \mathfrak{n}^{-1}\cdot\mathfrak{a}$ $(i = 1, \cdots, r)$. Hence $a \in (a_1, \cdots, a_r) \subseteq \mathfrak{n}^{-1}\cdot\mathfrak{a} \subseteq \mathfrak{a}_P$, i. e. $\overline{\mathfrak{a}_P} \subseteq \mathfrak{a}_P$. Since evidently $\mathfrak{a}_P \subseteq \overline{\mathfrak{a}_P}$, we get $\overline{\mathfrak{a}_P} = \mathfrak{a}_P$.

*Lemma* 5.10. *If* $\mathfrak{a}$ *is a c-ideal, then*

$$\mathfrak{a}_P = \overline{\mathfrak{o}_P\mathfrak{a}} = \overline{\mathfrak{a}\mathfrak{o}_P} = \overline{\mathfrak{o}_P\mathfrak{a}\mathfrak{o}_P}.$$

Proof. $\mathfrak{a}_P = \bigcup \mathfrak{n}^{-1}\cdot\mathfrak{a} \subseteq \overline{\bigcup \mathfrak{n}^{-1}\mathfrak{a}} = \overline{(\bigcup \mathfrak{n}^{-1})\mathfrak{a}} = \overline{\mathfrak{o}_P\mathfrak{a}} \subseteq \overline{\mathfrak{a}_P} = \mathfrak{a}_P$ Hence $\mathfrak{a}_P = \overline{\mathfrak{o}_P\mathfrak{a}}$. Similarly we get $\mathfrak{a}_P = \overline{\mathfrak{a}\mathfrak{o}_P} = \overline{\mathfrak{o}_P\mathfrak{a}\mathfrak{o}_P}$.

*Theorem* 5.1. *The closure* $\overline{\mathfrak{A}}$ *of an* $\mathfrak{o}_P$*-ideal* $\mathfrak{A}$ *is also an* $\mathfrak{o}_P$*-ideal.*

Proof. Let $a$ be any element of $\overline{\mathfrak{A}}$ and $c$ any element of $\mathfrak{o}_P$. There exists a $c$-ideal $(a_1, \cdots, a_r)$, $a_i \in \mathfrak{A}$, containing $a$, and $\mathfrak{n}^{-1}$ containing $c$. Then $ca \in \mathfrak{n}^{-1}\cdot(a_1, \cdots, a_r) = \overline{\bigcup_i \mathfrak{n}^{-1}\mathfrak{o}a_i\mathfrak{o}} = (b_1, \cdots, b_s)$, where $b_j \in \bigcup_i \mathfrak{n}^{-1}\mathfrak{o}a_i\mathfrak{o} \subseteq \mathfrak{A}$. Hence $ca \in \overline{\mathfrak{A}}$; similarly $ac \in \overline{\mathfrak{A}}$. Since there exists a regular element $\lambda$ such that $\mathfrak{A}\lambda \subseteq \mathfrak{o}_P$, we get by Lemma 5.3 $\overline{\mathfrak{A}}\lambda \subseteq \overline{\mathfrak{o}_P} = \mathfrak{o}_P$.

*Theorem* 5.2. *If* $\mathfrak{A}$ *is an* $\mathfrak{o}_P$*-ideal contained in* $\mathfrak{o}_P$, *then* $\mathfrak{a} = \mathfrak{A}\frown\mathfrak{o}$ *is an* $\mathfrak{o}$*-ideal and* $\overline{\mathfrak{A}} = (\bar{\mathfrak{a}})_P$.

Proof. Since $\mathfrak{A} \supseteq \mathfrak{a}$, we get $\mathfrak{A} \supseteq \mathfrak{n}^{-1}\mathfrak{a}$, $\overline{\mathfrak{A}} \supseteq \mathfrak{n}^{-1}\cdot\bar{\mathfrak{a}}$. Hence $\overline{\mathfrak{A}} \supseteq (\bar{\mathfrak{a}})_P$. Let $a$ be any element of $\overline{\mathfrak{A}}$. Then $a \in (a_1, \cdots, a_r)$, $a_i \in \mathfrak{A}$ $(i = 1, \cdots, r)$. There exists an $\mathfrak{n}$ such that $\mathfrak{n}a_i \subseteq \mathfrak{o}$ and $\subseteq \mathfrak{A}$, therefore, $\mathfrak{n}a_i \subseteq \mathfrak{a}$ $(i = 1, \cdots, r)$. Hence $a_i \in \mathfrak{n}^{-1}\cdot\bar{\mathfrak{a}}$, $a \in (a_1, \cdots, a_r) \subseteq \mathfrak{n}^{-1}\cdot\bar{\mathfrak{a}} \subseteq (\bar{\mathfrak{a}})_P$, i. e. $\overline{\mathfrak{A}} \subseteq (\bar{\mathfrak{a}})_P$.

*Corollary. If* $\mathfrak{A}$ *is a c-*$\mathfrak{o}_P$*-ideal (closed* $\mathfrak{o}_P$*-ideal) in* $\mathfrak{o}_P$, *then there exists a c-ideal* $\mathfrak{a}$ *such that* $\mathfrak{A} = \mathfrak{a}_P$.

*Theorem* 5.3. *The c-*$\mathfrak{o}_P$*-ideals* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \cdots$ *form a group* $G_P$ *with respect to the product* $\mathfrak{A}\cdot\mathfrak{B} = \overline{\mathfrak{A}\mathfrak{B}}$. $G_P$ *is homomorphic to the group* $G$ *of all c-ideals* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots$.

