Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 3, No. 1–2, Series A

On Uniform Homeomorphism between Two Uniform Spaces

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(Received September 15, 1952)

The purpose of this paper is to study conditions in order that two complete uniform spaces are uniformly homeomorphic.

We concerned ourselves with the same problem in a previously published paper.[1] There we characterized a point by a family of uniform coverings, and for that the condition of the lattice of uniform basis as well as the proofs of theorems was considerably complicated and unnatural. By using a family of families of uniform coverings in the place of a family of uniform coverings, in this paper we show that the condition 1) in the previous paper and a condition, weaker than 2) there are sufficient for the conditions of the lattice of uniform basis which defines the uniform complete space up to a uniform homeomorphism, and we simplify proofs of propositions.

We concern ourselves with the lattice $L(R)$ of uniform basis of a complete uniform space R[2], satisfying the following conditions,

1) *if* $\mathfrak{U}$, $\mathfrak{V} \in L(R)$, *then* $\mathfrak{U} \overset{\vee}{\wedge} \mathfrak{V} \in L(R)$,
2) *if* $\mathfrak{U} \in L(R)$, *then for an arbitrary open set* U_0, *there exists* $\mathfrak{M}(U_0, \mathfrak{U})$ *in* $L(R)$
   such that i) $M \in \mathfrak{M}(U_0, \mathfrak{U})$ *implies* $M \not\supseteq U_0$,
   ii) $U \in \mathfrak{U}$ *and* $U \cap U_0 = \phi$ *imply* $U \subset M \in \mathfrak{M}(U_0, \mathfrak{V})$ *for some* M.

1) is the same condition as 1) in the previous paper.

2) is weaker than 2′) in the previous paper and accordingly than 2) there.

Definition. We denote by $\{\mu\}(p)$ the set of all the families μ of uniform coverings such that
for every nbd (=neighbourhood) $U(p)$ of p, there exists $\mathfrak{M} \in \mu$ for which $U(p) \not< M$ for all $M \in \mathfrak{M}$.

Lemma 1. $\{\mu\}(p)$ *satisfies the following two conditions*

I) *for every* $\mathfrak{U} \in L(R)$, *there exist* $\mathfrak{U}' \in L(R)$ *and* $\mathfrak{A} \in L(R)$ *such that*
$\mathfrak{U}' < \mathfrak{U}$, $\mathfrak{A} \not> \mathfrak{U}'$;
$\mathfrak{U}' \not< \mathfrak{P} \vee \mathfrak{A}$ *and* $\mathfrak{U}' \not< \mathfrak{Q} \vee \mathfrak{A}$ *imply* $\mathfrak{U} \not< \mathfrak{P} \vee \mathfrak{Q}$;
$\mathfrak{U}' \not< \mathfrak{A} \vee \mathfrak{M}$ *for every* μ *and some* $\mathfrak{M} \in \mu$,

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1) On conditions in order that two uniform spaces are uniformly homeomorphic, this journal, Vol. 2, No. 2, 1952.

2) $L(R)$ is a family of open uniform coverings.

II) *for every* $\mathfrak{U}\in L(R)$, *there exists* $\mathfrak{U}'\in L(R)$ *such that* $\mathfrak{U}'<\mathfrak{U}$; *if* $\mathfrak{U}'\not<\mathfrak{N}\vee\mathfrak{M}$ *for every* μ *and some* $\mathfrak{M}\in\mu$ *and if* $\mathfrak{U}'\not<\mathfrak{P}\vee\mathfrak{M}$ *for every* μ *and some* $\mathfrak{M}\in\mu$, *then* $\mathfrak{U}\not<\mathfrak{P}\vee\mathfrak{N}$.

Proof. $\{\mu\}(p)$ satisfies I).

