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Exact Sequences $\sum_p(K, L)$ and their Applications

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§ 0. Introduction

Homotopy classifications of mappings of an n dimensional finite cell complex K^n into an n-sphere S^n or an $(n-1)$-sphere S^{n-1} and the corresponding extension theorems were solved by H. Hopf and N. E. Steenrod [5] respectively. Introducing the cohomotopy group, E. Spanier [4] unified these results in an exact sequence, while J. H. C. Whitehead [9] gave a general and constructive method to obtain an exact sequence, starting with a certain sequence of homomorphisms.

In this paper, we shall define exact sequences $\sum_p(K)$ by applying Whitehead's method to the cohomotopy group of a complex K (§1). It is proved that $\sum_p(K)$ are invariances of homotopy type of complex K (§2), and that, as its special case, $\sum_0(K)$ may be regarded as a generalization of Spanier's sequence (§3, 4, 5). $\sum_p(K)$ are also utilized to obtain a homotopy classification theorem and a corresponding extension theorem concerning mappings of a certain kind of an $(n+2)$-dimensional complex into S^n (§6). Furthermore we determine the n-th cohomotopy group of an A_n^2-polyhedron in terms of its cohomology system (§7). At the end of this paper it is shown that two A_n^2-polyhedra are of the same homotopy type if and only if their Spanier's sequences are properly isomorphic.

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§ 1. Exact sequences $\sum_p(K, L)$

In the first place, let us define an exact sequence $\sum$ abstractly, following J. H. C. Whitehead [9].

Let r be an arbitrary fixed integer, and let (C, A) be the following sequence of groups and homomorphisms;

$$(1,1)\quad C^{r-1}\xrightarrow{\beta^{r-1}}A^r\xrightarrow{j^r}C^r\longrightarrow\cdots\cdots\longrightarrow A^q\xrightarrow{j^q}C^q\xrightarrow{\beta^q}A^{q+1}\xrightarrow{j^{q+1}}C^{q+1}\longrightarrow\cdots\cdots,$$

where C^q, A^q are arbitrary abelian groups and q is an integer such that $q\geqq r$. In this sequence it is assumed that $j^qA^q=\overset{-1}{\beta^q}(0)$ for any $q\geqq r$, but $\beta^{q-1}C^{q-1}=\overset{-1}{j^q}(0)$ is not always assumed. If we denote $d^q=j^{q+1}\beta^q$: $C^q\rightarrow C^{q+1}$, we have

$d^{q+1}d^q=0$. Let Z^q be $\overset{-1}{d^q}(0)$, then we have $d^{q-1}C^{q-1}\subset Z^q$. Now we define three groups Γ^q, Π^q, H^q with homomorphisms as follows:

$$(1.2)\qquad \Gamma^q = \overset{-1}{j^q}(0),\quad \Pi^q = A^q/\beta^{q-1}C^{q-1},\quad H^q = Z^q/d^{q-1}C^{q-1}$$

As to homomorphisms we define

i) $\mathfrak{b}^q$: $H^q\to\Gamma^{q-1}$. Let $z\in Z^q$ be a representative of a class of H^q, then $d^qz=j^{q+1}\beta^q z=0$, so that $\beta^q z\in \overset{-1}{j^{q+1}}(0)=\Gamma^{q+1}$. Since $\beta^q(d^{q-1}C^{q-1})=\beta^q j^q\beta^{q-1}C^{q-1}=0$, a mapping $z\to\beta^q z$ induces a homomorphism $\mathfrak{b}^q$: $H^q\to\Gamma^{q+1}$.

ii) $\mathfrak{i}^q$: $\Gamma^q\to\Pi^q$. If $\gamma\in\Gamma^q$, γ is an element of A^q. Thus we define $\mathfrak{i}^q$ such that γ corresponds to a class of Π^q containing γ.

iii) $\mathfrak{j}^q$: $\Pi^q\to H^q$. Let $a\in A^q$ be a representative of $\bar{a}\in\Pi^q$, then $d^qj^qa=j^{q+1}\beta^qj^qa=0$, so that $j^qa\in Z^q$. Since $j^q\beta^{q-1}C^{q-1}=d^{q-1}C^{q-1}$, a correspondence $\bar{a}\to\{j^qa\}$, a class of H^q containing j^qa, induces a homomorphism $\mathfrak{j}^q$: $\Pi^q\to H^q$.

As a direct consequence of our definition we have, as is shown in [9],

Lemma 1. *The sequence*

$$(1.3)\quad \Sigma:\ \Gamma^r \xrightarrow{\mathfrak{i}^r} \Pi^r \longrightarrow \cdots\cdots \longrightarrow \Gamma^q \xrightarrow{\mathfrak{i}^q} \Pi^q \xrightarrow{\mathfrak{j}^q} H^q \xrightarrow{\mathfrak{b}^q} \Gamma^{q+1} \longrightarrow \cdots\cdots$$

is exact.

Next, we shall apply the above result to the cohomotopy group.

Let K be a complex, the subcomplex of which is denoted by L, and let y be a fixed point of a k-sphere S^k. If dim$(K-L)\leqq n$ and if $n\leqq 2k-2$, we can define an addition among all the homotopy classes of mappings f: $(K, L)\to(S^k, y)$, following Borsuk-Spanier. Thus we have the k-dimensional cohomotopy group, which is designated by $\pi^k(K, L)$. Refer to E. Spanier [4] for detailed account. From now on we shall use terminologies and notations in [4], and it is assumed in §§1–6 that (K, L) is a complex pair with dim$(K-L)\leqq n$.

Let p be an arbitrary fixed integer, and let $r(p)$ be the smallest integer satisfying

$$(1.4)\qquad r = r(p) \geqq \mathrm{Max}\left\{\frac{n}{2}+1-p,\ 3-2p\right\}.$$

Let us define C^q, A^q, β^q, j^q in (1.1) as follows:

$$C^q = C^q_p(K, L) = \pi^{p+q}(\bar{K}^q, \bar{K}^{q-1}) \qquad (q\geqq r(p)-1),$$
$$A^q = A^q_p(K, L) = \pi^{p+q}(K, \bar{K}^{q-1}) \qquad (q\geqq r(p)),$$
$$\beta^q = \beta^q_p(K, L) = \Delta:\ \pi^{p+q}(\bar{K}^q, \bar{K}^{q-1})\to\pi^{p+q+1}(K, \bar{K}^q) \qquad (q\geqq r(p)-1),$$
$$j^q = j^q_p(K, L) = i^\#:\ \pi^{p+q}(K, \bar{K}^{q-1})\to\pi^{p+q}(\bar{K}^q, \bar{K}^{q-1}) \qquad (q\geqq r(p)),$$

where $\bar{K}^q=K^q\cup L$, and Δ is the usual coboundary operator of the cohomotopy group and $i^\#$ is the homomorphism induced by the inclusion map i: $(\bar{K}^q, \bar{K}^{q-1})\to(K, \bar{K}^{q-1})$.[1] Let us remember here that groups and homomorphismus defined

1) In the following, for the sake of brevity, we shall call a homomorphism between cohomotopy groups induced by an inclusion "inclusion homomorphism".

above are not meaningless under the restriction in dimensions, which are indicated in the round brackets.

Since the sequence

$$\pi^{p+q}(K, \bar{K}^{q-1}) \xrightarrow{i\#} \pi^{p+q}(\bar{K}^q, \bar{K}^{q-1}) \xrightarrow{\Delta} \pi^{p+q+1}(K, \bar{K}^q)$$

is exact, we have $j^q A^q = \overset{-1}{\beta^q}(0)$. Thus groups and homomorphisms, $\Gamma^q = \Gamma^q_p(K, L)$, $\Pi^q = \Pi^q_p(K, L)$ $\mathrm{H}^q = \mathrm{H}^q_p(K, L)$ and $\mathfrak{i}^q = \mathfrak{i}^q_p(K, L)$, $\mathfrak{j}^q = \mathfrak{j}^q_p(K, L)$, $\mathfrak{h}^q = \mathfrak{h}^q_p(K, L)$ can be defined for any $q \geqq r(p)$. From Lemma 1 we have

Theorem 1. *The sequence* $\sum = \sum_p(K, L)$:

$$\Gamma^r_p(K, L) \xrightarrow{\mathfrak{i}^r_p} \Pi^r_p(K, L) \longrightarrow \cdots\cdots \longrightarrow \Gamma^q_p(K, L) \xrightarrow{\mathfrak{i}^q_p} \Pi^q_p(K, L) \xrightarrow{\mathfrak{j}^q_p} \mathrm{H}^q_p(K, L) \xrightarrow{\mathfrak{h}^q_p} \Gamma^{q+1}_p(K, L) \longrightarrow \cdots\cdots$$

is exact, where $r = r(p) = Max\left(\frac{n}{2} + 1 - p,\ 3 - 2p\right)$.

§2. Properties of Γ, Π, H.

In this section we shall establish formal properties of $\Gamma^q_p(K, L)$, $\Pi^q_p(K, L)$ and $\mathrm{H}^q_p(K, L)$.

I) Consider the diagram

$$\begin{array}{ccccccc}
\pi^{p+q-1}(\bar{K}^{q-1}, \bar{K}^{q-2}) & \xrightarrow{\Delta} & \pi^{p+q}(K, \bar{K}^{q-1}) & \xrightarrow{i\#} & \pi^{p+q}(K, \bar{K}^{q-2}) & \xrightarrow{i\#} & \pi^{p+q}(\bar{K}^{q-1}, \bar{K}^{q-2}) \\
\| & & \| & & \| & & \| \\
C^{q-1}_p(K, L) & \xrightarrow{\beta^{q-1}_p} & A^q_p(K, L) & & A^{q-1}_{p+1}(K, L) & \xrightarrow{\mathfrak{j}^{q-1}_{p+1}} & C^{q-1}_{p+1}(K, L)
\end{array}$$

in which the upper sequence is exact. Then from the definition of $\Gamma^{q-1}_{p+1}(K, L)$, $\Pi^q_p(K, L)$ we have immediately

(2.1) $$\mathfrak{l}^q_p : \Pi^q_p(K, L) \cong \Gamma^{q-1}_{p+1}(K, L)$$

for any $q \geqq r(p)$, where $\mathfrak{l}^q_p$ is the homomorphism induced by the inclusion homomorphism $\pi^{p+q}(K, \bar{K}^{q-1}) \to \pi^{p+q}(K, \bar{K}^{q-2})$.

