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The Structures of neighbourhood systems and the types of convergences

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1. Dircted systems. Let X be a set and $>$ be a binary relation between two elements of X. We say that $(X, >)$ is a directed system, if the following conditions are satisfied:

1. $x > x_1 > x_2 \rightarrow x > x_2$,
2. $\underset{x_1}{\forall} \underset{x_2}{\forall} \underset{x}{\exists} x > x_1, x > x_2$.

When X' is a subset of X, the two notions "X' is cofinal in X", "X' is residual in X" are defined as follows:

$$X' \operatorname{cof} X \equiv \underset{x}{\forall} \underset{x_1}{\exists} x_1 > x, \quad x_1 \in X',$$

$$X' \operatorname{res} X \equiv \underset{x}{\exists} \underset{x_1}{\forall} x_1 > x \rightarrow x_1 \in X'.$$

If σ is a mapping from a directed system X to X, satisfying the condition $\underset{x}{\forall} \sigma(x) > x$, then σ is said a increasing transformation of X. It is easy to see that

$$X' \operatorname{cof} X \equiv \underset{\sigma}{\exists} \operatorname{ran} \sigma \subset X',$$

where $\operatorname{ran} \sigma$ denotes the range of σ. Hereafter we use "ran" is this sense.

2. Ordering. If ρ is a mapping from a directed system X to another directed system Y, satisfying the condition

$$\underset{y}{\forall} \underset{x_0}{\exists} \underset{x}{\forall} x > x_0 \rightarrow \rho(x) > y,$$

then ρ is said a divergent transformation from X to Y. Particularly, a divergent transformation from the naturally ordered natural numbers to the naturally ordered positive numbers is a divergent sequence.

If there exists a divergent transformation from X to Y, then we define $X > Y$.

This order is transitive and reflexive. The direct product $X \otimes Y > X, Y$, because the projections are divergent transformations. Accordingly this order is also directed. Generally

$$X_1 \otimes X_2 \otimes \cdots > X_1, X_2, \cdots$$

Hence any number of directed systems have an upper bound.

If we define

$$X \sim Y \equiv X > Y, Y > X,$$

then $\sim$ is a congruence. Each class which is decided by this congruence is called a "type". We write the type of X by $\tau(x)$.

Lemma 1. X' cof $X \rightarrow X' \sim X$.

Proof. If X' cof X, there exists a increasing transformation σ of X such that ran $\sigma \subset X'$. This σ is a divergent transformation from X to X'. Also the identity is a divergent transformation from X' to X.

Lemma 2. *If X is a countable directed system, X has the greatest element, or has the same type to the naturally ordered natural numbers.*

Proof. We assume $X = \{x_1, x_2, x_3, \cdots\}$. Put

$$\begin{aligned} &a = x_1, \\ &a = x^{\vee} a, \\ &a = x^{\vee} a, \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots, \end{aligned}$$

where $^{\vee}$ denotes an upper bound. Then

$$a_1 < a_2 < a_3 < \cdots.$$

This sequence $\{a_n\}$ is cofinal in X. If $\{a_n\}$ is consists of a finite number of elements, X has the greatest element. Otherwise, $\{a_n\}$ is congruent to the naturally ordered natural numbers, and hence by Lemma 1, X has the same type to the naturally ordered natural numbers.

3. Mappings from a directed system. If φ is a mapping from a directed system X to a space R and if A is a subset of R, we define

$$\varphi \text{ ult } A \equiv \underset{x_0}{\exists} \underset{x}{\forall}\, x > x_0 \rightarrow \varphi(x) \in A,$$

$$\varphi \text{ div } A \equiv \underset{x_0}{\forall} \underset{x}{\exists}\, x > x_0, \quad \varphi(x) \in A.$$

It is easy to see that

$$\varphi \text{ ult } A \equiv \underset{X'}{\exists} X' \text{ res } X, \qquad \varphi(X') \subset A,$$

$$\begin{aligned} \varphi \text{ div } A &\equiv \underset{X'}{\exists} X' \text{ cof } X, \qquad \varphi(X') \subset A \\ &\equiv \underset{\sigma}{\exists} \text{ ran}(\varphi \circ \sigma) \subset A. \end{aligned}$$

4. The structures of neighbourhood systems and the types of convergences. We study the relation between a neighbourhood space for which a neighbourhood relation "nbd" is given and a convergence space for which a convergence relation "conv" is given. Here we confine the base of convergence within a directed system X. If all directed systems are taken, that is the usul convergence.