Proof. We shall show first $\mathfrak{a}_P\cdot\mathfrak{b}_P = \overline{\mathfrak{a}_P\mathfrak{b}_P} = (\mathfrak{a}\cdot\mathfrak{b})_P$. $\overline{\mathfrak{a}_P\mathfrak{b}_P} = \overline{\overline{\mathfrak{o}_P\mathfrak{a}}\,\overline{\mathfrak{b}\mathfrak{o}_P}} = \overline{\mathfrak{o}_P\mathfrak{a}\mathfrak{b}\mathfrak{o}_P} = \overline{\mathfrak{o}_P(\mathfrak{a}\cdot\mathfrak{b})\mathfrak{o}_P} = (\mathfrak{a}\cdot\mathfrak{b})_P$ The mapping $\mathfrak{a} \to \mathfrak{a}_P$ is a homomorphism of $G$ into $G_P$. If $\mathfrak{A} \subseteq \mathfrak{o}_P$ then there exists $\mathfrak{a}$ such that $\mathfrak{A} = \mathfrak{a}_P$. If $\mathfrak{C}$ is not contained in $\mathfrak{o}_P$, then there exists $\mathfrak{a}_P$ such that $\mathfrak{a}_P\cdot\mathfrak{C} \subseteq \mathfrak{o}_P$, therefore, $\mathfrak{a}_P\cdot\mathfrak{C} = \mathfrak{b}_P$ and $\mathfrak{C} = \mathfrak{a}_P^{-1}\cdot\mathfrak{b}_P = (\mathfrak{a}^{-1}\cdot\mathfrak{b})_P$. Hence any element of $G_P$ is an image of an element of $G$.

We have easily.

*Theorem* 5.4. *If the three axioms $A_1$, $A_2$ and $A_3$ hold for c-ideals then that is so for the c-$\mathfrak{o}_P$-ideals.*

*Theorem* 5.5. *Let $\mathfrak{a}$ be an integral c-ideal. Then $\mathfrak{a}_P=\mathfrak{o}_P$ if and only if $\mathfrak{a}$ is coprime to P.*

Proof. If $\mathfrak{a}$ is coprime to $P$, then $\mathfrak{a}^{-1}\subseteq\mathfrak{o}_P$ $\mathfrak{a}_P=\overline{\mathfrak{o}_P\mathfrak{a}}\supseteq\overline{\mathfrak{a}^{-1}\mathfrak{a}}=\mathfrak{o}$, hence $\mathfrak{a}_P\supseteq\mathfrak{o}_P$ and $\mathfrak{a}_P=\mathfrak{o}_P$. If, conversely, $\mathfrak{a}_P=\mathfrak{o}_P$, then $1\in\mathfrak{a}_P$, $\mathfrak{n}1\subseteq\mathfrak{a}$ with an $\mathfrak{n}$ coprime to $P$. This means $\mathfrak{a}$ is coprime to $P$.

*Theorem* 5.6. *If an integral c-ideal $\mathfrak{a}$ contains a product $\mathfrak{c}=\Pi_\nu\cdot\mathfrak{p}_\nu$ of prime ideals contained in P, then $\mathfrak{a}_P\cap\mathfrak{o}=\mathfrak{a}$.*

Proof. It is obvious that $\mathfrak{a}_P\cap\mathfrak{o}\supseteq\mathfrak{a}$. Let $a$ be any element of $\mathfrak{a}_P\cap\mathfrak{o}$. Then there exists an $\mathfrak{n}$ coprime to $P$ such that $\mathfrak{n}a\subseteq\mathfrak{a}$. Since $\mathfrak{n}$ and $\mathfrak{c}$ have no common divisor, $(\mathfrak{n},\mathfrak{c})=\overline{\mathfrak{n}\cup\mathfrak{c}}=\mathfrak{o}$. From $\mathfrak{c}a\subseteq\mathfrak{a}$, $\mathfrak{n}a\subseteq\mathfrak{a}$ and $(\mathfrak{c}\cup\mathfrak{n})a\subseteq\bar{\mathfrak{a}}$, it follows that $a\in\mathfrak{o}a\subseteq\bar{\mathfrak{a}}=\mathfrak{a}$, i. e. $\mathfrak{a}_P\cap\mathfrak{o}\subseteq\mathfrak{a}$.

*Theorem* 5.7. *The P-component $\mathfrak{p}_P$ of $\mathfrak{p}$ in P is a prime c-$\mathfrak{o}_P$-ideal, and any prime c-$\mathfrak{o}_P$-ideal is the P-component of* some *$\mathfrak{p}$ in P.*

Proof. If $\mathfrak{p}\in P$ then $\mathfrak{p}_P\neq\mathfrak{o}_P$. If there exists a $c$-$\mathfrak{o}_P$-ideal $\mathfrak{A}$ such that $\mathfrak{p}_P\subset\mathfrak{A}\subseteq\mathfrak{o}_P$, then $\mathfrak{A}=\mathfrak{a}_P$ with $\mathfrak{a}=\mathfrak{A}\cap\mathfrak{o}$, therefore, $\mathfrak{p}\subseteq\mathfrak{p}_P\cap\mathfrak{o}\subseteq\mathfrak{a}\subseteq\mathfrak{o}$, $\mathfrak{p}\neq\mathfrak{a}$, and $\mathfrak{a}=\mathfrak{o}$. Hence $\mathfrak{A}=\mathfrak{o}_P$. Thus $\mathfrak{p}_P$ is maximal and therefore prime. Since the group $G_P$ of all the $c$-$\mathfrak{o}_P$-ideals is generated by all the $\mathfrak{p}_P(\mathfrak{p}\in P)$, it is clear that the set of $\mathfrak{p}_P(\mathfrak{p}\in P)$ is the totality of prime $c$-$\mathfrak{o}_P$-ideals.