II) Let $p \geqq 1$, then $C^q_p(K, L) = \pi^{p+q}(\bar{K}^q, \bar{K}^{p-1}) = 0$, so that we have $\Gamma^q_p(K, L) = A^q_p(K, L) = \pi^{p+q}(K, \bar{K}^{q-1})$ for any $q \geqq r$. Since the sequence

$$\pi^{p+q-1}(\bar{K}^{q-1}, L) \xrightarrow{\Delta} \pi^{p+q}(K, \bar{K}^{q-1}) \xrightarrow{j\#} \pi^{p+q}(K, L) \xrightarrow{i\#} \pi^{p+q}(\bar{K}^{q-1}, L)$$

is exact and since $\pi^{p+q-1}(\bar{K}^{q-1}, L) = 0$, $\pi^{p+q}(\bar{K}^{q-1}, L) = 0$ for $p \geqq 1$, we have $\pi^{p+q}(K, \bar{K}^{q-1}) \cong \pi^{p+q}(K, L)$. Thus we have

$(2.2)_1$ $$\mathfrak{k}^q_p : \Gamma^q_p(K, L) \cong \pi^{p+q}(K, L)$$

for $p \geqq 1$ and for $q \geqq r(p)$, where $\mathfrak{k}^q_p$ is the homomorphism induced by the inclusion homomorphism $\pi^{p+q}(K, \bar{K}^{q-1}) \to \pi^{p+q}(K, L)$.

III) From (2.1) and (2.2), we have

$(2.2)_2$ $$\mathfrak{k}^{q-1}_{p+1}\mathfrak{l}^q_p : \Pi^q_p(K, L) \cong \pi^{p+q}(K, L)$$

for $p \geqq 0$ and for $q \geqq r(p)$.

IV) Let $q \geqq n$, then we have $\pi^{p+q}(K, \bar{K}^q) = \pi^{p+q}(K, K) = 0$. From the exactness of the sequence

$$\pi^{p+q}(K, \bar{K}^q) \xrightarrow{j^{\#}} \pi^{p+q}(K, \bar{K}^{q-1}) \xrightarrow{i^{\#}} \pi^{p+q}(\bar{K}^q, \bar{K}^{q-1}),$$

it is concluded that $i^{\#}$ is isomorphic into, so that

$$(2.3)_1 \qquad \Gamma_p^q(K, L) = 0$$

for any $q \geqq \mathrm{Max}(n, r(p))$.

V) From (2.1) and $(2.3)_1$, we have

$$(2.3)_2 \qquad \Pi_p^q(K, L) = 0$$

for any $q \geqq \mathrm{Max}(n+1, r(p))$.

VI) By definition $\pi^{p+q}(\bar{K}^q, \bar{K}^{q-1})$ is isomorphic onto $C^q(K, L: (p+q)^q)$,[2] the q-dimensional cochain group, for $q \geqq 2-2p$. We denote this isomorphism by $\bar{\psi}_p^q$. Then it was proved by Spanier [4] that the commutativity holds in the diagram

$$\begin{array}{ccc} \pi^{p+q}(\bar{K}^q, \bar{K}^{q-1}) & \xrightarrow{\Delta} & \pi^{p+q+1}(\bar{K}^{q+1}, \bar{K}^q) \\ \bar{\psi} \downarrow \wr\Vert & & \bar{\psi} \downarrow \wr\Vert \\ C^q(K, L; (p+q)^q) & \xrightarrow{E\delta} & C^{q+1}(K, L; (p+q+1)^{q+1}) \end{array}$$

for $q \geqq 2-2p$, where δ is the coboundary operator of the cochain group, and E is the suspension of the coefficient group. From this fact, we have easily

$$(2.4) \qquad \psi_p^q: \mathrm{H}_p^q(K, L) \approx \mathfrak{H}^q(K, L; (p+q)^q)$$

for $q \geqq 3-2p$, where ψ_p^q is the homomorphism induced by the isomorphism $\bar{\psi}_p^q$.

VII) Let $n \geqq r(p)$, then from Theorem 1 the sequence

$$\Gamma_p^n(K, L) \xrightarrow{\mathfrak{i}_p^n} \Pi_p^n(K, L) \xrightarrow{\mathfrak{j}_p^n} \mathrm{H}_p^n(K, L) \xrightarrow{\mathfrak{h}_p^n} \Gamma_p^{n+1}(K, L)$$

is exact, and from $(2.3)_1$ we have $\Gamma_p^n(K, L) = 0$, $\Gamma_p^{n+1}(K, L) = 0$. Therefore we have

$$(2.5) \qquad \mathfrak{j}_p^n: \Pi_p^n(K, L) \approx \mathrm{H}_p^n(K, L).$$

From (2.1), (2.4) and (2.5), it is concluded that

$$(2.6) \qquad \psi_p^n \mathfrak{j}_p^n \mathfrak{l}_p^{n^{-1}}: \Gamma_{p+1}^{n-1}(K, L) \simeq \mathfrak{H}^n(K, L; (n+p)^n)$$

for $n \geqq 3-2p$.

Thus we have proved

Lemma 2.

$$\begin{array}{rll} \mathfrak{l}_p^q: \Pi_p^q(K, L) \approx \Gamma_{p+1}^{q-1}(K, L) & \textit{for} & q \geqq r(p), \\ \mathfrak{k}_p^q: \Pi_p^q(K, L) \approx \pi^{p+q}(K, L) & \textit{for} & p \geqq 1 \ \textit{and} \ q \geqq r(p), \\ \mathfrak{k}_{p+1}^{q-1}\mathfrak{l}_p^q: \Pi_p^q(K, L) \approx \pi^{p+q}(K, L) & \textit{for} & p \geqq 0 \ \textit{and} \ q \geqq r(p), \\ \Gamma_p^q(K, L) = 0 & \textit{for} & q \geqq \textit{Max}(n, r(p)), \\ \Pi_p^q(K, L) = 0 & \textit{for} & q \geqq \textit{Max}(n+1, r(p)), \end{array}$$

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2) We denote by q^p the p-th homotopy group $\pi_p(S^q)$ of a q-sphere S^q.

$$\psi_p^q: \quad \mathrm{H}_p^q(K, L) \cong \mathfrak{H}^q(K, L; (p+q)^q) \quad \text{for} \quad q \geqq 3-2p$$
$$\psi_p^n \mathfrak{j}_p^n {\mathfrak{l}_p^n}^{-1}: \Gamma_{p+1}^{n-1}(K, L) \cong \mathfrak{H}^n(K, L; (n+p)^n) \quad \text{for} \quad n \geqq 3-2p$$

§3. Invariance of $\sum_p (K, L)$

Let (K, L) and (K', L') be complex pairs with $\dim(K-L) \leqq n$ and with $\dim(K'-L') \leqq n$ respectively. Let us consider a cellular map $f: (K', L') \to (K, L)$, then f induces homomorphisms

$$_C\mathfrak{f}_p^{q\#}: C_p^q(K, L) \longrightarrow C_p^q(K', L'),$$
$$_A\mathfrak{f}_p^{q\#}: A_p^q(K, L) \longrightarrow A_p^q(K', L')$$

for each $q \geqq r(p)$, in virtue of $f(\bar{K}'^q) \subset \bar{K}^q$. And we have[3)]

$$\beta_p^{q\prime}\ {}_C\mathfrak{f}_p^{q\#} = {}_A\mathfrak{f}_p^{q\#}\ \beta_p^q,$$
$$j_p^{q\prime}\ {}_A\mathfrak{f}_p^{q\#} = {}_C\mathfrak{f}_p^{q\#}\ j_p^q,$$

so that $_A\mathfrak{f}_p^{q\#}$ induces homomorphisms $_\Gamma\mathfrak{f}_p^q: \Gamma_p^q(K, L) \to \Gamma_p^q(K', L')$ and $_\Pi\mathfrak{f}_p^q: \Pi_p^q(K, L) \to \Pi_p^q(K', L')$, and $_C\mathfrak{f}_p^{q\#}$ induces a homomorphism $_H\mathfrak{f}_p^q: \mathrm{H}_p^q(K, L) \to \mathrm{H}_p^q(K', L')$. Then it is seen that

$$(3.1) \qquad \begin{cases} \mathfrak{i}_p^{q\prime}\ {}_\Gamma\mathfrak{f}_p^q = {}_\Pi\mathfrak{f}_p^q\ \mathfrak{i}_p^q, \\ \mathfrak{j}_p^{q\prime}\ {}_\Pi\mathfrak{f}_p^q = {}_H\mathfrak{f}_p^q\ \mathfrak{j}_p^q, \\ \mathfrak{b}_p^{q\prime}\ {}_H\mathfrak{f}_p^q = {}_\Gamma\mathfrak{f}_p^{q+1}\ \mathfrak{b}_p^q. \end{cases}$$

Lemma 3. *If* $f, g: (K', L') \to (K, L)$ *are homotopical maps, we have* $_\Gamma\mathfrak{f}_p^q = {}_\Gamma\mathfrak{g}_p^q$, $_\Pi\mathfrak{f}_p^q = {}_\Pi\mathfrak{g}_p^q$, *and* $_H\mathfrak{f}_p^q = {}_H\mathfrak{g}_p^q$.

Proof. In virtue of the assumption there exists a map $F: (K' \times I, L' \times I) \to (K, L)$ such that

$$F_0 = F | K' \times 0 = f,$$
$$F_1 = F | K' \times 1 = g,$$

where I denotes the interval between 0 and 1. Further it may be assumed without loss of generality that F is cellular (i. e. $F(K'^{q-1} \times I) \subset K^q$ for any q) [8].

i) $_\Gamma\mathfrak{f} = {}_\Gamma\mathfrak{g}$. If $\gamma \in \Gamma_p^q(K, L)$, we have $\gamma \in \pi^{p+q}(K, \bar{K}^{q-1})$ and $i^\# \gamma = 0$. Since the sequence

$$\pi^{p+q}(K, \bar{K}^q) \xrightarrow{j^\#} \pi^{p+q}(K, \bar{K}^{q-1}) \xrightarrow{i^\#} \pi^{p+q}(\bar{K}^q, \bar{K}^{q-1})$$

is exact, γ belongs to $j^\# \pi^{p+q}(K, \bar{K}^q)$, so that a map $t: (K, \bar{K}^q) \to (S^{p+q}, y)$ can be taken as a representative of γ. Since $F: (K' \times I, \bar{K}'^{q-1} \times I) \to (K, K^q)$, we have

$$tF: (K' \times I, \bar{K}'^{q-1} \times I) \longrightarrow (S^{p+q}, y).$$

Therefore $\{tF_0\} = \mathfrak{f}^\# \gamma$ and $\{tF_1\} = g^\# \gamma$ represent the same element of $\pi^{p+q}(K', \bar{K}'^{q-1})$. This proves $_\Gamma\mathfrak{f} = {}_\Gamma\mathfrak{g}$.

ii) $_\Pi\mathrm{f} = {}_\Pi\mathrm{g}$. The commutativity holds in the diagram

3) We agree that $\mathfrak{i}, \mathfrak{j}, \beta, i, j, b$ in the complex pair (K', L') are denoted by $i', \mathfrak{j}', \beta', i, j', b'$.