Let $p\in U\in\mathfrak{U}$. Take $\mathfrak{U}'\in L(R)$ such that $\mathfrak{U}'<\mathfrak{U}$, $S^2(p,\mathfrak{U}')\subset U$.[3]

Let $p\in U'\in\mathfrak{U}'$. Then $\mathfrak{U}'\not<\mathfrak{A}$ holds for $\mathfrak{M}(U',\mathfrak{U}')=\mathfrak{A}$. If $\mathfrak{U}'\not<\mathfrak{P}\vee\mathfrak{A}$ and $\mathfrak{U}'\not<\mathfrak{Q}\vee\mathfrak{A}$, then since $U_0'\in\mathfrak{U}'$ and $S(p,\mathfrak{U}')\frown U_0'=\phi$ imply $U_0'\subset A\in\mathfrak{A}$ for some A, there exists $U_0'\in\mathfrak{U}'$ such that $S(p,\mathfrak{U}')\frown U_0'\neq\phi$; $U_0'\not\subset P$ for all $P\in\mathfrak{P}$. Since for such U_0', $U_0\subset U$ holds, we get $U\not\subset P$ for all $P\in\mathfrak{P}$. Similarly we get $U\not\subset Q$ for all $Q\in\mathfrak{Q}$. Therefore $\mathfrak{U}\not<\mathfrak{P}\vee\mathfrak{Q}$.

It is obvious that $\mathfrak{U}'\not<\mathfrak{A}\vee\mathfrak{M}$ for every μ and some $\mathfrak{M}\in\mu$.

$\{\mu\}(p)$ satisfies II).

Let $p\in U\in\mathfrak{U}$. Take $\mathfrak{U}'\in L(R)$ such that $\mathfrak{U}'<\mathfrak{U}$, $S^2(p,\mathfrak{U}')\subset U$. If $\mathfrak{U}'\not<\mathfrak{N}\vee\mathfrak{M}$ for every μ and some $\mathfrak{M}\in\mu$, then $U'\not\subset N$ holds for some $U'\in\mathfrak{U}'$ such that $U'\frown S(p,\mathfrak{U}')\neq\phi$ and for every $N\in\mathfrak{N}$.

To see this we assume the contrary. Obviously $\{\mathfrak{M}(U_0,\mathfrak{U}')\,|\,U_0\subset S(p,\mathfrak{U}')$, U_0: nbd of $p\}=\mu\in\{\mu\}(p)$. For every $\mathfrak{M}\in\mu$, $S(p,\mathfrak{U}')\frown U'=\phi$ and $U'\in\mathfrak{U}'$ imply $U'\subset M\in\mathfrak{M}$ for some M; hence from the assumption $\mathfrak{U}'<\mathfrak{M}\vee\mathfrak{N}$ for every $\mathfrak{M}\in\mu$, which is a contradiction. Therefore from $U'\frown S(p,\mathfrak{U}')\neq\phi$, $U'\not\subset N$ for all $N\in\mathfrak{N}$, we get $U'\subset U$ and $U\not\subset N$ for all $N\in\mathfrak{N}$.

Similarly we get $U\not\subset P$ for all $P\in\mathfrak{P}$. Hence $\mathfrak{U}\not<\mathfrak{P}\vee\mathfrak{N}$ holds,

Lemma 2. *If* $\{\mu\}$ *is a family of families of uniform coverings satisfying the condition I) in Lemma 1, then for every* $\mathfrak{U}_0\in L(R)$, *there exists* $U_0\in\mathfrak{U}_0$ *such that for every* $\mu\in\{\mu\}$, *there exists* $\mathfrak{M}\in\mu$ *for which* $U_0\not\subset M$ *holds for all* $M\in\mathfrak{M}$.

Proof. Take $\mathfrak{U}$ such that $\mathfrak{U}^{**}<\mathfrak{U}_0$.

By the condition I), take $\mathfrak{U}'<\mathfrak{U}$ and $\mathfrak{A}>\mathfrak{U}'$ such that $\mathfrak{U}'\not<\mathfrak{P}\vee\mathfrak{A}$ and $\mathfrak{U}'\not<\mathfrak{Q}\vee\mathfrak{A}$ imply $\mathfrak{U}\not<\mathfrak{P}\vee\mathfrak{Q}$.