*Theorem* 5.8. *Let $\mathfrak{a}$ be a c-ideal. Then $\mathfrak{a}=\bigcap_\mathfrak{p}\mathfrak{a}_\mathfrak{p}$, where $\mathfrak{p}$ runs over all the prime c-ideals of $\mathfrak{o}$. More generally, $\mathfrak{a}_P=\bigcap_{\mathfrak{p}\in P}\mathfrak{a}_\mathfrak{p}$.*

Proof. It is evident that $\mathfrak{a}_P\subseteq\bigcap_{\mathfrak{p}\in P}\mathfrak{a}_\mathfrak{p}$. Let $a$ be any element of $\bigcap_{\mathfrak{p}\in P}\mathfrak{a}_\mathfrak{p}$, then there exists $\mathfrak{n}(\mathfrak{p})$ coprime to $\mathfrak{p}$ such that $\mathfrak{n}(\mathfrak{p})a\subseteq\mathfrak{a}$. Since from $\mathfrak{m}a\subseteq\mathfrak{a}$, $\mathfrak{m}'a\subseteq\mathfrak{a}$ it follows that $(\mathfrak{m},\mathfrak{m}')a\subseteq\mathfrak{a}$, there exists the greatest ideal $\mathfrak{m}_0$ such that $\mathfrak{m}_0a\subseteq\mathfrak{a}$. Being $\mathfrak{m}_0\supseteq\mathfrak{n}(\mathfrak{p})$, $\mathfrak{m}_0$ is coprime to $\mathfrak{p}$, hence to $P$, therefore, $a\in\mathfrak{a}_P$, i. e. $\bigcap_{\mathfrak{p}\in P}\mathfrak{a}_\mathfrak{p}\subseteq\mathfrak{a}_P$.

*Definition.* A subsemigroup $S'$ of $S$ is called an $\mathfrak{o}$-*semigroup* when it is a closed $\mathfrak{o}$-set containing $\mathfrak{o}$.

A subsemigroup $S'$ is an $\mathfrak{o}$-semigroup if and only if for any finite elements $c_1,\cdots,c_n$ in $S'$ the $c$-ideal $(1,c_1,\cdots,c_n)$ generated by $1,c_1,\cdots,c_n$ is contained in $S'$.

For example $\mathfrak{o}_P$ is an $\mathfrak{o}$-semigroup in S.

*Lemma* 5.11. *Let $S'$ be an $\mathfrak{o}$-semigroup. If $S'$ is not contained in $\mathfrak{o}_\mathfrak{p}$, then $S'\supseteq\mathfrak{p}^{-1}$.*

Proof. There exists an element $c$ contained in $S'$ but not in $\mathfrak{o}_{\mathfrak{p}}$. $(1,c)^{-1}$ is integral and contained in $\mathfrak{p}$, if not $\mathfrak{n}=(1,c)^{-1}$ is coprime to $\mathfrak{p}$, hence $\mathfrak{o}_{\mathfrak{p}}\supseteq\mathfrak{n}^{-1}\ni c$, a contradiction. We get, therefore, $S'\supseteq(1,c)\supseteq\mathfrak{p}^{-1}$.

*Theorem* 5.9. *$\mathfrak{o}_{\mathfrak{p}}$ is a maximal $\mathfrak{o}$-semigroup in S. And any $\mathfrak{o}$-semigroup $\neq S$ is contained in some $\mathfrak{o}_{\mathfrak{p}}$.*

Proof. If there exists an $\mathfrak{o}$-semigroup $S'\neq\mathfrak{o}_{\mathfrak{p}}$ containing $\mathfrak{o}_{\mathfrak{p}}$, then by Lemma 5.11 $S'\supseteq\mathfrak{p}^{-1}$, hence $S'\supseteq\mathfrak{P}^{-1}$ where $\mathfrak{P}=\mathfrak{p}_{\mathfrak{p}}$. $S'$ contains all the powers of $\mathfrak{P}$, i. e. all the $c$-$\mathfrak{o}_{\mathfrak{p}}$-ideals, therefore all the $\mathfrak{o}_{\mathfrak{p}}$-ideals. Since any element of $S$ is contained in some $\mathfrak{o}_{\mathfrak{p}}$-ideal, $S'=S$. Any $\mathfrak{o}$-semigroup $S'(\neq S)$ is contained in some $\mathfrak{o}_{\mathfrak{p}}$: $S'\subseteq\mathfrak{o}_{\mathfrak{p}}$, for if not, by Lemma 5.11 $S'$ contains all the $\mathfrak{p}^{-1}$, hence all the $\mathfrak{o}$-ideals, and it follows that $S'=S$.

*Theorem* 5.10. *Any $\mathfrak{o}$-semigronp $S'(\neq S)$ coincides with some $\mathfrak{o}_P$.*

Proof. Let $P$ be the set of all $\mathfrak{p}$'s such that $\mathfrak{o}_{\mathfrak{p}}\supseteq S'$. We shall show that $S'=\mathfrak{o}_P$. If $S'\neq\mathfrak{o}_P$ then $\mathfrak{o}_P\supset S'$, and there exists an element $a$ contained in $\mathfrak{o}_P$, but not in $S'$. Obviously $\mathfrak{o}_P\supseteq(1,a)\supset\mathfrak{o}$ and $(1,a)^{-1}\subset\mathfrak{o}$. Let $(1,a)^{-1}=\mathfrak{q}_1\cdot\cdots\cdot\mathfrak{q}_r$ be the prime factorization of $(1,a)^{-1}$. Since $(1,a)=\mathfrak{q}_1^{-1}\cdot\cdots\cdot\mathfrak{q}_r^{-1}$ is not contained in $S'$, for some prime factor $\mathfrak{q}$ of $(1,a)^{-1}$, $\mathfrak{q}^{-1}$ is not contained in $S'$. Since $\mathfrak{q}^{-1}\subseteq(1,a)\subseteq\mathfrak{o}_P\subseteq\mathfrak{o}_{\mathfrak{p}}(\mathfrak{p}\in P)$ and $\mathfrak{q}^{-1}\not\subseteq\mathfrak{o}_{\mathfrak{q}}$, $\mathfrak{q}$ is not contained in $P$. Therefore, $S'$ is not contained in $\mathfrak{o}_{\mathfrak{q}}$, hence by Lemma 5.11 $S'\supseteq\mathfrak{q}^{-1}$. This is a contradiction.

