Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 2, No. 2, Series A

Homological Structure of Fibre Bundles

By Tatsuji KUDO

(Received Dec. 15, 1951)

1. To find relations existing among the homological characters of the bundle space, of the base space, and of the fibre of a given fibre bundle is an important problem in topology.

In the preceding paper [9] the author, in connection with this problem, gave a new formulation of the so-called Leray's algorism [10] on the one hand, and generalized two theorems of Samelson concerning to homogeneous spaces to theorems of fibre bundles (see §2 below) on the other hand.

The purpose of the present paper is to give them more detailed accounts and to derive almost all theorems in our direction.[1] In part II characteristic groups and characteristic isomorphisms are defined for arbitrary set systems, and their fundamental properties are given.[2] In this form they reveal a close bearing on the theory of Morse, classifying cycles according to critical levels. Moreover they may be applied to homotopy as well as cohomotopy theories. In particular, if applied to cohomotopy theory, they give a formal answer to the classification problem of maps of an $(n+r)$-complex into an n-sphere for arbitrary r but for sufficiently large n (II, §4). In part III results of part II are applied to fibre bundles over a complex. In part IV various formulas concerning to $\cup$- and $\cap$-multiplications are given, and the theorems of Gysin [5], of Thom-Chern-Spanier [23], [3], and of Wang [24] are generalized. In part V $\circ$- and $\square$-multiplications are introduced and as application several important theorems about homological trivialness are given, some of of which[3] seem to be contained in the results announced by Hirsch [6].

2. To explain our problem we shall give here some theorems about homological triviality.

Theorem A. (*Künneth' theorem*) *If A is the product complex of two complexes B and F, the cohomology ring $H^*(A)$ is isomorphic to the Kronecker product $H^*(B)\otimes H^*(F)$ of the cohomology rings $H^*(B)$, $H^*(F)$ of B, F respectively, where the rational number field is taken as the coefficient ring.*

---

1) Major parts of this paper (Part II-IV) were published in Japanese in March, 1951.

2) Another abstract formulation of Leray's algorism was obtained by H. Cartan, J. Leray [30], and J. L. Koszul [8].

3) Theorem 22 and Theorem 23. In the case of homogeneous space these theorems are consequences of the results proved by Koszul [8].

For a product bundle $A=B\times F$, the cohomology ring of the bundle space A is therefore completely determined by the knowledge of those of the base space B, and of the fibre F.

But besides product bundles there are many classes of fibre bundles for which the same proposition hold. For example we have the following Samelson's theorem [15]:

Theorem B. *If a compact connected Lie group G acts on a sphere S transitively, and the isotropy group* $U(G/U=S^n)$ *is connected, then*

(*a*) *if n is odd,* $H^*(G)\approx H^*(S^n\times U)$,[4]

(*b*) *if n is even,* $H^*(G)\approx H^*(S^{2n-1}\times \Pi)$, *where* Π *is a product space of several odd dimensional spheres such that* $H^*(U)\approx H^*(S^{n-1}\times \Pi)$.

Sacrificing the multiplicative observation, the conclusions (a), (b) of Theorem B reduce respectively to

$$\mathfrak{P}_G(t) = \mathfrak{P}_{S^n}(t)\times\mathfrak{P}_U(t)\,, \tag{2.1}$$

$$\mathfrak{P}_G(t) = \mathfrak{P}_{S^{2n-1}}(t)\times\mathfrak{P}_{\Pi}(t)\,, \tag{2.2}$$

where $\mathfrak{P}_U(t)=\mathfrak{P}_{S^{n-1}}(t)\times\mathfrak{P}_{\Pi}(t)$, and where $\mathfrak{P}_M(t)$ denotes the Poincaré polynomial of M. In this reduced form the case (a) of Theorem B was generalized by the author to the following theorem [9], (V, §6):

Theorem C. *If A is a fibre bundle over an odd dimensional homology sphere B* ($\dim B=n, \mathfrak{P}_B(t)=1+t^n$), *and if the group of the bundle G is a compact connected Lie group, then*

$$\mathfrak{P}_A(t) = \mathfrak{P}_B(t)\times\mathfrak{P}_F(t)\,. \tag{2.3}$$

Incidentally in important applications (2.3) plays also the role of (2.2). In fact *we may calculate the Poincare polynomials of the closed simple groups belonging to the main four classes as well as of the Stiefel manifolds by making use of Theorem C alone.*

We call a fibre bundle $\mathfrak{F}=\{F, G, B, A, \psi, \phi_U\}$ *homologically trivial* when it satisfies (2.3). It is known that the following classes of fibre bundles are also homologically trivial:

(2.4) Fibre bundles over an acyclic complex (III, §5),

(2.5) Even dimensional sphere-bundles over a complex (Chern-Spanier [3]; (IV, §8)),

(2.6) Fibre bundles over an n-sphere with fibre F such that $H^p(F)=0$ ($p\gg n-1$), or with fibre F which is a d-dimensional homology sphere with $d>n-1$ (Wang [24]; (IV, §9)),

(27) Let A be a compact connected Lie group, and F a closed connected

4) H* is the same as above.

subgroup. The coset space $A/F=B$ is a fibre bundle of the type $\{F, F, B, A, \dots, \dots\}$. We shall call it simply a *homogeneous space.* Then a homogeneous space over a homological Γ-manifold B ($H^*(B)\approx H^*(\Pi)$, where Π is a product space of several cdd dimensional spheres) (Koszul [8].

The non-trivial characteristic groups measure a deviation from homological trivialness.

Part I
Preliminaries

1. Fibre bundles. In this paper by a *fibre bundle* we mean a coordinate bundle in the sense of Steenrcd [21], and we use the notations $\mathfrak{F}=\{F, G, B, A, \psi, \phi_U\}$, $'\mathfrak{F}=\{'F, 'G, 'B, 'A, '\psi, '\phi'_U\}$, etc. to denote fibre bundles, where $F, G, B, A, \psi, U, \phi_U$ are respectively the *fibre, group of the bundle, base space, bundle space: projection, coordinate neighborhoods, and coordinate function of* $\mathfrak{F}$.[5] By a fibre bundle of type (F_0, G_0) (type (F_0, G_0, B_0)) we mean a fibre bundle $\mathfrak{F}$ such that $F=F_0$, $G=G_0$ ($F=F_0$, $G=G_0$, $B=B_0$). $\mathfrak{F}$ is a *d-sphere bundle* (an *orientable d-sphere-bundle*) if it is of type (S^d, O_{d+1}) (type (S^d, R_d)), where S^d is the unit d-sphere, O_{d+1} is the orthogonal group of $d+1$ variables, and $R_d=O^+_{d+1}$ is the rotation gronp of S^d.[6] $\mathfrak{F}$ is a *principal fibre bundle,* if it is of type (G, G), where G acts on itself as the group of left translations. A map[7] $h: A\to 'A$ is a *bundle map* $h:\mathfrak{F}\to '\mathfrak{F}$, or it is *admissile,* if there exists a map $h: B\to 'B$ such that (1) $\bar{h}\psi='\psi h$, (2) $h|\psi^{-1}(x)$ is a homeomorphism onto $'\psi^{-1}(\bar{h}(x))$, where $x\in B$, (3) for any $U\ni x$, $'U\ni 'x=\bar{h}(x)$ the correspondence: $U\cap h^{-1}('U)\ni x\to '\phi^{-1}_{'U}, '_x h\phi_{U,x}\in G$ is a map, where $\phi_{U,x}: F\to A$ is defined as usual by $\phi_{U,x}(y)=\phi_U(x, y)$, $y\in F$. $\mathfrak{F}$ and $'\mathfrak{F}$ of the same type (F, G) ((F, G, B)) are *equivalent* (*equivalent in the restriced sense*) if there exists a bundle map $h:\mathfrak{F}\to '\mathfrak{F}$, such that $h: A\to 'A$ is a homeomorphism onto (if moreover the induced map $h: B\to 'B=B$ is an identity map).

F is a *product bundle,* if $G=1$, $A=B\times F$, $\psi(x, y)=x$, $U=B$, $\phi_U(x, y)=x\times y$. If G is contained in a larger group $\bar{G}$ which operates on F, $\mathfrak{F}$ gives rise to a new fibre bundle $\bar{\mathfrak{F}}=\mathfrak{F}_{\bar{G}}$ by merely replacing G by $\bar{G}$. $\mathfrak{F}$ and $'\mathfrak{F}$ with $F='F$ are *$\bar{G}$-equivalent* (*$\bar{G}$-equivalent in the restricted sense*), if $\mathfrak{F}_{\bar{G}}$ and $'\mathfrak{F}_{\bar{G}}$ are definable and are equivalent (equivalent in the restricted sense). If $\mathfrak{F}$ is G-equivalent

5) Unless otherwise mentioned the definitions and terminologies are the same as may be found in the Steenrod's book [21], which we have in common as the standard text. We shall refer it to [S]. It would have been more convenient if we had used the same notations. But it was too cumbersome for the author to revice all of the notations in the old manuscript, in which we used the terminologies and notations in the Chern-Sun's paper [2]. We shall refer it to [CS].

6) Generally $\mathfrak{F}$ with an arc-wise connected G is called *orientable.*

7) A map of topological space into a topological space is always assumed to eb continuous.

in the restricted sense to the product bundle $A=B\times F$, it is roughly also called a *product bundle.* $\mathfrak{F}$ is a *portion of* $'\mathfrak{F}$ *over* $B_0\subset'B$, and is denoted by $'\mathfrak{F}|B_0$ if $F='F$, $G='G$, $B=B_0$, $A='\psi^{-1}(B_0)$, $\psi='\psi|A$, $U='U\cap B$, $\phi_U='\phi'_U|(U\times F)$. $\mathfrak{F}$ is a *product of* $'\mathfrak{F}$ *and* $''\mathfrak{F}$, and is denoted by $\mathfrak{F}='\mathfrak{F}\times''\mathfrak{F}$, if $F='F\times''F$, $G='G\times''G$, $gy=('g\times''g)\cdot('y\times''y)=('g\cdot'y)\times(''g\cdot''y)$ for $g\in G$, $B='B\times''B$, $A='A\times''A$, $\psi(a)=\psi('a\times''a)='\psi('a)\times''\psi(''a)$, $U='U\times''U$, $\phi_U(x,y)=\phi_U('x\times''x,'y\times''y)='\phi'_U('x,'y)\times''\phi''_U(''x,''y)$. $\mathfrak{F}$ is the *induced bundle of* $'\mathfrak{F}$ *by a given map* $f:B\to'B$, and is denoted by $\mathfrak{F}='\mathfrak{F}_f$, if $F='F$, $G='G$, $A=$[the graph of the many valued function $\bar{f}:B\to'A$[8]]=[the subset of $B\times'A$ consisting of points $(x,'a)$ such that $f(x)='\psi(a)$], $'\psi(x,'a)=x$, $U=f^{-1}('U)$, $\phi_U(x,y)=(x,\phi'_U(f(x),y))$. Let $\hat{\mathfrak{F}}$ be a fibre bundle of type (F,G,B). It is called a *universal fibre bundle of type* (F,G,B) if it has the following properties:

(i) For any $\mathfrak{F}$ of type (F,G,B) there exists a map $f:B\to\hat{B}$ such that the induced bundle $\hat{\mathfrak{F}}_f\approx\mathfrak{F}$[9],

(ii) If two maps $f_1,f_2:B\to\hat{B}$ are homotopic $(f_1\sim f_2)$, then $\hat{\mathfrak{F}}_{f_1}\approx\hat{\mathfrak{F}}_{f_2}$,

(iii) If $\hat{\mathfrak{F}}_{f_1}\approx\hat{\mathfrak{F}}_{f_2}$ for two maps $f_1,f_2:B\to\hat{B}$, then $f_1\sim f_2$. In the case when $F=G/U$ is a homogeneous space[10], and B is a polyhedron, the existence of a universal fibre bundle of type (F,G,B) was assured by Steenrod [S][11]. The following theorems, which were established by Chern-Sun [CS], are important in the sequel.

Theorem D. *Given two fibre bundles* $\mathfrak{F}$ *and* $'\mathfrak{F}$, *where* $'B$ *is compact. Consider the fibre bundle* $\mathfrak{F}\times I$,[12] *and let* $f:'\mathfrak{F}\times(0)\to\mathfrak{F}$ *be a bundle map. Then for any homotopy* $\bar{f}:'B\times I\to B$ *of the induced map* $\bar{f}_0:'B\times(0)\to B$, *there exists a bundle map inducing* $\bar{f}$ *and such that* $f|'\mathfrak{F}\times(0)=f_0$.

Theorem E. *Let* F *be a principal fibre bundle such that* $\pi_i(A)=0$ $(0\leqq i\leqq n)$. *Then it is a universal fibre bundle of type* (G,G,B), *where* B *is a polyhedron of dimension at most* n.

Theorem F. *The base space* $\hat{B}$ *of a universal fibre bundle of type* (G,G,B) *is at the same time the base space of a universal fibre bundle of type* (F,G,B) *and vice versa.*

The actual form of the universal fibre bundle of type (G,G,B^n) given by Steenrod [S] is $\{G, G, O_m/O_{n+1}\times G, O_m/O_{n+1},\dots,\dots\}$. where G is a compact Lie group and m is sufficiently large so that O_m contains $O_{n+1}\times G$.[13] If G is connected we may replace O_k by $R_{k-1}=O_k^+$.

---

8) $\bar{f}(x)='\psi^{-1}(f(x))\subset'A$ for $x\in B$.
9) "$\approx$", "$\sim$" read "equivalent in the restricted sense to", "equivalent to" respectively.
10) See (Introduction, § 2).
11) The existence of a universal fibre bundle of type (F,G,B) was proved independently by Chern-Sun [CS], when G is a linear group, F is a coset space of G, and B is a polyhedron.
12) I= the unit interval <0,1>.
13) See, [S], 7.5.

$'\mathfrak{F}$ is called the *associated principal fibre bundle* of $\mathfrak{F}$, if it is of type (G, G, B), $'A$ is the totality of bundle maps $'a : F \to \mathfrak{F}$,[14] $'\phi('a) = x$, where $'a(F) = F_x$, $'U = U$, $'\phi_U(x, g) = \phi_{U,x} \circ g$. In such a case we may define a map $\eta : 'A \times F \to A$ by $\eta('a, y) = 'a \cdot y$.[15] Finally if $F = G/U$ is a homogeneous space (Introduction, §2), the associated principal fibre $'\mathfrak{F}$ of $\mathfrak{F}$ together with the natural map $\nu : 'a \to 'ay$ determine a fibre bundle of type $(U, U, A, 'A, \nu, \dots)$.

2. Eilenberg-Steenrod's axioms for cohomology theory. We expect that the reader are familiar to the seven axioms for cohomology theory as were given by Eilenberg-Steenrod [4].[17] But for the sake of convenience we shall give them here but in a slightly different form; the excision axiom adopted here is the strong one.

Axiom I: *Identity map* $f: (X, A) \to (X, A)$[18] *induces the identity isomorphism* $f^*: H^p(X, A) \approx H^p(X, A)$.

Axiom II: *For* $f: (X, A) \to (Y, B)$, $g: (Y, B) \to (Z, C)$, *there holds the relation* $(gf)^* = f^* g^*$.

Axiom III: *For homotopic* $f, g: (X, A) \to (Y, B)$, $f^* = g^*$.

Axiom IV: *The following sequence of homomorphisms:*

$$(2.1) \qquad \longrightarrow H^p(X, A) \xrightarrow{j^*} H^p(X, B) \xrightarrow{i^*} H^p(A, B) \xrightarrow{\delta} H^{p+1}(X, A) \longrightarrow$$

is exact, where $j: (X, B) \to (X, A)$, $i: (A, B) \to (X, B)$ *are inclusion maps.*

Axiom V: *Let* $f_1: (X, A) \to (Y, C)$, $f_2: (X, B) \to (X, D)$, $f_3: (A, B) \to (C, D)$ *be induced from* $f: (X, A, B) \to (Y, C, D)$. *Then commutativity relations holds in the diagram:*

$$(2.2) \qquad \begin{array}{ccccccccc} \longrightarrow & H^p(Y, C) & \longrightarrow & H^p(Y, D) & \longrightarrow & H^p(C, D) & \longrightarrow & H^{p+1}(Y, C) & \longrightarrow \\ & \downarrow f_1^* & & \downarrow f_2^* & & \downarrow f_3^* & & \downarrow f_1^* & \\ \longrightarrow & H(X, A) & \longrightarrow & H(X, B) & \longrightarrow & H(A, B) & \longrightarrow & H^{p+1}(X, A) & \longrightarrow \end{array}$$

Axiom VI′: *For the identity map* $k: (X\text{-}Int\ A, A\text{-}Int\ A) \to (X, A)$, k^* *is an isomorphism onto (excision isomorphism).*

Axiom VII: *For a space consisting of a single point* P, $H^p(P) = 0$ $(p \neq 0)$, $H^0(P) =$ *a given group* ρ *(which is called coefficient group).*[19] [20]

3. Immediate consequences of the Eilenberg-Steenrod's axioms.

Lemma 1. $H^p(A, A) = 0$.

---

14) Here F is regarded as a fibre bundle of type (F, G) over a point.
15) See, [S], 8. 7.
16) See, [S], 9. 6.
17) See also [20], [16].
18) (X, A) is assumed to be a closed pair.
19) Remember that $H^p(X,A) = 0$ for $p < 0$.

Lemma 2. *If* $f:(X, A)\to(Y, B)$ *has a homotopy inverse, then* f^* *is an isomorphism onto. In particular if* (X, A), (Y, B) *have the same homotopy type* $H^p(X, A)\approx H^p(Y, B)$.

Lemma 3. *Let* $f:(X, A)\to(Y, B)$ *be the identity map, and let* φ_t $(0\leqq t\leqq 1)$ *be a deformation of* Y *such that* $\varphi_1(Y)\subset X$, $\varphi_t(B)\subset B$, $\varphi_1(B)\subset A$, $\varphi_t(X)\subset X$, $\varphi_t(A)\subset A$. *Then* f^* *is an isomorphism onto.*

Lemma 4. *If* A *is a deformation retract of* X, *then* $H^p(X, A)=0$.

Lemma 5. *Let* $X=X_1\cup X_2$; $X\supset A\supset X_1\cap X_2$, *and let* $\lambda_i:(X_i, X_i\cup A)\to(X, A)$, $\mu_i:(X_i, X_i\cap A)\to(X, X_j\cup A)$, $\nu_i:(X, A)\to(X, X_j\cup A)$ $((i, j)=(1, 2)$ *or* $(2.1))$ *be the identity maps. Then* μ_i^* *is an isomorphism onto,* $\omega_i=\nu_i^*(\mu_i^*)^{-1}$ *is an isomorphism into,* λ_i^* *is a homomorphism onto,* $H^p(X, A)=\omega_1H^p(X_1, X_1\cap A)+\omega_2H^p(X_2, X_2\cap A)$ *is a direct decomposition, and* $\lambda_i^*w_i=1$.

We give here only the proof of the last lemma. Since $(X\text{-Int }(X_j\cup A), (X_j\cup A)\text{-Int }X_j\cup A))=(X_i\text{-Int }(X_i\cap A), X_i\cap A\text{-Int }(X_i\cap A))$, by making use of the excision axiom (Axiom VI′) two times we see that μ_i^* is an isomorphism onto. By Axiom II $\mu_i^*=\lambda_i^*\nu_i^*$, $\lambda^*_i\nu_i^*(\mu_i^*)^{-1}=1$, $\lambda_i^*\omega_i=1$. Hence λ_i^* is a homomorphism onto and ω_i is an isomorphism into. It remains to prove that $H^p(X, A)=\omega_1H^p(X_1, X_1\cap A)+\omega_2H^p(X_2, X_2\cap A)$ is a direct decomposition. Firstly noticing that $H^p(X_i, X_i\cap A)\approx H^p(X_i\cup A, A)$ by the excision axiom, the sequence $H^p(X, X_i\cup A)\xrightarrow{\nu_j^*}H^p(X, A)\xrightarrow{\lambda_i^*}H^p(X_i, X_i\cap A)$ is exact. Hence if $\lambda_1^*w=\lambda_2^*w=0$ for $w\in H^p(X, A)$, then $w=\nu_1^*u$ for some $u\in H^p(X, X_2\cap A)$, hence $0=\lambda_1^*w=\lambda_1^*\nu_1^*u=\mu_1^*u$, implying that $u=0$, or $w=0$. Secondly $\omega_iu_i=\omega_ju_j$, $u_k\in H^p(X_k, X_k\cap A)$ $(k=1, 2)$ imply $\omega_iu_i=\omega_ju_j=0$, i.e. $\omega_1H^p(X_1, X_1\cap A)+\omega_2H^p(X_2, X_2\cap A)$ is a direct sum. For $u_i=\lambda_i^*\omega_iu_i=\lambda_i^*\omega_ju_j=\lambda_i^*\nu_j^*(\mu_j^*)^{-1}u_j=0$.

Now let $w\in H^p(X, A)$ be arbitrary. Then $\lambda_i^*(w-\omega_i\lambda_i^*w-\omega_j\lambda_j^*w)=\lambda_i^*w-\lambda_i^*w_i\lambda_i^*\omega-\lambda_i^*w_j\lambda_j^*w=\lambda_i^*w-\lambda_i^*w-0=0$, implying that $w=\omega_i\lambda_i^*w+\omega_j\lambda_j^*w$.

4. Mayer cochain complexes. By a Mayer cochain complex we mean a homomorphism sequence $\longrightarrow C^p\xrightarrow{\delta_p}C^{p+1}\xrightarrow{\delta_{p+1}}C^{p+2}\longrightarrow\cdots$, such that $\delta_{p+1}\delta_p=0$. It is denoted by $\{C^p; \delta_p\}$. The p-th cocycle group Z^p of a given Mayer complex $\{C^p; \delta_p\}$ is defined by $Z^p=[$the kernel of $\delta_p]\subset C^p$. Obviously $\delta_{p-1}C^{p-1}\subset Z^p$. The factor group $H^p=Z^p/\delta C^{p-1}$ is called the cohomology group of the Mayer cochain complex.

5. Faisceau of groups. Cohomology theory with local coefficients. Let X be a topological space, and $\{\sigma\}$ be a family of closed sets of X. If a group ρ_σ is associated with each σ, and if a homomorphism $\chi(\sigma', \sigma):\rho_\sigma\to\rho_{\sigma'}$ is associated with each pair (σ', σ) with $\sigma<\sigma'$, such that $\chi(\sigma'', \sigma')\chi(\sigma', \sigma)=\chi(\sigma'', \sigma)$ for $\sigma<\sigma'<\sigma''$, and such that $\chi(\sigma, \sigma)=1$, then, following J. Leray [10], we say that a *faisceau* of groups $\tilde{\rho}=\{\rho_\sigma, \chi(\sigma', \sigma)\}$ is given over X.