$$\begin{array}{ccc} \pi^{p+q}(K', \bar{K}'^{q-1}) & \xrightarrow{j^{\#}} & \pi^{p+q}(K', \bar{K}'^{q-2}) \\ \downarrow {}_A\mathfrak{f}^{q\#}_p & & \downarrow {}_A\mathfrak{f}^{q-1\#}_{p+1} \\ \pi^{p+q}(K, \bar{K}^{q-1}) & \xrightarrow{j'^{\#}} & \pi^{p+q}(K, K^{q-2}), \end{array}$$

where $j^{\#}$, $j'^{\#}$ are the inclusion homomorphisms. Therefore, if $\mathfrak{l}^q_p: \Pi^q_p(K, L) \to \Gamma^{q-1}_{p+1}(K, L)$, $\mathfrak{l}^{q\prime}_p: \Pi^q_p(K', L') \to \Gamma^{q-1}_{p+1}(K', L')$ are the homomorphisms induced by $j^{\#}$, $j'^{\#}$ respectively, we have

$$\mathfrak{l}^{q\prime}_p \; {}_{\Pi}\mathfrak{f}^q_p = {}_{\Gamma}\mathfrak{f}^{q-1}_{p+1} \mathfrak{l}^q_p .$$

By the same process with respect to g we have

$$\mathfrak{l}^{q\prime}_a \; {}_{\Pi}\mathfrak{g}^q_p = {}_{\Gamma}\mathfrak{g}^{q-1}_{p+1} \mathfrak{l}^q_p .$$

From these, together with i), we have

$$\mathfrak{l}^{q\prime}_p \; {}_{\Pi}\mathfrak{f}^q_p = \mathfrak{l}^{q\prime}_p \; {}_{\Pi}\mathfrak{g}^q_p \,{}^{4)} .$$

As $\mathfrak{l}^{q\prime}_p$ is an isomorphism[4] from (2.1), we have

$${}_{\Pi}\mathfrak{f}^q_p = {}_{\Pi}\mathfrak{g}^q_p .$$

iii) ${}_H\mathfrak{f} = {}_H\mathfrak{g}$. Let $\{a\} \in \pi^{p+q-1}(\bar{K}'^{q-1}, \bar{K}'^{q-2})$ and let $a: (\bar{K}'^{q-1}, \bar{K}'^{q-2}) \to (S^{p+q-1}, y)$ be a representative of $\{a\}$. Then we shall define a map $E^q_p(a): (\bar{K}'^{q-1} \times I, (\bar{K}'^{q-1} \times I)^{q-1}) \to (S^{p+q}, y)$ by

$$E^q_p(a)(x, t) = \varphi(a(x), t) \qquad \text{for } x \in \bar{K}'^{q-1},\ t \in I,$$

where $\varphi: S^{p+q-1} \times I \to S^{p+q}$ maps $(y \times I) \cup (S^{p+q-1} \times 0) \cup (S^{p+q-1} \times 1)$ into a point y and elsewhere topologically onto $S^{p+q-1} - y$. If $E^{q\#}_p: \pi^{p+q-1}(\bar{K}'^{q-1}, \bar{K}'^{q-2}) \to \pi^{p+q}(\bar{K}'^{q-1} \times I, (\bar{K}^{q-1} \times I)^{q-1})$ is a homomorphism such that $\{a\}$ corresponds to $\{E^q_p(a)\} \in \pi^{p+q}(\bar{K}'^{q-1} \times I, (\bar{K}'^{q-1} \times I)^{q-1})$, $E^{q\#}_p$ is evidently an isomorphism for $q \geqq 3 - 2p$ in virtue of Freudenthal's suspension theorem. Moreover $F | \bar{K}'^{q-1} \times I$ maps $\bar{K}'^{q-1} \times I$, $(\bar{K}'^{q-1} \times I)^{q-1}$ into $\bar{K}^q$, $\bar{K}^{q-1}$ respectively, so that it induces a homomorphism

$$F^{q\#}_p: \pi^{p+q}(\bar{K}^q, \bar{K}^{q-1}) \longrightarrow \pi^{p+q}(\bar{K}'^{q-1} \times I, (\bar{K}'^{q-1} \times I)^{q-1}) .$$

If we put $\xi^q_p = E^{q\#^{-1}}_p F^{q\#}_p$, ξ^q_p is a homomorphism $\pi^{p+q}(\bar{K}^q, \bar{K}^{q-1}) \to \pi^{p+q-1}(K'^{q-1}, K'^{q-2})$. Namely we have the homomorphism $\xi^q_p: C^q_p(K, L) \to C^{q-1}_p(K', L')$. Then it can be proved as in the classical chain homotopy theory that ξ^q_p has a property:[5]

$${}_G\mathfrak{f}^{q\#}_p - {}_G\mathfrak{g}^{q\#}_p = \xi^{q+1}_p d^q_p + d'^{q-1}_p \xi^q_p ,$$

so that ${}_H\mathfrak{f}^q_p = {}_H\mathfrak{g}^q_p$. This completes the proof of Lemma 3.

Theorem 2. *If two complex pairs (K, L), (K', L') with $dim(K-L) \leqq n$ and $dim(K'-L') \leqq n$ are of the same homotopy type, $\sum_p(K, L)$ and $\sum_p(K', L')$ are isomorphic. Namely, there exists a family of isomorphisms $\mathfrak{f} = \{{}_{\Gamma}\mathfrak{f}^q_p, {}_{\Pi}\mathfrak{f}^q_p, {}_H\mathfrak{f}^q_p\}$ such that the commutativity holds in each rectangle of the diagaam*

4) An isomorphism, without qoulification, will always mean an isomorphism onto.

5) Such a homorphism ξ is called a *homotopy operator for $f^{\#}$ and $g^{\#}$* in the classical theory of chain homotopy (cf. S. Lefschetz: Algebraic Topology (1942))

$$
(3.2)\qquad
\begin{array}{ccccccccc}
\Gamma^r_p(K,L) & \xrightarrow{\mathfrak{i}} & \Pi^r_p(K,L) & \longrightarrow & \cdots & \longrightarrow & \Gamma^q_p(K,L) & \xrightarrow{\mathfrak{i}} & \\
\downarrow {}_{\Gamma}\mathfrak{f} & \mathfrak{i}' & \downarrow {}_{\Pi}\mathfrak{f} & & & & \downarrow {}_{\Gamma}\mathfrak{f} & \mathfrak{i}' & \\
\Gamma^r_p(K',L') & \longrightarrow & \Pi^r_p(K',L') & \longrightarrow & \cdots & \longrightarrow & \Gamma^q_p(K',L') & \longrightarrow &
\end{array}
$$

$$
\begin{array}{ccccccc}
\Pi^q_p(K,L) & \xrightarrow{\mathfrak{j}} & \mathrm{H}^q_p(K,L) & \xrightarrow{\mathfrak{b}} & \Gamma^{q+1}_p(K,L) & \longrightarrow & \cdots\cdots \\
\downarrow {}_{\Pi}\mathfrak{f} & \mathfrak{j}' & \downarrow {}_{\mathrm{H}}\mathfrak{f} & \mathfrak{b}' & \downarrow {}_{\Gamma}\mathfrak{f} & & \\
\Pi^q_p(K',L') & \longrightarrow & \mathrm{H}^q_p(K,L) & \longrightarrow & \Gamma^{q+1}_p(K',L') & \longrightarrow & \cdots\cdots,
\end{array}
$$

where $r=r(p)\geqq \mathrm{Max}\left(\frac{n}{2}+1-p,\ 3-2p\right)$.

Proof. As was shown in [8], homotopy equivalences $f: (K', L')\to(K, L)$ and $g: (K, L)\to(K', L')$ can be assumed to be cellular. If ${}_{\Gamma}\mathfrak{f}^q_p: \Gamma^q_p(K, L)\to\Gamma^q_p(K', L')$, ${}_{\Pi}\mathfrak{f}^q_p: \Pi^q_p(K, L)\to\Pi^q_p(K', L')$, ${}_{\mathrm{H}}\mathfrak{f}^q_p: \mathrm{H}^q_p(K, L)\to\mathrm{H}^q_p(K', L')$ are isomorphisms induced by f, it is seen from (3.1) that the commutativity holds in each rectangle of (3.2).

Let ${}_{\Gamma}\mathfrak{g}^q_p: \Gamma^q_p(K', L')\to\Gamma^q_p(K, L)$, ${}_{\Pi}\mathfrak{g}^q_p: \Pi^q_p(K', L')\to\Pi^q_p(K, L)$, ${}_{\mathrm{H}}\mathfrak{g}^q_p: \mathrm{H}^q_p(K', L')\to\mathrm{H}^q_p(K, L)$ be homomorphisms induced by g and let ${}_{\Gamma}(\mathfrak{f}\mathfrak{g})^q_p: \Gamma^q_p(K, L)\to\Gamma^q_p(K, L)$, ${}_{\Pi}(\mathfrak{f}\mathfrak{g})^q_p: \Pi^q_p(K, L)\to\Pi^q_p(K, L)$, ${}_{\mathrm{H}}(\mathfrak{f}\mathfrak{g})^q_p: \mathrm{H}^q_p(K, L)\to\mathrm{H}^q_p(K, L)$ be homomorphisms induced by $fg: (K, L)\to(K, L)$. Then, since fg is homotopic to the identity, it follows from Lemma 3 that the homomorphisms ${}_{\Gamma}(\mathfrak{f}\mathfrak{g})^q_p$, ${}_{\Pi}(\mathfrak{f}\mathfrak{g})^q_p$, ${}_{\mathrm{H}}(\mathfrak{f}\mathfrak{g})^q_p$ are all the identities. As is easily seen, we have ${}_{\Gamma}(\mathfrak{f}\mathfrak{g})^q_p={}_{\Gamma}\mathfrak{g}^q_p\,{}_{\Gamma}\mathfrak{f}^q_p$, ${}_{\Pi}(\mathfrak{f}\mathfrak{g})^q_p={}_{\Pi}\mathfrak{g}^q_p\,{}_{\Pi}\mathfrak{f}^q_p$, ${}_{\mathrm{H}}(\mathfrak{f}\mathfrak{g})^q_p={}_{\mathrm{H}}\mathfrak{g}^q_p\,{}_{\mathrm{H}}\mathfrak{f}^q_p$. Thus ${}_{\Gamma}\mathfrak{g}^q_p$, ${}_{\Pi}\mathfrak{g}^q_p$, ${}_{\mathrm{H}}\mathfrak{g}^q_p$ are all onto and ${}_{\Gamma}\mathfrak{f}^q_p$, ${}_{\Pi}\mathfrak{f}^q_p$, ${}_{\mathrm{H}}\mathfrak{f}^q_p$ are all isomorphisms into. Again, using $gf\simeq 1: (K', L')\to(K', L')$, we see that ${}_{\Gamma}\mathfrak{g}^q_p$, ${}_{\Pi}\mathfrak{g}^q_p$, ${}_{\mathrm{H}}\mathfrak{g}^q_p$ are all isomorphisms into and ${}_{\Gamma}\mathfrak{f}^q_p$, ${}_{\Pi}\mathfrak{f}^q_p$, ${}_{\mathrm{H}}\mathfrak{f}^q_p$ are all onto. Thus Theorem 2 is established.