It is easily seen that all the $\mathfrak{o}$-semigroups, i. e. all the $\mathfrak{o}_P$ form a lattice with respect to inclusion relation. In fact

$$\mathfrak{o}_P\cup\mathfrak{o}_{P'}=\mathfrak{o}_{P\cap P'},\qquad \mathfrak{o}_P\cap\mathfrak{o}_{P'}=\mathfrak{o}_{P\cup P'}$$

From this we have the following

*Theorem* 5.11. *The set of all $\mathfrak{o}$-semigroups forms a Boolian algehra which is dual isomorphic to the Boolian algebra consisting of all subsets of the set of all prim c-ideals in $\mathfrak{o}$.*

## §6. Factorization of integral elements in a lattice-ordered gruppoid

Let $G$ be a gruppoid with units $e_i$, $e_k$, $\cdots$. Elements of $G$ will be denoted by small letters with or without suffices.

*Definition.* A gruppoid $G$ is called *lattice-ordered* when for any index $i$ and $k$,

1) $L_i=\{x\,|\,e_ix=x, x\in G\}$ forms a lattice,
2) $R_k=\{y\,|\,ye_k=y, y\in G\}$ forms a lattice,
3) $N_{ik}=L_i\cap R_k$ forms a sublattice of both $L_i$ and $R_k$.

In the following $a = a_{ik}$ will denote that $a \in N_{ik}$.

*Definition.* An element $a = a_{ik}$ is called *integral* if $a \leq e_i$ and $\leq e_k$.

In this section we shall study on factorization of integral elements in a lattice-ordered gruppoid $G$ with following conditions:

$P_1$: $a_{ik} \leq e_i$ implies $a_{ik} \leq e_k$, and conversely.

$P_2$: $L_i$ and $R_k$ are modular lattices.

$P_3$: If $a \leq b$ $(a, b \in L_i)$, $c \in R_i$, then $ca \leq cb$, and if $a \leq b$ $(a, b \in R_k)$,$c \in L_k$, then $ac \leq bc$.

$P_4$: There exists $\sup A$ for any non-void set $A$ consisting of integral elements in $L_i$, and similarly for $R_k$.

If there exists $\sup A$ for a subset $A$ of $L_i$, then there exists $\sup(aA)$ for any $a \in R_i$, and $a(\sup A) = \sup(aA)$. Analogously, if there exists $\sup B$ for a subset $B$ of $R_k$, then there exists $\sup(Bb)$ for any $b \in L_k$, and $\sup(Bb) = (\sup B)\, b$. Because, since $a(\sup A) \geq ax (a = a_{li})$ for any $x \in A$, $a(\sup A) \geq \sup(aA)$, therefore $a^{-1}\sup(aA) \geq \sup(a^{-1}aA) = \sup A$, $\sup(aA) \geq a(\sup A)$. Hence $\sup(aA) = a(\sup A)$. The other is analogously obtained.

If a subset $A$ of $L_i$ is bounded then there exists $\sup A$, and analogously for $R_k$. Because, there exists an element $c = c_{ik} \in L_i$ satisfying $x \leq c$ $(x \in A)$, $c^{-1}x \leq c^{-1}c = e_k$. Hence there exists $\sup(c^{-1}A)$ in $L_k$, and $c(\sup c^{-1}A)) = \sup(cc^{-1}A) = \sup A$. By this fact, $N_{ii}$ forms a *cl*-group. $N_{ii}$ is therefore a commutative group under multiplication.

If $a \leq b$ $(a = a_{ij}, b = b_{ik})$, then $b^{-1}a \leq b^{-1}b = e_k$, $b^{-1}a \in N_{kj}$, by the condition $P_1$ $b^{-1}a \leq e_j$, therefore $b^{-1} = b^{-1}aa^{-1} \leq e_j a^{-1} = a^{-1}$. Hence $L_i$ is anti-isomorphic to $R_i$ by the mapping $a \rightarrow a^{-1}$ $(a \in L_i, a^{-1} \in R_i)$. It is easy to see that if $a \leq b$, $a, b \in L_i$, $(a, b \in R_k)$, then these exists an integral element $c$ such that $a = bc$ $(a = cb)$.

*Theorem* 6.1. *By a mapping* $a \rightarrow c^{-1}ac$ $(c = c_{ij}, a \in N_{ii}) N_{ii}$ *and* $N_{jj}$ *are group-isomorphic under multiplicaton and lattice-isomorphic under ordering. And this mapping is uniquely determined independently of the choice of* $c \in N_{ij}$.

Proof. The first part of this theorem is obvious. Let $c'$ be any element of $N_{ij}$. Then $c'c^{-1} \in N_{ii}$. Since $N_{ii}$ is a commutative group, we have $c'c^{-1}a = ac'c^{-1}$, $c^{-1}ac = c'^{-1}ac'$.

*Definition.* An element $a'$ in $N_{jj}$ is called *conjunctive* to an element $a$ in $N_{ii}$ when there exists $c = c_{ij}$ satisfying $a' = c^{-1}ac$.

*Definition.* An integral element $a = a_{ij}$ is called *transposable* to an integral element $b = b_{kl}$ if there exists an integral element $c = c_{ik}$ such that $b = c^{-1}(c \cap a)$, $c \cup a = e_i$.

This relation is reflexsive and transitive, but not symmetric.

*Definition.* An integral element $a$ is called ***projective*** to an integral element $b$ when there exists a sequence of integral elements $a=c_0, c_1, \cdots, c_n, c_{n+1}=b$ in which for any two successive elements $c_i, c_{i+1}$ one is transposable to the other.