We consider in the followings only the case when X is a polyhedron and $\{\sigma\}$ is a family of all (closed) cells of X.[21] Let $K=\{\sigma_i^q\}$ be a cellular decomposition of X and K_0 its closed subcomplex. We define: C^q=the additive group of all linear forms $c^q=\sum a_i\sigma_i^q$ with $a_i\in\rho_{\sigma_i^q}$, or equivalently all function $a_i=c^q(\sigma_i^q)$ defined over all q-cells of $K-K_0$ and with values in $\rho_{\sigma_i^q}$. Define a homomorphism $\delta: C^q\ni c^q\to\delta c\in C^{q+1}$ by $\delta c(\sigma_j^{q+1})=\sum_i[\sigma_j^{q+1}:\sigma_i^q]\,\chi(\sigma_j^{q+1},\sigma_i^q)\,c^q(\sigma_i^q)$.[21] Remembering the property $\chi(\sigma_l^{q+2},\sigma_j^{q+1})\chi(\sigma_j^{q+1},\sigma_i^q)=\chi(\sigma_l^{q+2},\sigma_i^q)$, we can easily prove that $\delta\delta=0$, obtaining a Mayer cochain complex $\{C^q;\delta\}$. The cohomology groups of this Mayer complex are called the cohomology group with the *faisceau* of groups $\tilde{\rho}$ as coefficient domain, and are denoted by $H^q(K,\tilde{\rho})$,

Now assume that every $\chi(\sigma',\sigma)(\sigma'>\sigma)$ is an isomorphism onto, and K is a simplicial decomposition. Then, if $\sigma'\supset\sigma\ni x_0, x_1$, $\chi(x_0,\sigma)\chi(\sigma,x_1)=\chi(x_0,\sigma')\chi(\sigma',\sigma)\chi(\sigma,\sigma')\chi(\sigma',x_1)=\chi(x_0,\sigma')\chi(\sigma',x_1)$, where $\chi(\sigma,\sigma')$ denotes $[\chi(\sigma',\sigma)]^{-1}$ for $\sigma'>\sigma$. This enables us to define $\omega(x_0,x_1)=\chi(x_1,\sigma)\chi(\sigma,x_1)$ without ambiguity. ω has the property: $\omega(x_0x_1)\,\omega(x_1x_2)\dots\omega(x_qx_0)=1$ for $\sigma=(x_0x_1\dots x_q)$. From the above we easily see that the cohomology theory with *faisceau* of groups as coefficient domain is essentially[22] the cohomology theory with local coeffioient in the sense of N. E. Steenrod [18].

6. $\cup$- and $\cap$-products. The consideration of $\cup$- and $\cap$-products appear only in the last two parts, where the axiomatic treatment is abondoned. But it is not at all unusefal to enumerate the most important of their properties in an axiomatic form, following Steenrod [20].[23] The basic coefficient domain ρ of our oohomology theory is a ring (with unit 1) when we are dealing with $\cup$-product only. It is a field for both homology and cohomology theories when we are dealing with $\cup$- and $\cap$-products.

(6.1) *For $u^p\in H^p(X,A_1)$, $v^q\in H^q((X,A_2)$ where $X\supset A_1\cup A_2$, the $\cup$-product $u^p\cup v^q$ of u^p and v^p is an element of $H^{p+q}(X,A_1\cup A_2)$.*

(6.2) *The left multiplication $(u^p\cup): H^q(X,A_2)\to H^{p+q}(X,A_1\cup A_2)$ defined by $(u^p\cup)v^q=u^p\cup v^q$ is linear; similarly the right multiplication $(\cup v^q): H^p(X,A_1)\to H^{p+p}(X,A_1\cup A_2)$ defined by $(\cup v^q)u^p=u^p\cup v^q$ is linear.*

(6.3) *Let $f_i:(X,A_i)\to(Y,B_i)(i=1,2)$, $f_3:(X,A_1\cup A_2)\to(Y,B_1\cup B_2)$ be induced from $f:(X,A_1,A_2)\to(Y,B_1,B_2)$. Then $f_3^*(u^p\cup v^q)=f_1^*u^p\cup f_2^*v^q$, for $u^p\in H^p(Y,B_1)$, $v^q\in H^q(Y,B_2)$.*

(6.4) *If $v^q\in H^q(X)$, $(\cup v^q)$ maps the cohomology sequence of the triple (X,A,B) into itself homomorphically but raising the dimension by q, i.e. commutativity relations hold in the diagram:*

21) $\left[\sigma_i^{q+1}:\sigma_q\right]$ denotes the incidence number of σ_j^{q+1} and σ_i^q. For its axiomatic definition see [4], [16]. The cells which appear in this paper are homemrophs of the unit full-sphere. $[\ :\ ]$ is accordingly either $=0$ or ±1.

22) As for this formulation of local coefficients see also P. Olum's paper: Obstructions to extensions and homotopies, Annals of math.

23) As for the concrete definition of $\cup$- and $\cap$-products, see Lefschetz "algebraic topology".

$$\begin{array}{ccccccccc}
\longrightarrow & H^p(X,A) & \longrightarrow & H^p(X,B) & \longrightarrow & H^p(A,B) & \longrightarrow & H^{p+1}(X,A) & \longrightarrow \\
 & \downarrow (\cup v^q) & & \downarrow (\cup v^q) & & \downarrow (\cup v^q) & & \downarrow (\cup v^q) & \\
\longrightarrow & H^{p+q}(X,A) & \longrightarrow & H^{p+q}(X,B) & \longrightarrow & H^{p+q}(A,B) & \longrightarrow & H^{p+q+1}(X,A) & \longrightarrow
\end{array}$$

where $(\cup v^q): H^p(A, B) \to H^{p+q}(A, B)$ *is an abreviated notation of* $(\cup i^* v^q)$, i^* *being induced by the identity map* $i: A \to X$. *Such abreviation is always used.*
(6.4) *The left multiplication* $(u^p \cup)$ *has the analogus property.*
(6.5) $u^p \cup v^q = (-1)^{pq} v^q \cup u^p$ *for* $u^p \in H^p(X, A_1)$, $v^q \in H^q(X, A_2)$.
(6.6) $(u^q \cup v^q) \cup w^r = u^p \cup (v^q \cup w^r)$ for $u^p \in H^p(X, A_1)$, $v^q \in H^p(X, A_2)$, $w^r \in H^r(X, A_3)$, $X \supset A_1 \cup A_2 \cup A_3$.

The direct sum $H^*(X, A) = \sum_p H^p(X, A)$ thus becomes a ring with respect to the $\cup$-product and is called the *cohomology ring* of the pair (X, A). The commutation rule (6.5) holds in $H^*(X, A)$.
(6.7) Now let us assume that ρ is a field, and (X, A) is a polyhedral pair. $H^p(X, A)$ *and* $H_p(X, A)$ *are dual to each other*, i.e. inner product $\langle u^p, z^p \rangle$ of $u^p \in H^p(X, A)$, $z^p \in H_p(X, A)$ are defined in such a way that (i) $\langle u^p, z^p \rangle$ is bilinear, (ii) any homomorphism $\eta: H^p(X, A) \to \rho$ is representable in the form $\eta(u^p) = \langle u^p, z_p \rangle$ in a unique way, (iii) any homomorphism $\nu: H_p(Y, A) \to \rho$ is representable in the form $\nu(z^p) = \langle u^p, z^p \rangle$ in a unique way.
(6.8) *For a map* $f: (X, A) \to (Y, B)$, *the induced maps* $f^*: H^p(Y, B) \to H^p(X, A)$, $f^*: H_p(X, A) \to H_p(Y, B)$ *are dual to each other, i.e.* $\langle u^p, f^* z^p \rangle = \langle f^* u^p, z^p \rangle$ *for* $u^p \in H^p(Y, B)$, $z^p \in H^p(X, A)$.

Now we define the $\cap$-*product* $v^q \cap z^{p+q}$ of $v^q \in H^q(X)$ and $z^{p+q} \in H_{p+q}(X)$ by the relation[24]:
(6.9) $\langle u^p, v^q \cap z^{p+q} \rangle = \langle u^p \cup v^q, z^{p+q} \rangle$ *for every* $u^p \in H^p(X)$. Corresponding to (3.3) we have
(6.10) $f_*(f^* u^p \cap z^{p+q}) = u^p \cap f^* z^{p+q}$, *for* $u^p \in H^p(Y)$, $z^{p+q} \in H_{p+q}(X)$.

7. Composable and minimal elements.[25] Let X be a connected polyhedron, and let ρ be a field, as in the end of the last section. Then the cohomology ring $H^*(X)$ has a unit $1 \in H^0(X): 1 \cup u^p = u^p \cup 1 = u^p$ for any $u^p \in H^p(X)$, and $H^0(X) = \{a \cdot 1\}$, $a \in \rho$ (immediate consequence of the axioms). An element $u^p \in H^p(X)$, which can be generated from the elements of $H^1(X), \dots, H^{p-1}(X)$ by $\cup$-multiplication and addition, is called a *composable element* of $H^p(X)$. u^p is therefore of the form $u^p = \sum v_i \cup w_i$, where v_i, w_i are homogeneous and of positive dimensions. The totality of p-dimensional composable elements obviously constitutes a subspace $S^p(X)$ of $H^p(X)$. An element u^p of $H^p(X)$ which is not a composable element is called *primitive*. We can choose an

24) We need only the absolute case in this paper.
25) See, [7], [15], [6].

irreducible system of generators $\{1, u_1, u_2, \dots, u_l\}$ of the cohomology ring $H^*(X)$ consisting of homogeneous elements. Obviously the p-dimensional u_i's in the above system oonstitute a linearly independent representative system of $H^p(X)/S^p(X)$, and conversely if we choose for each p a linearly independent representative system of $H^p(X)/S^p(X)$, their union constitutes an irreducible system of generators of $H^*(X)$. Thus $l=\sum_{p\geq 1} \dim(H^p(X)/S^p(X))$. It is called the *rank* of X.

An element z^p of $H_p(X)$ is called *minimal*, if it is orthogonal to every element u^p of $S^p(X)$: $\langle u^p, z^p\rangle=0$: z^p is a minimal element of $H_p(X)$ if and only if it is contained in the annihilator $M_p(X)$ in $H_p(X)$ of $S^p(X)$. It is easily seen that the condition of minimality of u^p is evuivalent to the following condition:

(7.1) *If* $0<r<p$, $u^r \cap z^p=0$ *for everyu* $u^r \in H^r(X)$. *For the minimal elements the following facts are fundamental*:

(7.2) *Let* $f: X\to Y$ *be a mmap. Then* $f_*: H_p(X)\to H_p(Y)$ *maps* $M_p(X)$ *into* $M_p(Y)$.

(7.3) *If* $H_p(X)=0$ *for* $0<p<d$, *then any* $z^d \in H_d(X)$ *is minimal. In particular a d-dimensional homology element of a homology d-sphere is minimal.*

(7.4) *The rank of* X *is equal to* $\sum_{p\geq 1} \dim M_p(X)$.

(7.5) *Let* $\bar{K}_p \subset H_p(X)$ *be a subspace and* K_p *its annihilator in* $H^p(X)$. *If* $\bar{K}_p \cap M_p(X)=0$ *for each* $p>0$, $\{1, K_1, K_2, \dots, K_p, \dots\}$ *generates* $H^*(X)$.

We shall prove only (7.2). Let $z^p \in M_p(X)$. Then for each $u^p \in H^p(Y)$, $0<r<p$, $u^r \cap f_* z^p = f_*(f^* u \cap z^p)=0$ by (6.10).

8. The cohomology ring of a group manifold. Pontrjagin ring. Let X be a group manifold.[27] The structure of the cohomology ring $H^*(X)$ is investigated by H. Hopf [7] and H. Samelson [15]. We state here only such results which we need in Part V.[28] Let $H_*(X)=\sum_p H_p(X)$ be the total homology group. We can introduce in $H_*(X)$ a multiplication $z^p \circ z^q$ as follows: Let $\eta: X\times X\to X$ be defined by $\eta(x'\times x'')=x'x''$. If c^p and c^q are singular cycles of X, then $c^p\times c^q$ and $\eta(c^p\times c^q)$ are singular cycles of $X\times X$ and X respectively. The homology class of $\eta(c^p\times c^q)$ is determined by the homology classes z^p, z^q of c^p, c^q respectively, and is denoted by $z^p\circ z^q$. This product was first introduced by Pontrjagin [13] and is called the *Pontrjagin product*.

(8.1) $H_*(X)$ *becomes a ring* $\mathfrak{P}(X)$ *with respect to the Pontrjagin multiplication.*

(8.2) *Let* $M_*^+(X)=\sum_{p\geq 1} M_p(X)$ *and let* $\wedge(M_*^+(X))$ *be the Grassmann algebra*

26) See, Theorem C (Introduction).

27) In thi paper "group manifold of a compact connected Lie group".

28) For detail see [7], [15], [8].

over the field ρ *of the space* $M_*^+(X)$. *Then* $\wedge(M_*^+(X)) = \sum_p \wedge_p(M_*^+(X)) \approx \mathfrak{P}(X)$ *by the correspondence*: $\lambda(\xi_1 \wedge \xi_2 \ldots \wedge \xi_k) = \xi_1 \circ \xi_2 \circ \ldots \circ \xi_k$ $(\xi_i \in M_*^+(X))$.

The elements of $\sum_{p\geqslant 2} \wedge_p(M_*^+(X))$ are called *composable* (or more preoisely $\circ$-*composable*), and the elements of the annihilator $M^*(X)$ in $H^*(X)$ of $\sum_{p\geqslant 2} \wedge_p(M_*^+(X))$ are called *minimal* (or more precisely $\circ$-*minimal*). Obviously $M^*(X)$ is the direct sum of the spaces $M^p(X) = M^*(X) \cap H^p(X)$. We put $M_+^*(X) = \sum_{p\geqslant 1} M^p(X)$.

(8.3) $\wedge(M_+^*(X)) \approx H^*(X)$ *by the correspondence* $\lambda(\xi_1 \wedge \xi_2 \ldots \wedge \xi_k) = \xi_1 \cup \xi_2 \cup \ldots \cup \xi_k (\xi_i \in M_+^*(X))$. *Thus the linearly independent basis of* $M_+^*(X)$ *together with 1 constitute an irreducible system of generators of the ring* $H^*(X)$.

We use further the facts:

(8.4) $M_{2\nu}(X) = 0$ $(\nu > 0)$: *every non-trivial minimal homogeneous element of* $H_*(X)$ *is odd dimensional.*

(8.5) $M^{2\nu}(X) = 0$ $(\nu > 0)$; *every non-trivial minimal homogeneous element of* $H^*(X)$ *is odd dimensional.*

As a dual operation of the left $\circ$-multiplication $(z^p\circ): H_q(X) \to H_{p+q}(X)$, $(z^p\circ)z^q = z^p \circ z^q$, where $z^p \in H_p(X)$, $z^q \in H_q(X)$, we define an operation $(\sqsubset z^p)$: $H^{p+q}(X) \to H^q(X)$ by:

(8.6) $$\langle u_{\square}^{p+q} z^p, z^q \rangle = \langle u^{p+q}, z^p \circ z^q \rangle.$$

Then the following is the lemma 2.2 of [8]:

(8.7) *Let* $\mathfrak{a} \neq 0$ *be a subring of* $H^*(X)$. *If* $\mathfrak{a}$ *is stable under every* $(\sqsubset z)$, $z \in H_*(X)$, *then* $\mathfrak{a}$ *is generated by* 1 *and a subspace* V^* *of* $M_+^*(X)$, *i.e.* $\mathfrak{a}$ *contains* 1 *and is generated by the minimal elements belonging to* $\mathfrak{a}$.

Part II

1. Characteristic groups and characteristic isomorphisms of a set system.

Given a set system $\mathfrak{A} = \{A = A_n \supset A_{n-1} \supset \cdots \supset A_0 \supset A_{-1}\}$, we consider, for $q \geqslant q' \geqslant q''$, the cohomology sequence of the triple $(A_q, A_{q'}, A_{q''})$:

(1.1) $$\longrightarrow H^p(A_q, A_{q'}) \xrightarrow{\gamma_p^{(q,q',q'')}} H^p(A_q, A_{q''}) \xrightarrow{\beta_p^{(q,q',q'')}} H^p(A_{q'}, A_{q''}) \xrightarrow{\alpha_p^{(q,q',q'')}} H^{p+1}(A_q, A_{q'}) \longrightarrow.$$

For brevity we shall write $[q', q'')$, $(q, q']$, $[q']$, (q) instead of $(q'+1, q', q'')$, $(q, q', q'-1)$, $(q'+1, q', q'-1)$, $(q, q-1, -1)$ respectively. Furthermore we put $A_q = A_n$ $(q \geqslant n)$, $A_q = A_{-1}$ $(q \leqslant -1)$.

Now we define

(1.2) $$\mathfrak{C}^q(p-q) \equiv \mathfrak{C}^{p-q,q}(A) \equiv H^p(A_q, A_{q-1}),$$

(1.3) $$\mathfrak{Z}_k^q(p-q) = kernel\ \alpha_p^{(q+k+2,\,q)} = image\ \beta_p^{(q+k+2,\,q)}(k \geqslant -2),$$

(1.4) $$\mathfrak{B}_k^q(p-q) = kernel\ \gamma_p^{(q-1,\,q-k-2)} = image\ \alpha_{p-1}^{(q-1,\,q-k-2)}(k \geqslant -1).$$

Clearly $\mathfrak{Z}_k^q = \mathfrak{Z}_{k+1}^q = \cdots$, $\mathfrak{B}_k^q = \mathfrak{B}_k^q = \cdots$ for sufficiently large k: we shall denote them by $\mathfrak{Z}_\infty^q$, $\mathfrak{B}_\infty^q$ respectively. We see also that

(1.5) $$\mathfrak{C}^q = \mathfrak{Z}_{-2}^q \supset \mathfrak{Z}_{-1}^q \equiv \mathfrak{Z}^q \supset \mathfrak{Z}_0^q \supset \cdots \supset \mathfrak{Z}_\infty^q \supset \mathfrak{B}_\infty^q \supset \cdots \supset \mathfrak{B}_1^q \supset \mathfrak{B}_0^q \equiv \mathfrak{B}^q \supset \mathfrak{B}_{-1}^q = 0.$$

Let us prove for instace $\mathfrak{Z}_k^q \supset \mathfrak{B}_\infty^q$. Remembering that the transitivity relation holds in the diagram:

$$\begin{array}{ccc} H^p(A_q,\, A_{q-1}) & \xleftarrow{\ \beta_p^{(q+k+2,\,q)}\ } & H^p(A_{q+k+2},\, A_{q-1}) \\ & \nwarrow{\scriptstyle \alpha_{p-1}^{(q)}} & \Big\uparrow {\scriptstyle \alpha_{p-1}^{(q+k+2,\,q-1,\,-1)}} \\ & & H^{p-1}(A_{q-1},\, A_{-1}), \end{array}$$

we have $\mathfrak{Z}_k^q = image\ \beta_p^{(q+k+2,\,q)} \supset image\ \beta_p^{(q+k+2,\,q)}\alpha_{p-1}^{(q+k+2,\,q-1,\,-1)} = image\ \alpha_{p-1}^{(q)} = \mathfrak{B}_\infty^q$.

Theorem 1.

(1.6) $$\Phi_k : \mathfrak{Z}_{k-1}^q(p)/\mathfrak{Z}_k^q(p) \approx \mathfrak{B}_{k+1}^{q+k+2}(p-k-1)/\mathfrak{B}_k^{q+k+2}(p-k-1).$$

Theorem 2. *$H^{p,q}$ being the kernel of the injection homomorphism $H^{p+q}(A, A_{-1}) \supset H^{p+q}(A_q, A_{-1})$, we have*

(1.7) $$\Psi : H^{p+1,\,q-1}/H^{p,\,q} \approx \mathfrak{Z}_\infty^q(p)/\mathfrak{B}_\infty^q(p).$$

Lemma 1. *Assume that the transitivity relations hold in the diagram:*

$$\begin{array}{ccccc} & {\scriptstyle \theta_1'}\nearrow & G_4 & & \\ & & \Big\uparrow{\scriptstyle \psi} & & \\ G_1 & \xrightarrow{\ \theta_1\ } & G_2 & \xrightarrow{\ \varphi\ } & G_3\ , \\ & & \Big\uparrow{\scriptstyle \theta_2} & \nearrow{\scriptstyle \theta_2'} & \\ & & G_5 & & \end{array}$$

and that the homorphisms θ_1, φ; θ_2, ψ are exact. If we put $\Gamma = kernel\ \theta_1$, $\Gamma' = kernel\ \theta_1'$, $\Delta = kernel\ \theta_2$, $\Delta' = kernel\ \theta_2'$, we have $\Gamma'/\Gamma \approx \Delta'/\Delta$.

Proof: For $x \in \Gamma'$, $0 = \theta_1' x = \psi\theta_1 x$. Therefore, from the exactness of θ_2, ψ, $\theta_1 x = \theta_2 y$ for some $y \in G_5$. From the exactness of θ_1, φ, $\theta_2' y = \varphi\theta_1 x = 0$, implying that $y \in \Delta'$. Although y cannot be determined uniquely, but for two such y $\theta_2(y_1 - y_2) = 0$. Therefore $y_1 - y_2 \in \Delta$. Thus we obtain a homomorphism $\Gamma' \to \Delta'/\Delta$, which is easily verified to be an onto homomorphism. That the kernel of this homomorphism is Γ is also clear.

Proof of Theorem 1: We have only to apply Lemma 1 to the following diagram:

$$\begin{array}{ccccc}
 & & H^{p+q+1}(A_{q+k+1},\ A_q) & & \\
 & \nearrow^{\alpha_{p+q}^{(q+k+1,\ q)}} & \uparrow \beta_{p+q+1}^{(q+k+1,\ q)} & \gamma_{p+q+1}^{(q+k+2,\ q)} & \\
H^{p+q}(A_q,\ A_{q-1}) & \xrightarrow{} & H^{p+q+1}(A_{q+k+2},\ A_q) & \xrightarrow{} & H^{p+q+1}(A_{q+k+2},\ A_{q-1}). \\
 & \alpha_{p+q}^{(q+k+2,\ q)} & \uparrow \gamma_{p+q+1}^{(q+k+1,\ q)} & \nearrow \gamma_{p+q+1}^{(q+k+1,\ q-1)} & \\
 & & H^{p+q+1}(A_{q+k+2},\ A_{q+k+1}) & &
\end{array}$$

In the same way we can prove Theorem 2.

Considering the factor groups of the groups in (1.5) by $\mathfrak{B}_0^q$, we have

$$(1.8)\qquad \mathfrak{G}^q/\mathfrak{B}_0^q = \mathfrak{H}_{-2}^q \supset \mathfrak{H}_{-1}^q \equiv \mathfrak{H}^q \supset \cdots \supset \mathfrak{H}_\infty^q \supset \mathfrak{K}_\infty^q \supset \cdots \supset \mathfrak{K}_1^q \supset \mathfrak{K}_0^q = 0,$$

and the isomorphisms of Theorem 1, 2 become

$$(1.9)\qquad \Phi_k : \mathfrak{H}_{k-1}^q(p)/\mathfrak{H}_k^q(p) \approx \mathfrak{K}_{k+2}^{q+k+2}(p-k-1)/\mathfrak{K}_k^{q+k+2}(p-k-1),$$

$$(1.10)\qquad \Psi : H^{p+1,q-1}/H^{p,q} \approx \mathfrak{H}_\infty^q(p)/\mathfrak{K}_\infty^q(p).$$

Definition 1. For the set system $\mathfrak{A}$ we call $\mathfrak{H}_k^q(p)$, $\mathfrak{K}_k^q(p)$, $H^{p,q}$; Φ, Ψ the characteristic groups and the characteristic isomorphisms of $\mathfrak{A}$. In the case when one wishes to make explicit the fact that they are associated with $\mathfrak{A}$, they are written as $\mathfrak{H}_k^{p,q}(\mathfrak{A})$, $\mathfrak{K}_k^{p,q}(\mathfrak{A})$, $H^{p,q}(\mathfrak{A})$; $\Phi(\mathfrak{A})$, $\Psi(\mathfrak{A})$.