§4. Properties of $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{b}$.

In this section we shall prove several properties of $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{b}$.

Lemma 4. *In the diagram*

$$
\begin{array}{ccc}
\Gamma^{n-1}_0(K,L) & \xrightarrow{\mathfrak{i}} & \Pi^{n-1}_0(K,L) \\
\wr\!\!\parallel \downarrow \psi\mathfrak{j}\mathfrak{l}^{-1} & \Lambda & \wr\!\!\parallel \downarrow \mathfrak{k}\mathfrak{l} \\
\mathfrak{H}^n(K, L, (n-1)^n) & \longrightarrow & \pi^{n-1}(K,L)
\end{array}
$$

the commutativity holds, where $n\geqq 5$ *and* Λ *is the homomorphism which was given by E. Spanier* [4, §20].

Proof. $\mathfrak{l}\mathfrak{j}^{-1}$, $\mathfrak{i}$ and $\mathfrak{k}\mathfrak{l}$ are induced by the inclusion homomorphisms, $\pi^{n-1}(K,\bar{K}^{n-1})\to\pi^{n-1}(K,\bar{K}^{n-2})$, $\pi^{n-1}(K,\bar{K}^{n-2})\to\pi^{n-1}(K,\bar{K}^{n-2})$ and $\pi^{n-1}(K,\bar{K}^{n-2})\to\pi^{n-1}(K, L)$ respectively. So $\mathfrak{l}\mathfrak{k}\mathfrak{i}\mathfrak{l}\mathfrak{j}^{-1}$ is induced by the inclusion homomorphism: $\pi^{n-1}(K, \bar{K}^{n-1})\to\pi^{n-1}(K, L)$. Thus $\Lambda'=\mathfrak{k}\mathfrak{l}\mathfrak{i}(\psi\mathfrak{j}\mathfrak{l}^{-1})^{-1}$ is a homomorphism such that we describe below. $\{z\}\in\mathfrak{H}^n(K, L, (n-1)^n)$ is represented by a cocycle z such that for an n-cell σ^n, $z(\sigma^n)$ is an element of $\pi_n(S^{n-1})$. Let $\alpha: K\to S^{n-1}$ be a map such that $\alpha|\bar{K}^{n-2}=y$, and $\alpha|\sigma^n$ represents $z(\sigma^n)$. Then $\{z\}$ corresponds to $\{\alpha\}\in\pi^{n-1}(K, L)$ by Λ'. This is the definition of Λ. Thus $\Lambda'=\Lambda$, and so Lemma 4 is proved.

Lemma 5. *In the diagram*

$$\begin{array}{ccc} \Pi_0^q(K, L) & \xrightarrow{\ \mathfrak{j}\ } & \mathrm{H}_0^q(K, L) \\ \wr\!\!\parallel \big\downarrow \mathfrak{lk} & \bar{\phi} & \wr\!\!\parallel \big\downarrow \psi \\ \pi^q(K, L) & \longrightarrow & \mathfrak{H}^q(K, L;\ q^q), \end{array}$$

the commutativity holds, where $q \geqq \mathrm{Max}\left(\frac{n}{2}+1, 3\right)$, *and* $\bar{\phi}$ *is the natural homomorphism of the cohomotopy group into the cohomology group* [4, §17].

Proof. Since $\mathfrak{lk}$ is induced by the inclusion homomorphism: $\pi^q(K, \bar{K}^{q-1}) \rightarrow \pi^q(K, L)$ and since $\mathfrak{j}$ is induced by the inclusion homomorphism: $\pi^q(K, \bar{K}^{q-1}) \rightarrow \pi_q(\bar{K}^q, \bar{K}^{q-1})$, $\varphi\mathfrak{j}(\mathfrak{lk})^{-1}$ is a homomorphism, by which $a \in \pi^q(K, L)$ corresponds to an element of $\mathfrak{H}^q(K, L, q^q)$ containing $\bar{\varphi} j^{\#}(lk)^{\#-1}\, a$. This correspondence is nothing else but the definition of $\bar{\phi}$. This proves Lemma 5.

Lemma 6. *In the diagram*

$$(3.2)\qquad \begin{array}{ccccccc} \mathrm{H}_0^q(K, L) & \xrightarrow{\ \mathfrak{h}_0^q\ } & \Gamma_0^{q+1}(K, L) & \underset{\cong}{\xleftarrow{\ \mathfrak{l}_{-1}^{q+2}\ }} & \Pi_{-1}^{q+2}(K, L) & \xrightarrow{\ \mathfrak{j}_{-1}^{q+2}\ } & \mathrm{H}_{-1}^{q+2}(K, L) \\ \wr\!\!\parallel \big\downarrow \psi_0^q & & & \mathrm{Sq}^2 & & & \wr\!\!\parallel \big\downarrow \psi_{-1}^{q+2} \\ \mathfrak{H}^q(K, L;\ q^q) & & & \longrightarrow & & & \mathfrak{H}^{q+2}(K, L;\ (q+1)^{q+2}) \end{array}$$

the commutativity

$$\psi_{-1}^{q+2}\mathfrak{j}_{-1}^{q+2}\mathfrak{l}_{-1}^{q+1^{-1}}\mathfrak{h}_0^q = \mathrm{Sq}^2\psi_0^q$$

holds true, where $q \geqq \mathrm{Max}\left(\frac{n}{2}, 3\right)$.[6)]

Lemma 7. *In the diagram*

$$\begin{array}{ccccccc} \mathrm{H}_{-1}^q(K, L) & \xrightarrow{\ \mathfrak{h}_{-1}^q\ } & \Gamma_{-1}^{q+1}(K, L) & \underset{\cong}{\xrightarrow{\ \mathfrak{l}_{-2}^{q+2}\ }} & \Pi_{-2}^{q+2}(K, L) & \xrightarrow{\ \mathfrak{j}_{-2}^{q+2}\ } & \mathrm{H}_{-2}^{q+2}(K, L) \\ \wr\!\!\parallel \big\downarrow \psi_0^q & & & \mathrm{Sq}^2 & & & \wr\!\!\parallel \big\downarrow \psi_{-2}^{q+2} \\ \mathfrak{H}^q(K, L;\ (q-1)^q) & & & \longrightarrow & & & \mathfrak{H}^{q+2}(K, L;\ q^{q+2}) \end{array}$$

the commutativity

$$\psi_{-2}^{q+2}\mathfrak{j}_{-2}^{q+2}\mathfrak{l}_{-2}^{q+2}\mathfrak{h}_{-1}^q = \mathrm{Sq}^2\psi_0^q$$

holds true, where $q \geqq \mathrm{Max}\left(\frac{n}{2}+1, 5\right)$.

Before we prove Lemmas 6 and 7, let us consider more generally $\mathfrak{s}_p^q = \mathfrak{j}_{p-1}^{q+2}\mathfrak{l}_{p-1}^{q+2^{-1}}\mathfrak{h}_p^q$: $\mathrm{H}_p^q(K, L) \rightarrow \mathrm{H}_{p+1}^{q+2}(K, L)$.

In the diagram

$$(4.1)\qquad \begin{array}{ccccccc} & & \pi^{p+q}(\bar{K}^{q+1}, \bar{K}^{q-1}) & \xrightarrow{\ j_1^{\#}\ } & \pi^{p+q}(\bar{K}^q, \bar{K}^{q-1}) & & \\ & & \big\downarrow \Delta' & & \big\downarrow \Delta_1' & \searrow \Delta_1 & \\ \pi^{p+q}(\bar{K}^{q+1}, \bar{K}^q) & \xrightarrow{\ \Delta_2'\ } & \pi^{p+q+1}(K, \bar{K}^{q+1}) & \xrightarrow{\ j_2^{\#}\ } & \pi^{p+q+1}(K, \bar{K}^q) & \xrightarrow{\ i_2^{\#}\ } & \pi^{p+q+1}(\bar{K}^{q+1}, \bar{K}^q) \\ & \searrow \Delta_2 & \big\downarrow i^{\#} & & & & \\ & & \pi^{p+q+1}(\bar{K}^{q+2}, \bar{K}^{q+1}), & & & & \end{array}$$

---

6) In the following, the group multiplication with respect to squaring operation Sq^2 is always defined such that the product of the generator and itself is the generator.

let $\varDelta'$, $\varDelta_1$, $\varDelta_1'$, $\varDelta_2$, $\varDelta_2'$ be the coboundary operators, and let $i^{\#}$, $i_2^{\#}$, $j_1^{\#}$, $j_2^{\#}$ be the inclusion homomorphisms. Then the commutativity holds in a rectangle and two triangles in (4.1). If $\{a\}$ is an element of H_q^p which is represented by $a\in\pi^{p+q}(\dot{\bar{K}}^q, \dot{\bar{K}}^{q-1})$, we have $\varDelta_1 a=0$, and $\mathfrak{S}_p^q\{a\}\in \mathrm{H}_{p-1}^{q+2}(K, L)$ is represented by $i^{\#}j_2^{\#^{-1}}\varDelta_1'a$. In virtue of the exactness of $j_1^{\#}$, $\varDelta_1$, there exists an element $b\in\pi^{p+q}(\dot{\bar{K}}^{q+1}, \dot{\bar{K}}^{q-1})$ such that $j_1^{\#}b=a$. And we have

$$\begin{aligned} j_2^{\#}(\varDelta'b-j_2^{\#^{-1}}\varDelta_1'a) &= j_2^{\#}\varDelta'b-\varDelta_1'a = \varDelta_1'j_1^{\#}b-\varDelta_1'a \\ &= \varDelta_1'a-\varDelta_1'a=0. \end{aligned}$$

From the exactness of $\varDelta_2'$, $j_2^{\#}$, there exists an element $c\in\pi^{p+q}(\dot{\bar{K}}^{q+1}, \dot{\bar{K}}^{q})$ such that $\varDelta'b-j_2^{\#-1}\varDelta_1'a=\varDelta_2'c$. And we have

$$i^{\#}\varDelta'b-i^{\#}j_2^{\#^{-1}}\varDelta_1'a = i^{\#}\varDelta_2'c=\varDelta_2 c$$

Therefore we have $\{\varDelta b\}=\{i^{\#}j_2^{\#^{-1}}\varDelta_1'a\}=\mathfrak{S}_p^q\{a\}$, where $\varDelta=i^{\#}\varDelta'$: $\pi^{p+q}(\dot{\bar{K}}^{q+1}, \dot{\bar{K}}^{q-1})\to\pi^{p+q+1}(\dot{\bar{K}}^{q+2}, \dot{\bar{K}}^{q+1})$. Thus we establish

$$\text{(4.2)}\qquad \mathfrak{S}_p^q\{a\}=\{\varDelta j_1^{\#^{-1}}a\}.$$

Proof of Lemma 6. Let M^{q+2} be a complex $\mathrm{S}^q\cup e^{q+2}$, where e^{q+2} is attached to S^q by an essential map: $\dot{E}^{q+2}\to\mathrm{S}^q$. In this complex, it is easily seen from (4.2) and from [5, §20] that $\mathfrak{S}_0^q$ corresponds to Sq^2 by ψ. A proof for a general complex is given as follows.