*Lemma* 6.1. *If two lattice-quotients $a/a'$ and $b/b'$ of $L_i$ are projective, then integral elements $c=a^{-1}a'$ and $d=b^{-1}b'$ are projective in the sense of above definition.*

Proof. If $a/ac$ is transposable to $b/bd$ and $b\leq a$, then $b\cup ac=a$, $b\cap ac=bd$. Since there exists an integral element $f$ such that $b=af$, $bd=afd$, we have $f\cup c=e$, $f\cap c=fd$, $d=f^{-1}(f\cap c)$. Hence $c$ is transposable to $d$. By induction we complete the proof.

A factorization of an integral element and its refinement are defined as in §1.

*Theorem* 6.2. (*Refinement theorem*) *Two factorizations of an integral element in G have such two refinements that there is a one-to-one correspondence between their factors, and the paired factors are projective to each other.*

Proof. Let $a=a_1\cdots a_r=b_1\cdots b_s$ be two factorizations of an integral element $a\in L_i$. Put $A_\mu=a_1\cdots a_\mu$, $B_\nu=b_1\cdots b_\nu$. Since $A_\mu$ and $B_\nu$ are contained in $L_i$, we get two chains in $L_i$ such that $e_i=A_0>A_1>\cdots>A_r=a$ and $e_i=B_0>B_1>\cdots>B_s=a$. By Jordan-Hölder-Schreier theorem in a modular lattice, we get two refinements of the same length $e_i=A_0'>A_1'>\cdots>A_n'=a$, $e_i=B_0'>B_1'>\cdots>B_n'=a$ such that $A'_{\mu-1}/A_\mu'$ is projective to $B'_{\nu-1}/B_\nu'$ in pairs. Hence by Lemma 6.1 $a_\mu'=A'^{-1}_{\mu-1}A_\mu'$ is projective to $b_\nu'=B'^{-1}_{\nu-1}B_\nu'$, and $a=\Pi^n_{\mu=1}a_\mu'=\Pi^n_{\nu=1}b_\nu'$.

If we assume the ascending chain condition for integral elements in $R_k$, then the descending chain condition holds in $L_i$ for integral elements that contain any fixed integral element. Let $a=a_{ik}$ be any fixed integral element of $L_i$ and $a_1\geq a_2\geq\cdots\geq a$ $(a_i\leq e_i)$ infinite descending chain in $L_i$. Since $a_1^{-1}a\leq a_2^{-1}a\leq\cdots\leq a^{-1}a=e_i$ is an ascending chain in $R_k$, we get $a_n^{-1}a=a_{n+\nu}^{-1}a$, hence $a_n=a_{n+\nu}$ $(\nu=1,2,\cdots)$.

*Definitiun.* An integral element of $G$ is called ***reducible*** if it is equal to a product of two integral elements not equal to units of $G$. An integral element of $G$, which is not reducible, is called ***irreducible.***

From the refinement theorem we get:

*Theorem* 6.3. *Suppose that the ascending chain condition holds for integral elements in all $L_i$ and $R_k$. Then any integral element in G is decomposed into a product of finite irreducible elements of G. Moreover such a factoriza-*

*tion is uniquely determined apart from the projectivity of their factors.*

In the following we shall assume:

$P_5$: $N_{ik}$ contains an integral element.

$P_6$: There exists for any element $a \in N_{ik}$ an element $c \in N_{ii}$ such that $c \leq a$. (It follows from this $c' = a^{-1}ca \leq a$, $c' \in N_{kk}$)

*Lemma* 6.2. *Let $q = q_{ij}$ be an irreducible element of G. If a is a maximal element in $N_{ii}$ such that $a \leq q$, then a is a prime element of $N_{ii}$.*

Proof. Put $a = bc$, where $b$ and $c$ are both integral in $N_{ii}$ and not equal to $e_i$. Since $c \cup q = e_i$ we have $q \geq bc \cup bq = b > a$. This is a contradiction.

*Definition.* The element $a$ in Lemma 6.2 is called a *prime element corresponding to q.*

*Definition.* Let $p = p_{ii}$ and $p' = p'_{jj}$ be two prime elements corresponding to irreducible elements $q = q_{il}$ and $q' = q'_{jk}$ respectively. Then $q$ is called *similar* to $q'$ when $p$ is conjunctive to $p'$. Symbol: $q \cong q'$.

*Lemma* 6.3. *Let an irreducible element $q = q_{ij}$ be transposable to an irreducible element $q' = q'_{kl}$, i.e. $c \cup q = e_i$ $(c = c_{ik})$, $q' = c^{-1}(c \cap q)$. Then $p' = c^{-1}pc \in N_{kk}$ is a prime element corresponding to $q'$, where $p = p_{ii}$ is a prime element corresponding to q.*

Proof. $cq' = c \cap q \geq c \cap p \geq pc = cp'$, and $q' \geq p'$.

*Lemma* 6.4. *Two projective irreducible elements are similar.*

*Theorem* 6.4. *If $p \in N_{ij}$ and $q \in N_{jk}$ are irreducible elements, then $pq = q'p'$ where $p \cong p'$, $q \cong q'$. $q' \in N_{il}$, $p' \in N_{lk}$.*

Proof. If $p \cong q$ then we may take $q' = p$ and $p' = q$. We now suppose $p$ is not similar to $q$. Let $P$ and $Q$ be prime elements corresponding to $p$ and $q$ respectively. We have $pq \leq pQ = Q'p$ where $Q' = pQp^{-1}$. Evidently $pq \leq pq \cup Q' \leq e_i$. Now we shall show that $pq < pq \cup Q' < e_i$. First if $pq = pq \cup Q'$ then $Q' \leq pq$, $Q' \leq p$. Hence $p \geq p \cup Q' = e_i$. This is a contradiction. Next, if $pq \cup Q' = e_i$, then by modularity of $L_i$ we have $p = e_i \cap p = (pq \cup Q') \cap p = pq \cup (Q' \cap p)$. If $Q' = p \cap Q'$ then $p \geq Q'$, a contradiction. Since $Q' > p \cap Q' \geq Q'p$ we get $p \cap Q' = Q'p$. Hence $p = pq \cup Q'p = pq \cup pQ = pq$. This is a contradiction. Hence $q' = pq \cup Q'$ is irreducible. And we have $pq = p \cap q' = q'p'$, where $p \cup q' = e_i$.