nbd given, conv is defined by

$$T_{nc}.\quad \varphi \text{ conv } a \equiv \underset{N}{\forall}\, N \text{ nbd } a \to \varphi \text{ ult } N$$

conv given, nbd is defined by

$$T_{cn}.\quad N \text{ nbd } a \equiv \underset{\varphi}{\forall}\, \varphi \text{ conv } a \to \varphi \text{ ult } N$$

We study the condition for nbd and conv to be mutually reversible by these transformations. For this pourpose we put

$$T'_{nc}.\quad (\underset{N}{\forall}\, N \text{ nbd } a \to \varphi \text{ ult } N) \to \varphi \text{ conv } a\,,$$

$$T''_{nc}.\quad \varphi \text{ conv } a \to (\underset{N}{\forall}\, N \text{ nbd } a \to \varphi \text{ ult } N)\,,$$

$$o_n.\quad [\underset{\varphi}{\forall}\, (\underset{M}{\forall}\, M \text{ nbd } a \to \varphi \text{ ult } M) \to \varphi \text{ ult } N] \to N \text{ nbd } a\,;$$

$$T'_{cn}.\quad (\underset{\varphi}{\forall}\, \varphi \text{ conv } a \to \varphi \text{ ult } N) \to N \text{ nbd } a\,,$$

$$T''_{cn}.\quad N \text{ nbd } a \to (\underset{\varphi}{\forall}\, \varphi \text{ conv } a \to \varphi \text{ ult } N)\,,$$

$$o_c.\quad [\underset{N}{\forall}\, (\underset{\psi}{\forall}\, \psi \text{ conv } a \to \psi \text{ ult } N) \to \varphi \text{ ult } N] \to \varphi \text{ conv } a\,.$$

It is easy to see that

$$\left.\begin{array}{l} T''_{nc} \to T''_{cn} \\ T'_{nc} \end{array}\right\} \to o_c$$

$$\left.\begin{array}{l} T'_{nc} \\ o_n \end{array}\right\} \to T'_{cn}\,,$$

$$\left.\begin{array}{l} T''_{cn} \to T''_{nc} \\ T'_{cn} \end{array}\right\} \to o_n$$

$$\left.\begin{array}{l} T'_{cn} \\ o_c \end{array}\right\} \to T'_{nc}\,.$$

From above we have

Theorem 1. *Each of nbd satisfying o_n and conv satisfying o_c turns to one another by T_{nc}, T_{cn} respectively and returns to itself by T_{nc} and T_{cn}, T_{cn} and T_{nc} respectively.*

o_c is reformed as follows:

$$(\underset{A}{\forall}\, \varphi \text{ div } A \to \underset{\psi}{\exists}\, \text{ran } \psi \subset A\,,\ \psi \text{ conv } a) \to \varphi \text{ conv } a\,.$$

This means a star-convergence.

Next we take the neighbourhood system of an additive topology:

1_n. $R \operatorname{nbd} a$,

2_n. $M \supset N \operatorname{nbd} a \rightarrow M \operatorname{nbd} a$,

3_n. $M, N \operatorname{nbd} a \rightarrow MN \operatorname{nbd} a$.

We study the relation between these conditions and o_n. We can easily verify the next lemma.

Lemma 3. $T'_{cn} \rightarrow 1_n$; T'_{cn}, $T''_{cn} \rightarrow 2_n$, 3_n.

Theorem 2. $o_n \rightarrow 1_n$, 2_n, 3_n.

Proof. If we define conv by T_{nc} from nbd satisfying o_n, by Theorem 1. T_{cn} is satisfied by the first nbd. Accordingly, by Lemma 3, 1_n, 2_n, 3_n are hold.

Theorem 3. *If* $\underset{a}{\forall} X > \tau\{N : N \operatorname{nbd} a\}$, $2_n \rightarrow o_n$.

Proof. o_n is equivalent to

$$\bar{A} \, \overline{\operatorname{nbd}} \, a \rightarrow \underset{\varphi}{\exists} \, \varphi \operatorname{div} A, \underset{N}{\forall} N \operatorname{nbd} a \rightarrow \varphi \operatorname{ult} N,$$

where $\bar{A}$ means the complement of A and $\overline{\operatorname{nbd}}$ means the negation of nbd. This formula is also equivalent to

$$\bar{A} \, \overline{\operatorname{nbd}} \, a \rightarrow \underset{\varphi}{\exists} \operatorname{ran} \varphi \subset A, \ \underset{N}{\forall} N \operatorname{nbd} a \rightarrow \varphi \operatorname{ult} N.$$