Let $\{a\}\in\mathrm{H}_0^q(K, L)$ be an element which is represented by $a\in\pi^q(\dot{\bar{K}}^q, \dot{\bar{K}}^{q-1})$ and let a be represented by a map f: $(\dot{\bar{K}}^q, \dot{\bar{K}}^{q-1})\to(\mathrm{S}^q, y)$. Using the notations in (4.1) we have $\varDelta_1 a=0$, so that a belongs to the image of $j_1^{\#}$. Thus f can be extended to a map $\bar{f}$: $(\dot{\bar{K}}^{q+1}, \dot{\bar{K}}^{q-1})\to(\mathrm{S}^q, y)$. Since $\pi_{q+1}(M^{q+2})=0$, $\bar{f}$ can be extended again to a map $\bar{\bar{f}}$: $(\dot{\bar{K}}^{q+2}, \dot{\bar{K}}^{q+1})\to(M^{q+2}, \mathrm{S}^q)$. Then we have a diagram

$$\begin{array}{ccccc}
\pi^q(\dot{\bar{K}}^q, \dot{\bar{K}}^{q-1}) & \xleftarrow{j_1^{\#}} & \pi^q(\dot{\bar{K}}^{q+1}, \dot{\bar{K}}^{q-1}) & \xrightarrow{\Delta} & \pi^{q-1}(\dot{\bar{K}}^{q+2}, \dot{\bar{K}}^{q+1}) \\
\uparrow f^{\#} & & \uparrow \bar{f}^{\#} & & \uparrow \bar{\bar{f}}^{\#} \\
\pi^q(M^q, M^{q-1}) & \xleftarrow{j_1^{\#}} & \pi^q(M^{q+1}, M^{q-1}) & \xrightarrow{\Delta} & \pi^{q+1}(M^{q+2}, M^{q+1}),
\end{array}$$

where the commutativity holds in each rectangle. If $\{a_0\}$ is an element of $\mathrm{H}_0^q(M^{q+2})$ which is represented by the generator a_0 of $\pi^q(M^q, M^{q-1})$, we have

$$\mathrm{Sq}^2\psi\{a_0\}=\psi\mathfrak{S}_0^q\{a_0\},$$

from the fact that Lemma 6 holds in M^{q+2}. Therefore we have

$$\bar{\bar{f}}^*\mathrm{Sq}^2\psi\{a_0\}=\bar{\bar{f}}^*\psi\mathfrak{S}_0^q\{a_0\}.$$

Moreover we have

$$\begin{aligned} \bar{\bar{f}}^*\mathrm{Sq}^2\psi\{a\} &= \mathrm{Sq}^2\bar{\bar{f}}^*\psi\{a_0\} \\ &= \mathrm{Sq}^2\psi\{f^{\#}a_0\}=\mathrm{Sq}^2\psi\{a\}, \end{aligned}$$

and

$$\begin{aligned} \bar{\bar{f}}^*\psi\mathfrak{S}_0^q\{a_0\} &= \bar{\bar{f}}^*\psi\{\varDelta j_1^{\#^{-1}}a_0\}=\psi\{\bar{\bar{f}}^{\#}\varDelta j_1^{\#^{-1}}a_0\} \\ &= \psi\{\varDelta\bar{f}^{\#}j_1^{\#^{-1}}a_0\}=\psi\{\varDelta j_1^{\#^{-1}}f^{\#}a_0\} \\ &= \psi\{\varDelta j_1^{\#^{-1}}a\}=\psi\mathfrak{S}_0^q\{a\}. \end{aligned}$$

Therefore we have

$$\mathrm{Sq}^2\, \psi = \psi \mathfrak{s}_0^q .$$

This proves Lemma 6.

Proof of Lemma 7. It is easily verified that Lemma 7 holds in a special complex $N^{q+2}=M^{q+2}\cup e^{q+1}$, where e^{q+1} is attached to $S^q \subset M^{q+2}$ by a map $\dot{E}^{q+1}\to S^q$ of degree 2. Now, let K be an arbitrary complex. $\{a\}\in \mathrm{H}_{-1}^{q}(K, L)$ is represented by an element $a\in \pi^{q-1}(\bar{K}^q, \bar{K}^{q-1})$, which is represented by a map $f': (\bar{K}^q, \bar{K}^{q-1})\to(S^{q-1}, y)$. Then it may be assumed that $f'\simeq\eta\cdot f$, where f is a map $(\bar{K}^q, \bar{K}^{q-1})\to(S^q, y)$ and η is an essential map $(S^q, y)\to(S^{q-1}, y)$. Since $\varDelta_1 a=0$, f' can be extended to a map $\bar{f}': (\bar{K}^{q+1}, \bar{K}^{q-1})\to(S^{q-1}, y)$. Thus for a $(q+1)$-cell σ^{q+1}, $f\,|\,\dot{\sigma}^{q+1}$ is a map $\dot{\sigma}^{q+1}\to S^q$ of even degree. If we consider S^q as the q-sphere of N^{q+1}, f can be extended to a map $\bar{f}: (\bar{K}^{q+1}, \bar{K}^{q-1})\to(N^{q+1}, N^{q-1})$. Since $\pi_{q+1}(N^{q+2})=0$, $\bar{f}$ can be extended to a map $\bar{\bar{f}}: (\bar{K}^{q+2}, \bar{K}^{q+1})\to(N^{q+2}, N^{q+1})$. Then in the diagram

$$\begin{array}{ccccc}
\pi^{q-1}(\bar{K}^q, \bar{K}^{q-1}) & \xrightarrow{j_1^{\#}} & \pi^{q-1}(\bar{K}^{q+1}, \bar{K}^{q-1}) & \xrightarrow{\Delta} & \pi^q(\bar{K}^{q+2}, \bar{K}^{q+1}) \\
\uparrow f^{\#} & & \uparrow \bar{f}^{\#} & & \downarrow \bar{\bar{f}}^{\#} \\
\pi^{q-1}(N^q, N^{q-1}) & \xrightarrow{j_1^{\#}} & \pi^{q-1}(N^{q+1}, N^{q-1}) & \xrightarrow{\Delta} & \pi^q(N^{q+2}, N^{q+1}),
\end{array}$$

the commutativity holds in each rectangle. If $\{a_0\}$ is an element of $\mathrm{H}_{-1}^{q}(N^{q+2})$ which is represented by the generator a_0 of $\pi^{q-1}(N^q, N^{q-1})$, we have $a=f^{\#}a_0$. From the consideration that Lemma 7 holds in N^{q+2}, Lemma 7 can be easily deduced in a general complex through an analogous way as Lemma 6, by the aids of (4.2) and of $a=f^{\#}a_0$.

§5. Exact sequence of E. Spanier

Let $n\geqq 6$ and let us consider the diagram

$$\text{(5.1)}\quad \begin{array}{ccccccc}
\Pi_0^{n-2}(K, L) & \xrightarrow{\mathfrak{j}} & \mathrm{H}_0^{n-2}(K, L) & \xrightarrow{\mathfrak{h}} & \Gamma_0^{n-1}(K, L) & \xrightarrow{\mathfrak{i}} & \Pi_0^{n-1}(K, L) \\
\wr\|\downarrow \mathfrak{kl} & & \wr\|\downarrow \psi & & \wr\|\downarrow \psi\mathfrak{j}\mathfrak{l}^{-1} & & \wr\|\downarrow \mathfrak{kl} \\
\pi^{n-2}(K, L) & \xrightarrow{\bar{\phi}} & \mathfrak{H}^{n-2}(K, L; (n-2)^{n-2}) & \xrightarrow{\mathrm{Sq}^2} & \mathfrak{H}^{n}(K, L; (n-1)^{n}) & \xrightarrow{\Lambda} & \pi^{n-1}(K, L)
\end{array}$$

$$\begin{array}{cccccccccc}
\xrightarrow{\mathfrak{j}} & \mathrm{H}_0^{n-1}(K, L) & \xrightarrow{\mathfrak{h}} & \Gamma_0^{n}(K, L) & \xrightarrow{\mathfrak{i}} & \Pi_0^{n}(K, L) & \xrightarrow{\mathfrak{j}} & \mathrm{H}_0^{n}(K, L) & \xrightarrow{\mathfrak{h}} & \Gamma_0^{n+1}(K, L) \\
 & \wr\|\downarrow \psi & & \wr\|\downarrow & & \wr\|\downarrow \mathfrak{kl} & & \wr\|\downarrow \psi & & \wr\|\downarrow \\
\xrightarrow{\bar{\phi}} & \mathfrak{H}^{n-1}(K, L; (n-1)^{n-1}) & \to & 0 & \to & \pi^{n}(K, L) & \xrightarrow{\bar{\phi}} & \mathfrak{H}^{n}(K, L; n^{n}) & \to & 0 .
\end{array}$$

This diagram has the following properties:

i) The upper sequence $\sum_0(K, L)$ is exact by Theorem 1,

ii) the vertical homomorphisms are all isomorphisms in virtue of Lemma 2,

iii) the commutativity holds in each rectancgle by Lemmas 4, 5 and 6.

Therefore the lower sequence of (5.1) is also exact. Thus we have

Theorem 3. (E. Spanier) *Let* (K, L) *be a complex pair with* $dim(K-L) \leqq n$ $(n \geqq 6)$. *Then we have the exact sequence*

$$\pi^{n-2}(K, L) \xrightarrow{\bar{\phi}} \mathfrak{H}^{n-2}(K, L; (n-2)^{n-2}) \xrightarrow{Sq^2} \mathfrak{H}^n(K, L; (n-1)^n) \xrightarrow{\Lambda} \pi^{n-1}(K, L)$$
$$\xrightarrow{\bar{\phi}} \mathfrak{H}^{n-1}(K, L; (n-1)^{n-1}) \longrightarrow 0 \longrightarrow \pi^n(K, L) \xrightarrow{\bar{\phi}} \mathfrak{H}^n(K, L; n^n) \longrightarrow 0$$

Remark. We see that this theorem is proved for $n \geqq 5$, if $\pi^{n-1}(K, L)$ is discarded.

§6. Homotopy classification of mappings of certain complex K into an $(n-2)$-sphere S^{n-2}.