*Lemma* 6.5. *If the ascending chain condition holds for integral elements in one fixed $L_i$, then it holds in any $L_k$.*

Proof. Let $a_1 \leq a_2 \leq \cdots$ be an ascending chain in $L_k$ and $c = c_{ik}$ an integral element. Then $ca_\nu \leq ce_k = c \leq e_i$ ($\nu = 1, 2, \cdots$), and $ca_1 \leq ca_2 \leq \cdots$ is an ascending chain of integral elements in $L_i$. Hence there exists an $n$ such that $ca_n = ca_{n+\nu}$, $a_n = a_{n+\nu}$, $\nu = 1, 2, \cdots$.

*Theorem* 6.5. *If we assume the ascending chain condition for integral elements in one fixed $L_i$ and one fixed $R_k$, then any integral element in $G$ is decomposed into a product of finite irreducible elemnts is $G$, And this factorization is uniquely determined apart from its similarity. Moreover the factors in this product is commutative within their similarity.*

## §7. Gruppoid of normal ideals

Let $S$ be a semigroup with unity quantity 1, and $\mathfrak{o}_0$ be a fixed order of $S$. In this section we shall take out a system of orders which are equivalent to $\mathfrak{o}_0$. The term an "order" will denote an "order equivalent to $\mathfrak{o}_0$". Hence two orders are equivalent to each other.

*Definition.* Let $\mathfrak{o}$ and $\mathfrak{o}'$ be two orders of $S$, and $\mathfrak{a}$ be an ideal such that the left and the right orders of $\mathfrak{a}$ are $\mathfrak{o}$ and $\mathfrak{o}'$ respectively. $\mathfrak{a}$ is called *normal*, if both $\mathfrak{o}$ and $\mathfrak{o}'$ are maximal.

If $\mathfrak{o}$ is a maximal order and $\mathfrak{a}$ is a left (or right) $\mathfrak{o}$-ideal, then the inverse ideal $\mathfrak{a}^{-1}$ of $\mathfrak{a}$ is normal.

*Definition.* Let $\mathfrak{a}$ and $\mathfrak{b}$ be two normal ideals. A product $\mathfrak{a}\mathfrak{b}$ is called *proper* if and only if the right order of $\mathfrak{a}$ coincides with the left order of $\mathfrak{b}$.

In what follows the term "product" of normal ideals is used only in this sense.

*Definition.* Let $\mathfrak{o}$ be a maximal order. A left or a right $\mathfrak{o}$-ideal is called a *v-ideal*, if $\mathfrak{a}^* = (\mathfrak{a}^{-1})^{-1} = \mathfrak{a}$.

For example the inverse ideal $\mathfrak{a}^{-1}$ of $\mathfrak{a}$ is a $v$-ideal. A $v$-ideal is normal.

The mapping $\mathfrak{a} \rightarrow \mathfrak{a}^*$ has the following properties:

$$\mathfrak{a} \subseteq \mathfrak{a}^*, \qquad \mathfrak{a}^{**} = \mathfrak{a}^*,$$

$\mathfrak{a} \subseteq \mathfrak{b}$ implies $\mathfrak{a}^* \subseteq \mathfrak{b}^*$, if $\mathfrak{a}$ and $\mathfrak{b}$ have the same left or right order.

If $\mathfrak{a}$ and $\mathfrak{b}$ are left (or right) $\mathfrak{o}$-ideals, then the meet (intersection) and the join (set-union) are also. And it is easily verified that

$$(\mathfrak{a}^* \cap \mathfrak{b}^*)^* = \mathfrak{a}^* \cap \mathfrak{b}^*,$$
$$(\mathfrak{a} \cup \mathfrak{b})^* = (\mathfrak{a}^* \cup \mathfrak{b})^* = (\mathfrak{a} \cup \mathfrak{b}^*)^* = (\mathfrak{a}^* \cup \mathfrak{b}^*)^*.$$

If $\mathfrak{a}\mathfrak{b}$ is a proper product of two normal ideals, then

$$\mathfrak{a}^*\mathfrak{b}^* \subseteq (\mathfrak{a}\mathfrak{b})^*$$

Let $\mathfrak{a}, \mathfrak{b}$ be $\mathfrak{o}$-$\mathfrak{o}'$-ideal, $\mathfrak{o}'$-$\mathfrak{o}''$-ideal respectively. Then

$$\mathfrak{a}\mathfrak{b}(\mathfrak{a}\mathfrak{b})^{-1} \subseteq \mathfrak{o}, \qquad \mathfrak{b}(\mathfrak{a}\mathfrak{b})^{-1} \subseteq \mathfrak{a}^{-1} = \mathfrak{a}^{*-1},$$
$$\mathfrak{b}(\mathfrak{a}\mathfrak{b})^{-1}\mathfrak{a}^* \subseteq \mathfrak{a}^{*-1}\mathfrak{a}^* \subseteq \mathfrak{o}', \qquad (\mathfrak{a}\mathfrak{b})^{-1}\mathfrak{a}^* \subseteq \mathfrak{b}^{-1} = \mathfrak{b}^{*-1},$$

hence $(\mathfrak{a}\mathfrak{b})^{-1}\mathfrak{a}^*\mathfrak{b}^* \subseteq \mathfrak{o}''$, $\mathfrak{a}^*\mathfrak{b}^* \subseteq ((\mathfrak{a}\mathfrak{b})^{-1})^{-1} = (\mathfrak{a}\mathfrak{b})^*$.