Definition 2. A map $f: A \to A'$ is called a map of the set system $\mathfrak{A}$ into $\mathfrak{A}'$ and is denoted by $f: \mathfrak{A} \to \mathfrak{A}'$, if $f(A_q) \subset A_q'$ for each q, where $\mathfrak{A}' = \{A' = A_n' \supset A_{n-1}' \supset \cdots \supset A_{-1}'\}$.

Theorem 3. *$f: \mathfrak{A} \to \mathfrak{A}'$ obviously induces the homomorphisms;*

$$(1.11)\qquad \begin{aligned} f^\# : \mathfrak{H}_k^{p,q}(\mathfrak{A}') &\longrightarrow \mathfrak{H}_k^{p,q}(\mathfrak{A}), \\ \mathfrak{K}_k^{p,q}(\mathfrak{A}') &\longrightarrow \mathfrak{K}_k^{p,q}(\mathfrak{A}), \\ H^{p,q}(\mathfrak{A}') &\longrightarrow H^{p,q}(\mathfrak{A}), \end{aligned}$$

and hence the homomorphisms

$$\begin{aligned} f^\# : \mathfrak{H}_{k-1}^{p,q}(\mathfrak{A})/\mathfrak{H}_k^{p,q}(\mathfrak{A}) &\longrightarrow \mathfrak{H}_{k-1}^{p,q}(\mathfrak{A})/\mathfrak{H}_k^{p,q}(\mathfrak{A}) \\ \mathfrak{K}_{k+1}^{p,q}(\mathfrak{A})/\mathfrak{K}_k^{p,q}(\mathfrak{A}') &\longrightarrow \mathfrak{K}_{k+1}^{p,q}(\mathfrak{A})/\mathfrak{K}_k^{p,q}(\mathfrak{A}) \end{aligned}$$

commute with the characteristic isomorphism:

$$\begin{aligned} f^\# \Phi(\mathfrak{A}') &= \Phi(\mathfrak{A}) f^\#, \\ f^\# \Psi(\mathfrak{A}') &= \Psi(\mathfrak{A}) f^\#. \end{aligned}$$

The proof is omitted (cf, the proof of Theorem 4).

Definition 3. Two maps $f_0, f_1: \mathfrak{A} \to \mathfrak{A}'$ are called *homotopic* (in the weak sense), if there exists a map $F: A \times I \to A'$ such that $F(a, 0) = f_0(a)$, $F(a, 1) = f_1(a)$, $F(A_q \times I) \subset A_{q+1}'$.

Theorem 4. *If $f_0, f_1: \mathfrak{A} \to \mathfrak{A}'$ are homotopic, then $f_0^\# = f_1^\#$, $k \geqq -1$.*

Proof: It is sufficient to prove $f_0^{\#}\varsigma - f_1^{\#}\varsigma \in \mathfrak{B}_0^{p,q}(\mathfrak{A})$ for any $\mathfrak{Z}_{-1}^{p,q}(\mathfrak{A}')$. If we define $\lambda : A\times I \to A$, $\mu_i : A \to A\times I$ by $\lambda(a\times t)=a$, $\mu_i(a)=a\times(i)$, commutativity relations hold in the diagram

$$\begin{array}{ccccc} H^{p+q}(A_q', A_{q-1}') & \xleftarrow{\quad\quad\quad\quad i^* \quad\quad\quad\quad} & & & H^{p+q}(A_{q+1}', A_{q-1}') \\ \downarrow f_i^* & & & & F^* \downarrow \\ H^{p+q}(A_q, A_{q-1}) & \xrightarrow{j_*} & H^{p+q}(A_q, A_{q-2}) & \underset{\mu_i^*}{\overset{\lambda^*}{\rightleftarrows}} & H^{p+q}(A_q\times I \ \ A_{q-2}\times I), \end{array}$$

and λ^*, μ_i^* are inverses of each other. Choosing $\xi \in H^{p+q}(A_{q+1}', A_{q-1}')$ such that $\varsigma = i^*\xi$, we have $j^*f_i^*\varsigma = j^*f_i^*i^*\xi = \mu_i^*F^*\xi = (\lambda^*)^{-1}F^*\xi$. Therefore $j^*(f_0^*\varsigma - f_1^*\varsigma) = 0$, $f_0^*\varsigma - f_1^*\varsigma \in \mathfrak{B}_0^{p,q}(\mathfrak{A})$, $f_0^{\#} = f_1^{\#}(k \geqslant -1)$.

2. Leray's relation for Poincaré polynomials. In this section we take the rational number field as the coefficient group of our cohomology theory. We define three kinds of polynomials as follows:

$$\mathfrak{P}(t, s) = \sum_{p, q\geqslant 0} t^p s^q \rho(\mathfrak{H}^{p,q}), \quad \mathfrak{E}(t, s) = \sum_{p, q\geqslant 0} t^p s^q \rho(H^{p+1, q-1}/H^{p,q}),$$

$\mathfrak{D}_k(t, s) = \sum\limits_{p\geqslant k+1,\ q\geqslant 0} t^{p-(k+1)} s^q \rho(\mathfrak{H}_{k-1}^{p,q}/\mathfrak{H}_k^{p,q})$, where $\rho(\mathfrak{m})$ denotes the rank of $\mathfrak{m}$.

The following relation among these polynomials was given by J. Leray [11]: If $\mathfrak{H}_{-1}^{p,q} = \mathfrak{H}_\infty^{p,q}$ for any p, q,

$$\mathfrak{P}(t, s) = \mathfrak{E}(t, s) + \sum_{k=0}^{\infty} (t^{k+1} + s^{k+2})\mathfrak{D}_k(t, s). \tag{2.1}$$

Proof: $\sum\limits_{k=0}^{\infty} t^{k+1}\mathfrak{D}_k(t, s) = \sum\limits_{k=0}^{\infty} \sum\limits_{p\geqslant k+1,\ q\geqslant 0} t^p s^q \rho(\mathfrak{H}_{k-1}^{p,q}/\mathfrak{H}_k^{p,q}) = \sum\limits_{p\geqslant 1,\ q\geqslant 0} t^p s^q \rho(\mathfrak{H}^{p,q}/\mathfrak{H}^{p,q})$
$+ \sum\limits_{p\geqslant 2,\ q\geqslant 0} t^p s^q \rho(\mathfrak{H}_0^{p,q}/\mathfrak{H}_1^{p,q}) + \sum\limits_{p\geqslant 3,\ q\geqslant 0} t^p s^q \rho(\mathfrak{H}_1^{p,q}/\mathfrak{H}_2^{p,q}) + \cdots = \sum\limits_{p\geqslant 1,\ q\geqslant 0} t^p s^q \rho(\mathfrak{H}^{p,q})$
$-[\sum\limits_{q\geqslant 0} t^1 s^q \rho(\mathfrak{H}_0^{1,q}) + \sum\limits_{q\geqslant 0} t^2 s^q \rho(\mathfrak{H}_1^{2,q}) + \cdots] = \mathfrak{P}(t, s) - \sum\limits_{q\geqslant 0} [t^p s^q \rho(\mathfrak{H}_\infty^{0,q}) + t^1, s^q \rho(\mathfrak{H}_\infty^{1,q})$
$+ t^2 s^q \rho(\mathfrak{H}_\infty^{2,q}) + \cdots] = \mathfrak{P}(t, s) - \sum\limits_{p, q\geqslant 0} t^p s^q \rho(\mathfrak{H}^{p,q})$.

$\sum\limits_{k=0}^{\infty} s^{k+2}\mathfrak{D}_k(t, s) = \sum\limits_{k=0}^{\infty} \sum\limits_{p\geqslant k+1,\ q\geqslant 0} t^{p-(k+1)} s^{q+k+2} \rho(\mathfrak{K}_{k+1}^{p-k-1, q+k+2}/\mathfrak{K}_k^{p-k-1, q+k+2})$
$= \sum\limits_{k=0}^{\infty} \sum\limits_{p\geqslant 0,\ q\geqslant k+2} t^p s^q (\mathfrak{K}_{k+1}^{p,q}/\mathfrak{K}_k^{p,q}) = \sum\limits_{p\geqslant 0,\ q\geqslant 2} t^p s^q (\mathfrak{K}_1^{p,q}) + \sum\limits_{p\geqslant 0,\ q\geqslant 3} t^p s^q \rho(\mathfrak{K}_2^{p,q}/\mathfrak{K}_1^{p,q}) + \cdots$
$= \sum\limits_{p\geqslant 0} t^p s^2 \rho(\mathfrak{K}_1^{p,2}) + \sum\limits_{p\geqslant 0} t^p s^3 \rho(\mathfrak{K}_2^{p,3}) + \cdots = \sum\limits_{p\geqslant 0,\ q\geqslant 2} t^p s^q \rho(\mathfrak{K}_\infty^{p,q}) = \sum\limits_{p\geqslant 0,\ q\geqslant 0} t^p s^q \rho(\mathfrak{K}_\infty^{p,q})$.

$\sum\limits_{k=0} (t^{k+1} + s^{k+2})\mathfrak{D}_k(t, s) = \mathfrak{P}(t, s) - \sum\limits_{p, q\geqslant 0} t^p s^q \rho(\mathfrak{H}_\infty^{p,q}) + \sum\limits_{p, q\geqslant 0} t^p s^q \rho(\mathfrak{K}_\infty^{p,q}) = \mathfrak{P}(t, s)$
$- \sum\limits_{p, q\geqslant 0}^{\infty} t^p s^q \rho(\mathfrak{H}^{p,q}/\mathfrak{K}_\infty^{p,q}) = \mathfrak{P}(t, s) - \sum\limits_{p, q\geqslant 0} t^p s^q \rho(H^{p+1, q-1}/H^{p,q}) = \mathfrak{P}(t, s) - \mathfrak{E}(t, s)$.

3. Results for homology theory. The discussions and theorems in the preceeding sections may apply to homology groups, homotopy groups, as well

29) This is the case in Part III.

as cohomotopy groups. For later use we state here the results for homology groups briefly. The sequence corresponding to (1.1) is the homology sequence of the triple $(A_q, A_{q'}, A_{q''})$:

$$(3.1)\qquad \begin{aligned} &\longleftarrow \bar{H}^p(A_q, A_{q'}) \xleftarrow{\bar{\gamma}_p^{(q,\, q',\, q'')}} \bar{H}^p(A_q, A_{q''}) \\ &\xleftarrow{\bar{\beta}_p^{(q,\, q',\, q'')}} \bar{H}^p(A_{q'}, A_{q''}) \xleftarrow{\bar{\alpha}_p^{(q,\, q',\, q'')}} \bar{H}^{p+1}(A_q, A_{q'}) \longleftarrow . \end{aligned}$$

We define:

(3.2) $\bar{\mathfrak{C}}^q(p-q) \equiv \bar{\mathfrak{C}}^{p-q,\, q}(\mathfrak{A}) = \bar{H}^p(A_q, A_{q-1})$, $\bar{\mathfrak{Z}}^q_k(p-q) \equiv \bar{\mathfrak{Z}}^{p-q,\, q}_k(\mathfrak{A}) =$ ***kernel*** $\bar{\alpha}_{p-1}^{(q-1,\, q-k-2)} =$ ***image*** $\bar{\gamma}_p^{(q-1,\, q-k-2)}$, $\bar{\mathfrak{B}}^q_k(p-q) \equiv \bar{\mathfrak{B}}^{p-q,\, q}_k(\mathfrak{A}) =$ ***kernel*** $\bar{\beta}_p^{(q+k+2,\, q)}$ $=$ ***image*** $\bar{\alpha}_p^{(q+k+2,\, q)}$.

$$(3.3)\qquad \bar{\mathfrak{C}}^q = \bar{\mathfrak{Z}}^q_{-1} \supset \bar{\mathfrak{Z}}^q_0 \equiv \bar{\mathfrak{Z}}^q \supset \bar{\mathfrak{Z}}^q_1 \supset \cdots \supset \bar{\mathfrak{Z}}^q_\infty \supset \bar{\mathfrak{B}}^q_\infty \supset \cdots \supset \bar{\mathfrak{B}}^q_0 \supset \bar{\mathfrak{B}}^q_{-1} \equiv \bar{\mathfrak{B}}^q \supset \bar{\mathfrak{B}}^q_2 = 0$$

Theorem 1′.

$$(3.4)\qquad \bar{\Phi}:\ \bar{\mathfrak{B}}^q_k(p)/\bar{\mathfrak{B}}^q_{k-1}(p) \approx \bar{\mathfrak{Z}}^{q+k+2}_k(p-k-1)/\bar{\mathfrak{Z}}^{q+k+2}_{k+1}(p-k-1).$$

Theorem 2′. *$\bar{H}^{p,q}$ being the image of the injection homomorphism $\bar{H}^{p+q}(A_q, A_{-1}) \to \bar{H}^{p+q}(A, A_{-1})$ we have*

$$(3.5)\qquad \bar{\Psi}^{-1}:\ \bar{H}^{p,q}/\bar{H}^{p+1,\,q-1} \approx \bar{\mathfrak{Z}}^q_\infty(p)/\bar{\mathfrak{B}}^q_\infty(p).$$

$$(3.6)\qquad \bar{\mathfrak{C}}^q/\bar{\mathfrak{B}}^q \supset \bar{\mathfrak{H}}^q_0 \equiv \bar{\mathfrak{H}}^q \supset \bar{\mathfrak{H}}^q_1 \supset \cdots \supset \bar{\mathfrak{H}}^q_\infty \supset \bar{\mathfrak{K}}^q_\infty \supset \cdots \supset \bar{\mathfrak{K}}^q_0 \supset \bar{\mathfrak{K}}^q_{-1} = 0.$$

Similarly Theorem 3′, 4′. corresponding to Theorem 3, 4. hold for homology theory.

If A_q are polyhedra and we are based upon suitable coefficient groups for instance the rational number field for both cohomology and homology groups, we have the following dualities: $\mathfrak{C}^q(p)$ and $\bar{\mathfrak{C}}^q(p)$ are dual to *each other*; $\mathfrak{Z}^q_k(p)$, $\mathfrak{B}^q_k(p)$, *are annihilators of* $\bar{\mathfrak{B}}^q_k(p)$, $\bar{\mathfrak{Z}}^q_k(p)$ *respectively.* $H^{p+q}(A)$ *and* $\bar{H}^{p+q}(A)$ *are dual to each other*; $H^{p,q}$ *and* $\bar{H}^{p,q}$ *are annihilators of each other.*

4. Application to cohomotopy groups. Theorem 1–4 also apply to cohomotopy groups [16], under certain restriction of dimensions. Let K be an n-complex and let K^q be its q-section ($q=-1, 0, \cdots, n$). Let us put $A_q = K^q$ such that we obtain a set system $\mathfrak{A} = \{K = K^n \supset \cdots \supset K^{-1} = 0\}$.

Let $\pi^q(X, Y)$ denote the q-th relative cohomotopy group. Then $\mathfrak{C}^q(p) \equiv \pi^{p+q}(K^q, K^{q-1})$ for $2(p+q)-1 > q$, and $\mathfrak{Z}^q(p)$ for $2(p+q)-1 > n$, are defined; $\mathfrak{B}^q(p)$ for $2(p+q)-1 > q-1$, $H^{p,q} \equiv \pi^{p,q}$ for $2(p+q)-1 > n$, are also defined. Thus in these cases where $2(p+q) > n+2$ and $q = n$, or $2(p+q) > n+1$ and $q \leqq n-1$, we are not restricted in applying Theorem 1–4. In the remainder we assume that p, q range over the domain refered to above.

First of all it should be remarked that the following facts are known[30]:

(5.1) $\mathfrak{G}^q(p) \equiv \pi^{p+q}(K^q, K^{q-1}) \approx C^q(K, (p+q)^q)$ *where* $(l)^h$ *denotes the h-th homotopy group of the l-sphere* S^l.

(5.2) $(l)^h = 0$ *for* $h < l$, *hence* $\mathfrak{G}^q(p) = 0$ *for* $p > 0$; $(h)^h \approx I$; $(h)^{h+1} \approx I_2$ *for* $h \geqq 3$; $(h)^{h+2} \approx I_2$ *for* $h \geqq 2$.

(5.3) $H^q(p) \approx H^q(K, (p+q)^q)$,

(5.4) $\pi^{p+q}(K) = \pi^{1, p+q-1} \supset \pi^{0, p+q} \supset \cdots \supset \pi^{p+q-n, n} = 0$,

(5.5) $\mathfrak{H}^n(0) = \mathfrak{H}^n_{-1}(0) = \mathfrak{H}^n_\infty(0)$, $\mathfrak{H}^{n-1}(0) = \mathfrak{H}^{n-1}_{-1}(0) = \mathfrak{H}^{n-1}_\infty(0)$,

$\mathfrak{H}^{n-2}(0) = \mathfrak{H}^{n-2}_{-1}(0) \supset \mathfrak{H}^{n-2}_0(0) = \mathfrak{H}^{n-2}_\infty(0)$,

(5.6) $\mathfrak{K}^q_\infty(0) = 0$, $\mathfrak{K}^q_\infty(-1) = \mathfrak{K}^q_1(-1)$, $\mathfrak{K}^q_\infty(-2) = \mathfrak{K}^q_2(-2)$,

(5.7) *We also denote by* Φ *the following composite homomorphism*: $\mathfrak{H}^q_{-1}(0) \longrightarrow \mathfrak{H}^q_{-1}(0)/\mathfrak{H}^q_0(0) \longrightarrow \mathfrak{K}^{q+2}_1(-1)/\mathfrak{K}^{q+2}_0(-1) = \mathfrak{K}^{q+2}_1(-1) = \mathfrak{K}^{q+2}_\infty(-1) \subset \mathfrak{H}^{q+2}_{-1}(-1)$. *If in virtue of the isomorphism of* (5.3) *we substitute* $H^q(K, I)$, $H^{q+2}(K, I_2)$ *for* $\mathfrak{H}^q_{-1}(0)$, $\mathfrak{H}^{q+2}_{-1}(-1)$ *respectively, the homomorphism* Φ: $\mathfrak{H}^q_{-1}(0) \longrightarrow \mathfrak{H}^{q+2}_{-1}(-1)$ *coresponds to the Steenrod's squaring homomorphism*: Sq_{q-2}: $H^q(K, I) \longrightarrow H^{q+2}(K, I_2)$, *so that* $\mathfrak{H}^q_0(0)$ *and* $\mathfrak{K}^{q+2}_1(-1)$ *may be identified to Kernel* $\{Sq_{q-2}\} \subset H^q(K, I)$, *Image* $\{H^q(K, I)\}$ *respectively.*

Keeping in mind the above remark, we have:

(i) $\quad \pi^n(K) \approx \pi^n(K)/\pi^{0,n} \approx \mathfrak{H}^n_\infty(0)/\mathfrak{K}^n_\infty(0) = \mathfrak{H}^n(0) \approx H^n(K, I) \quad (n \geqq 3)$.[31]

This is the Hopf-Whitney's classification theorem of maps of an n-complex into an n-sphere

(ii)
$$\pi^{1, n-2}/\pi^{0, n-1} \approx \mathfrak{H}^{n-1}_\infty(0)/\mathfrak{K}^{n-1}_\infty(0) = \mathfrak{H}^{n-1}(0),$$
$$\pi^{0, n-1}/\pi^{-1, n} \approx \mathfrak{H}^n_\infty(-1)/\mathfrak{K}^n_\infty = \mathfrak{H}^n(-1)/\mathfrak{K}^n_{-1}(-1),$$

or
$$\pi^{n-1}(K)/\pi^{0, n-1} \approx H^{n-1}(K, I),$$
$$\pi^{0, n-1} \approx H^n(K, I_2)/Sq_{n-4}H^{n-2}(K, I), \quad (n > 5).$$[31]

This is the Steenrod's classification theorem of maps of an n-complex into an $(n-1)$-sphere.

(iii)
$$\pi^{n-2}(K) = \pi^{1, n-3},$$
$$\pi^{1, n-3}/\pi^{0, n-2} \approx \mathfrak{H}^{n-2}_\infty(0)/\mathfrak{K}^{n-2}_\infty(0) = \mathfrak{H}^{n-2}_0(0),$$
$$\pi^{0, n-2}/\pi^{-1, n-1} \approx \mathfrak{H}^{n-1}_\infty(-1)/\mathfrak{K}^{n-1}_\infty(-1) = \mathfrak{H}^{n-1}/(-1)/\mathfrak{K}^{n-1}_1(-1),$$
$$\pi^{-1, n-1}/\pi^{-2, n} \approx \mathfrak{H}^n_\infty(-2)/\mathfrak{K}^n_\infty(-2) = \mathfrak{H}^n(-2)/\mathfrak{K}^n_2(-2),$$

30) See [16], [25].

31) Our method does not aply to lower dimensional cases.

or

$$\pi^{n-2}(K)/\pi^{0,n-2} \approx Kernel\ \{Sq_{n-4}\} \subset H^{n-2}(K, I),$$
$$\pi^{0,n-2}/\pi^{-1,n-1} \approx H^{n-1}(K, I_2)/Sq_{n-5}H^{n-3}(K, I),$$
$$\pi^{-1,n-1} \approx H(K, I_2)/\Gamma^{32)}.$$

In order to determine the group Γ or $\mathfrak{K}_2^n(-2)$ we must consider:

$$\Phi: \mathfrak{H}_0^{n-3}(0) \longrightarrow \mathfrak{K}_2^n(-2)/\mathfrak{K}_1^n(-2) \subset \mathfrak{H}^n(-2)/\mathfrak{K}_1^n(-2),$$

$$\Phi: \mathfrak{H}^{n-2}(-1) \longrightarrow \mathfrak{K}_1^n(-2)/\mathfrak{K}_0^n(-2) = \mathfrak{K}_1^n(-2) \subset \mathfrak{H}^n(-2),$$

or

$$\Phi': H_0^{n-3}(K, I) \longrightarrow H^n(K, I_2)/K_1^n(-2),$$

$\Phi'': H^{n-2}(K, I_2) \longrightarrow K_1^n(-2)\,(H^n(K, I_2),$ where $H_0^{n-3}(K, I) =$ Kernel $\{Sq_{n-5}\} \subset H^{n-3}(K, I)$.

Part III
Fibre bundles over a complex

1. Let $\mathfrak{F}=\{F, G; B, A, \psi, \varphi_U\}$ be a fibre bundle over a complex B, i.e. a finite polyhedron with a definite cellular decomposition $B=\{\sigma_i^q\}$.[21] We denote the q-section of B by B^q, and the inverse image $\psi^{-1}(\Omega)$ of Ω by $\tilde{\Omega}$, where Ω is any subset of B. Let B_0 a given subcomplex of B, and put $A_q=\widetilde{B^q \cup B_0}$. A_q's form a set system $\mathfrak{A}=\{A=A_n \supset A_{n-1} \supset \cdots \supset A_0 \supset A_{-1}\}$, the characteristic groups and isomorphisms of which are now precisely analysed. The method is axiomatic, and is closely parallel to the one which was used when Eilenberg-Steenrod proved the coincidence of the axiomatic cohomology groups of a complex with the ordinary one calculated from its cellular structure.