Now we show that if $U',U''\in\mathfrak{U}'$ and $U',U''\not\subset A\in\mathfrak{A}$ for every A, then $S(U',\mathfrak{U})\frown U''\neq\phi$. We assume the contrary. Putting $\mathfrak{M}(U',\mathfrak{U})=\mathfrak{P}$, $\mathfrak{M}(U'',\mathfrak{U})=\mathfrak{Q}$, we get $\mathfrak{U}'\not<\mathfrak{P}\vee\mathfrak{A}$, $\mathfrak{U}'\not<\mathfrak{Q}\vee\mathfrak{A}$. $\mathfrak{U}<\mathfrak{P}\vee\mathfrak{Q}$ is obvious from the assumption, which contradicts the condition I).

Taking U_0 such that $S^2(U',\mathfrak{U})\subset U_0\in\mathfrak{U}_0$ for this U', we see that $\mathfrak{U}'\ni U''\not\subset A\in\mathfrak{A}$ for every A implies $U''\subset U_0$. Hence if there exists $\mu\in\{\mu\}$ such that $U_0\subset M$ holds for every $\mathfrak{M}\in\mu$ and for some $M\in\mathfrak{M}$, for every $U''\in\mathfrak{U}'$ we get either $U''\subset A\in\mathfrak{A}$ for some A or $U''\subset U_0$, *i.e.* $\mathfrak{U}'<\mathfrak{A}\vee\mathfrak{M}$ for every $\mathfrak{M}\in\mu$, which contradicts the condition I). Thus this lemma is proved.

3) $S^2(p,\mathfrak{U}')=S(S(p,\mathfrak{U}'),\mathfrak{U}')$.

Lemma 3. *If for $\{\mu\}$, a family of families of uniform coverings, and for $\mathfrak{U}$, $\mathfrak{U}'$, the condition II) holds, then $S(U', \mathfrak{U}')_\frown U'' \neq \phi$ holds for $U', U'' \in \mathfrak{U}$ such that for every $\mu \in \{\mu\}$ there exists $\mathfrak{M} \in \mu$ such that $U' \not\subset M \in \mathfrak{M}$ for every M, and for every $\mu \in \{\mu\}$ there exists $\mathfrak{M} \in \mu$ such that $U'' \not\subset M \in \mathfrak{M}$ for every M.*

Proof. Let us assume the contrary and put $\mathfrak{M}(U', \mathfrak{U}) = \mathfrak{P}$, $\mathfrak{M}(U'', \mathfrak{U}) = \mathfrak{N}$, then we get $\mathfrak{U}' \not< \mathfrak{P} \vee \mathfrak{M}$ for every $\mu \in \{\mu\}$ and for some $\mathfrak{M} \in \mu$ and $\mathfrak{U}' \not< \mathfrak{N} \vee \mathfrak{M}$ for every $\mu \in \{\mu\}$ and for some $\mathfrak{M} \in \mu$. For these $\mathfrak{P}$ and $\mathfrak{N}$ $\mathfrak{U} < \mathfrak{P} \vee \mathfrak{N}$ is obvious, which contradicts the condition II).

Lemma 4. *$\{\mu\}(p)$ is a maximum set satisfying the condition I).*

Proof. Let $\nu \notin \{\mu\}(p)$, then there exists a nbd U of p such that $U \subset N \in \mathfrak{N}$ for every $\mathfrak{N} \in \nu$ and for some N.

Take $\mathfrak{U}'$ such that $S^2(p, \mathfrak{U}') \subset U$, and put $\{\mathfrak{M}(U_0, \mathfrak{U}') | U_0 \subset S(p, \mathfrak{U}'), U_0$: nbd of $p\} = \mu$, then $\mu \in \{\mu\}(p)$. Taking an arbitrary $U' \in \mathfrak{U}'$, we see that if $U'_\frown S(p, \mathfrak{U}') \neq \phi$, then $U' \subset N \in \mathfrak{N}$ for every $\mathfrak{N} \in \nu$ and for some N and that if $U'_\frown S(p, \mathfrak{U}') = \phi$, then $U' \subset M \in \mathfrak{M}$ for every $\mathfrak{M} \in \mu$ and for some M. Hence from Lemma 2, the condition I) is not satisfied by $\{\{\mu\}(p), \nu\}$. Thus this proposition is valid.