On the other hand, we have from 2_n

$$\bar{A} \, \overline{\operatorname{nbd}} \, a \rightarrow \underset{N}{\forall} N \operatorname{nbd} a \rightarrow NA \neq 0$$

Therefore, for o_n to be hold, it is sufficient to prove

$$(\underset{N}{\forall} N \operatorname{nbd} a \rightarrow NA \neq 0) \rightarrow \underset{\varphi}{\exists} \operatorname{ran} \varphi \subset A, \ \underset{N}{\forall} N \operatorname{nbd} a \rightarrow \varphi \operatorname{ult} N.$$

We shall prove the above formula. From this assumption $\{NA : N \operatorname{nbd} a\}$ is a collection of non-null sets. By Zermelo's axiom, there exists *a* function α which choices a element from each NA. Also, by the assumption of this theorem, there exists a divergent transformation ρ from X to $\{N : N \operatorname{nbd} a\}$.
Then

$$\operatorname{ran}(\alpha \circ \rho) \subset A$$

and

$$\underset{N}{\forall} N \operatorname{nbd} a \rightarrow \underset{x_0}{\exists} \underset{x}{\forall} x > x_0 \rightarrow \rho(x) \subset N.$$

If we notice that $(\alpha \circ \rho)(x) = \alpha(\rho(x)) \in \rho(x)$,

$$\underset{N}{\forall} N \operatorname{nbd} a \rightarrow \underset{x_0}{\exists} \underset{x}{\forall} x > x_0 \rightarrow (\alpha \circ \rho)(x) \in N,$$

that is

$$\underset{N}{\forall} N \operatorname{nbd} a \rightarrow (\alpha \circ \rho) \operatorname{ult} N.$$

Hence the existence of φ is assured by $\alpha \circ \rho$.

By Theorem 1, 3, we have

Theorem 4. *If* nbd *gives an additive topology and if* $\underset{a}{\forall} X > \tau$ $\{N : N \text{ nbd } a\}$, nbd *returns to itself by* T_{nc} *and* T_{cn}.

Next we study the relation between nbd, conv and the closure operator f. We put

$$T_{cf}.\quad a \in f(A) \equiv \underset{\varphi}{\exists} \operatorname{ran} \varphi \subset A,\ \varphi \operatorname{conv} a,$$
$$T_{fn}.\quad N \operatorname{nbd} a \equiv a \bar{\in} f(\bar{N}).$$

Hence

$$T_{cf} T_{fn}.\quad N \operatorname{nbd} a \equiv \overline{\underset{\varphi}{\exists} \operatorname{ran} \varphi \subset \bar{N},\ \varphi \operatorname{conv} a}$$
$$\equiv \underset{\varphi}{\forall} \varphi \operatorname{conv} a \rightarrow (\operatorname{ran} \varphi) N \neq 0.$$

On the other hand

$$T_{cn}.\quad N \operatorname{nbd} a \equiv \underset{\varphi}{\forall} \varphi \operatorname{conv} a \rightarrow \varphi \operatorname{ult} N.$$

The right side of $T_{cf} T_{fn}$ is weaker than that of T_{cn}. But if conv satisfies

$$1_c.\quad \varphi \operatorname{conv} a \rightarrow (\varphi \circ \sigma) \operatorname{conv} a,$$

both are mutually equivalent and $T_{cf} T_{fn} = T_{cn}$, because we have from 1_c,

$$(\underset{\varphi}{\exists} \operatorname{conv} a,\ \varphi \operatorname{div} \bar{N}) \rightarrow (\underset{\varphi}{\exists} \varphi \operatorname{conv} a,\ \operatorname{ran} \varphi \subset \bar{N}).$$

We put

$$T_{nf}.\quad a \in f(A) \equiv \bar{A}\, \overline{\operatorname{nbd}}\, a.$$

It is evident that nbd and f are mutually reversible by T_{nf}, T_{fn}. Since $T_{nc} \rightarrow 1_c$, we have

$$T_{nc} T_{cf} T_{fn} = T_{nc} T_{cn}.$$

Hence $T_{nc} T_{cn} = 1$ (1 means that nbd returns to itself) is equivalent to $T_{nc} T_{cf} T_{fn} = 1$. This is also equivalent to $T_{nc} T_{cf} = T_{nf}$, since nbd and f are mutually reversible by T_{nf}, T_{fn}.

Hence we have next

Theorem 5. nbd *returns to itself by* T_{nc} *and* T_{cn} *if and only if* nbd *and* conv *defined by* T_{nc} *give the same topology.*

From Theorem 4, 5, we have

Thorem 6. *If* nbd *gives an additive topology and if* $\underset{a}{\forall} X > \tau$ $\{N : N \text{ nbd } a\}$, nbd *is equivalent to a convergence with the base* X.