2. The case: $\mathbf{B=E^q}$. If $B=E^q$, $\mathfrak{F}$ is a product bundle, and without loss of generality we may assume that $A=\tilde{E}^q=B\times F$. Let the northern and southern hemi-spheres of S^r be denoted by E_+^r, E_-^r respectively. Then the equator is $S^{r-1}=E_+^r \cap E_-^r$. In this section we take $B_0=0$. Consider the cohomology sequence of the triple $(\tilde{E}_+^r, \tilde{S}^{r-1}, \tilde{E}_-^{r-1})$:

$$\longrightarrow H^{p+r+1}(\tilde{E}_+^r, \tilde{E}_-^{r-1}) \longrightarrow H^{p+r-1}(\tilde{S}^{r-1}, \tilde{E}_-^{r-1}) \xrightarrow{\delta} H^{p+q}(\tilde{E}_+^r, \tilde{S}^{r-1})$$
$$\longrightarrow H^{p+r}(\tilde{E}_+^r, \tilde{E}_-^{r-1}) \longrightarrow.$$

Since $H^p(\tilde{E}_+^r, \tilde{E}_-^r)=0$ by (I, §3, Lemma 4), the coboundary homomorphism δ in the above sequence is an isomorphism onto. On the other hand, we obtain an excision isomorphism $H^{p+r-1}(\tilde{S}^{r-1}, \tilde{E}_-^{r-1}) \approx H^{p+r-1}(\tilde{E}_+^{r-1}, \tilde{S}^{r-2})$ by Axiom VI' (I, §2). Combining these, we obtain isomorphisms:

32) This is the only group which is not known to be calculable from a given simplicial decomposition of K.

(2.1) $$g_{r-1}: H^{p+r-1}(\tilde{E}_+^{r-1}, \tilde{S}^{r-2}) \approx H^{p+r}(\tilde{E}_+^r, \tilde{S}_-^{r-1}),$$

(2.2) $$g^q: g_{q-1}g_{q-2}\cdots g_0: H^p(\tilde{E}_+^0) \longrightarrow H^{p+q}(\tilde{E}_+^q, \tilde{S}^{q-1}).$$

Futher it is easy to see that the injection homomorphism $i^*: H^p(\tilde{E}_+^q) \rightarrow H^p(\tilde{E}_+^0)$ is an isomorphism onto, and combining it with g^q we obtain the following isomorphism onto:

(2.3) $$h^q: H^p(\tilde{E}_+^q) \longrightarrow H^{p+q}(\tilde{E}_+^q, \tilde{S}^{q-1}).$$

Since $\tilde{E}_+^0$ is homeomorphic with the fibre F, we have

(2.4) $$H^{p+q}(\tilde{E}_+^q, \tilde{S}^{q-1}) \approx H^p(F).$$

In particular, considering the case when H consists of a single point,

(2.5) $H^p(E_+^q, S^{q-1}) = 0$ $(p \neq q)$, otherwise $\approx H^0(E_+^0)$ $(=\rho$: the coefficient group).

3. The structure of $H^{p+q}(A_q, A_{q-1}) \equiv \mathfrak{C}^q(p)$. We put $u_i^q = \sigma_i^q - \tilde{\sigma}_i^q$. Let λ_i^q: $H^{p+q}(A_q, A_{q-1}) \rightarrow H^{p+q}(\tilde{\sigma}_i^q, \dot{\sigma}_i^q)$, μ_i^q: $H^{p+q}(A_q, \widetilde{(B^q - u_i^q) \cup B_0}) \rightarrow H^{p+q}(\tilde{\sigma}_i^q, \dot{\sigma}_i^q)$, ν_i^q: $H^{p+q}(A_q, \widetilde{(B^q - u_i^q) \cup B_0}) \rightarrow H^{p+q}(A_q, A_{q-1})$ be the homomorphisms induced by identity maps Then μ_i^q is an isomorphism onto for $\sigma_i^q \notin B_0$, and $w_i^q = \nu_i^q(\mu_i^q)^{-1}$ is definable. Since $\mu_i^q = \lambda_i^q \nu_i^q$, $\lambda_i^q w_i^q = 1$; consequently w_i^q is an isomorphism into and λ_i^q is a homomorphism onto. Now applying the argument in the proof of Lemma 5 (I, §3), we see that

(3.1) $$H^{p+q}(A_q, A_{q-1}) = \sum w_i^q H^{p+q}(\tilde{\sigma}_i^q, \dot{\sigma}_i^q)$$

is a direct decomposition. Now let $\tilde{f}_i$: $(\tilde{E}_+^q, \tilde{S}^{q-1}) \rightarrow (\tilde{\sigma}_i^q, \dot{\sigma}_i^q)$ be an admissible map, the existence of which is assured by the Feldbau's theorem. Then

(3.2) $$h_i^q = (\tilde{f}_i^*)^{-1} h^q \tilde{f}_i^*: H^p(\tilde{\sigma}_i^q) \longrightarrow H^{p+q}(\tilde{\sigma}_i^q, \dot{\sigma}_i^q)$$

is an isomorphism onto. By (3.1), (3.2) we obtain:

Proposion 1. *The elements of $H^{p+q}(A_q, A_{q-1}) \equiv \mathfrak{C}^q(p)$ are of the form $\sum_i w_i^q h_i^q(a_i^p)$, where $a_i^p \in H^p(\tilde{\sigma}_i^q)$, and the summation is ranged over all i with $\sigma_i^q \notin B_0$.*

4. Mayer cochain complexes $\{\mathfrak{C}^q(p); \alpha_{p+q}^{[q]}\}$ and $\{L^q(p, \mathfrak{F}); l_{p,q}\}$. Since $\alpha_{p+q+1}^{[q+1]} \alpha_{p+q}^{[q]} = \alpha_{p+q+1}^{[q+1, q-1]} \gamma_{p+q+1}^{[q]} \alpha_{p+q}^{[q]} = 0$,

(4.1) $$\longrightarrow \mathfrak{C}^q(p) \xrightarrow{\alpha_{p+q}^{[q]}} \mathfrak{C}^{q+1}(p) \xrightarrow{\alpha_{p+q+1}^{[q+1]}} \mathfrak{C}^{q+2}(p) \longrightarrow$$

is a Mayer cochain complex for each fixed p. The corresponding cohomology groups are $\mathfrak{H}^q(p) = \mathfrak{H}_{-1}^q(p)$.

We define now another Mayer cochain complex. For any closed cell σ of B, we define $\rho_\sigma = H^p(\tilde{\sigma})$. If $\sigma < \sigma'$, the identity map $\tilde{\sigma} \rightarrow \tilde{\sigma}'$ induces a homomorphism

$\chi(\sigma, \sigma'): \rho_{\sigma'} \to \rho_{\sigma}$. But since for a point $x \in \sigma$ both of the injection homomorphisms $H^p(\tilde{\sigma}') \to H^p(x)$, $H^p(\tilde{\sigma}) \to H^p(x)$ are isomorphisms onto, so is $\chi(\sigma, \sigma')$. We define $\chi(\sigma', \sigma) = \{\chi(\sigma, \sigma')\}^{-1}$. Then it is easy to see that $\{\rho_\sigma ; \chi(\sigma', \sigma)\}$ is a *faisceau* of groups over B. The Mayer cochain complex corresponding to this *faisceau* of groups is denoted by $\{L^q(p, \mathfrak{F}), l_{p,p}\}$.

Our goal is the following theorem:

Theorem 5. *The two Mayer complexes* $\{\mathfrak{E}^q(p), \alpha^{[q]}_{p+q}\}$, $\{L^q(p, \mathfrak{F}), l_{p,q}\}$ *are equivalent: more precisely, there exists an isomorphism* $\kappa: L^q(p, F) \to \mathfrak{E}^q(p)$, *such that*

$$\alpha^{[q]}_{p+q}\kappa = \kappa l_{p,q}. \tag{4.2}$$

κ *is actually given by* $\kappa: L^q(p, \mathfrak{F}) \ni \sum a_i^p \sigma_i^q \to \sum w_i^q h_i^q a_i^q \in \mathfrak{E}^q(p)$.

Proof: It is sufficient to prove, for each $a_i^p \sigma_i^q \in L^q(p, \mathfrak{F})$,

$$\alpha^{[q]}_{p+q}\kappa(a_i^p \sigma_i^q) = \kappa l_{p,q}(a_i^p \sigma_i^q), \text{ or}$$

$$\alpha^{[q]}_{p+q} w_i^q h_i^q a_i^q = \textstyle\sum_j w_j^{q+1} h_j^{q+1}([\sigma_j^{q+1}) \sigma_i^q]\,\chi(\sigma_j^{q+1}, \sigma_i^q) a_i^p), \text{ or}$$

$$\lambda_j^{q+1} a^{[q]}_{p+q} w_i^q h_i^q a_i^p = h_j^{q+1}([\sigma_j^{q+1} : \sigma_i^q]\,\chi(\sigma_j^{q+1}, \sigma_i^q) a_i^p), \text{ for each } j \text{ with } \sigma_j^q \notin B_0. \tag{4.3}$$

Condition (4.3) may be further simplified. Consider the diagram:

$$\begin{array}{ccccc}
H^{p+q}(A_q, A_{q-1}) & & \xrightarrow{\alpha^{[q]}_{p+q}} & & H^{p+q+1}(A_{q+1}, A_q) \\
\uparrow \omega_i^q & \mu_i^q & \searrow \nu_i^q & \uparrow \delta & \downarrow \lambda_j^{p+1} \\
H^{p+q}(\tilde{\sigma}_i^q, \dot{\sigma}_i^q) & \xleftarrow[\delta]{} & H^{p+q}(A_q, \overline{(B^q - u_i^q) \cup B_0}) & & \\
 & & \downarrow \theta^* & & \\
 & & H^{p+q}(\dot{\sigma}_j^{q+1}) & \xrightarrow[\delta]{} & H^{p+q+1}(\tilde{\sigma}_j^{q+1}, \dot{\sigma}_j^{q+1}).
\end{array}$$

Since commutativity relations hold in the diagram, (4.3) reduces to:

$$\delta\theta^*(\mu_i^q)^{-1} h_i^q a_i^p = h_j^{q+1}([\sigma_i^{q+1} : \sigma_i^q]\,\chi(\sigma_j^{q+1}, \sigma_i^q) a_i^p). \tag{4.4}$$

(i) The case: $[\sigma_j^{q+1} : \sigma_i^q] = 0$. (4 4) becomes:

$$\delta\theta^*(\mu_i^q)^{-1} b_i = 0. \quad \text{for} \quad b_i \in H^{p+q}(\tilde{\sigma}_i^q, \dot{\sigma}_i^q). \tag{4.5}$$

Consider the diagram:

$$\begin{array}{ccccc}
H^{p+q}(\tilde{\sigma}_i^q, \dot{\sigma}_i^q) & \xleftarrow{\mu_i^q} & H^{p+q}(A_q, \overline{(B^q - u_i^q) \cup B_0}) & \xrightarrow{\theta^*} & H^{p+q}(\dot{\sigma}_j^{q+1}) \\
\nwarrow \varphi_1^* & & \downarrow \varphi_2^* & & \nearrow \varphi_3^* \\
 & & H^{p+q}(\overline{\dot{\sigma}_j^{q+1} \cup \sigma_i^q}, \overline{\dot{\sigma}_j^{q+1} \cup \dot{\sigma}_i^q}), & &
\end{array}$$

where φ's are identity maps. Commutativity relations hold in the diagram; φ_1^* is an excision isomorphism; $\varphi_3^* = 0$. Consequently, $\theta^*(\mu_i^q)^{-1} = \varphi_3^*(\varphi_1^*)^{-1} = 0$, which proves (4.5).

(ii) The case $[\sigma_j^{q+1} : \sigma_i^q] = \pm 1$. Commutativity relations hold in the following diagram:

$$\begin{array}{ccccc} & \mu_i^q & & \theta^* & \\ H^{p+q}(\tilde{\sigma}_i^q, \dot{\tilde{\sigma}}_i^q) & \longleftarrow & H^{p+q}(A_q, \widetilde{(B^q-u_i^q)\cup B_0}) & \longrightarrow & H^{p+q}(\dot{\tilde{\sigma}}_j^{q+1}) \\ & \nwarrow \varphi_4^* & \uparrow \varphi_5^* & & \downarrow \delta \\ & & H^{p+q}(\tilde{\sigma}_j^{q+1}, \widetilde{(\dot{\sigma}_j^{q+1}-u_i^q)}) & \underset{\delta}{\longrightarrow} & H^{p+q+1}(\tilde{\sigma}_j^{q+1}, \dot{\tilde{\sigma}}_j^{q+1}), \end{array}$$

where φ's are identity maps.

Since φ_4^* is an exision isomorphism, (4.4) reduces to:

$$(4.6) \qquad \delta(\varphi_4^*)^{-1}h_i^q a_i^p = [\sigma_j^{q+1}: \sigma_i^q]\, h_j^{q+1}\chi(\sigma_j^{q+1}, \sigma_i^q)\, a_i^p.$$

Now let us consider the case $[\sigma_j^{q+1}: \sigma_i^q]=+1$. In this case $'f_j=f_j|(E_+^q, S^{q-1}) \sim f_i$. Representing $\tilde{\sigma}_i^q$ as a product space $\sigma_i^q\times F$, let $\tilde{f}_i(x, y)=(f_i(x), \bar{g}_i(x, y))$, $'f_j(x, y)=('f_j(x), \bar{g}_j(x, y))$, where $(x, y)\in \tilde{E}_+^q$. Then, since E_+^q is contractible, without loss of generality, we may assume $\bar{g}_i(x, y)=g_iy$, $\bar{g}_j(x, y)=g_jy$ where g_i, $g_j\in G$. Further we may assume $'f_j=f_i \cdots f_i^*$, f_j^* is not changed by doing so. Thus assumed we have $\tilde{f}_i={}'\tilde{f}_j\tilde{g}$, where $\tilde{g}$ is an automorphism of $\tilde{E}_+^q$ defined by: $\tilde{g}: \tilde{E}_+^q \ni (x, y)\to(x, g_j^{-1}g_iy)=(x, gy)\in \tilde{E}_+^q$. It is obvious that, $h^q\tilde{g}^*=\tilde{g}^*h^q$: $H^p(\tilde{E}^q)\to H^{q+q}(\tilde{E}^q, \tilde{S}^{q-1})$. Hence $\delta(\varphi_4^*)^{-1}h_i=\delta(\varphi_4^*)^{-1}(\tilde{f}_i^*)\, h^q\, \tilde{f}_i^*=\delta(\varphi_4^*)('\tilde{f}_j^*)^{-1}$ $(\tilde{g}^*)^{-1}\, h^q\tilde{g}^{*\prime}\tilde{f}^*=\delta(\varphi_4^*)\,('\tilde{f}_j^*)\, h^{q\prime}\tilde{f}^*$. Therefore (4.6) reduces to:

$$(4.7) \qquad \delta(\varphi_4^*)^{-1}\,('\tilde{f}_j^*)^{-1}\, h^{q\prime}\tilde{f}_j^* = h_j^{q+1}\chi(\sigma_j^{q+1}, \sigma_i^q).$$

The proof of (4.7) is easy.

Finally let $[\sigma_j^{q+1}: \sigma_i^q]=-1$. Let τ be defined by $\tau(\xi_0, \cdots, \xi_q, \xi_{q+1})=(\xi_0, \cdots, -\xi_q, \xi_{q+1})$, and put $\bar{f}_j= f_j\tau$ and $\bar{h}_j^{q+1}=(\bar{f}_j^*)^{-1}\bar{h}^q\bar{f}_j^*$. Then $\bar{h}_j^{q+1}=(f_j\tau)^{*-1}h^q(f_j\tau)$ $=(f_j^*)^{-1}(\tau^*)^{-1}\, h^q\tau^*f_j^*$. But since the latter τ^* operating on $H^{p+q}(\tilde{E}_+^{q+1})$ is the identity, while the former $(\tau^*)^{-1}=\tau^*$ operating on $H^{p+q}(\tilde{E}_+^{q+1}, \tilde{S}^q)$ only changes sign, $\delta(\varphi_4^*)^{-1}\,\bar{h}_i^q a_i^p=\bar{h}_j^{q+1}\chi(\sigma_j^{q+1}, \sigma_i^q)\, a_i^p=-h_j^{q+1}\chi(\sigma_j^{q+1}, \sigma_i^q)\, a_i^p$, which proves (4.6). Theorem 5 is thus proved.

5. Orientable case (G=arc-wise connected). The group $L^q(p, \mathfrak{F})$ is isomorphic to the ordinary cochain group $C^q(B-B_0, H^p(F))$ of B with the p-th cohomology group of the fibre as coefficient domain. The isomorphism is realized as follows:

Let φ_i: $\sigma_i^q\times F\to\tilde{\sigma}_i^q$ be any admissible map, and put $\varphi_{i,x}$: $F\ni y\to\varphi_{i,x}(y)$ $=\varphi_i(x, y)$. $\varphi_{i,x}$ induces obviously isomorphism $\varphi_{i,x}^*$: $H^p(\tilde{\sigma}_i^q)\approx H^p(F)$, not depending on a special choice of $x\in\sigma^q$. Then $\bar{\kappa}$: $L^q(p, \mathfrak{F})\ni\sum a_i^p\sigma_i^q\to\sum(\varphi_{i,x}^* a_i^p)\,\sigma_i^q\in C^q$ $(B-B_0, H^p(F))$ is the desired isomorphism. Now and throughout the remainder of this paper **we assume that $\mathfrak{F}$ is orientable, i. e. G is arc-wise connected.** Then the above isomorphism $\bar{\kappa}$ is indedendent of a special choice of φ_i. Let φ_i' be another admissible map, then the map $\sigma_i^q\ni x\to\varphi_{i,x}^{-1}\varphi_{i,x}^1=g_{ix}\in G$ is homotopic to the constant map $\sigma_i^q\ni x\to e\in G$, where e denotes the unit element of G. Hence

$\overset{*}{\varphi}_{i,x}=\overset{*\prime}{\varphi}_{i,x}$, as desired. From this we can prove that $\delta=\bar{\kappa}l_{p,q}\bar{\kappa}^{-1}$: $C^q(B-B_0, H^p(F))\to C^{q+1}(B-B_0, H^p(F))$ is an ordinary coboundary operator. Thus we have:

Theorem 6. *The Mayer cochain complex* $\{L^q(p,\mathfrak{F}); l_{p,q}\}$, *and hence* $\{\mathfrak{C}(p), d_{p+q}^{[q]}\}$, *is equivalent to the Mayer complex* $\{C^q(B-B_0, H^p(F)), \delta\}$. *In particular* $\mathfrak{Z}^q(p)$, $\mathfrak{B}^q(p)$, $\mathfrak{H}^q(p)$ *are isomorphic to* $Z^q(B-B_0, H^p(F))$, $B^q(B-B_0, H^p(F))$, $H^q(B-B_0, H^p(F))$ *respectively. In particular, if our basic coefficient domain is a field,* $\mathfrak{H}^q(p)$ *is isomorphic to the Kronecker product* $H^q(B-B_0)\otimes H^p(F)$.

6. Invariance theorem. The characteristic groups and isomorphisms in the preceeding sections depend not only on the fibre bundle $\mathfrak{F}$ but also on a cellular decomposition (K, K_0) of the base space (B, B_0). Therefore their strict notations should have been $\mathfrak{H}_k^{p,q}(\mathfrak{F}; K, K_0)$, etc. Now let us consider for a given $\mathfrak{F}$ various cellular decompostions of (B, B_0) and find the relations among the corresponding characteristic groups and isomorphisms.

Theorem 7. *With each pair* $\{(K, K_0), ('K, 'K_0)\}$ *of cellular decompositions of* (B, B_0), *we may associate isomorphisms* $w\{(K, K_0), ('K, 'K_0)\}$: $\mathfrak{H}_k^{p,q}(\mathfrak{F}; 'K, 'K_0)\to\mathfrak{K}_k^{p,q}(\mathfrak{F}; K, K_0)$, $\mathfrak{K}_k^{p,q}(\mathfrak{F}, 'K, 'K_0)\to\mathfrak{K}_k^{p,q}(\mathfrak{F}, K, K_0)$, $H^{p,q}(\mathfrak{F}, 'K, 'K_0)\to H^{p,q}(\mathfrak{F}; K, K_0)$ *such that* $\Phi(\mathfrak{F}; K, K_0)w\{(K, K_0), ('K, 'K_0)\}=w\{(K, K_0), ('K, 'K_0)\}\Phi(\mathfrak{F}; 'K, 'K_0)$, $\Psi(\mathfrak{F}; K, K_0)w\{(K, K_0), ('K, 'K_0)\}=w\{(K, K_0), ('K, 'K_0)\}\Psi(\mathfrak{F}; 'K, 'K_0)$, *and such that* $w\{(K, K_0), ('K, 'K)\}w\{('K, 'K_0), (''K, ''K_0)\}=w\{(K, K_0), (''K, ''K_0)\}$.

Proof: Let $f:\mathfrak{F}\to'\mathfrak{F}$ be a bundle map, which induces a cellular map $\bar{f}:(K, K_0)\to('K, 'K_0)$, where (K, K_0), $('K, 'K_0)$ are given cellular decompositions of (B, B_0), $('B, 'B_0)$ respectively. Then f induces a map $f^{\natural}$ of the set system $\mathfrak{A}=\mathfrak{A}(\mathfrak{F}; K, K_0)$ into the set system $'\mathfrak{A}=\mathfrak{A}('\mathfrak{F}; 'K, 'K_0)$. By Theorem 3 (II, §1) $f^{\natural}:\mathfrak{A}\to'\mathfrak{A}$ induces in turn a homomorphism $f^{\#}$: $\mathfrak{H}_k^{p,q}('\mathfrak{F}; 'K, 'K_0)\to\mathfrak{H}_k^{p,q}(\mathfrak{F}; K, K_0)$, etc., commuting with Φ, Ψ. Let f_1: $\mathfrak{F}\to'\mathfrak{F}$ be another bundle map which induces a cellular map $\bar{f}_1$: $(K, K_0)\to('K, 'K_0)$, and which is connected to $f_0=f$ by a continuous family of bundle maps: $\mathfrak{F}\to'\mathfrak{F}$ $(0\leqq t\leqq 1)$. We assume further that each f_t induces $\bar{f}_t$: $(B, B_0)\to('B, 'B_0)$. $\bar{f}_t$ need not be cellular, but we may suppose $f_t(K^q)\subset'K^{q+1}$ for each q. For, if otherwise, we first replace $\bar{f}_t$ by $\bar{g}_t$ with the above property by deformation, and then replace f_t by g_t which induces $\bar{g}_t$ by Theorem D (I. §1). Thus assumed, f_t induces a homotopy of $f_0^{\natural}, f_1^{\natural}$: $\mathfrak{A}\to'\mathfrak{A}$, and by Theorem 4 (II, §1) we obtain:

$$f_0^{\#}=f_1^{\#}. \tag{6.1}$$

We first consider the case: $\mathfrak{F}='\mathfrak{F}$, $B_0=B_0$, $('K, 'K_0)>(K, K_0)$ ($('K, 'K_0)$ is a subdivision of (K, K_0)). Let $\bar{g}_1$ be a cellular map $('K, 'K_0)\to(K, K_0)$ which is homotopic to the identity $\bar{g}_0$: $(B, B_0)\to(B, B_0)$ the homotopy being

$\bar{g}_t$.[33] By Theorem D, there exists a continuous family of bundle maps g_t which induces $\bar{g}_t$ and such that g_0 is the identity 1. Obviously $g_1('\mathfrak{A})\subset\mathfrak{A}$, and induces $g_1^{\#}$: $\mathfrak{H}_k^{p,q}(\mathfrak{F}, 'K, 'K_0)\to\mathfrak{H}_k^{p,q}(\mathfrak{F}, 'K, 'K_0)$, etc.; $1^h(\mathfrak{A})\subset'\mathfrak{A}$ and induces $1^{\#}$: $\mathfrak{H}_k^{p,q}(\mathfrak{F}, K, K_0)\to\mathfrak{H}_k^{p,q}(\mathfrak{F}; K, K_0)$ etc. Now, since $g_1^h 1_1^h(\mathfrak{A})\subset\mathfrak{A}$ and $1^h g_1^h('\mathfrak{A})\subset'\mathfrak{A}$ are homotopic to the identities $1: \mathfrak{A}\to\mathfrak{A}$, $1: '\mathfrak{A}\to'\mathfrak{A}$ respectively, by Theorem 4 $1^{\#}g^{\#}=1$, $g^{\#}1^{\#}=1$.[34] Therefore $1^{\#}$ and $g^{\#}$ are both isomorpeisms onto, and inverses to each other. Since 1 (and hence $1^{\#}$) depends only on $\mathfrak{F}: K, K_0$; $'K, 'K_0$, so is $g_1^{\#}$. We put $w\{(K, K_0), ('K, 'K_0)\}=1^{\#}$, $w\{('K, 'K_0), (K, K_0)\}=g_1^{\#}$. Obviously if $(''K, ''K_0)>('K, 'K_0)>(K, K_0)$, $w\{(K, K_0), ('K, 'K_0)\}$ $w\{('K, 'K_0), (K, K_0)\}=w\{(''K, ''K_0), (K, K_0)\}$. Now let (K, K_0), $(\bar{K}, \bar{K}_0)$ be arbitrary two decompostions of (B, B_0). Let $('K, 'K_0)>(K, K_0)$ be so fine that the identity map $(B, B_0)\to(B, B_0)$ can be approximated by a cellular map $\bar{\psi}$: $('K, 'K_0)\to(K, K_0)$.[33] By making use of Theorem D, a bundle map $\psi: \mathfrak{F}\to\mathfrak{F}$ which induces ψ can be defined, inducing homomorphisms $\bar{\psi}^{\#}$: $\mathfrak{H}_k^{p,q}(\mathfrak{F}, \bar{K}, \bar{K}_0)\to\mathfrak{H}_k^{p,q}(\mathfrak{F}, 'K, 'K_0)$ etc. By (6.1) $\psi^{\#}$ is independent of the special ψ. We put $w\{(\bar{K}, \bar{K}_0)(K, K_0)\}=w\{(\bar{K}, \bar{K}_0), ('K, 'K_0)\}\, w\{'K, 'K_0), (K, K_0)\}$. It is indedendent of the subivision chosen. Now by the discussion used in the special case: $('K, 'K_0)>(K, K_0)$, we see that $w\{(K, K_0), (\bar{K}, \bar{K}_0)\}\, w\{(\bar{K}, \bar{K}_0)('K, 'K_0)\}=1$, showing that $w\{(\bar{K}, \bar{K}_0), (K, K_0)\}$'s are isomorphisms onto. That $w\{(K, K_0), ('K, 'K_0)\}\, w\{('K, 'K_0), (''K, ''K_0)\}=w\{(K, K_0), (''K, ''K_0)\}$, and that w commute with Φ, Ψ, are obvious, and the theorem is proved.