Let $n \geqq 7$. Applying Lemmas 2 and 7 to the exact sequence $\sum_{-1}(K, L)$, we have a diagram

$$\begin{array}{ccccccccc}
H^{n-2}_{-1}(K, L) & \xrightarrow{\mathfrak{h}} & \Gamma^{n-1}_{-1}(K, L) & \xrightarrow{\mathfrak{i}} & \Pi^{n-1}_{-1}(K, L) & \xrightarrow{\mathfrak{j}} & H^{n-1}_{-1}(K, L) & \xrightarrow{\mathfrak{h}} & \Gamma^n_{-1}(K, L) \\
\wr\| \downarrow \psi & & \wr\| \downarrow \psi\mathfrak{j}\mathfrak{l}^{-1} & & & & \wr\| \downarrow \psi & & \wr\| \downarrow \\
\mathfrak{H}^{n-2}(K, L; (n-3)^{n-2}) & \xrightarrow{Sq^2} & \mathfrak{H}^n(K, L; (n-2)^n) & & & & \mathfrak{H}^{n-2}(K, L; (n-2)^{n-1}) & \rightarrow & 0
\end{array}$$

in which the commutativity holds. From this we see that $\Pi^{n-1}_{-1}(K, L)$ is a group extension[7] of $\mathfrak{H}^n(K, L; (n-2)^n)/Sq^2\mathfrak{H}^{n-2}(K, L; (n-3)^{n-2})$ by $\mathfrak{H}^{n-1}(K, L; (n-2)^{n-1})$. And we have $\mathfrak{l}: \Pi^{n-1}_{-1}(K, L) \simeq \Gamma^{n-2}_0(K, L)$

(6.1) $\Gamma^{n-2}_0(K, L)$ *is a group extension of* $\mathfrak{H}^n(K, L; (n-2)^n)/Sq^2\, \mathfrak{H}^{n-2}(K, L; (n-3)^{n-2})$ *by* $\mathfrak{H}^{n-1}(K, L; (n-2)^{n-1})$.

Now, let us assume (K, L) to be a complex pair such that $Sq^2: \mathfrak{h}^{n-2}(K, L; (n-2)^{n-1}) \rightarrow \mathfrak{H}^n(K, L; (n-2)^n)$ is onto. Then from (6.1), we have

(6.2) $$\psi\mathfrak{j}\mathfrak{l}^{-1}: \Gamma^{n-2}_0(K, L) \cong \mathfrak{H}^{n-1}(K, L; (n-2)^{n-1}).$$

Consider the diagram

(6.3)
$$\begin{array}{ccccccc}
\Pi^{n-3}_0(K, L) & \xrightarrow{\mathfrak{j}} & H^{n-3}_0(K, L) & \xrightarrow{\mathfrak{h}} & \Gamma^{n-2}_0(K, L) & & \\
\wr\| \downarrow \mathfrak{kl} & & \wr\| \downarrow \psi & & \wr\| \downarrow \psi\mathfrak{j}\mathfrak{l}^{-1} & & \\
\pi^{n-3}(K, L) & \xrightarrow{\bar{\phi}} & \mathfrak{H}^{n-3}(K, L; (n-3)^{n-3}) & \xrightarrow{Sq^2} & \mathfrak{H}^{n-1}(K, L; (n-2)^{n-1}) & & \\
\xrightarrow{\mathfrak{i}} \Pi^{n-2}_0(K, L) & \xrightarrow{\mathfrak{j}} & H^{n-2}_0(K, L) & \xrightarrow{\mathfrak{h}} & \Gamma^{n-1}_0(K, L) & & \\
\wr\| \downarrow \mathfrak{kl} & & \wr\| \downarrow \psi & & \wr\| \downarrow \psi\mathfrak{j}\mathfrak{l}^{-1} & & \\
\longrightarrow \pi^{n-2}(K, L) & \xrightarrow{\bar{\phi}} & \mathfrak{H}^{n-2}(K, L; (n-2)^{n-2}) & \xrightarrow{Sq^2} & \mathfrak{H}^n(K, L; (n-1)^n). & &
\end{array}$$

This diagram has the following properties:

i) The upper sequence $\sum_0(K, L)$ is exact by Theorem 1,

ii) the vertical homomorphisms are all isomorphisms in virtue of Lemma 2 and (6.2). It should be noted that for $n=7$ the first vertical homomorphism is meaningless,

---

7) Let A, C be groups and let B be a subgroup of A. If there exists a homomorphism of A onto C with kernel B, we call that A is a group extension of B by C.

iii) the commutativity holds in each rectangle by Lemmas 4, 5 and 6. Therefore the lower sequence of (6.3) is exact, so that we have

Theorem 4. *Let $n \geqq 7$ and let K be a complex pair with $dim(K-L) \leqq n$ such that* $\mathrm{Sq}^2: \mathfrak{H}^{n-2}(K, L, I_2) \to \mathfrak{H}^n(K, L; I_2)$[8] *is onto. Then $\pi^{n-2}(K, L)$ has a subgroup isomorphic to $\mathfrak{H}^{n-1}(K, L; I_2)/\mathrm{Sq}^2\mathfrak{H}^{n-3}(K, L; I)$ and the factor group by this subgroup is isomorphic to the kernel of* $\mathrm{Sq}^2: \mathfrak{H}^{n-2}(K, L; I) \to \mathfrak{H}^n(K, L; I_2)$.

Furthermore we have the corresponding extension theorem;

Theorem 5. *Let $n \geqq 8$. Let K be an n-dimensional complex such that* $\mathrm{Sq}^2: \mathfrak{H}^{n-2}(K, I_2) \to \mathfrak{H}^n(K, I_2)$ *is onto and let L be its $(n-3)$-dimensional subcomplex. In order that a map $f: L \to S^{n-3}$ is extendable to K, it is necessary and sufficient that there exists $u \in \mathfrak{H}^{n-3}(K; I)$ such that*

$$f^* \{s^{n-3}\} = i^* \{u\},$$

and

$$\mathrm{Sq}^2 \{u\} = 0,$$

where $\{s^{n-3}\}$ is the generator of $\mathfrak{H}^{n-3}(S^{n-3}; I)$, $f^*: \mathfrak{H}^{n-3}(S^{n-3}; I) \to \mathfrak{H}^{n-3}(L; I)$, $i^*: \mathfrak{H}^{n-3}(K; I) \to \mathfrak{H}^{n-3}(L; I)$ are the homomorphisms induced by f and the injection $i: L \to K$ respectively, and Sq^2 *is the homomorphism of $\mathfrak{H}^{n-3}(K; I)$ to $\mathfrak{H}^{n-1}(K, I_2)$.*

Proof. Necessity. Let $\bar{f}: K \to S^{n-3}$ be an extension of f, and let $\{u\} = \bar{f}^* \{s^{n-3}\}$. Then we have

$$i^* \{u\} = i^* \bar{f}^* \{s^{n-3}\} = (\bar{f} i)^* \{s^{n-3}\} = f^* \{s^{n-3}\}.$$

As from (6.3) the sequence

$$\pi^{n-3}(K) \xrightarrow{\bar{\phi}} \mathfrak{H}^{n-3}(K; I) \xrightarrow{\mathrm{Sq}^2} \mathfrak{H}^{n-1}(K; I_2)$$

is exact, we have

$$\mathrm{Sq}^2 \{u\} = \mathrm{Sq}^2 \bar{f}^* \{s^{n-3}\} = \mathrm{Sq}^2 \bar{\phi} \{\bar{f}\} = 0,$$

Sufficiency. As we have $\mathrm{Sq}^2 \{u\} = 0$, from (6.3) there exists $\{g\} \in \pi^{n-3}(K)$ such that $\bar{\phi} \{g\} = \{u\}$. Since the commutativity holds in the diagram

$$\begin{array}{ccc} \pi^{n-3}(K) & \xrightarrow{\bar{\phi}} & \mathfrak{H}^{n-3}(K; I) \\ \downarrow i^{\#} & & \downarrow i^* \\ \pi^{n-3}(L) & \xrightarrow{\bar{\phi}} & \mathfrak{H}^{n-3}(L; I), \end{array}$$

we have

$$\begin{aligned} \bar{\phi} \{f\} = f^* \{s^{n-3}\} &= i^* \{u\} = i^* \bar{\phi} \{g\} \\ &= \bar{\phi} i^{\#} \{g\} = \bar{\phi} \{gi\}. \end{aligned}$$

As L is $(n-3)$-dimensional, $\bar{\phi}$ is an isomorphism from Theorem 3, so that we have $\{f\} = \{gi\}$. Namely we have

$$f \simeq gi \simeq g|L.$$

8) I_h denotes a cyclic group of order h, and I denotes a free cyclic group.

Since $g|L$ has an extension $g: K \to S^{n-3}$, f can be also extended to K in virtue of the homotopy extension property.

§7. The n-th cohomotopy group of an A_n^2-polyhedron

Let K be an $(n+2)$-dimensional complex with $\pi_i(K)=0$ for $i \leqq n-1$. According to J. H. C, Whitehead, we refer to such a complex as an A_n^2-polyhedron [7]. In this section, we shall calculate the n-th cohomotopy group of an A_n^2-polyhedron in terms of its cohomology system.

First we prove

Lemma 8. *Let $n \geqq 5$. Let K be an A_n^2-polyhedron, then we have*

$$\pi^n(K) \cong \Gamma_0^n(K) \oplus \mathfrak{H}^n(K\,;\,I)^{9)}$$

where $\Gamma_0^n(K)$ *is a group extension of* $\mathfrak{H}^{n+2}(K\,;\,I_2)/\mathrm{Sq}^2\,\mathfrak{H}^n(K\,;\,I_2)$ *by* $\mathfrak{H}^{n+1}(K\,;\,I_2)$.

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
\mathrm{H}_0^{n-1}(K) & \xrightarrow{\mathfrak{h}} & \Gamma_0^n(K) & \xrightarrow{\mathfrak{i}} & \Pi_0^n(K) & \xrightarrow{\mathfrak{j}} & \mathrm{H}_0^n(K) & \xrightarrow{\mathfrak{h}} & \Gamma_0^{n+1}(K) \\
\wr\Vert\downarrow\psi & & & & \wr\Vert\downarrow\mathfrak{f}\mathfrak{l} & & \wr\Vert\downarrow\psi & & \wr\Vert\downarrow\psi\mathfrak{j}\mathfrak{l}^{-1} \\
\mathfrak{H}^{n-1}(K\,;\,I) & & & & \pi^n(K) & \xrightarrow{\bar{\phi}} & \mathfrak{H}^n(K\,;\,I) & \xrightarrow{\mathrm{Sq}^2} & \mathfrak{H}^{n-2}(K\,;\,I_2)\,.
\end{array}$$

This diagram has the following properties:

i) The upper sequence is exact by Theorem 1,

ii) the vertical homomorphisms are all isomorphisms by Lemma 2,

iii) the commutativity holds in each rectangle by Lemmas 5 and 6,

iv) as K is $(n-1)$-connected, $\mathfrak{H}^{n-1}(K\,;\,I)=0$ so that $\mathfrak{i}$ is isomorphism into,

v) $\mathfrak{H}^n(K\,;\,I)$ is free abelian because K is $(n-1)$-connected, and $\mathfrak{H}^{n+2}(K\,;\,I_2)$ is finite, so that the kernel of Sq^2 is isomorphic to $\mathfrak{H}^n(K\,;\,I)$.