If $\{\mu\}$ is a family of families of uniform coverings satisfying the condition I) and II), then for $U(\mathfrak{U}) = \cup \{U | U \in \mathfrak{U}$; for every $\mu \in \{\mu\}$, there exists $\mathfrak{M} \in \mu$ such that $U \not\subset M \in \mathfrak{M}$ for every $M\}$, $\{U(\mathfrak{U}) | \mathfrak{U} \in L(R)\}$ is a cauchy filter from Lemma 2 and from Lemma 3. Since R is complete, $\{U(\mathfrak{U})\}$ converges to a point p. Then obviously $\{\mu\} \subset \{\mu\}(p)$ holds. If moreover $\{\mu\}$ is a maximum set satisfying I), then $\{\mu\} = \{\mu\}(p)$.

Definition. We denote by $\mathfrak{L}(R)$ the set of all $\{\mu\}$, maximum sets satisfying I), II) *i.e.* the set of all $\{\mu\}(p)$. Obviously, there exists a one-to-one correspondence between R and $\mathfrak{L}(R)$. We denoted by $\mathfrak{L}(A)$ the image of a subset A of R in $\mathfrak{L}(R)$ by this correspondence.

Definition. We call a covering $\{\mathfrak{L}(U_\alpha)\}$ of $\mathfrak{L}(R)$ a uniform covering of $\mathfrak{L}(R)$ when $\{\mathfrak{L}(U_\alpha)\}$ satisfies tne following condition: There exists $\mathfrak{U} \in \mathfrak{L}(R)$ such that if $\{\mu\}(p_\alpha) \notin \mathfrak{L}(U_\alpha)$, then we can choose certain $\{\mu\}(p_1)$, $\{\mu\}(p_2)$ from $\{\{\mu\}(p_\alpha)\}$ so that there exist μ_1 in $\{\mu\}(p_1)$ and μ_2 in $\{\mu\}(p_2)$ for which $\mathfrak{M}_1 \in \mu_1$ and $\mathfrak{M}_2 \in \mu_2$ imply $\mathfrak{U} < \mathfrak{M}_1 \vee \mathfrak{M}_2$.

Lemma 5. *In order that $\{\mathfrak{L}(U_\alpha)\}$ is a uniform covering of $\mathfrak{L}(R)$, it is necessary and sufficient that $\{U_\alpha\}$ is a uniform covering of R.*

Proof. Firstly we prove that if $\{U_\alpha\}$ is a uniform covering of R, then $\{\mathfrak{L}(U_\alpha)\}$ is a uniform covering of $\mathfrak{L}(R)$. We take $\mathfrak{U} \in L(R)$ such that $\mathfrak{U}^{\Delta**} < \{U_\alpha\}$.

Let $\{\mu\}(p_\alpha) \notin \mathfrak{L}(U_\alpha)$, then $p_\alpha \notin U_\alpha$. Take an arbitrary point p_1 from $\{p_\alpha\}$, and let $S^3(p_1, \mathfrak{U}) \subset U_\alpha$, then $p_2 = p_\alpha \notin S^3(p_1, \mathfrak{U})$ holds. For $\{\mathfrak{M}(U', \mathfrak{U}) \mid U' \subset S(p_i, \mathfrak{U})$; U' is a nbd of $p_i\} = \mu_i$ $(i=1,2)$, $\mu_i \in \{\mu\}(p_i)$ is obvious. If $\mathfrak{M}_i$ are arbitrary elements of μ_i $(i=1,2)$, then for an arbitrary $U' \in \mathfrak{U}$, we get either $U' \frown S(p_1, \mathfrak{U}) = \phi$ or $U' \frown S(p_2, \mathfrak{U}) = \phi$, that is, either $U' \subset M_1 \in \mathfrak{M}_1$ or $U' \subset M_2 \in \mathfrak{M}_2$. Therefore $\mathfrak{U} < \mathfrak{M}_1 \vee \mathfrak{M}_2$.

Conversely we show that if $\{U_\alpha\}$ is not a uniform covering of R, then $\{\mathfrak{L}(U_\alpha)\}$ is not a uniform covering of $\mathfrak{L}(R)$.