By Theorem 7, we may unite $\mathfrak{H}_k^{p,q}(F; K, K_0)$, etc. into $\mathfrak{H}_k^{p,q}(F, B, B_0)\equiv\{\mathfrak{H}_k^{p,q}(\mathfrak{F}; K^{(\alpha)}, K_0^{(\alpha)}), w\{(K^{(\beta)}, K_0^{(\beta)}), (K^{(\alpha)}, K_0^{(\alpha)})\}\}$ etc. An element u of $\mathfrak{H}_k^{p,q}(F; B, B_0)$ is of the form: $u=\{\cdots, u^{(\alpha)}, \cdots\}$, $u^{(\alpha)}\in\mathfrak{H}_k^{p,q}(\mathfrak{F}; K^{(\alpha)}, K_0^{(\alpha)})$, $u^{(\beta)}=w\{(K^{(\beta)}, K_0^{(\beta)}), (K^{(\alpha)}, K_0^{(\alpha)})\}\, u^{(\alpha)}$.

The influence under a general bundle map f: $\mathfrak{F}\to'\mathfrak{F}$ may be discussed in an obvious way.

8. Results for homology theory. All the results in the preceding sections may be similarly stated in terms of homology theory. In particular:

Theorem 6. *If G is arc-wise connected, the Mayer chain complex $\{\mathfrak{C}^q(p), \bar{d}_{p+q-1}^{[q-1]}\}$ is equivalent to the Mayer chain complex $\{\bar{C}^q(B, B_0; \bar{H}^p(F)), \partial\}$. In particular $\bar{\mathfrak{Z}}^q(p)$, $\bar{\mathfrak{B}}^q(p)$, $\mathfrak{H}^q(p)$ are isomorphic to $\bar{Z}^q(B, B; \bar{H}^p(F))$, $\bar{B}^q(B, B_0; \bar{H}^p(F))$, $\bar{H}^q(B, B_0; \bar{H}^p(F))$ respectively. In particular, if our basic coefficient domain is a field, $\bar{H}^q(p)$ is isomorphic to theKronecker product $\bar{H}^q(B, B_0)\otimes\bar{H}^p(F)$.*

9. Corollaries to Theorem 6 (and 6′). Throughout the remainder of this paper B, F are assumed to be connected polyhedra of dimensions n and d respectively, and G is arc-wise connected.

33) We may assume that $g_t('K^q)\subset K^{q+1}$. See, J. H. C. Whitehead [26].

34) $(gf)^{\#}=f^{\#}g^{\#}$.

Cor. 1. *If* $H^p(F)=0$, *in particular if* $p<0$ *or* $p>d$, *then* $\mathfrak{E}^q(p)=0$.

Cor. 2 $H^p(A_{q'}, A_q)=0$ $(q'\geqq q\geqq p)$.

Proof: $H^p(A_q, A_q)=0$, $H^p(A_{q+1}, A_q)\equiv\mathfrak{E}^{q+1}(p-q-1)=0$, $H^p(A_{q+2}, A_q)=0$ from the exactness of $0=H^p(A_{q+2}, A_{q+1})\rightarrow H^p(A_{q+2}, A_q)\rightarrow H^p(A_{q+1}, A_q)=0$, and so on.

The intuitive meaning of Cor. 2 is better understood in the dual form:

(9.1) $\bar{H}^p(A_{q'}, A_q)=0$ $(q'\geqq q\geqq p)$. (9.1) implies:

(9.2) (*Taking* $B_0=0$), *any* p*-cycle in* A *is homologous to a* p*-cycle in* A_p; *any* $(p-1)$*-cycle which is homologous to* 0 *in* A *is already homologous to* 0 *in* A_p.

Cor. 3. $H^p(A_q, A_{q'})=0$ $(p-d-1\geqq q\geqq' q)$.

Cor. 4. $\mathfrak{Z}^q_{n-q-2}(p)=\mathfrak{Z}^q_{p-1}(p)=\mathfrak{Z}^q_\infty(p)$, $\mathfrak{B}^q_{d-p}(p)=\mathfrak{B}^q_{q-1}(p)=\mathfrak{B}^q_\infty(p)$.

In particular, $\mathfrak{Z}^q(0)=\mathfrak{Z}^q_\infty(0)$, $\mathfrak{B}^q_d(0)=\mathfrak{B}^q_\infty(0)$; $\mathfrak{Z}^q_{d-1}(d)=\mathfrak{Z}^q_\infty(d)$, $\mathfrak{B}^q(d)=\mathfrak{B}^\infty(d)$.

Cor. 5. *If* $H^p(F)=0$ *for* $0<p<d$,

(9.3) $$\mathfrak{Z}^q(0)=\mathfrak{Z}^q_\infty(0)\supset\mathfrak{B}^q_\infty(0)=\mathfrak{B}^q_d(0)\supset\mathfrak{B}^q_{d-1}(0)=\mathfrak{B}^q(0),$$

(9.4) $$\mathfrak{Z}^q(d)=\mathfrak{Z}^q_{d-1}(d)\supset\mathfrak{Z}^q_{d-1}(d)=\mathfrak{Z}^q_\infty(d)\supset\mathfrak{B}^q_\infty(d)=\mathfrak{B}^q(d).$$

Consequently

(9.5) $$\mathfrak{Z}^{q-d-1}(d)/\mathfrak{Z}^{q-d-1}_\infty(d)/=\mathfrak{Z}^{q-d-1}_{d-2}(d)/\mathfrak{Z}^{q-d-1}_{d-1}(d)\approx\mathfrak{B}^q_d(0)/\mathfrak{B}^q_{d-1}(0)$$
$$=\mathfrak{B}^q_\infty(0)/\mathfrak{B}^q(0).$$

Cor. 6. *From the definition,*

(9.6) $$H^p(A, A_{-1})=H^{p+1,-1}\supset H^{p,0}\supset\cdots\supset H^{p-n,n}=0.$$

Besides,

(9.7) $$H^p(A, A_{-1})=H^{d+1,p-d-1},\ (9.8)\ H^{0,p}=0.$$

In particular, if $H^p(F)=0$ *for* $0<p<d$,

(9.8) $$H^p(A, A_{-1})=H^{d+1,p-d-1}\supset H^{d,p-d}=\cdots=H^{1,p-1}\supset H^{0,p}=0.$$

Consequently,

(9.9) $$H^{q-1}(A)/H^{1,q-1}=H^{d+1,q-d-2}/H^{d,q-d-1}\approx\mathfrak{Z}^{q-d-1}_\infty(d)/\mathfrak{B}^{q-d-1}_\infty(d)$$
$$=\mathfrak{Z}^{q-d-1}_\infty(d)/\mathfrak{B}^{q-d-1}(d).$$

(9.10) $$\mathfrak{Z}^q(0)/\mathfrak{B}^q_\infty(0)=\mathfrak{Z}^q_\infty(0)/\mathfrak{B}^q_\infty(0)\approx H^{1,q-1}/H^{0,q}=H^{1,q-1}.$$

Cor. 7. *Assvme that* ρ *is a field,* $B_0=0$. *If* $H^q(B)=0$ *for* $0<q<n$, *it is obvious that* $\mathfrak{Z}^q(p)=\mathfrak{Z}^q_\infty(p)=\mathfrak{B}^q_\infty(p)=\mathfrak{B}^q(p)$.

For $q=0$ *or* $=n$, *we have*

$$\mathfrak{Z}^0(p)=\mathfrak{Z}^0_{n-3}(p)\supset\mathfrak{Z}^0_{n-2}(p)=\mathfrak{Z}^0_\infty(0)\supset\mathfrak{B}^0_\infty(p)=\mathfrak{B}^0(p),$$

$$\mathfrak{Z}^n(p)=\mathfrak{Z}^n_\infty(p)\supset\mathfrak{B}^n_\infty(p)=\mathfrak{B}^n_{n-1}(p)\supset\mathfrak{B}^n_{n-2}(p)=\mathfrak{B}^n(p).$$

Consequently non-trivial characteristic isomorphism Φ *is*:

(9.11) $\mathfrak{Z}^0(p)/\mathfrak{Z}^0_\infty(p)=\mathfrak{Z}^0_{n-3}(p)/\mathfrak{Z}_{n-2}(p)\approx\mathfrak{B}^n_{n-1}(p-n+1)/\mathfrak{B}^n_{n-2}(p-n+1)$
$=\mathfrak{B}^n_\infty(p-n+1)/\mathfrak{B}^n(p-n+1)$.

Since $H^p(A)=H^{p+1,-1}\supset H^{p,0}=\cdots=H^{p-n+2,n-1}\supset H^{p-n,n}=0,$ *non-trivial characteristic isomorphisms* Ψ *are*:

(9.12) $\mathfrak{Z}^n(p-n+1)/\mathfrak{B}^n_\infty(p-n+1)=\mathfrak{Z}^n_\infty(p-n+1)/\mathfrak{B}^n_\infty(p-n+1)$
$\approx H^{p-n+2,n-1}/H^{p-n+1,n}=H^{p+1,0}$,

(9.13) $H^p(A)/H^{p,0}=H^{p+1,-1}/H^{p,0}\approx Z^0_\infty(p)/B^0_\infty(p)=Z^0_\infty(p)/B^0(p)$.

Cor. 8. *If* $H^{p+q}(A,A_{-1})=0$, *then* $\mathfrak{Z}^q_\infty(p)=\mathfrak{B}^q_\infty(p)$.

10. The case: $p=0$, $B_0=0$. By Theorem 6, and Cor. 4. $H^q(B)\approx H^q(B,H^0(F))\approx\mathfrak{H}^q(0)=\mathfrak{H}^q_\infty(0)$. By Cor. 6, $H^{0,q}=0$. Thus the characteristic isomorphism Ψ: $H^{1,q-1}/H^{0,q}\approx\mathfrak{H}^q_\infty(0)/\mathfrak{K}^q_\infty(0)$ reduces to $H^q(A)\supset H^{1,q-1}\approx\mathfrak{H}^q_\infty(0)/\mathfrak{K}^q_\infty(0)$, inducing a homomorphism:

(10.1) $$\mathfrak{H}^q(0)\longrightarrow H^q(A).$$

It can be easily verified that the homomorphism (10.1) is equivalent to the homomorphism:

(10.2) ψ^*: $H^q(B)\to H^q(A)$, *induced form the projection* ψ: $A\to B$. Therefore identifying $\mathfrak{H}^q(0)$ to $H^q(B)$ (if necessary), we see

(10.3) $\mathfrak{K}^q_\infty(0)=Kernel\ \psi^*$, (10.4) $H^{1,q-1}=Image\ \psi^*$.

11. The case $q=0$, $B_0=0$. By Theorem 6 and Cor. 4, $H^p(F)\approx H^0(B.H^p(F))\approx\mathfrak{H}^0(p)\supset\mathfrak{H}^0_\infty(p)\supset\mathfrak{K}^0_\infty(p)=0$. By Cor. 6, $H^{p+1,-1}=H^p(A)$. Thus the characteristic isomorphism Ψ: $H^{p+1,-1}/H^{p,0}\approx\mathfrak{H}^0_\infty(p)/\mathfrak{K}^0_\infty(p)$ reduces to $H^p(A)/H^{p,0}\approx\mathfrak{H}^0_\infty(p)$, inducing a homomorphism:

(11.1) $$H^p(A)\longrightarrow\mathfrak{H}^0(p).$$

It can easily be seen that the homomorphism (11.1) is equivalent to the homomorphism:

(11.2) i^*: $H^p(A)\to H^p(F_x)$, *induced by the injection* i: $F_x\to A$, *where* $A\supset F_x$ *is a fibre at a fixed point* x *of* B.

Therefore, identifying $\mathfrak{H}^0(p)$ to $H^p(F_{x_0})$ (if necessary), we see

(11.3) $\mathfrak{H}^0_\infty(p)=Image\ i^*$, (11.4) $H^{p,0}=Kernel\ i^*$.

Similarly identifying $\bar{\mathfrak{H}}^0(p)$ to $\bar{H}^p(F)$,

(11.3)′ $\bar{\mathfrak{K}}^0_\infty(p)=Kernel\ i_*$, (11.4)′ $\bar{H}^{p,0}=Image\ i_*$.

As for the characteristic isomorphism Φ, by Cor. 4, $\Phi:\mathfrak{H}^0_{p-2}(p)/\mathfrak{H}^0_{p-1}(p)\approx \mathfrak{K}^{p+1}_p(0)/\mathfrak{K}^{p+1}_{p-1}(0)$ reduces to

$$(11.5)\qquad \Phi:\mathfrak{H}^0_{\#}(p)/\mathfrak{H}^0_{\infty}(p)\approx \mathfrak{K}^{p+1}_{\infty}(0)/\mathfrak{K}^{p+1}_{\#}(0),\ \textit{where we put}$$
$$\mathfrak{H}^0_{\#}(p)=\mathfrak{H}^0_{p-2}(p),\ \mathfrak{K}^{p+1}_{\#}(0)=\mathfrak{K}^{p+1}_{p-1}(0).$$

Dually, putting $\bar{\mathfrak{K}}^0_{\#}(p)=\bar{\mathfrak{K}}^0_{p-2}(p)$, $\bar{\mathfrak{H}}^{p+1}_{\#}(0)=\bar{\mathfrak{H}}^{p+1}_{p-1}(0)$, we have

$$(11.5)'\qquad \bar{\Phi}:\bar{\mathfrak{K}}^0_{\infty}(p)/\bar{\mathfrak{K}}^0_{\#}(0)\approx \bar{\mathfrak{H}}^{p+1}_{\#}(0)/\bar{\mathfrak{H}}^{p+1}_{\infty}(0).$$

The properties of $\mathfrak{H}^0_{\#}(p)$, $\mathfrak{K}^{p+1}_{\#}(0)$; $\bar{\mathfrak{K}}^0_{\#}(p)$, $\bar{\mathfrak{H}}^{p+1}_{\#}$ will be investigated later.

12. The case $p=d, B_0=0, \bar{H}^d(F)\approx\rho$. By the dual forms of Theorem 6, Cor. 4, and Cor. 7, we have $\bar{H}^q(B)\approx\bar{H}^q(B,\bar{H}^q(F))\approx\bar{\mathfrak{H}}^q(d)=\bar{\mathfrak{H}}^q_{\infty}(d)$, and $\bar{H}^{d+1,q-1}=0$. Thus the characteristic isomorphism $\Psi^{-1}:\bar{H}^{d,q}/\bar{H}^{d+1,q-1}\approx\bar{\mathfrak{H}}^q_{\infty}(d)/\bar{\mathfrak{K}}^q_{\infty}(d)$ reduces to $\bar{H}^{d,q}\approx\bar{\mathfrak{H}}^q(d)/\bar{\mathfrak{K}}^q_{\infty}(d)$ inducing a homomorphism

$$(11.1)\qquad \tilde{\psi}:\bar{\mathfrak{H}}^q(d)\longrightarrow\bar{H}^{q+d}(A).$$

By Cor. 6, the characteristic isomorphism $\bar{\Phi}:\bar{\mathfrak{K}}^q_{d-1}(d)/\bar{\mathfrak{K}}^q_{d-2}(d)\approx\bar{\mathfrak{H}}^{q+1}_{d-1}(0)/\bar{\mathfrak{H}}^{q+1}_{d}(0)$, reduces to

$$(12.2)\qquad \bar{\mathfrak{K}}^q_{\infty}(d)/\bar{\mathfrak{K}}^q_{d-2}(d)\approx\bar{\mathfrak{H}}^{q+1}_{d-1}(0)/\bar{\mathfrak{H}}^{q+1}_{\infty}(0).$$

The meanings of (12.2) and (12.1) will be given later (IV, §6; §7).

13. Leray's relation. According to Theorem 6, in case where ρ is a field Leray's relation (2.1) (II, §2) becomes;

$$(13.1)\qquad \mathfrak{P}_F(t)\cdot\mathfrak{P}_B(s)=\mathfrak{E}(t,s)+\sum_{k=0}^{\infty}(t^{k+1}+s^{k+2})\mathfrak{D}_k(t,s).$$

In particular putting $t=s$, we have:

$$(13.2)\qquad \mathfrak{P}_F(t)\cdot\mathfrak{P}_B(t)=\mathfrak{P}_A(t)+(1+t)\sum_{k=0}^{\infty}t^{k+1}\mathfrak{D}_k(t,t).$$

Immediate consequences of (13.2) are:

(13.3) $\chi_F\cdot\chi_B=\chi_A$, *where χ_M denotes the Euler characteristic of M*,

(13.4) $\mathfrak{P}_F(t)\cdot\mathfrak{P}_B(t)\geqq\mathfrak{P}_A(t)$ (*Leray's inequality*),

(13.5) $\mathfrak{P}_F(t)\cdot\mathfrak{P}_B(t)=\mathfrak{P}_A(t)$ if and only if $\mathfrak{H}^q(p)=\mathfrak{H}^q_{\infty}(p)$ for any p, q.

Part IV

Multiplicative Properties of characteristic groups

1. In this part the set system $\mathfrak{A}$ is the same as in the preceding part. We shall assume further that B_0 is void; the group of the bundle G is a connected compact Lie group, and the fibre F is a homogeneous space $F=G/U$. Under this

circumstance, given an arbitrary simplicial decomposition of B, A may be simplicially decomposed in such a way that $A_q(q=0, 1, \dots, n)$ are all subcomplexes of A. Cohomology theory and $\cup$-product appearing here are those introduced into polyhedra in the usual way.[35]

2. The product of two fibre bundles. Let $\mathfrak{F}='\mathfrak{F}\times''\mathfrak{F}$, and let $'B=\{'\sigma_i^q\}$, $'A=\{'\tau_k^p\}$, $''B=\{''\sigma_l^s\}$, $''A=\{''\tau_l^r\}$ be cellular decompositions such that $'A_q$, $''A_s$ are subcomplexes of $'A$, $''A$ respectively. Then clearly $B='B\times''B=\{'\sigma_i^q\times''\sigma_j^s\}$, $A='A\times''A=\{'\tau_k^p\times''\tau_l^r\}$ are cellular decompositions of B, A respectively, and A_t's are subcomplexes of A. In fact, the support of a point $'a\times''a\in A$ being the product $'\tau_k^p\times''\tau_l^r$ of the supports $'\tau_k^p$, $''\tau_l^r$ of $'a$, $''a$ respectively, and the projections $'\psi('\tau_k^p)$, $''\psi(''\tau_l^r)$ being contained in the supports $'\sigma_i^q$, $''\sigma_j^s$ of $'\psi('a)$, $''\psi(''a)$ respectively, the actual form of $A_t=\tilde{B}^t$ is $A_t=\cup_{q+s\leqq t}\, 'A_q\times''A_s$. Then it is easy to prove:

(2.1) *If $'\bar{u}$, $''\bar{u}$ are cochains of $'A$, $''A$ respectively, such that $'\bar{u}|'A_q=0$, $''\bar{u}|''A_s=0$, then $'\bar{u}\times''\bar{u}$ is a cochain of A such that $('\bar{u}\times''\bar{u})|A_{q+s+1}=0$.*

Now we define the following multiplications:

$$'\mathfrak{C}^q(p)\times''\mathfrak{C}^s(r)\subset\mathfrak{C}^{q+s}(p+r), \tag{2.2}$$

$$H^p('A)\times H^r(''A)\subset H^{p+r}(A). \tag{2.3}$$

As for (2.2): Let $'u\in'\mathfrak{C}^q(p)\equiv H^{p+q}('A_q, 'A_{q-1})$, $''v\in''\mathfrak{C}^s(r)\equiv H^{q+s}(''A_s, ''A_{s-1})$ be represented respectively by cochains $'\bar{u}$, $''\bar{v}$ of $'A$, $''A$ such that $\delta'\bar{u}|'A_q=0$, $'\bar{u}|'A_{q-1}=0$; $\delta''\bar{v}|''A_s=0$, $''\bar{v}|A_{s-1}=0$.
Then $\bar{w}='\bar{u}\times''\bar{v}$ is a cochain of A such that $\delta\bar{w}|A_{q+s}=0$, $\bar{w}|A_{q+s-1}=0$, representing an element w of $H^{p+r+q+s}(A_{q+s}, A_{q+s-1})\equiv\mathfrak{C}^{q+s}(p+r)$. That w is determined uniquely by $'u$ and $''v$ is easily verified also by making use of (2.1). w is denoted by $u\times v$. As for (2.3): this is the ordinary multiplication.