From Lemma 2, Theorem 6, we have

Theorem 7. *If the neighbourhood system of an additive topology satisfies the first countability axiom, the neighbourhood system is equivalent to a sequencial convergence.*

If we define conv from nbd and define nbd_1 from that conv,

$$\left.\begin{matrix} T''_{nc} \to T''_{cn} \\ T'_{cn_1} \end{matrix}\right\} \to (N \ \mathrm{nbd}\ a \to N \ \mathrm{nbd}_1\ a)\,.$$

Accordingly, the neighbourhood system of a point is extended.

Theorem 8. *We define* nbd_1, nbd_2 *from* nbd *by two processes*

$$\mathrm{nbd} \to \mathrm{conv}_1 \to \mathrm{nbd}_1 \ \textit{with base}\ X\,,$$
$$\mathrm{nbd} \to \mathrm{conv}_2 \to \mathrm{nbd}_2 \ \textit{with base}\ Y\,.$$

In this case, if $Y > X$, *then*

$$N \ \mathrm{nbd}_2\, a \to N \ \mathrm{nbd}_1\, a\,.$$

Proof. Let ρ be a divergent transformation from Y to X. If φ ult N, then

$$\underset{x_0}{\exists}\ \underset{x}{\forall}\ x > x_0 \to \varphi(x) \in N\,.$$

While

$$\underset{x_0}{\forall}\ \underset{y_0}{\exists}\ \underset{y}{\forall}\ y > y_0 \to \rho(y) > x_0\,.$$

Hence

$$\underset{y_0}{\exists}\ \underset{y}{\forall}\ y > y_0 \to (\varphi \circ \rho)(y) \in N.$$

This means $(\varphi \circ \rho)$ ult N. That is

$$\varphi \ \mathrm{ult}\ N \to (\varphi \circ \rho) \ \mathrm{ult}\ N\,.$$

Hence

$$(\underset{N}{\forall}\ N \ \mathrm{nbd}\ a \to \varphi \ \mathrm{ult}\ N) \to (\underset{N}{\forall}\ N \ \mathrm{nbd}\ a \to (\varphi \circ \rho) \ \mathrm{ult}\ N)\,.$$

That is

$$(*) \quad \varphi \ \mathrm{conv}_1\, a \to (\varphi \circ \rho) \ \mathrm{conv}_2\, a\,.$$

After this preparation, we go to the proof of this theorem. It is sufficient that we lead to a contradiction from

$$(\underset{\psi}{\forall}\ \psi \ \mathrm{conv}_2\, a \to \psi \ \mathrm{ult}\ N)\,, \qquad \varphi \ \mathrm{conv}_1\, a\,, \qquad \varphi \ \mathrm{div}\ N.$$

From $\varphi \ \mathrm{div}\ \bar{N}$, there exists a increasing transformation σ such that $\mathrm{ran}(\varphi \circ \sigma) \subset \bar{N}$. From $\varphi \ \mathrm{conv}_1\, a$, $(\varphi \circ \sigma) \ \mathrm{conv}_1\, a$, because T_{nc_1} implies $\varphi \ \mathrm{conv}_1\, a \to (\varphi \circ \sigma) \ \mathrm{conv}_1\, a$. Using (*), $(\varphi \circ \sigma \circ \rho) \ \mathrm{conv}_2\, a$. From $\underset{\psi}{\forall}\ \psi \ \mathrm{conv}_2\, a \to \psi \ \mathrm{ult}\ N$, $(\varphi \circ \sigma \circ \rho)$ ult N. On the other hand, from $\mathrm{ran}(\varphi \circ \sigma) \subset \bar{N}$, $\mathrm{ran}(\varphi \circ \sigma \circ \rho) \subset \bar{N}$. These two results lead us to a contradiction.

Theorem 9. *If $Y > X$ and the neighbourhood system of an additive topology is equivalent to the convergence defined by T_{nc} with the base X, then it is also equivalent to the convergence defined by T_{nc} with the base Y.*

Proof. By Theorem 8,

$$N \text{ nbd } a \rightarrow N \text{ nbd}_2\, a \rightarrow N \text{ nbd}_1\, a \rightarrow N \text{ nbd } a\,.$$

Hence

$$N \text{ nbd } a \rightleftarrows N \text{ nbd}_2\, a\,.$$

By Theorem 5, we get the statement.

Theorem 6 gives a sufficient type of X for an additive topology and a star-convergence to be equivalent. It remains to decide a necessary and sufficient type of X.

References

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