We get easily

$$(\mathfrak{a}\mathfrak{b})^* = (\mathfrak{a}^*\mathfrak{b})^* = (\mathfrak{a}\mathfrak{b}^*)^* = (\mathfrak{a}^*\mathfrak{b}^*)^*$$

*Lemma* 7.1. *Let* $\mathfrak{o}$ *be a maximal order and* $\mathfrak{a}$ *be a left* $\mathfrak{o}$*-ideals. Then* $(\mathfrak{a}\mathfrak{a}^{-1})^* = \mathfrak{o}$.

Proof. Since $(\mathfrak{a}\mathfrak{a}^{-1})(\mathfrak{a}\mathfrak{a}^{-1})^{-1} \subseteq \mathfrak{o}$, $\mathfrak{a}^{-1}(\mathfrak{a}\mathfrak{a}^{-1})^{-1} \subseteq \mathfrak{a}^{-1}$ and by maximality of $\mathfrak{o}$ $(\mathfrak{a}\mathfrak{a}^{-1})^{-1} \subseteq \mathfrak{o}$, we get

$$\mathfrak{o} = \mathfrak{o}^* \supseteq (\mathfrak{a}\mathfrak{a}^{-1})^* = ((\mathfrak{a}\mathfrak{a}^{-1})^{-1})^{-1} \supseteq \mathfrak{o}^{-1} = \mathfrak{o}, \qquad (\mathfrak{a}\mathfrak{a}^{-1})^* = \mathfrak{o}.$$

*Theorem* 7.1. *The set of all* $v$*-ideals in* $S$ *forms a lattice-ordered groupoid with respect to product*: $(\mathfrak{a}\mathfrak{b})^*$, *join*: $(\mathfrak{a}\cup\mathfrak{b})^*$ *and meet*: $\mathfrak{a}\cap\mathfrak{b}$, *where the product* $\mathfrak{a}\mathfrak{b}$ *is proper and* $\mathfrak{a}$ *and* $\mathfrak{b}$ *have the same left or right order for join and meet.*

Let $\mathfrak{o}_i$, $\mathfrak{o}_j$, $\mathfrak{o}_k, \cdots$ be the maximal orders of $S$ and $\mathfrak{L}_i$ be the set of all left $\mathfrak{o}_i$-ideals, $\mathfrak{R}_k$ the set of all right $\mathfrak{o}_k$-ideals and $\mathfrak{R}_{ik} = \mathfrak{L}_i \cap \mathfrak{R}_k$. Let us suppose that there exists a mapping of $\mathfrak{L}_i$ into itself for all $i$ and a mapping of $\mathfrak{R}_k$ into itself for all $k$ with the following properties:

1) $\mathfrak{a} \subseteq \bar{\mathfrak{a}}$ ($\bar{\mathfrak{a}}$, image of $\mathfrak{a}$)

2) $\bar{\bar{\mathfrak{a}}} = \bar{\mathfrak{a}}$

3) If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}_i$ or $\mathfrak{a}, \mathfrak{b} \in \mathfrak{R}_k$, then $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\bar{\mathfrak{a}} \subseteq \bar{\mathfrak{b}}$.

We suppose further that the mapping of $\mathfrak{L}_i$ and the mapping of $\mathfrak{R}_k$ induce the same mapping of $\mathfrak{R}_{ik}$ into itself, and

4) $\bar{\mathfrak{a}}\,\bar{\mathfrak{b}} \subseteq \overline{\mathfrak{a}\mathfrak{b}}$, if $\mathfrak{a}\mathfrak{b}$ is a proper product of normal ideals.

*Lemma* 7.2. *Let* $\mathfrak{a}$ *be a left (or right)* $\mathfrak{o}_i$*-ideal. Then*

$$\overline{\mathfrak{a}^{-1}} = \mathfrak{a}^{-1}, \qquad \mathfrak{a} \subseteq \bar{\mathfrak{a}} \subseteq \mathfrak{a}^* \ (\mathfrak{a}^* = (\mathfrak{a}^{-1})^{-1})$$

Proof. Let $\mathfrak{a} \in \mathfrak{L}_i$, $\mathfrak{a}^{-1} \in \mathfrak{R}_{ji}$, From

$$\mathfrak{a}\overline{\mathfrak{a}^{-1}} \subseteq \overline{\mathfrak{a}\mathfrak{o}_j}\,\overline{\mathfrak{a}^{-1}} \subseteq \overline{\mathfrak{a}\mathfrak{o}_j\mathfrak{a}^{-1}} = \overline{\mathfrak{a}\mathfrak{a}^{-1}} \subseteq \bar{\mathfrak{o}_i} = \mathfrak{o}_i$$

we get $\overline{\mathfrak{a}^{-1}} \subseteq \mathfrak{a}^{-1}$, hence $\overline{\mathfrak{a}^{-1}} = \mathfrak{a}^{-1}$. Since $\bar{\mathfrak{a}}\mathfrak{a}^{-1} \subseteq \overline{\mathfrak{a}\mathfrak{o}_j}\,\overline{\mathfrak{a}^{-1}} \subseteq \mathfrak{o}_i$ we get $\mathfrak{a} \subseteq \bar{\mathfrak{a}} \subseteq (\mathfrak{a}^{-1})^{-1}$.

*Definition.* An ideal $\mathfrak{a}$ in $\mathfrak{L}_i$ or $\mathfrak{R}_k$ is called a *closed ideal* (*c-ideal*) if $\bar{\mathfrak{a}} = \mathfrak{a}$.