By making use of (2.1) and repeating the above arguments we have:

Theorem 8. *For $k\geqq 0$,*

$$'\mathfrak{Z}_k^q\ (p)\times''\mathfrak{Z}_{k-1}^s(r)\subset\mathfrak{Z}_{k-1}^{q+s}(p+r), \tag{2.4}$$

$$'\mathfrak{B}_k^q\ (p)\times''\mathfrak{Z}_{k-1}^s(r)\subset\mathfrak{B}_k^{q+s}(p+r), \tag{2.5}$$

$$'\mathfrak{Z}_{k-1}^q(p)\times''\mathfrak{B}_k^s(r)\ \subset\mathfrak{B}_k^{q+s}(p+r), \tag{2.6}$$

$$'H^{p,q}\ \times\ ''H^{r,s}\ \subset H^{p+r-1,q+s+1} \tag{2.7}$$

Since $'\mathfrak{Z}_{-1}^q(p)\times''\mathfrak{B}_0^s(r)\subset\mathfrak{B}_0^{q+s}(p+r)$, $'\mathfrak{B}_0^q(p)\times''\mathfrak{Z}_{-1}^s(r)\subset\mathfrak{B}_0^{q+s}(p+r)$, in (2.4), (2.5), (2.6) we may replace $\mathfrak{Z}$, $\mathfrak{B}$ by $\mathfrak{H}$, $\mathfrak{R}$ respectively.

Obviously the above multiplications are bilinear, and we may define for $'u_k\in'\mathfrak{H}_k^q(p)$, the following homomorphisms:

35) See Lefschetz's text. See also [1].

$$(2.8)_l \qquad ('u_k\times):\ ''\mathfrak{H}^{s}_{k-1}(r)/''\mathfrak{H}^{s}_{k}(r) \longrightarrow \mathfrak{H}^{q+s}_{k-1}(p+r)/\mathfrak{H}^{q+s}_{k}(p+r),$$

$$(2.9)_l \qquad ('u_k\times):\ ''\mathfrak{K}^{s}_{k+1}(r)/''\mathfrak{K}^{s}_{k}(r) \longrightarrow \mathfrak{K}^{q+s}_{k+1}(p+r)/\mathfrak{K}^{q+s}_{k}(p+r),$$

$$(2.10)_l \qquad ('u_\infty\times):\ ''\mathfrak{H}^{s}_{\infty}(r)/''\mathfrak{K}^{s}_{\infty}(r) \longrightarrow \mathfrak{H}^{q+s}_{\infty}(p+r)/\mathfrak{K}^{q+s}_{\infty}(p+r).$$

Further we may define:

$$(2.11)_l \qquad ('u_\infty\times);\ ''H^{r+1,s-1}/''H^{r,s} \longrightarrow H^{p+r+1,q+s-1}/H^{q+r,q+s},$$

directly using cochains. Then we have:

Theorem 9_l. *For* $'u_k\in{}'\mathfrak{H}^{q}_{k}(p)$,

$$(2.12)_l \qquad (-1)^{p+q}\,('u_k\times)\Phi_k^{\pm1} = \Phi_k^{\pm1}('u_k\times),$$

$$(2.13)_l \qquad ('u^\infty\times)\Psi^{\pm1} = \Psi^{\pm1}('u_\infty\times).$$

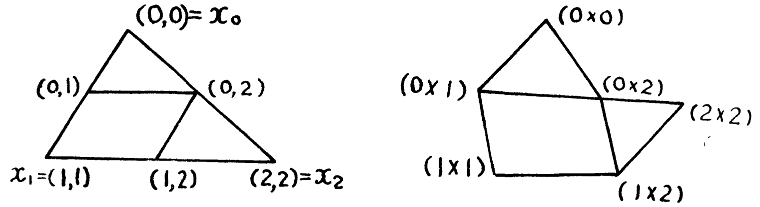
Replacing the left multiplications in $(2.8)_l\sim(2.12)_l$ by the right multiplications $(\times''v)$, we obtain in the same way the corresponding definitions and formulas $(2.1)_r\sim(2.12)_r$. In particular:

Theorem 9_r. *For* $''v_k\in{}''\mathfrak{L}^{s}_{k}(r)$,

$$(2.12)_r \qquad (\times''v_k)\,\Phi_k^{\pm1} = \Phi_k^{\pm1}(\times''v_k),$$

$$(2.13)_r \qquad (\times''v_\infty)\Psi^{\pm1} = \Psi^{\pm1}(\times''v_\infty).$$

3. $\vee$-product. Given a fibre boundle $\mathfrak{F}$ we consider the self-product $'\mathfrak{F}=\mathfrak{F}\times\mathfrak{F}$. Let the decomposition of B be simplicial. We first notice that there exists a map $g_1: B\to{}'B=B\times B$ homotopic to the diagonal may $g_0: B\to{}'B$[36] such that $g^1(B^q)\subset{}'B^q$. The following map h is an example of such g_1. Let the vertices of B be simply ordered. h maps each ordered simpley $\sigma=(x_0x_1\dots x_q)$ Onto $\{(x_0)\times(x_0x_1\dots x_q)\}\cup\{(x_0x_1)\times(x_1x_2\dots x_q)\}\cup\dots\cup(x_0x_1\dots x_q)\times(x_q)\}$ semi-linearly as indicated by the following figure:



Let $g: B\times I\to{}'B$ be a homotopy connecting g_0 to g_1. By Theorem D, the induced bundles (I, §1) $'\mathfrak{F}_{g_0}\times I$ and $'\mathfrak{F}_g$ are equivalent in the restricted sense. But, since the bundle space of $'\mathfrak{F}_{g_0}\times I$ is a subset $\hat{A}$ of $B\times{}'A\times I$ defined by $\hat{A}=\{(x,{}'a,t)\,|\,g_0(x)={}'\psi('a)\}$, and the bundle space of $'\mathfrak{F}_g$ is a subset $\check{A}$ of $B\times I\times{}'A$ defined by $A=\{[x,t,{}'a]\,|\,g(x,t)={}'\psi('a)\}$, the map $\chi:\hat{A}\to\check{A}$, giving the above equivalence may be written as: $\chi(x,{}'a,t)=[x,t,\xi_{x,t}('a)]$, for $'a\in{}'F_{g_0(x)}$,

36) For $x\in B$, $g_0(x)=x\times x\in{}'B$. See [3].

where $\xi_{x,t}: {}'F_{g_0(x)} \to {}'F_{g(x,t)}$ is a homeomorphism, and $\xi_{x,t}$ reduces to the identity when $t=0$. By making use of $\xi_{x,t}$'s, we define a map $f: A\times I \to {}'A$ by $f(a,t) = \xi_{x,t}(f_0(a))$ for $a \in F_x$, where f_0 is the diagonal map: $A \to {}'A = A\times A$. Obviously ${}'\psi f(a,t) = g(x,t) = g_1(\psi a, t)$; $f_1: A \to {}'A$ defined by $f_1(a) = f(a,1)$ satisfies $f_1(A_q) \subset {}'A_q$, inducing a map of the set systems $f_1^h: \mathfrak{A} \to {}'\mathfrak{A}$. By Theorem 3, f^h induces the homomorphisms $f_1^{\#}$ of the characteristic groups:

$$(3.1) \qquad f_1^{\#}: {}'H_{k-1}^{q+s}(p+r) \longrightarrow H_{k-1}^{q+s}(p+r),$$

$$(3.2) \qquad f_1^{\#}: {}'K_k^{q+s}(p+r) \longrightarrow K_k^{q+s}(p+r),$$

$$(3.3) \qquad f_1^{\#}: {}'H^{p+r-1,q+s+1} \longrightarrow H^{p+r-1,q+s+1},$$

$$(3.3)' \qquad f_1^{\#}: {}'H^{p+r-1,q+s+1}/{}'H^{p+r,q+s} \longrightarrow H^{p+r-1,q+s+1}/H^{p+r,p+q}.$$

We give the following definitions: For $u \in \mathfrak{H}^q(p)$, $v \in \mathfrak{H}^s(r)$; $a \in H^p(A)$, $b \in H^r(A)$; $u_\infty \in \mathfrak{H}_\infty^q(p)$, $c \in H^{r-1,s+1}/H^{r,s}$, $c_1 \in H^{p-1,q+1}/H^{p,q}$, we define

$$u\vee v = f_1^{\#}(u\times v) \in \mathfrak{H}^{q+1}(p+r),\ a\vee b = f_1^{\#}(a\times b) \in H^{q+s}(A),$$

$$u_\infty \vee c = f_1^{\#}(v_\infty \times c) = H^{p+r-1,q+s+s}/H^{p+r,q+s},\ c_1\vee c = f_1^{\#}(c_1\times c) \in H^{p+r+1,q+s+1}/H^{p+r,q+s}.$$

Then in virtue of (2.4)-(2.7), (5.1)-(3.3), we have:

Theorem 10.

$$(3.4) \qquad \mathfrak{H}_{k-1}^q(p) \vee \mathfrak{H}_{k-1}^s(r) \subset \mathfrak{H}_{k-1}^{q+s}(p+r),$$

$$(3.5) \qquad \mathfrak{K}_k^q\ (p) \vee \mathfrak{H}_{k-1}^s(r) \subset \mathfrak{K}_k^{q+s}(p+r),$$

$$(3.6) \qquad \mathfrak{H}_{k-1}^q(p) \vee \mathfrak{K}_k^s\ (r) \subset \mathfrak{K}_k^{q+s}(p+r),$$

$$(3.7) \qquad H^{p,q} \vee H^{r,s} \subset H^{p+r-1,q+s-1}.$$

Further if we define $(u_k\vee)$, $(\vee u_k)$ for $u_k \in \mathfrak{H}_k^q(p)$ in the same way $(u_k\times)$, $(\times u_k)$ were defined in the preceding section, we have in virtue of Theorem 9 and (3.1)-(3.3)':

Theorem 11.

$$(3.8)_l \quad (-1)^{p+q}(u_k\vee)\Phi_k^{\pm 1} = \Phi_k^{\pm 1}(u_k\vee), \qquad (3.8)_r \quad (\vee u_k)\Phi_k^{\pm 1} = \Phi_k^{\pm 1}(\vee u_k),$$

$$(3.9)_l \quad (v_\infty\vee)\Psi^{\pm 1} = \Psi^{\pm 1}(u_\infty\vee), \qquad (3.8)_r \quad (\vee v_\infty)\Psi^{\pm 1} = \Psi^{\pm 1}(\vee u_\infty).$$

We remark that if we denote by θ the following composite homomorphism:

$$(3.10) \qquad \theta: \mathfrak{H}_\infty^q(p) \longrightarrow \mathfrak{H}_\infty^q(p)/\mathfrak{K}_\infty^q(p) \longrightarrow H^{p-1,q+1}/H^{p,q},$$

we have:

$$(3.11) \qquad u_\infty \vee c = \theta(u_\infty)\vee c,$$

$$(3.11) \qquad \theta(u_\infty \vee v_\infty) = \theta(v_\infty)\vee\theta(v_\infty), \ \text{ for } u \in \mathfrak{H}_\infty^q(p),\ v \in \mathfrak{H}_\infty^q(r).$$

Now let us prove that the product $\vee$ does not depend on a special g_1. For this purpose let g_{-1} be another g_1, i.e. a map $g_{-1}: B \to {}'B$ which is homotopic to the diagonal map g_0 and such that $g_{-1}(B^q)\subset 'B^q$. Let f_{-1} be defined in the same way as f_1 was defined from g_1. Then it is sufficient to prove in virtue of Theorem 4 and the definition of the product $\vee$ that f_{-1}^{h} is homotopic to f_1^{h} in the sense of (II, §1). Let $\bar{g}: B\times I_- \to 'B$, $\bar{\chi}: '\mathfrak{F}_{g_0}\times I_- \to '\mathfrak{F}_{\bar{g}}$, $\xi_{x,t}: 'F_{g_0w}\to 'F_{g(x,t)}$ maps used to define f_{-1}, where I_- is the interval $\langle -1, 0\rangle$.

The interval $\langle -1, 1\rangle$ being denoted by E, let $\tilde{g}: B\times E\to 'B$, $\tilde{\chi}: \mathfrak{F}_{g_0}\times E\to '\mathfrak{F}_{\tilde{g}}$ be defined by: $\tilde{g}=\bar{g}$, $\tilde{\chi}=\bar{\chi}$, for $t\in I_-$; $\tilde{g}=g$, $\tilde{\chi}=\chi$, for $t\in I$. Since $g_{\pm 1}(B^q)\subset 'B^q$, the homotopy $\tilde{g}$ connecting g_{-1} to g_1 may be deformed into a homotopy k such that $k(B^q, t)\subset 'B^{q+1}$ for each t.[33] Let l be a deformation of $\tilde{g}$ to k: l is a map $l: B\times E\times I\to 'B$ such that $l(x,t,0)=\tilde{g}(x, t)$, $l(x, t, 1)=k(x, t)$, $l(x, -1, s)=g_{-1}(x)$, $l(x, 1, s)=g_1(x)$. Then by Theorem D the equivalence $\tilde{\chi}: '\mathfrak{F}_{g_0}\times E\to '\mathfrak{F}_{\tilde{g}}$ may be extended to an equivalence $\theta: \mathfrak{F}_{g_0}\times E\times I\to \mathfrak{F}_l$, such that, when we define $\tilde{\xi}_{xt}$, $\eta_{x,t,s}$ by $\tilde{\chi}(x, 'a, t)=[x, t, \tilde{\xi}_{x,t}('a)]$, $\theta(x, 'a, t, s)=[x, t, s, \eta_{x,t,s}('a)]$ for $'a\in 'F_{g_0(x)}$, we have $\eta_{x,t,0}=\tilde{\xi}_{x,t}$, $\eta_{x,\pm 1,s}=\xi_{x,\pm 1}$. Noticing that $f_{\pm 1}: A\to 'A$ are given by $f_{\pm 1}(a)=\xi_{x,\pm 1}(f_0(a))$, $a\in F_x$, we easily see that the map: $A\times E\ni (a, t)\to \eta_{x,t,1}(f_0(a))\subset 'A$ gives the desired homotopy of f_1^{h} to f_1^{h}.

4. Relation to the ordinary $\cup$-product. Let f_1 be the same as defined in the preceding section. The map $f_1^{h}: \mathfrak{A}\to '\mathfrak{A}$ induces a homophism $f_1^{\#}: '\mathfrak{C}^q(p)\to \mathfrak{C}^q(p)$. Let $u\in \mathfrak{C}^q(p)$, $v\in \mathfrak{C}^s(r)$, and define $u\underset{f_1}{\vee} v\in \mathfrak{C}^{q+s}(p+r)$ by $u\underset{f_1}{\vee} v=f_1^{\#}(u\times v)$.

Lemma. *Let $\hat{B}$ be a subcomplex of B, and let $\hat{\mathfrak{F}}=\mathfrak{F}|\hat{B}$. Let $i: \hat{A}\to A$ be the inclusion map, and let $i^{\#}: \mathfrak{C}^q(p)\to \hat{\mathfrak{C}}^q(p)$ be the induced homomorphism* (II, §1). *Let $\hat{f}$; $\hat{A}\to \hat{A}\times \hat{A}$ be obtained from f_1 by restricting the range of definition as well as the range of values. Then we have:*

$$i^{\#}(u\vee_{f_1} v)=(i^{\#}u\vee_{\hat{f}} i^{\#}v),\ \ for\ \ u\in \mathfrak{C}^q(p),\ v\in \mathfrak{C}^s(r). \tag{4.1}$$

Proof: Let $j: \hat{A}\times \hat{A}\to A\times A$ be the inclusion map. Since it is easily seen that $j^{\#}(u\times v)=i^{\#}u\times i^{\#}v$, we have $i^{\#}(u\vee_{f_1} v)=i^{\#}f_1^{\#}(u\times v)=(f_1 i)^{\#}(u\times v)=(j\hat{f})^{\#}(u\times v)=\hat{f}^{\#}j^{\#}(u\times v)=\hat{f}^{\#}(i^{\#}u\times i^{\#}v)=i^{\#}u\vee_{\hat{f}} i^{\#}v$.

We apply this lemma to the case where $\hat{B}=\sigma_k^{q+s}$. We notice that $i^{\#}: \mathfrak{C}^{q+s}(p+r)\to \hat{\mathfrak{C}}^{q+s}(p+r)$ is identical with the homomorphism (III, §3) $\lambda_k^{q+s} H^{p+q+r+s}(A_{q+s}, A_{q+s-1})\to H^{p+q+r+s}(\bar{\sigma}_k^{q+s}, \dot{\bar{\sigma}}_k^{q+s})$. If we identify $\mathfrak{C}^q(p)$ to $C^q(B, H^p(F))$ in virtue of the isomorphism in Theorem 6, from (4.1) we have

$$\lambda_k^{q+s}(a_i^p\sigma_i^q\vee_{f_1} a_j^r\sigma_j^s)=i^{\#}(a_i^p\sigma_i^q)\vee_{\hat{f}} i^{\#}(a_j^r\sigma_j^s),\ \text{for}\ a_i^p\sigma_i^q\in C^q(B, H^p(F)),\ a_j^r\sigma_j^s\in C^s(B, H^r(F)). \tag{4.2}$$

Now let us assume that a fixed simple ordering of the vertices of B is given and let us take $g_1=h$: the special one which was given in the preceding section as an example. We notice that $H^p(F)$ and $H^r(F)$ are paired to $H^{p+r}(F)$ by the ordinary (Čech-Whitney's) $\cup$-product, and that with respect to this pairing

$C^q(B, H^p(F)$ and $C^s(B, H^r(F))$ are paired to $C^{q+s}(B, H^{p+r}(F))$ in the usual way: by making use of the functional notation we define $u \cup v \in C^{q+s}(B, H^{p+r}(F))$ for $u \in C^q(B, H^p(F))$, $v \in C^s(B, H^r(F))$ by $(u \cup v)(x_0 x_1 \dots x_{q+s}) = u(x_0 x_1 \dots x_q) \cup v(x_q \dots x_{q+s})$, where $\sigma = (x_0 x_1 \dots x_{q+s})$ is an ordered simplex. We shall show that two multiplications $\vee_{f_1}$ and $\cup$ coincide up to sign For this purpose it is sufficient to prove that $\lambda_k^{p+q}((a_i^p\sigma_i^q) \vee_{f_1} (a_j^r\sigma_j^s)) = \pm \lambda_k^{p+q}((a_i^p\sigma_i^q) \cup (a_j^r\sigma_j^s))$, i.e.

$$(4.3) \qquad i^{\#}(a_i^p\sigma_i^q) \vee_f i^{\#}(a_j^r\sigma_j^s) = \pm \lambda_k^{p+q}((a_i^p \cup a_j^r)(\sigma_i^q \cup \sigma_j^s)).$$

There are several cases. If either σ_i^q or σ_j^s is not contained in σ_k^{p+q}, both sides of (4.3) vanish. If both σ_i^q and σ_j^s are contained in σ_k^{p+q}, a slight consideration shows that (4.3) reduces to

$$(4.4) \qquad (a_i^p\sigma_i^q) \vee_{\hat{f}} (a_j^r\sigma_j^s) = \pm (a_i^p \cup a_j^r)(\sigma_i^q \cup \sigma_j^s), \quad \text{whre}$$

$$a_i^p\sigma_i^q \in C^q(\hat{B}, H^p(F)), \; a_j^r\sigma_j^s \in C^s(\hat{B}, H^r(F)).$$

(i) If $\sigma_i^q \cup \sigma_j^s = 0$, dim $[(\text{Image } g_1) \cap (\sigma_i^q \times \sigma_j^s))] \leqslant q+s$. Hence it is easily seen that both sides of (4.4) vanish.

(ii) If $\sigma_i^q \cup \sigma_j^s \neq 0$, we may assume without loss of generality that $\sigma_i^q \cup \sigma_j^s = \sigma_k^{q+s}$. Since $\mathfrak{F}$ is a product bundle, we may assume that $\hat{A} = \hat{B} \times F$. Then $\hat{f} : \sigma_k^{q+s} \times F \to \sigma_k^{q+s} \times \sigma_k^{q+s} \times F \times F$ may be represented by $\hat{f}(x, y) = (h(x), \xi_x(d(y)))$, where $\xi_x : F \times F \to F \times F$ is an automorphism which is homotopic to the identity and where $d : F \to F \times F$ is the diagonal map. Let $\tilde{d} : F \to F \times F$ be a cellular map which is homotopic to d. Since σ_k^{q+s} is contractible $\hat{f}$ is homotopic to the map $\eta(x, y) = (h(x), \tilde{d}(y))$, the projection $[x \to h(x)]$ remaining fixed, (4.4) reduces to

$$(4.5) \qquad \eta^{\#}[(a_i^p\sigma_i^q) \times (a_j^r\sigma_j^s)] = (a_i^p \cup a_j^r)\sigma_k^{q+s}.$$

Since $a_i^p\sigma_i^q$, $a_j^r\sigma_j^s$ may be represented by cochains of the form $\sigma_i^q \times c_i^p$, $\sigma_j^s \times c_j^r$ respectively, $(a_i^p\sigma_i^q) \times (a_j^r\sigma_j^s)$ is represented by the cochain $(\sigma_i^q \times c_i^p) \times (\sigma_j^s \times c_j^r) = (-1)^{ps}(\sigma_j^q \times \sigma_j^s) \times (c_i^p \times c_j^r)$. Hence $(a_i^p\sigma_i^q) \vee_{f_1} (a_j^r\sigma_j^s)$ is represented by the cochain $(-1)^{ps} h^*(\sigma_i^q \times \sigma_j^s) \times d^*(c_i^p \times c_j^r)$, which represents $(-1)^{ps}(a_i^p \cup a_j^r)\sigma_k^{q+s}$.

Theorem 12. *Identifying* $\mathfrak{H}^q(p)$ *to* $H^q(B, H^p(F))$, *we have*

$$(4.6) \qquad u \vee v = (-1)^{ps} u \cup v, \; \textit{for} \; u \in \mathfrak{H}^q(p), \; v \in \mathfrak{H}^s(r),$$

$$(4.7) \qquad a \vee b = a \cup b, \; \textit{for} \; a \in H^p(A), \; b \in H^r(A).$$

Proof : (4.6) is the immediate consequence of the above discussion. (4.7) is the immediate consequence of [1].[36]

By this theorem, Theorem 10 becomes:

36) See [1].

Theorem 10 ($\cup$).

(4. 8) $$\mathfrak{H}^q_{k-1}(p) \cup \mathfrak{H}^s_{k-1}(r) \subset \mathfrak{H}^{q+s}_{k-1}(p+r),$$

(4. 9) $$\mathfrak{K}^q_k\ (p) \cup \mathfrak{H}^s_{k-1}(r) \subset \mathfrak{K}^{q+s}_k(p+r),$$

(4.10) $$\mathfrak{H}^q_{k-1}(p) \cup \mathfrak{K}^s_k\ (r) \subset \mathfrak{K}^{q+s}_k(p+r),$$

(4.11) $$H^{p,q} \cup H^{r,s} \subset H^{p+r-1,q+s+1}.$$

Corollary to Theorem 12. *If ρ is a field, we have* $\mathfrak{H}^q(p)=\mathfrak{H}^q(0)\vee\mathfrak{H}^0(p)$.

5. Algebras $\mathfrak{H}$ and $H(A)$. We consider the direct sums $\mathfrak{H}=\sum_{p,q}\mathfrak{H}^q(p)$, $H(A)=\sum_p H^p(A)$. They are algebras with respect to $\vee$- (or $\cup$-) multiplication.[37] In particular $H(A)$ is the ordinary cohomology ring of A. The following facts are only the restatement of the results in the preceding sections:

(5.1) In virtue of (3.4) in Theorem 10, $H^0=\sum_p H^0(p)$, $H(0)=\sum_q H^q(0)$, $H_k=\sum_{p,q}\mathfrak{H}^q_k(p)$, and $\mathfrak{H}_\infty=\sum_{p,q}\mathfrak{H}^q_\infty(p)$ are subalgebras of $\mathfrak{H}$, and so are their intersections $\mathfrak{H}^0_k=\mathfrak{H}^0\cap\mathfrak{H}_k$, $\mathfrak{H}^0_\infty=\mathfrak{H}^0\cap\mathfrak{H}_\infty$.