Let $\mathfrak{U}$ be an arbitrary uniform covering of $L(R)$, then since $\{U_\alpha\}$ is not a uniform covering, $\mathfrak{U} \not< \{U_\alpha\}$ holds; hence there exists $U \in \mathfrak{U}$ such that $U \frown U_\alpha^c \neq \phi$ for all $U_\alpha \in \{U_\alpha\}$. Take $p_\alpha \in U \frown U_\alpha^c$ for each U_α, then $\{\mu\}(p_\alpha) \notin \mathfrak{L}(U_\alpha)$. We choose $\{\mu\}(p_1)$ and $\{\mu\}(p_2)$ from $\{\{\mu\}(p_\alpha)\}$ in an arbitrary way. If $\mu_i \in \{\mu\}(p_i)$ $(i=1,2)$, then since U is a nbd of p_1 and of p_2, there exist $\mathfrak{M}_i \in \mu_i$ $(i=1,2)$ such that $U \not\subset M_i \in \mathfrak{M}_i$ for every M_i $(i=1,2)$. Therefore $\mathfrak{U} \not< \mathfrak{M}_1 \vee \mathfrak{M}_2$.

By this lemma we see that $\mathfrak{L}(R)$ with the uniform coverings defined above is a uniform space, being uniformly homeomorphic with R. Since the uniform space $\mathfrak{L}(R)$ is defined only by the lattice-order $<$ from $L(R)$, if R_1 and R_2 are complete uniform spaces, a lattice isomorphism between $L(R_1)$ and $L(R_2)$ generates a uniform homeomorphism between $\mathfrak{L}(R_1)$ and $\mathfrak{L}(R_2)$. Hence we get the following theorem.

Theorem. *In order that two complete uniform spaces R_1 and R_2 are uniformly homeomorphic, it is necessary and sufficient that $L(R_1)$ and $L(R_2)$ are lattice-isomorphic, where $L(R_1)$ and $L(R_2)$ are lattices of uniform bases satisfying conditions* 1), 2).

Corollary 1. *If R is a complete uniform space, then the uniform topology of R is characterized by any lattice $L(R)$ of uniform basis of R satisfying the condition* 1) *of Theorem and*

2′) *if $\mathfrak{U} \in L(R)$, then for an arbitrary open set U_0, there exists $\mathfrak{M}$ in $L(R)$ such that* i) *$M \in \mathfrak{M}$ implies $\mathfrak{M} \not\supset U_0$,* ii) *$U \in \mathfrak{U}$, $\bar{U} \not\supset U_0$ imply $U \in \mathfrak{M}$.*

This 2′) is the condition 2′) in the previous paper.

Corollary 2. *If R is a complete uniform space without isolated point, then the uniform topology of R is characterized by any lattice $L(R)$ of uniform basis of R satisfying the condition* 1) *and*

2″) *if $\mathfrak{U}, \mathfrak{V} \in L(R)$, then for an arbitrary nbd U of p, there exists $\mathfrak{M}$ in $L(R)$ such that $\mathfrak{M} < \mathfrak{V}$ in U, and $\mathfrak{M} > \mathfrak{U}$ in U^c.*

Corollary 3. *If R is a complete uniform space without isolated point, then the uniform topology of R is characterized by any lattice $L(R)$ of uniform basis of R satisfying the condition* 1) *and*

2‴) *if $\mathfrak{U} \in L(R)$, then for an arbitrary nbd U_0 of p, there exists $\mathfrak{M}$ in*

$L(R)$ *such that* i) $S(p, \mathfrak{M}) \subset U_0$, ii) $U \cap U_0 = \phi$, $U \in \mathfrak{U}$ *imply* $U \in \mathfrak{M}$.

Corollary 4. *If R is a complete uniform space, then the uniform topology of R is characterized by any lattice $L(R)$ of uniform basis of R satisfying the condition* 1) *and*

2$''''$) *if* $\mathfrak{U} \in L(R)$, *then for an arbitrary nbd U of p, there exists a uniform covering* $\{U', U''\} = \mathfrak{U}'$ *such that* $p \in U' \subset U(p)$, $U'' \neq R$ *and* $\mathfrak{U}' \wedge \mathfrak{U} \in L(R)$.