*Lemma* 7.3. *Suppose that if* $a \in \overline{\mathfrak{a}\cup\mathfrak{b}}$ $(\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}_i)$ *then there exists* $b \in \bar{\mathfrak{b}}$ *such that* $\overline{\mathfrak{a}\cup\mathfrak{o}a} = \overline{\mathfrak{a}\cup\mathfrak{o}b}$. *The set* $L_i$ *of all closed ideals in* $\mathfrak{L}_i$ *forms a modular lattice under join* $\overline{\mathfrak{a}\cup\mathfrak{b}}$ *and meet* $\mathfrak{a}\cap\mathfrak{b}$ $(\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}_i)$.

Proof. $L_i$ forms evidently a lattice. Now assume that

$$\mathfrak{a} \supset \mathfrak{b}, \quad \overline{\mathfrak{a} \cup \mathfrak{c}} = \overline{\mathfrak{b} \cup \mathfrak{c}} \quad (\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in L_i)$$

Let $a$ be an element contained in $\mathfrak{a}$, but not in $\mathfrak{b}$. Since $a \in \mathfrak{a} \subseteq \overline{\mathfrak{b} \cup \mathfrak{c}}$, there exists an element $c$ such that

$$\overline{\mathfrak{b} \cup \mathfrak{o}c} = \overline{\mathfrak{b} \cup \mathfrak{o}a} \neq \mathfrak{b}, \qquad c \in \mathfrak{c}$$

Hence $c \in \overline{\mathfrak{b} \cup \mathfrak{o}a} \subseteq \overline{\mathfrak{b} \cup \mathfrak{a}} = \bar{\mathfrak{a}} = \mathfrak{a}$, $c \notin \mathfrak{b}$, and therefore we get $\mathfrak{a} \cap \mathfrak{c} \supset \mathfrak{b} \cap \mathfrak{c}$, i.e. $L_i$ is modular.

In the following we shall assume:

1) The maximal orders are regular.

2) Ascending chain condition holds for integral closed left $\mathfrak{o}_i$-ideals (or right $\mathfrak{o}_k$-ideals).

3) A prime c-ideal of $\mathfrak{o}_i$ is maximal (as a two-sided $c$-ideal).

4) Any $c$-ideal is normal.

5) The lattice $L_i(R_i)$ of all closed left (right) $\mathfrak{o}_i$-ideals is modular.

Then the set $N_{ii} = L_i \cap R_i$ of all closed two-sided $\mathfrak{o}_i$-ideals forms a $cl$-group.

*Lemma* 7.4. *If* $\mathfrak{a} = \mathfrak{a}_{ik} \in N_{ik} = L_i \cap N_k$ then

$$\overline{\mathfrak{a}\mathfrak{a}^{-1}} = \mathfrak{o}_i, \qquad \overline{\mathfrak{a}^{-1}\mathfrak{a}} = \mathfrak{o}_k$$

Proof. $\overline{\mathfrak{a}\mathfrak{a}^{-1}}$ is a closed two-sided $\mathfrak{o}_i$-ideal, and

$$((\overline{\mathfrak{a}\mathfrak{a}^{-1}})^{-1})^{-1} = (\mathfrak{a}\mathfrak{a}^{-1})^* = \mathfrak{o}_i,$$

hence $\overline{\mathfrak{a}\mathfrak{a}^{-1}} = \mathfrak{o}_i$, because the closed two-sided $\mathfrak{o}_i$-ideals form a group. Similarly we have $\overline{\mathfrak{a}^{-1}\mathfrak{a}} = \mathfrak{o}_k$.

*Lemma* 7.5. Every $c$-ideal is a $v$-ideal.

Proof. Let $\mathfrak{a} = \mathfrak{a}_{ik}$ be a $c$-ideal. Then

$$\mathfrak{a}^* = \overline{\overline{\mathfrak{a}\mathfrak{a}^{-1}}\mathfrak{a}^*} = \overline{\mathfrak{a}\mathfrak{a}^{-1}\mathfrak{a}^*} = \overline{\mathfrak{a}\mathfrak{a}^{-1}(\mathfrak{a}^{-1})^{-1}} = \mathfrak{a}$$

*Lemma* 7.6. There exists an integral $\mathfrak{o}_i$-$\mathfrak{o}_k$-ideal.

Proof. $\mathfrak{c} = (\mathfrak{o}_k\mathfrak{o}_i)^{-1}$ is integral, since $\mathfrak{o}_i\mathfrak{o}_k \supset \mathfrak{o}_i$.

*Theorem* 7.2. *The set $G$ of all c-ideals in $S$ forms a lattice-ordered gruppoid with the properties $P_1$—$P_6$ in* §6 *with respect to a product*: $\mathfrak{a}_{ik} \cdot \mathfrak{b}_{kl} = \overline{\mathfrak{a}_{ik}\mathfrak{b}_{kl}}$, *join*: $(\mathfrak{a}_{ik}, \mathfrak{b}_{jl}) = \overline{\mathfrak{a}_{ik} \cup \mathfrak{b}_{jl}}$ *and meet* $\mathfrak{a}_{ik} \cap \mathfrak{b}_{jl}$, *where $i = j$ or $k = l$ for join and meet.*

From the results obtained in §6 we get the following theorems:

*Theorem* 7.3. *By a mapping* $\mathfrak{a}' \to \mathfrak{c}^{-1}\mathfrak{a}\mathfrak{c}$ $(\mathfrak{c} \in N_{ik}, \mathfrak{a} \in N_{ii}, \mathfrak{a}' \in N_{kk})$ *two lattice-ordered groups $N_{ii}$ and $N_{kk}$ are group-isomorphic and lattice-isomorphic. And* *this mapping is uniquely determined independenty of the choice of* $\mathfrak{c}$ *in* $N_{ik}$.

*Theorem* 7.4. *Any integral c-ideal is decomposed into a product of finite irreducible c-ideals. And this factorization is uniquely determined apart from its similarity. Moreover the factors in this product is commutative within their similarity.*

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