From these facts and from (6.1) we have immediately Lemma 8.

We shall determine $\pi^n(K)$ more precisely.

Let $(a_1, \cdots, a_m)$ be a system of independent generators of $\mathfrak{H}^{n+2}(K\,;\,I)$, where a_i is of order σ_i if $i \leqq t$ and a_i is of infinite order if $t+1 \leqq j \leqq m$. Further let σ_i be a power of a prime $\neq 2$ if $i \leqq s\,(\leqq t)$ and let σ_i be a power of 2 if $s+1 \leqq i \leqq t$. Then ${}_2(\mathfrak{H}^{n-2}(K\,;\,I))$ is generated by $\left(\frac{1}{2}\sigma_{s+1}a_{s+1}, \cdots\cdots, \frac{1}{2}\sigma_t a_t\right)$ and $\mathfrak{H}^{n+2}(K\,;\,I_2) = (\mathfrak{H}^{n+2}(K\,;\,I))_2$ is generated by $(\bar{a}_{s+1}, \cdots\cdots, \bar{a}_m)$, where $\bar{a}_i$ is the class of a_i[10]. Let $A(K)$ be a group extension of $\mathfrak{H}^{n+2}(K\,;\,I_2)/\mathrm{Sq}^2\,\mathfrak{H}^n(K\,;\,I_2)$ by ${}_2(\mathfrak{H}^{n+2}(K\,;\,I))$ determined by the relations:

$$\begin{cases} 2a_i = \mu\bar{a}_i\,, & \text{if } \sigma_i = 2\,, \\ 2a_i = 0\,, & \text{otherwise}\,, \end{cases}$$

---

9) If A, B are any abelian groups, $A \oplus B$ will always denote their direct sum.

10) Let G be an abelian group, then $G_2 = G/2G$, and ${}_2G$ is the subgroup ot G which consists of all the element g such that $2g=0$.

for $i=s+1, \cdots\cdots, t$, where $(a_{s+1}, \cdots\cdots, a_t)$ are representatives in A for $\left(\frac{1}{2}\sigma_{s+1}a_{s+1}, \cdots\cdots, \frac{1}{2}\sigma_t a_t\right)$ and μ is the natural homomorphism $\mathfrak{H}^{n+2}(K; I_2) \to \mathfrak{H}^{n+2}(K; I_2)/\mathrm{Sq}^2\mathfrak{H}^n(K; I_2)$. Then we have

Theorem 6. *Let $n \geq 5$ and let K be an A_n^2-polyhedron. Then the n-th cohomotopy group $\pi^n(K)$ is given in terms of its cohomology system as follows:*

(7.1) $$\pi^n(K) \cong \mathfrak{H}^n(K; I) \oplus (\mathfrak{H}^{n+1}(K; I))_2 \oplus A(K).$$

Before we proceed to prove this theorem, we shall remember two following definitions.

1) *An elementary A_n^2-polyhedron.* This is one of the following kinds [2], [6]:

i) $B_1^r = S^r$ $(r=n, n+1, n+2)$,

ii) $B_2(\sigma) = S^n \cup e^{n+1}$, where e^{n+1} is attached to S^n by a map $\dot{E}^{n+1} \to S^n$ of degree σ, a power of a prime,

iii) $B_3(\tau) = S^{n+1} \cup e^{n+2}$, where e^{n+2} is attached to S^{n+1} by a map $\dot{E}^{n+2} \to S^{n+1}$ of degree τ, a power of a prime,

iv) $B_4 = S^n \cup e^{n+2}$, where e^{n+2} is attached to S^n by an essential map $\dot{E}^{n+2} \to S^n$,

v) $B_5(2^p) = S^n \cup e^{n+1} \cup e^{n+2}$, where e^{n+1} is attached to S^n by a map $\dot{E}^{n+1} \to S^n$ of degree 2^p and e^{n+2} is attached to S^n by an essential map $\dot{E}^{n+2} \to S^n$,

vi) $B_6(2^q) = (S^n \vee S^{n+1}) \cup e^{n+2}$ [11]. where e^{n+2} is attached to $S^n \vee S^{n+1}$ by a map $\dot{E}^{n+2} \to S^n \vee S^{n+1}$ of the form $a+b$; a is an essential map $\dot{E}^{n+2} \to S^n$ and b is a map $\dot{E}^{n+2} \to S^{n+1}$ of degree 2^q.

vii) $B_7(2^p, 2^q) = B_6(2^q) \cup e^{n+1}$, where e^{n+1} is attached to S^n in $B_6(2^q)$ by a map $\dot{E}^{n+1} \to S^n$ of degree 2^p.

2) *A normal A_n^2-polyhedron.* We mean by this a polyhedron which consists of a collection of elementary A_n^2-polyhedra with a single point in common.

Proof of Theorem 6. Note that there exists a normal A_n^2-polyhedron which is of the same homotopy type as K [2] [6], and for two elementary A_n^2-polyhedra B, B' we have

$$\pi^n(B \vee B') \cong \pi^n(B) \oplus \pi^n(B'),$$
$$\mathfrak{H}^{n+1}(B \vee B'; I))_2 \cong (\mathfrak{H}^{n+1}(B; I))_2 \oplus (\mathfrak{H}^{n+1}(B'; I))_2,$$
$$\mathfrak{H}^n(B \vee B'; I) \cong \mathfrak{H}^n(B; I) \oplus \mathfrak{H}^n(B'; I),$$

and
$$A(B \vee B') \cong A(B) \oplus A(B'),$$

Then we see that it is sufficient to prove Theorem 6 for each A_n^2-polyhedron.

First we shall calculate the left hand of (7.1), the n-th cohomotopy group, for each elementary A_n^2-polyhedron. It follows from cohomological computation that

11) We denote by $A \vee B$ the union of two spaces A and B with a single point in common.

i) $\pi^n(B_1^n) \simeq I$, $\pi^n(B_1^{n+1}) \simeq I_2$, $\pi^n(B_1^{n+2}) \simeq I_2$,

ii) $\pi^n(B_2(\sigma)) = 0$ if σ is a power of a prime $\neq 2$,

$\simeq I_2$ if σ is a power of 2,

iii) $\pi^n(B_3(\tau)) = 0$ if τ is a power of a prime $\neq 2$,

$\pi^n(B_3(\tau))/I_2 \simeq I_2$ if τ is a power of 2,

iv) $\pi^n(B_4) \simeq I$, v) $\pi^n(B_5(2^p)) \simeq I_2$,

vi) $\pi^n(B_6(2^q)) \simeq I_2 \oplus I$, vii) $\pi^n(B_7(2^p, 2^q)) \simeq I_2 \oplus I_2$.

Moreover as for the group extension of iii), we have

iii)$'_1$ $\pi^n(B_3(\tau)) \simeq I_2 \oplus I_2$ if τ is $2^p (p>1)$,

iii)$'_2$ $\simeq I_4$ if τ is 2.

iii)$'_1$ follows from arguments similar to those used in the proof of Lemma 3.6 in P. J. Hilton [3], and iii)$'_2$ is the result due to M. G. Barratt and G. F. Paetcher [1].

Second, if we calculate the right hand of (7.1) for each elementary A_n^2-polyhedron, we shall easily find the same group as the above. Thus Theorem 6 is true.

Remark. Compare Theorem 6 with the one due to Hilton [3] with respect to the determination of the $(n+2)$-nd homotopy group of an A_n^2-polyhedron in terms of its homology system.

§8. Homotopy type of an A_n^2-polyhedron

J. H. C. Whitehead explained how the homotopy type of an A_n^2-polyhedron can be described in terms of cohomology [7]. We shall again deal with this problem in this section.

Let $n \geq 3$, and let K be an A_n^2-polyhedron. Let $\sum^n(K)$ be the part of $\sum_0(K)$ which begins with $H^n(K)$:

$$\mathrm{H}_0^n(K) \xrightarrow{\mathfrak{b}} \Gamma_0^{n+1}(K) \xrightarrow{\mathfrak{i}} \Pi_0^{n+1}(K) \xrightarrow{\mathfrak{j}} \mathrm{H}_0^{n+1}(K) \longrightarrow \cdots\cdots.$$

Then it follows from Lemma 2 that

$$\Gamma_0^{n+1}(K) \simeq (\mathrm{H}_0^{n+2}(K))_2, \quad \Gamma_0^{n+2}(K) = 0$$

and $\Gamma_0^i(K)$, $\Pi_0^i(K)$, $\mathrm{H}_0^i(K)$ are all zero for any $i>n+2$. On the other hand, let $\sum(K)$ be the exact sequence of J. H. C. Whitehead which is defined by his using the homotopy group (cf. [9] Chap III), and let $\sum_{n+2}(K)$ be the part of $\sum(K)$ which begins with $\mathrm{H}_{n+2}(K)$:

$$\mathrm{H}_{n+2}(K) \xrightarrow{\mathfrak{b}} \Gamma_{n+1}(K) \xrightarrow{\mathfrak{i}} \Pi_{n+1}(K) \xrightarrow{\mathfrak{j}} \mathrm{H}_{n+1}(K) \longrightarrow \cdots\cdots$$

It is known that

$$\Gamma_{n+1}(K) \simeq (\mathrm{H}_n(K))_2, \quad \Gamma_n = 0$$

and $\Gamma_i(K)$, $\Pi_i(K)$, $\mathrm{H}_i(K)$ are all zero for $i<n$.