(5.2) In virtue of (3.5), (3.6) in Theorem 10, $\mathfrak{K}_k=\sum_{p,q}\mathfrak{K}^q_k(p)$ is an ideal of $\mathfrak{H}_{k-1}$. Hence $\mathfrak{E}_k=\mathfrak{H}_{k-1}/\mathfrak{K}_k$ is an algebra, multiplication being induced from that of $\mathfrak{H}_{k-1}$.[38] $\mathfrak{K}_k(0)=\mathfrak{K}_k\cap\mathfrak{K}(0)$ and $\mathfrak{K}_\infty(0)=\mathfrak{K}_\infty\cap\mathfrak{K}(0)$ are ideals of $\mathfrak{H}_\infty(0)=\mathfrak{H}_{k-1}(0)=\mathfrak{H}(0)$.

(5.3) If we put $H^{p,*}=\sum_q H^{p,q}$, $H^{*,q}=\sum_p H^{p,q}$, in virtue of (3.7) in Theorem 10, $H^{1,*}$ is a subalgebra of $H(A)$; $H^{*,1}$ is an ideal of $H(A)$.

(5.4) Consider the direct sum $\mathfrak{G}=\sum_{p,q}H^{p+1,q-1}/H^{p,q}$ and introduce in it a multiplication $\vee$ by making use of representatives. This is possible in virtue of (3.7) in Theorem 10. Then $\mathfrak{G}\approx\mathfrak{H}_\infty/\mathfrak{K}_\infty$ is a ring isomorphism according to (3.12).

6. $\wedge$-product. In this section the coefficient ring ρ is a field. Under this assumption $\bar{\mathfrak{H}}=\sum_{p,q}\bar{\mathfrak{H}}^q(p)$ is dual to $\mathfrak{H}$, and $\bar{H}(A)=\sum_p H^p(A)$ is dual to $H(A)$. Then $\wedge$-product is defined as follows: Let $v\in\mathfrak{H}^s(r)$, $z\in\bar{\mathfrak{H}}^{q+s}(p+r)$. Then $v\wedge z$ is an element of $\bar{\mathfrak{H}}^q(p)$ satisfying the following equation: $\langle u, v\wedge z\rangle=\langle u\vee v, z\rangle$, for every $u\in\mathfrak{H}^q(p)$; let $b\in H^r(A)$, $z\in\bar{H}^{p+r}(A)$. Then $b\wedge z$ is an element of $\bar{H}^p(A)$ satisfying the following equation: $\langle a, b\wedge z\rangle=\langle a\wedge b, z\rangle$, for every $a\in H^p(A)$.

Identifying $\bar{\mathfrak{H}}^q(p)$ to $\bar{H}^q(B, \bar{H}^p(F))$, we see from Theorem 12 that $\wedge$-product is essentially the same with the ordinary $\cap$-product. From Theorem 10 we have:

37) $\mathfrak{H}$ is a doubly graded ring in the sense of H. Cartan.

38) When we denote by Δ_k the following composite homomorphism: $\mathfrak{E}_k=\mathfrak{H}_{k-1}/\mathfrak{K}_k\to\mathfrak{H}_{k-1}/\mathfrak{H}_k\to\mathfrak{K}_{k+1}/\mathfrak{K}_k\to\mathfrak{H}_{k-1}/\mathfrak{K}_k=\mathfrak{E}_k$, $\mathfrak{E}_k$ becomes a graded ring with differential operator Δ_k, the cohomology ring of which is isomorphic to $\mathfrak{E}_{k+1}$. See [8], [30].

Theorem 13.

$$\mathfrak{H}^{q}_{k-1}(r) \wedge \bar{\mathfrak{K}}^{q+s}_{k-1}(p+r) \subset \bar{\mathfrak{K}}^{q}_{k-1}(p), \tag{6.1}$$

$$\mathfrak{H}^{s}_{k-1}(r) \wedge \bar{\mathfrak{H}}^{q+s}_{k}(p+r) \subset \bar{\mathfrak{H}}^{q}_{k}\ (p), \tag{6.2}$$

$$\mathfrak{K}^{s}_{k}\ (r) \wedge \mathfrak{H}^{q+s}_{k-1}(p+r) \subset \bar{\mathfrak{K}}^{q}_{k-1}(p), \tag{6.3}$$

$$H^{r,s} \wedge \bar{H}^{p+r,q+s} \subset \bar{H}^{p+1,q-1}. \tag{6.4}$$

We define for $u_k \in \mathfrak{H}^{s}_{k}(r)$ the homomorphism $(u_k \wedge): \bar{\mathfrak{H}}^{q+s}_{k}(p+r)/\bar{\mathfrak{H}}^{q+s}_{k+1}(p+r) \to \bar{\mathfrak{H}}^{q}_{k}(p)/\bar{\mathfrak{H}}^{q}_{k+1}(p)$ directly in virtue of Theorem 13, or we define it as the dual homomorphism of $(\vee u_k): \mathfrak{K}^{q}_{k+1}(p)/\mathfrak{K}^{q}_{k}(p) \to \mathfrak{K}^{q+s}_{k+1}(p+r)/\mathfrak{K}^{q+s}_{k}(p+r)$.
The other homomorphisms being similarly defined, we have:

Theorem 14.

$$(u_k \wedge)\bar{\Phi}^{+1}_{k} = \bar{\Phi}^{+1}_{k}(u_k \wedge), \tag{6.5}$$

$$(u_\infty \wedge)\Psi^{\pm 1} = \Psi^{\pm 1}(u_\infty \wedge), \quad \text{where } u_\infty \in \mathfrak{H}_\infty \text{ or } u_\infty \in \mathfrak{H}_\infty/\mathfrak{K}_\infty. \tag{6.6}$$

Consider the following diagram:

$$\begin{array}{ccccc} \mathfrak{H}^{s}_{\infty}(r) & \xrightarrow{\theta_1} & \mathfrak{H}^{s}_{\infty}(r)/\mathfrak{K}^{s}_{\infty}(r) & \xrightarrow{\Psi^{-1}} & H^{r+1,s-1}/H^{r,s} \\ \downarrow (\wedge z_\infty) & & \downarrow (\wedge z_\infty) & & \downarrow (\wedge \bar{\Psi}\theta_2 z_\infty) \\ \bar{\mathfrak{H}}^{q}_{\infty}(p) & \xrightarrow{\theta_2} & \bar{\mathfrak{H}}^{q}_{\infty}(p)/\bar{\mathfrak{K}}^{q}_{\infty}(p) & \xrightarrow{\bar{\Psi}} & \bar{H}^{p,q}/\bar{H}^{p+1,q-1}, \end{array} \tag{6.7}$$

where $z_\infty \in \bar{\mathfrak{H}}^{q+s}_{\infty}(p+r)$; θ_i are natural homorphisms; $(\wedge z_\infty)$ and $(\wedge \bar{\Psi}\theta_2 z_\infty)$ are defined in virtue of Theorem 13. Then we have:

Theorem 15. *Commutativity relations hold in the above diagram* (6.7).

Proof: $(\wedge z_\infty)\theta_1 = \theta_2(\wedge z_\infty)$ is obvious. Let us prove that $\Psi^{-1}(v) \wedge \bar{\Psi}\theta_2 z_\infty = \bar{\Psi}(v \wedge z_\infty)$ for $v \in \mathfrak{H}^{s}_{\infty}(r)/\mathfrak{K}^{s}_{\infty}(r)$. In virtue of (5.4) for any $a \in H^{p+1,q-1}/H^{p,q}$ we have $\langle a, \bar{\Psi}(v \wedge z_\infty)\rangle = \langle \Psi(a), v \wedge \theta_2 z_\infty\rangle = \langle \Psi(a) \vee v, \theta_2 z_\infty\rangle = \langle \Psi(a \vee \Psi^{-1}(v)), \theta_2 z_\infty\rangle = \langle a \vee \Psi^{-1}(v), \bar{\Psi}\theta_2 z_\infty\rangle = \langle a, \Psi^{-1}(v) \wedge \bar{\Psi}\theta_2 z_\infty\rangle$.

By (II, §9) $\bar{\Psi}: \bar{\mathfrak{H}}^{n}_{\infty}(d)/\bar{\mathfrak{K}}^{n}_{\infty}(d) \approx \bar{H}^{d,n}/\bar{H}^{d+1,n-1}$ reduces to $\bar{\Psi}: \bar{\mathfrak{H}}^{n}(d) \approx H^{n+d}(A)$. Let $Z^{n,d} \in \bar{\mathfrak{H}}^{n}(d)$, $z^{n+d} \in H^{n+d}(A)$ correspond to each other under this isomorphism. Let us consider the case where $r=0$, $p=d$, $s+q=n$. Then in virtue of (III, §9) (6.7) reduces to:

$$\begin{array}{ccccccc} \mathfrak{H}^{n-q}(0) & \xrightarrow[\theta_1]{} & \mathfrak{H}^{n-q}(0)/\mathfrak{K}^{n-q}_{\infty}(0) & \xrightarrow[\Psi^{-1}]{} & H^{1,n-q-1} & \xrightarrow[\theta_3]{} & H^{n-q}(A) \\ \downarrow (\wedge Z^{d,n}) & & \downarrow (\wedge Z^{d,n}) & & \downarrow (\wedge Z^{n+d}) & & \downarrow (\wedge Z^{n+d}) \\ \bar{\mathfrak{H}}^{q}(d) & \xrightarrow{\theta_2} & \mathfrak{H}^{q}(d)/\mathfrak{K}^{q}(d) & \xrightarrow{\bar{\Psi}} & \bar{H}^{d,q} & \xrightarrow{\theta_4} & H^{q+d}(A), \end{array} \tag{6.8}$$

where θ_3, θ_4 are identity isomorphisms.

Now if B is an orientable manifold, so is A, since F is assumed to be a homogeneous space (IV, §1). Thus if we take as $Z^{n,d}$ the generator of $\bar{\mathfrak{H}}^{n}(d)$. according to Poincaré duality theorem[39] we see that

39) See [5]. 40) See (III, §10).

$$(\wedge Z^{n,d}) : \mathfrak{H}^{n-q}(0) \to \mathfrak{H}^q(d), \text{ and } (\wedge z^{n+d}) : H^{n-q}(A) \to \bar{H}^{d+q}(A)$$

are isomorphisms onto. On the other hand, since the composite homomorphism $\theta_3\Psi^{-1}\theta_1 : \mathfrak{H}^{n-q}(0) \to H^{n-q}(A)$ may be regarded as the induced homomorphism $\psi^* : H^{n-q}(B) \to H^{n-q}(A)$ (III, §10), the composite homomorphism $\theta_4\bar{\Psi}\theta_2 : \bar{\mathfrak{H}}^q(d) \to \bar{H}^{q+d}(A)$ may be written as:

$$\theta_4\Psi\theta_2 = (\wedge Z^{n+d})\,\psi^*(\wedge Z^{n,d})^{-1}. \tag{6.9}$$

Thus we see that the homorphism $\tilde{\psi} : \bar{H}^q(B) \to \bar{H}^{q+d}(A)$ observed in (III, §12) is in this case identical with the generalized Hopf's inverse homomorphism. We shall call $\tilde{\psi}$ the *Hopf's inverse homomorphism* also in the general case where B is an arbitrary polyhedron and ρ is an arbitrary ring. Thus $\mathfrak{K}^q_\infty(d)$, $\bar{H}^{d,q}$ *are regarded respectively as the kernel and the image of the Hopf's inverse homomorphism.*

7. A theorem of Gysin. Consider the characteristic isomorphism:

(7.1) $\bar{\Phi} : \bar{\mathfrak{K}}^q_\infty(d)/\bar{\mathfrak{K}}^q_{d-2}(d) \approx \bar{\mathfrak{H}}^{q+d+1}_{d-1}(0)/\bar{\mathfrak{H}}^{q+d+1}_\infty(0)$, which was already considered in (III, §12).

This induces a homomorphism

$$G : \bar{\mathfrak{K}}^q_\infty(d) \longrightarrow \bar{\mathfrak{H}}^{q+d+1}(0)/\bar{\mathfrak{H}}^{q+d+1}_\infty(0), \text{ or} \tag{7.2}$$

$$G : \mathfrak{K}^q_\infty(d) \longrightarrow \bar{H}^{q+d+1}(B)/\psi_*\bar{H}^{q+d+1}(A). \tag{7.3}$$

This homomorphism was first considered by W. Gysin, and will be called *Gysin's homomorphism* also in general cases.

Now let us assume that $\bar{H}^p(F)=0$ for $0<p<d$. Then by Cor. 5 (III, §9), we have $\bar{\mathfrak{H}}^{q+d+1}_{d-1}(0) = \bar{\mathfrak{H}}^{q+d+1}(0)$ and $\bar{\mathfrak{K}}^q(d) = 0$. Thus we have a generalized Gysin's theorem:

Theorem 16. *If* $\bar{H}^p(F)=0$ *for* $0<p<d$, *then*

(7.4) $\bar{\mathfrak{K}}^q_\infty(d) \approx \bar{H}^{q+d+1}(B)/\psi_* H^{q+d+1}(A)$,

(7.5) $\bar{\mathfrak{K}}^q_\infty(d) =$ [*the kernel of the Hopf's inverse homomorphism* $\tilde{\psi} : \bar{H}^q(B) \to \bar{H}^{q+d}(A)$].

8. Theorems of Thom [23] and Chern-Spanier [2]. Let us assume that $H^p(F)=0$ for $0<p<d$. Then by Cor. 5, 6 (III, §9), we have

$$\Phi : \mathfrak{H}^{q-d-1}(d)/\mathfrak{H}^{q-d-1}_\infty(d) \approx \mathfrak{K}^q_\infty(0) \subset \mathfrak{H}^q(0), \tag{8.1}$$

$$\Psi^{-1} : \mathfrak{H}^q(0)/\mathfrak{K}^q_\infty(0) \approx H^{1,q-1} \subset H^q(A), \tag{8.2}$$

$$\Psi : H^{q-1}(A)/H^{1,q-2} \approx \mathfrak{H}^{q-d-1}_\infty(d) \subset H^{q-d-1}(d). \tag{8.3}$$

Therefore we obtain the following exact sequence;

$$(8.4)\qquad \begin{array}{ccccccc} H^{q-1}(A) & \xrightarrow[\Psi]{} & \mathfrak{H}^{q-d-1}(d) & \xrightarrow[\Phi]{} & \mathfrak{H}^{q}(0) & \xrightarrow[\Psi^{-1}]{} & H^{q}(A) \longrightarrow \\ \cup & & \cup & & \cup & & \cup \\ H^{1,q-2} & & \mathfrak{H}_{\infty}^{q-d-1}(d) & & \mathfrak{K}_{\infty}^{q}(0) & & H^{1,q} \end{array}$$

Thus identifying $\mathfrak{H}^{q}(d)$ and $\mathfrak{H}^{q}(0)$ to $H^{q}(B)$, in virtue of (III, §10) we have:

Theorem 17. *If $H^{p}(F)=0$ for $0<p<d$, we have the following exact sequence*:

$$(8.5)\qquad H^{q-1}(A)\xrightarrow{\Psi} H^{q-d-1}(B)\xrightarrow{\Phi} H^{q}(B)\xrightarrow{\psi^{*}} H(A)\longrightarrow .$$

By (3.8) in Theorem 11, commutativity relation holds in the following diagram:

$$\begin{array}{ccc} \mathfrak{H}^{0}(d) & \longrightarrow & \mathfrak{H}^{d+1}(0) \\ \downarrow (\vee u) & & \downarrow (\vee u) \\ \mathfrak{H}^{q-d-1}(d) & \longrightarrow & \mathfrak{H}^{q}(0) \end{array}\quad , \textit{ where } u\in \mathfrak{H}^{q-d-1}(0).$$

Thus we have:

Theorem 18. *Put $\Omega=\Phi(1)$, where $1\in H^{0}(B)$. Then $\Phi: H^{q-d-1}(B)\rightarrow H^{q}(B)$ may be represented in the following form*:

$$(8.6)\qquad \Phi(u) = u\cup \Omega .$$

The above theorems were announced by Thom and independently by Chern-Spanier in case where $\mathfrak{F}$ is a d-sphere bundle. They stated also: *if $\mathfrak{F}$ is a d-sphere bundle, Ω is the image under the natural homomorphism $H^{d+1}(B, I)\rightarrow H^{d+1}(B)$ of the Whitney-Steenrod's characteristic cohomology class of the sphere bundle $\mathfrak{F}$. If ρ is a field and d is even, Ω vanishes according to the well-known properties of characteristic classes*[41], *hence Φ are all null homomorphisms.*

Thus

Theorem 19. *An even dimensional sphere bundle $\mathfrak{F}$ is homologically trivial.*

We shall not prove it in this paper, since direct proof will be given in another paper.

9. Theorems of Wang [24]. Let $\bar{H}^{q}(B, \bar{H}^{p}(F))=0$ for $0<q<n$, and let $\bar{H}^{n}(B, \bar{H}^{p}(F))\approx\bar{H}^{p}(F)$: for example let ρ be a field and let B be a homology sphere. Then as Cor. 7 (III, §9) we have:

$$(7.1)\qquad \begin{aligned} &\bar{\Phi}: \bar{\mathfrak{H}}^{n}(p-n+1)/\bar{\mathfrak{H}}^{n}_{\infty}(p-n+1)\approx \bar{\mathfrak{K}}^{0}_{\infty}(p)\subset \bar{\mathfrak{H}}^{0}(p),\\ &\bar{\Psi}^{-1}: \bar{H}^{p+1}(A)/\bar{H}^{p+1,0}\approx \bar{\mathfrak{H}}^{n}_{\infty}(p-n+1)\subset \bar{\mathfrak{H}}^{n}(p-n+1),\\ &\bar{\Psi}: \bar{\mathfrak{H}}^{0}(p)/\bar{\mathfrak{K}}^{0}_{\infty}(p)\approx \bar{H}^{p,0}\subset \bar{H}^{p}(A). \end{aligned}$$

Thus we have the following exact sequence:

$$(7.2)\qquad \longrightarrow \bar{H}^{p+1}(A)\xrightarrow{\bar{\Psi}^{-1}} \bar{H}^{n}(p-n+1)\xrightarrow{\bar{\Phi}} \bar{H}^{0}(p)\xrightarrow{\bar{\Psi}} \bar{H}^{p}(A)\longrightarrow .$$

41) [21], [29].

Identifying $\bar{H}^0(p)$ and $\bar{H}^n(p)$ to $\bar{H}^p(F)$, in virtue of (III, §11) we have:

Theorem 20. *Under the assumptions given at the beginning of this section we have the following exact sequence*:

$$(9.3)\qquad \longrightarrow \bar{H}^{p+1}(A) \longrightarrow \bar{H}^{p-n+1}(F) \xrightarrow{\bar{\Phi}} \bar{H}^p(F) \xrightarrow{\Psi = i^*} \bar{H}^p(F) \longrightarrow .$$

If further $\bar{H}^p(F)=0$ for $p \geqq n-1$, or if $d > n-1$ and $\bar{H}^p(F)=0$ for $0<p<d$, then $\bar{\Phi}$ are obviously null homomorphisms. Thus we have

$$(9.4)\qquad \bar{H}^p(A) \approx \bar{H}^p(F) \oplus \bar{H}^{p-n}(F) \quad \text{in these cases.}$$

Theorem 20 is thus a generalization of the theorems of Wang.

Part V

Further properties of characteristic groups and isomorphisms

1 The general assumptions throughout this part are the same as in the preceding part. Since we are dealing with polyhedral spaces only, we may choose the singular homology and cohomology theories as our basic theories.

2. $\circ$-multiplication. Let us define a kind of multiplication which is a generalization of Pontrjagin multiplication. Let $'\mathfrak{F} = \{G, G; B, 'A, '\psi, '\varphi_U\}$ be the principal fibre bundle of $\mathfrak{F}$. Notice that a point $'a \in G_x = '\psi^{-1}(x) \subset 'A$ is an admissible map $'a: F \to F_x$. Let $\eta: 'A \times F \to A$ be defined by $\eta('a, y) = 'ay$ for $'a \in 'A$ and $y \in F$, where $'ay$ is the image of y under $'a$. Given a singular q-chain ς^q of $'A$ and a singular p-chain a^p of F, the singular $(p+q)$-chain $\eta(\varsigma^q, a^p)$ of A is denoted by $\varsigma^q \circ a^p$. Obviously $\varsigma^q \circ a^p$ is bilinear and subjects to the usual boundary formula:

$$(2.1)\qquad \partial(\varsigma^q \circ a) = \partial \varsigma^q \circ a^p + (-1)^q \varsigma^q \circ \partial a^p .$$

In particular: if ς^q, a^p are singular cycles, so is $\varsigma^q \circ a^p$; if moreover either ς^q or a^p is a boundary, so is $\varsigma^q \circ a^p$. Thus a bilinear multiplication $\bar{H}^q(A') \circ \bar{H}^p(F) \subset \bar{H}^{p+q}(A)$ may be defined by taking representatives. More generally, for any pair of subcomplexes $B \subset C \subset D$, we may define a bilinear multiplication (or a pairing) $\bar{H}^p('\tilde{B}, '\tilde{C}) \circ \bar{H}^r(F) \subset H^{p+r}(\tilde{B}, \tilde{C})$. In particular $\bar{H}^p('A_q, 'A_{q'}) \circ \bar{H}^r(F) \subset H^{p+r}(A_q, A_{q'})$.

Obviously commutativity relations hold in the following diagram:

$$(2.2)\qquad \begin{array}{ccccccccc} \to \bar{H}^p('A_{q'}, 'A_{q''}) & \to & \bar{H}^p('A_q, 'A_{q''}) & \to & \bar{H}^p('A_q, 'A_{q'}) & \to & \bar{H}^{p-1}('A_{q'}, 'A_{q''}) & \to \\ \downarrow (\circ a^r) & & \downarrow (\circ a^r) & & \downarrow (\circ a^r) & & \downarrow (\circ a^r) & \\ \to \bar{H}^{p+r}(A_{q'}, A_{q''}) & \to & \bar{H}^{p+r}(A_q, A_{q''}) & \to & \bar{H}^{p+r}(A_q, A_{q'}) & \to & \bar{H}^{p+r}(A_{q'}, A_{q''}) & \to . \end{array}$$

From (2.2) we may derive the following formulas:

$$(2.3)\quad '\mathfrak{Z}^q_k(p) \circ \bar{H}^r(F) \subset \mathfrak{Z}^q_k(p+r), \qquad (2.4)\quad '\mathfrak{B}^q_k(p) \circ \bar{H}^r(F) \subset \mathfrak{B}^q_k(p+r),$$

(2.3) $'\bar{H}^{p,q}\circ\bar{H}^{r}(F) \subset \bar{H}^{p+r,q}$.

Since in particular $'\mathfrak{B}^{q}(p)\circ H(F)\subset\mathfrak{B}^{q}(p+r)$, we may replace $\mathfrak{Z}$, $\mathfrak{B}$ in (2.3), (2.4) by $\mathfrak{H}$, $\mathfrak{K}$ respectively:

(2.6) $'\bar{\mathfrak{H}}^{q}_{k}(p)\circ\bar{H}^{r}(F) \subset \bar{\mathfrak{H}}^{q}_{k}(p+r)$, (2.7) $'\bar{\mathfrak{K}}^{q}_{k}(p)\circ\bar{H}^{r}(F) \subset \bar{\mathfrak{K}}^{q+r}_{k}(p+r)$.