Assume that K' is also an A_n^2-polyhedron. We shall then define proper isomorphisms of $\Sigma^n(K)$ to $\Sigma^n(K')$, $\Sigma_{n+2}(K)$ to $\Sigma_{n+2}(K')$, $\Sigma_{n+2}(K)$ to $\Sigma^n(K')$ and $\Sigma^n(K)$ to $\Sigma_{n+2}(K')$. Since all of these can be defined in the same manner, we shall here denote only the definition of a proper isomorphism of the last one. $\Sigma^n(K)$ is called to be properly isomorphic to $\Sigma_{n+2}(K')$ if and only if there exists a family of isomorphisms $\rho=\{{}_\Gamma\rho, {}_\Pi\rho, {}_H\rho\}$ such that the commutativity holds in each rectangle of the diagram

$$\begin{array}{ccccccccccc}
H_0^n(K) & \longrightarrow & \Gamma_0^{n+1}(K) & \longrightarrow & \Pi_0^{n+1}(K) & \longrightarrow \cdots \longrightarrow & \Pi_0^{n+2}(K) & \longrightarrow & H_0^{n+2}(K) & \longrightarrow 0 \longrightarrow \\
\downarrow {}_H\rho^n & & \downarrow {}_\Gamma\rho^{n+1} & & \downarrow {}_\Pi\rho^{n+1} & & \downarrow {}_\Pi\rho^{n+2} & & \downarrow {}_H\rho^{n+2} & \\
H_{n+2}(K') & \longrightarrow & \Gamma_{n+1}(K') & \longrightarrow & \Pi_{n+1}(K') & \longrightarrow \cdots \longrightarrow & \Pi_n(K') & \longrightarrow & H_n(K') & \longrightarrow 0 \longrightarrow
\end{array}$$

and such that ${}_\Gamma\rho^{n+1}$ is identified to the homomorphism induced by ${}_H\rho^{n+2}$ if we make the identification $\Gamma_0^{n+1}(K)=(H_0^{n+1}(K))_2$ and $\Gamma_{n+1}(K')=(H_n(K'))_2$. Then we denote $\Sigma^n(K)\approx\Sigma_{n+2}(K')$ and call that ρ is a proper isomorphism of $\Sigma^n(K)$ to $\Sigma_{n+2}(K')$. The following Lemma 9 is proved by the arguments similar to those used in the proof of Theorem 16 in [9].

Lemma 9. *Two A_n^2-polyhedra K and K' are of the same homotopy type if and only if $\Sigma_{n+2}(K)\approx\Sigma_{n+2}(K')$.*

Now we shall define a "co-polyhedron" P^* of a normal A_n^2-polyhedron P as follows. As for elementary one, we define:

i) $B_1^{n*}=B_1^{n+2}$, $B_1^{n+1*}=B_1^{n+1}$, $B_1^{n+2*}=B_1^n$,

ii) $B_2(\sigma)^*=B_3(\sigma)$, iii) $B_3(\tau)^*=B_2(\tau)$,

iv) $B_4^*=B_4$, v) $B_5(2^p)^*=B_6(2^p)$,

vi) $B_6(2^q)^*=B_5(2^q)$, vii) $B_7(2^p, 2^q)=B_7(2^q, 2^q)$.

When P is a normal A_n^2-polyhedron, the "co-polyhedron" P^* of P is the one which is obtained by replacing each elementary A_n^2-polyhedron B of with its "co-polyhedron" B^*. Then P^* is also a normal A_n^2-polyhedron which is $\dim P^*=2n-\dim P+2$, and we have $P^{**}=P$. Furthermore we have

Lemma 10. *For any normal A_n^2-polyhedron P and its "co-polyhedron" P^*, $\Sigma^n(P)$ is properly isomorphic to $\Sigma_{n+2}(P^*)$. (If $n\geq 4$, we have $\Sigma^{n-1}(P)\approx\Sigma_{n+3}(P)$ more strongly.)*

Proof. As for elementary A_n^2-polyhedra, we assert this Lemma by inspectation, This can be shown easily, so that we will merely list the following table of homotopy groups and cohomotopy groups.

$\pi^i \backslash B$	B_1^n	B_1^{n+1}	B_1^{n+2}	$B_2(\sigma)$	$B_3(\tau)$	B_4	$B_5(2^p)$	$B_6(2^q)$	$B_7(2^p, 2^q)$	
π^n	I	I_2	I_2	$0^{\dagger}$ I_2	$0^{\dagger\dagger}$ $I_2 \oplus I_2$ I_4	I	I_2	$I_2 \oplus I$	$I_2 \oplus I_2$	π_{n+2}
π^{n+1}	0	I	I_2	I_σ	$0^{\dagger\dagger\dagger}$ I_2	0	$I_{2^{p+1}}$	0	$I_{2^{p+1}}$	π_{n+1}
π^{u+2}	0	0	I	0	I_τ	I	I	I_{2^q}	I_{2^q}	π_n
	B_1^{n*}	B_1^{n+1*}	B_1^{n+2*}	$B_2(\sigma)^*$	$B_3(\tau)^*$	B_4^*	$B_5(2^p)^*$	$B_6(2^q)^*$	$B_7(2^p, 2^q)^*$	$B^* \backslash \pi_i$

† $\pi^n(B_2(\sigma)) \cong \pi_{n+2}(B_2(\sigma)^* \cong I_2$ if σ is a power of 2,
$= 0$ otherwise.

†† $\pi^n(B_3(\tau)) \cong \pi_{n+2}(B_3(\tau)^*) \cong I_4$ if $\tau = 2$,
$\cong I_2 \oplus I_2$ if τ is a power of 2 and $\neq 2$,
$= 0$ otherwise.

††† $\pi^{n+1}(B_3(\tau)) \cong \pi_{n+1}(B_3(\tau)^*) \cong I_2$ if τ is a power of 2,
$= 0$ otherwise.

Generally, as for a normal A_n^2-polyhedron, Lemma 10 follows from that it is true for elementary A_n^2-polyhedra, and from the following fact: If B_1, B_2 be elementary A_n^2-polyhedia, the following theorems hold for Γ_0^i, Π_0^i, H_0^i; Γ_i, Π_i, H_i; $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{b}$ $(i=n, n+1, n+2)$ [2] [3]:

$$\Gamma_0(B_1 \vee B_2) \cong \Gamma_0^i(B_1) \oplus \Gamma_0^i(B_2),$$
$$\Pi_i(B_1 \vee B_2) \cong \Pi_i(B_1) \oplus \Pi_i(B_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots ;$$
$$\mathfrak{i} = \mathfrak{i}_1 + \mathfrak{i}_2, \quad \mathfrak{b} = \mathfrak{b}_1 + \mathfrak{b}_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots ,$$

where $\mathfrak{i}_1$, $\mathfrak{b}_2$, ... are the homomorphisms $\mathfrak{i}$, $\mathfrak{b}$, ... for B_1, B_2, ... respectively.

Finary corresponding to Lemma 9, we have

Lemma 11. *Two A_n^2-polyhedra K and K' are of the same homotopy type if and only if $\sum^n(K) \approx \sum^n(K')$.*

Proof. Necessity. Let K and K' are of the same homotopy type, and let $f: K' \to K$ be a homotopy equivalence of K and K'. Then if ${}_\Gamma\mathfrak{f}_0^i: \Gamma_0^i(K) \to \Gamma_0^i(K')$, ${}_\Pi\mathfrak{f}_0^i: \Pi_0^i(K) \to \Pi_0^i(K')$, ${}_\mathrm{H}\mathfrak{f}_0^i: \mathrm{H}_0^i(K) \to \mathrm{H}_0^i(K')$ are homomorphisms induced by f, it follows from Theorem 2 that $\mathfrak{f} = \{{}_\Gamma\mathfrak{f}, {}_\Pi\mathfrak{f}, {}_\mathrm{H}\mathfrak{f}\}$ is an isomorphism of $\sum^n(K)$ onto $\sum^n(K')$. Therefore it is sufficient to prove that ${}_\Gamma\mathfrak{f}_0^{n+1}$ is identified with the homomorphism induced by ${}_\mathrm{H}\mathfrak{f}_0^{n+2}$ when we make the identification $\theta: \Gamma_0^{n+1}(K) = (\mathrm{H}_0^{n+2}(K))_2$ and $\theta': \Gamma_0^{n+1}(K') = (\mathrm{H}_0^{n+2}(K'))_2$. Let $\lambda: \mathfrak{H}^{n+2}(K; I) \to \mathfrak{H}^{n+2}(K; I_2)$, $\lambda': \mathfrak{H}^{n+2}(K'; I) \to \mathfrak{H}^{n+2}(K'; I_2)$ be the natural homomorphisms, then λ and λ' are onto and we have

$$\theta = \psi_0^{n+2^{-1}} \lambda^{-1} \psi_{-1}^{n+2} \mathfrak{j}_{-1}^{n+2} \mathfrak{l}_{-1}^{n+2}, \quad \theta' = \psi_0'^{n+2^{-1}} \lambda' \psi'^{n+2}_{-1} \mathfrak{j}'^{n+2}_{-1} \mathfrak{l}'^{n+2}_{-1}.$$

Since the commntativities:

$${}_\Pi\mathfrak{f}\mathfrak{l} = \mathfrak{l}'{}_\Gamma\mathfrak{f}, \quad {}_\mathrm{H}\mathfrak{f}\mathfrak{j} = \mathfrak{j}'{}_\Pi\mathfrak{f}, \quad f^*\psi = \psi'{}_\mathrm{H}\mathfrak{f}, \quad f^*\lambda = \lambda' \cdot \mathfrak{f}^*$$

hold, we have easily

$$\theta'\, {}_{\Gamma}\mathfrak{f}_0^{n+} = {}_{H}\mathfrak{f}_0^{n+2}\theta\,.$$

Thus $\mathfrak{f}$ is a proper isomorphism.

Sufficiency. Let P and P' be normal A_n^2-polyhedra which are of the same homotopy type as K and K' respectively. Then we have

$$\sum^n(K) \approx \sum^n(P), \quad \sum^n(K') \approx \sum^n(P').$$

Thus we have

$$\sum^n(P) \approx \sum^n(P') \tag{8.1}$$

by the assumption of the sufficiency.

Let P^* and P'^* be " co-polyhedra " of P and P' respectively. Then it follows from Lemma 10 that

$$\sum^n(P) \approx \sum_{n+2}(P^*), \quad \sum^n(P') \approx \sum_{n+2}(P'^*). \tag{8.2}$$

From (8.1) and (8.2), we have

$$\sum_{n+2}(P^*) \approx \sum_{n+2}(P^{q*})$$

so that P^* and P'^* are of the same homotopy type in virtue of Lemma 9. Since P^* and P'^* are normal, we see $P^*=P'^*$. Thus we have $P=P^{**}=P'^{**}=P'$. Therefore K and K' are of the same homotopy type.

Let $S^n(K)$ denote the part of Spanier's exact sequence which begins with $\mathfrak{H}^n(K\,;I)$:

$$\mathfrak{H}^n(K\,;I) \xrightarrow{Sq_2} \mathfrak{H}^{n+2}(K\,;I_2) \xrightarrow{\Lambda} \pi^{n+1}(K) \xrightarrow{\bar\phi} \mathfrak{H}^{n+1}(K\,;I) \to 0 \to \pi^{n+2}(K) \xrightarrow{\bar\phi} \mathfrak{H}^{n+2}(K\,;I) \to 0$$

We shall now define a proper isomorphism of $S^n(K)$ onto $S^n(K')$ in the similar way as the definition of an proper isomorphism of $\sum^n(K)$ onto $\sum^n(K')$. Then the following theorem is the direct consequence of Theorem 3 and Lemma 11.

Theorem 7. *Let $n \geqq 3$.*[12] *Two A_n^2-polyhedra K and K' are of the same homotopy type if and only if their Spanier's sequence $S^n(K)$, $S^n(K')$ are properly isomorphic.*

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12) cf. Remarks of § 5.