The following relations are also obvious:

(2.8) $\bar{\Phi}^{\pm1}_{k}(\circ a) = (\circ a)\bar{\Phi}^{\pm1}_{k}$, (2.9) $\bar{\Psi}^{\pm1}(\circ a^{p}) = (\circ a^{p})\bar{\Psi}^{\pm1}$.

When ρ is a field $\square$-multiplication is defined starting from $\circ$-multiplication in the same way as $\cap$-product was defined from $\cup$-product: for $u\in H^{p+r}(\tilde{C},\tilde{D})$, $a^{r}\in\bar{H}^{r}(F)$, the product $u\circ a$ is an element of $H^{p}('\tilde{C},'\dot{D})$ such that $\langle u\square a, z\rangle = \langle u, z\square a\rangle$ for every $z\in\bar{H}^{p}('C,'\dot{D})$. Corresponding to (2.5)-(2.9) we have:

(2.5)′ $H^{p+r,q}\circ\bar{H}^{r}(F) \subset {}'H^{p,q}$,

(2.6)′ $\mathfrak{H}^{q}_{k}(p+r)\square\bar{H}^{r}(F) \subset {}'\mathfrak{H}^{q}_{k}(p)$, (2.7)′ $\mathfrak{K}^{q}_{k}(p+r)\square\bar{H}^{r}(F) \subset {}'\mathfrak{K}^{q}_{k}(p)$,

(2.8)′ $\Phi^{\pm1}_{k}(\square a^{p}) = (\square a^{p})\Phi^{\pm1}_{k}$, (2.9)′ $\Psi^{\pm1}(\square a^{p}) = (\square a^{p})\Psi^{\pm1}$.

3. Let $\varphi^{q}_{i}=\varphi_{i} : \sigma^{q}_{i}\to{}'A$ be a slicing map. Then since φ_{i} maps $(\sigma^{q}_{i}, \dot{\sigma}^{q}_{i})$ into $('\sigma^{q}_{i}, '\dot{\sigma}^{q}_{i})$, it represents an element $\varepsilon^{q}_{i}=\varepsilon_{i}$ of $\bar{H}^{q}('\tilde{\sigma}^{q}_{i}, \tilde{\dot{\sigma}}^{q}_{i})$, as well as an element $\bar{\lambda}^{q}_{i}\varepsilon_{i}$ of $\bar{H}^{q}('A_{q}, 'A_{q-1})$ (II, §3). ε_{i}, $\bar{\lambda}^{q}_{i}\varepsilon_{i}$ do not depend on a special choice of φ_{i}, since G is arc-wise connected.

It is easily seen that the correspondence $\bar{H}^{p}(F)\ni a^{p}\to\varepsilon_{i}\circ a^{p}\in\bar{H}^{p+q}(\tilde{\sigma}^{q}_{i}, \tilde{\dot{\sigma}}^{q}_{i})$ gives an isomorphism $\bar{H}^{p}(F)\approx\bar{H}^{p+q}(\tilde{\sigma}^{q}_{i}, \tilde{\dot{\sigma}}^{q}_{i})$, and that:

(3.1) *the correspondence* $\bar{C}^{q}(B,\bar{H}^{p}(F))\ni\sum a^{p}_{i}\sigma^{q}_{i}\to\sum(\bar{\lambda}^{q}_{i}\varepsilon_{i})\circ a^{q}_{i}\in\bar{H}^{p+q}(A_{q}, A_{q-1})\equiv\bar{\mathfrak{C}}^{q}(p)$ *gives the isomorphism* $\bar{C}^{q}(B.\bar{H}^{p}(F))\approx\bar{\mathfrak{C}}^{q}(p)$ *in Theorem* 6′. Hence (3.2) $'\bar{\mathfrak{C}}^{q}(0)\circ\bar{H}^{p}(F)=\bar{\mathfrak{C}}^{q}(p)$. We shall often identify $\bar{C}^{q}(B,\bar{H}^{p}(F))$ to $\mathfrak{C}^{q}(p)$, and use the symbol in several ways: for instance $\sigma^{q}_{i}\in\bar{H}^{q}('A_{q}, 'A_{q-1})$ implies $\sigma^{q}_{i}=\bar{\lambda}^{q}_{i}\varepsilon_{i}$, $\sigma^{q}_{i}\in\bar{H}^{q}(A_{q}, A_{q-1})$ implies $\sigma^{q}_{i}=(\bar{\lambda}^{q}_{i}\varepsilon_{i})\circ1$ where $1\in H^{0}(F)$.

The partial map $\varphi_{i}|\tilde{\dot{\sigma}}^{q}_{i}$ represents an element of $\bar{H}^{q-1}('\tilde{\dot{\sigma}}^{q}_{i})$ (as well as of $\bar{H}^{q-1}('A_{q-1})$), which is the image $\partial\varepsilon_{i}$ (the image $\bar{\omega}^{(q-1,-)}_{q-1}\bar{\lambda}^{q}_{i}\varepsilon_{i}$) under the boundary homomorphism $\partial:\bar{H}^{q}('\tilde{\sigma}^{q}_{i},'\tilde{\dot{\sigma}}^{q}_{i})\to\bar{H}^{q-1}('\tilde{\dot{\sigma}}_{q})$ $(\bar{\Phi}^{(q-1,-)}_{q-1}:\bar{H}^{q}('A_{q}, 'A_{q-1}\to\bar{H}^{q-1}('A_{q-1}))$. For the sake of simplicity we shall denote $\bar{\omega}^{(q-1,-)}_{q-1}\bar{\lambda}^{q}_{i}\varepsilon_{i}$ by $\dot{\varphi}_{i}$. Then we have by (2.2):

$$\bar{\omega}^{(q-1,-)}_{p+q-1}(\textstyle\sum_{i}a^{p}_{i}\sigma^{q}_{i}) = \sum_{i}\dot{\varphi}_{i}a_{i}. \tag{3.3}$$

4. From now on we shall assume that ρ is a field. In this case we obviously have:

(4.1) $'\bar{\mathfrak{Z}}^{q}(0)\circ\bar{H}^{p}(F) = \bar{\mathfrak{Z}}^{q}(p)$, (4.2) $'\bar{\mathfrak{B}}^{q}(0)\circ\bar{H}^{p}(F) = \mathfrak{B}^{q}(p)$,

(4.3) $'\bar{\mathfrak{H}}^{q}(0)\circ H^{p}(F) = \bar{\mathfrak{H}}^{q}(p)$.

More generally we have:

(4.4) $'\mathfrak{H}^q(p)\circ\bar{H}^r(F)=[' \mathfrak{H}(0)\bar{H}^p(G)]\circ\bar{H}^r(F)=\bar{\mathfrak{H}}^q(0)\circ['\bar{\mathfrak{H}}^0(p)\circ\bar{H}^r(F)]$.

(4.3) is the dual form of Corollary to Theorem 12 (IV, §4) in the following sense: (4.5) for any $\zeta'\in\mathfrak{H}^q(0)$, $\zeta''\in\mathfrak{H}^0(p)$, $z'\in{}'\bar{\mathfrak{H}}^q(0)$, $z''\in H^0(F)$ we have $<\zeta'\vee\zeta'', z'\circ z''> = <\zeta', z'><\zeta'', z''>$, where $<\zeta', z'>$, $<\zeta'', z''>$ may be understood after suitable identifications.

5. The homomorphism $\alpha_{p+q-1}^{[q-1,-1)}$. Consider the pairing $\bar{H}^{q-1}('A_{q-1})\circ\bar{H}^p(F)\subset\bar{H}^{p+q-1}(A_{q-1})$. This enables us to define a pairing of $H^q(B, \bar{H}^{q-1}('A_{q-1}))$ and $\bar{H}^q(B, \bar{H}^p(F))$ $(\approx\mathfrak{H}^q(p))$ into $\bar{H}^{p+q-1}(A_{q-1})$: explicitly $KI(\sum\zeta_i^{q-1}\sigma_i^q, \sum a_i^p\sigma_i^q)=\sum\zeta_i^{q-1}\circ a_i^p\in\bar{H}^{p+q-1}(A_{q-1})$, where $\sum\zeta_i^{q-1}\sigma_i^q\in Z^q(B, \bar{H}^{q-1}('A_{q-1}))$, $\sum a_i^p\sigma_i^q\in\bar{Z}^q(B, \bar{H}^p(F))$.

Now it is obvious that $\alpha_{q-1}^{[q-1,-1)}: {}'\mathfrak{C}^q(0)\to\bar{H}^{q-1}('A_{q-1})$ corresponds to $\sum\dot{\varphi}_i\sigma_i^q$ under the isomorphism Hom $\{'\mathfrak{C}^q(0)\to\bar{H}^{q-1}('A_{q-1})\}\approx C^q(B, \bar{H}^p(F))$, where as in §3 $\dot{\varphi}_i=\bar{\partial}_{p-1}^{[q-1,-1)}\bar{\lambda}_i^q\mathfrak{z}_i\in\bar{H}^{q-1}('A_{q-1})$. Since $\alpha_{q-1}^{[q-1,-1)}$ reduces to null homomorphism on $'B^q(0)$, $\sum\dot{\varphi}_i\sigma_i^q$ is a cocycle $\in z^q(B, \bar{H}^{q-1}('A_{q-1}))$.

On the other hand we observed in (3.3) that in the following diagram:

$$\begin{array}{ccc}\bar{H}^{p+q}(A_q, A_{q-1}) & \xrightarrow{\bar{\partial}_{p+q-1}^{[q-1,-1)}} & \bar{H}^{p+q-1}(A_{q-1})\\ \wr\wr & \nearrow_{KI(\sum\dot{\varphi}_i\sigma_i^q\cdot)} & \\ \bar{C}^q(B, \bar{H}^p(F)) & & \end{array}$$

$\bar{\partial}_{p+q-1}^{[q-1,-1)}$ corresponds to the homomorphism $KI(\sum\dot{\varphi}_i\sigma_i^q, \cdot)$. Since $\alpha_{p+q-1}^{[q-1,-1)}$ reduces to null homomorohism on $\mathfrak{B}^q(p)$ $(\subset\mathfrak{Z}_\infty^q(p))$ it induces a homomorphism $\alpha_p: \mathfrak{H}^q(p)\to\bar{H}^{p+q-1}(A_{q-1})$. It may be realized in the from $\alpha_pu=KI(\alpha', u)$, where $\alpha'\in H^q(B, \bar{H}^{q-1}('A_{q-1}))$ is represented by $\sum\dot{\varphi}_i\sigma_i^q$. If $\alpha'=0$, $\alpha_p=0$ for each p. Thus we have:

Theorem 21. *$'\mathfrak{H}^q(0)={}'\mathfrak{H}_\infty^q(0)$ implies $\mathfrak{H}^q(p)=\mathfrak{H}_\infty^q(p)$ for each p. In particular if $\mathfrak{F}$ is a principal fibre bundle, $\bar{\mathfrak{H}}^q(0)=\bar{\mathfrak{H}}_\infty^q(0)$ implies $\mathfrak{H}^q(p)=\mathfrak{H}_\infty^q(p)$ for each p.*

6. The groups $\mathfrak{H}_\#^0$, $\bar{\mathfrak{K}}_\#^0$, $\mathfrak{K}_\#(0)$ and $\bar{\mathfrak{H}}_\#(0)$. Consider the characteristic isomorphism

(6.1) $\Phi: \mathfrak{H}_{p-2}^0(p)/\mathfrak{H}_{p-1}^0(p)\approx\mathfrak{K}_p^{p+1}(0(/\mathfrak{K}_{p-1}^{p+1}(0)$, $\bar{\Phi}^{-1}: \bar{\mathfrak{K}}_{p-1}^0(p)/\bar{\mathfrak{K}}_{p-2}^0$
$\approx\bar{\mathfrak{H}}_{p-1}^{p+1}(0)/\bar{\mathfrak{H}}_p^{p+1}(0)$.

Since $\mathfrak{H}_{p-1}^0(p)=\mathfrak{H}_\infty^0(p)$, $\mathfrak{K}_p^{p+1}(0)=\mathfrak{K}_\infty^{p+1}(0)$, $\bar{\mathfrak{K}}_{p-1}^0(p)=\bar{\mathfrak{K}}_\infty^0(p)$ and $\bar{\mathfrak{H}}_p^{p+1}(0)=\bar{\mathfrak{H}}_\infty^{p+1}(0)$, putting $\mathfrak{H}_{p-2}^0(p)=\mathfrak{H}_\#^0(p)$, $\mathfrak{K}_{p-1}^{p+1}(0)=\mathfrak{K}_\#^{p+1}(0)$, $\bar{\mathfrak{K}}_{p-2}^0(p)=\bar{\mathfrak{K}}_\#^0(p)$ and $\bar{\mathfrak{H}}_{p-1}^{p+1}(0)=\bar{\mathfrak{H}}_\#^{p+1}(0)$, we have

(6.2) $\Phi: \mathfrak{H}_\#^0(p)/\mathfrak{H}_\infty^0(p)\approx\mathfrak{K}_\infty^{p+1}(0)/\mathfrak{K}_\#^{p+1}(0)$, $\bar{\Phi}^{-1}: \bar{\mathfrak{K}}_\infty^0(p)/\bar{\mathfrak{K}}_\#^0(p)\approx\bar{\mathfrak{H}}_\#^{p+1}(0)/\bar{\mathfrak{H}}_\infty^{p+1}(0)$.

The characteristic groups appearing in (6.2) are of particular importance as will be recognized in the following theorems. Putting $\mathfrak{H}_\#^0=\sum_p\mathfrak{H}_\#^0(p)$, $\bar{\mathfrak{K}}_\#^0=$

$\sum_p \bar{\mathfrak{K}}^0_{\#}(p), \ldots$, we have:

Theorem 22. *Let $\mathfrak{F}$ be a principal fibre bundle. Then*

(i) $\mathfrak{H}^0_{\#} = \mathfrak{H}^0_{\infty}$ *implies* $\mathfrak{H}^0 = \bar{\mathfrak{H}}^0_{\infty}$, *or equivalently*

(ii) $\bar{\mathfrak{K}}^0_{\#} = \bar{\mathfrak{K}}^0_{\infty}$ *implies* $\bar{\mathfrak{K}}^0_{\infty} = 0$, *or*

(iii) $\mathfrak{K}_{\#}(0) = \mathfrak{K}_{\infty}(0)$ *implies* $\mathfrak{K}_{\infty}(0) = 0$, *or*

(iv) $\bar{\mathfrak{H}}_{\#}(0) = \bar{\mathfrak{H}}_{\infty}(0)$ *implies* $\bar{\mathfrak{H}}(0) = \bar{\mathfrak{H}}_{\infty}(0)$.

Remark 1. Since $\mathfrak{H}^0_{\infty}$ is a subalgebra of $\mathfrak{H}^0$, Theorem 22 is obvious in case $\mathfrak{H}^0_{\#}$ generates $\mathfrak{H}^0$. The last condition is satisfied in the case of homogeneous (Koszul [8], Theorem 18. 3). The author could not generalize this to the corresponding theorem of fibre bundles.

Theorem 23. *If $\mathfrak{F}$ is a principal fibre bundle, we have:* $\mathfrak{H}^0_{\infty}(2\nu) = \mathfrak{H}^0(2\nu)$, *or eqvivalently* $\bar{\mathfrak{K}}^0_{\infty}(2\nu) = \bar{\mathfrak{K}}^0_{\#}(2\nu)$, *or* $\mathfrak{K}^{2\nu+1}_{\infty}(0) = \mathfrak{K}^{2\nu+1}_{\#}(0)$, *or* $\bar{\mathfrak{H}}^{2\nu+1}_{\infty}(0) = \bar{\mathfrak{H}}^{2\nu+1}_{\infty}(0)$.

Remark 2. Hirsch [6] states that "$\bar{\mathfrak{K}}^0_{\infty}/\bar{\mathfrak{K}}^0_{\#}$ may be regarded as a subspace of the group of minimal elements of $\bar{\mathfrak{H}}^0$. Such minimal elements are odd dimensional." This statement seems to contain both Theorem 22 and Theorem 23.

Theorem 24. $\mathfrak{H}^0_{\infty} = \mathfrak{H}^0$ *implies* $\mathfrak{H}_{\infty} = \mathfrak{H}$.

Proof: $\mathfrak{H}^q(p) = \mathfrak{H}^q(0) \vee \mathfrak{H}^0(p) = \mathfrak{H}^q_{\infty}(0) \vee \mathfrak{H}^0_{\infty}(p) \subset \mathfrak{H}^q_{\infty}(p) \subset \mathfrak{H}^q(p)$, in virtue of Cor. to Theorem 12 (III, §4), Cor. 5 (III, §9) and Theorem 10.

Theorem 25. *If $\mathfrak{F}$ is a principal fibre bundle, or a homological sphere bundle,* $\mathfrak{H}_{\infty}(0) = \mathfrak{H}(0)$ *implies* $\mathfrak{H}_{\infty} = \mathfrak{H}$.

Proof: In virtue of Theorem 22 and Cor. 5 (III, §9), $\mathfrak{H}_{\infty}(0) = \mathfrak{H}(0)$ imlies $\mathfrak{H}^0_{\infty} = \mathfrak{H}^0$. Hence in virtue of Theorem 24, we have $\mathfrak{H}_{\infty} = \mathfrak{H}$.

Remark 3. $\mathfrak{H}^0_{\infty} = \mathfrak{H}^0$ says that "$i^*: H(A) \to H(F_{x_0})$ is an onto homomor-
or equivalently "$i_*: \bar{H}(F_{x_0}) \to \bar{H}(A)$ is an isomorphism". $\mathfrak{H}_{\infty}(0) = \mathfrak{H}(0)$ says
phism," that "$\psi_*: \bar{H}(A) \to \bar{H}(B)$ is an onto homomorphism", or equivalently
"$\psi^*: H(B) \to H(A)$ is an isomorphism."

Theorem 26. *If B is an odd dimensional homology sphere, then* $\mathfrak{H} = \mathfrak{H}_{\infty}$.

Proof: If $\mathfrak{F}$ is a principal fibre bundle the proof is obvious according to Theorem 22, 23, 24. But Theorem 21 assures the validity of the theorem in the general case.

Remark 4. Theorem 24, 27 are in some sense generalizations of Samelson's theorems [15].

Theorem 27. *If $\mathfrak{F}$ is a principal fibre bundle, $\mathfrak{H}^0_k$ is generated by the subspace of $\mathfrak{H}^0_k$ consisting of $\circ$–minimal elements of* $\mathfrak{H}^0$.

The proofs of Theorem 22, 23, 27 will be given in the following sections.

7. Proofs of Theorem 22. We shall prove that $\bar{\mathfrak{K}}^0_{\infty} = \bar{\mathfrak{K}}^0_{\#}$ implies $\bar{\mathfrak{K}}^0_{\infty} = 0$. For $p \leqslant 1$, we have $\bar{\mathfrak{K}}^0_{\infty}(p) = \bar{\mathfrak{K}}^0_{\#}(p) = \bar{\mathfrak{K}}^0_{p-2}(p) = 0$.

Let us assume that $p > 1$, and that $\bar{\mathfrak{K}}^0_{\infty}(r) = 0$ (or equivalently $\mathfrak{H}^0_{\infty}(r) = \mathfrak{H}^0(r)$)

for $r<p$. In order to prove that $\bar{\mathfrak{K}}^0_\infty(p)=0$, it is sufficient to prove that $\bar{\mathfrak{K}}^0_\infty(p)$ contains no non-vanishing minimal element of $\bar{\mathfrak{H}}^0$. For if so, we have $\mathfrak{H}^0(p) = \mathfrak{H}^0_\infty(p)+[\textit{Composable elements of } \mathfrak{H}^0] \cap \mathfrak{H}^0(p)$. But since $\mathfrak{H}^0(r) = \mathfrak{H}^0_\infty(r)$ by assumption and since $\mathfrak{H}^0_\infty$ is a subalgebra, we have $\mathfrak{H}^0_\infty(p)=\mathfrak{H}^0(p)$. Or $\bar{\mathfrak{K}}^0_\infty(p)=0$. Assume on the contrary there exists a minimal element $0\neq z^p \in \bar{\mathfrak{K}}^0_\infty(p)=\bar{\mathfrak{K}}^0_\#(p) =\bar{\mathfrak{K}}^0_{p-2}(p)$, then there exists an integer k such that $0\leqslant k\leqslant p-2$, and such that $z\in\bar{\mathfrak{K}}^0_k(p)-\bar{\mathfrak{K}}^0_{k-1}(p)$.

Since $0<p-k-1<p$, by the definition of minimality we have $u\wedge z=0$ for every $u\in H^{p-k-1}(G)$. Now, denoting by $\{z\}$ the class mod $\bar{\mathfrak{K}}^0_{k-1}(p)$ of z, we choose a representative element $Z\in\bar{\mathfrak{H}}^{k+2}_k(p-k-1)$ from $\bar{\Phi}^{-1}(\{z\})$. Since $\bar{\mathfrak{H}}^{k+2}(p-k-1)=\bar{\mathfrak{H}}^{k+2}(0)\circ\bar{H}^{p-k-1}(G)$, Z may be written in the form $Z=\sum\zeta_i\circ a_i$ where $\zeta_i\in\bar{\mathfrak{H}}^{k+2}(0)$ and $\{a_i\}$ are a basis of $\bar{H}^{p-k-1}(G)$. Let $\{u_j\}$ be the dual basis of $H^{p-k-1}(G)$. Then by (4.5) $u_j\wedge Z=\sum\zeta_i\circ(u_j\wedge a_j)=\zeta_j$. On the other hand noticing that $u_j\in\mathfrak{H}^0(p-k-1)=\mathfrak{H}^0_\infty(p-k-1)$, in virtue of Theorem 14 we have $0=\bar{\Phi}^{-1}(\{u_j\wedge z\})=u_j\wedge\bar{\Phi}^{-1}(\{z\})=u_j\wedge\{Z\}=\{u_j\wedge Z\}=\{\zeta_j\}$. Consequently $\zeta_j\in\bar{\mathfrak{H}}^{k+2}_{k+1}(0)$. Thus in virtue of (2.6) we have $Z=\sum\zeta_i\circ a_i\in\bar{\mathfrak{H}}^{k+2}_{k+1}(p-k-1)$. But this implies that $\{Z\}=0$, hence $\{z\}=0$, contrary to the assumption.

8. Proof of Theorem 27. Since $\mathfrak{H}^0_k=\mathfrak{H}^0_\infty$ when we restrict ourselves to A_{k+2}, it is sufficient to prove that $\mathfrak{H}^0_\infty$ is generated by $\circ$-minimal element of $\mathfrak{H}^0$ belonging to $\mathfrak{H}^0_\infty$. First of all, $\mathfrak{H}^0_\infty$ is a subalgebra of $\mathfrak{H}^0$. $\mathfrak{H}^0_\infty$ is non-void since it contains $\mathfrak{H}^0(0)$. In vertue of (2.6)$'$, for any $a^p\in\bar{H}^p(G)$, $\bar{\mathfrak{H}}^0_\infty$ is stable under $(\Box a^p)$. On the other hand we easily see that $\circ$- and $\Box$-multiplications in the sense defined in this part coincide with those defined in (I, §8). Hence by (8.7) (I, §8), $\mathfrak{H}^0_\infty$ is generated by its subspace consisting of all $\circ$-minimal elements of $\mathfrak{H}^0$ contained in $\mathfrak{H}^0_\infty$.

9. Proof of Theorem 23. Since $\mathfrak{H}^0_{p-1}(r)=\mathfrak{H}_\infty(r)$ for $r<p+1$, and $\mathfrak{H}^0_{p-1}(p+1) =\mathfrak{H}_\#(p+1)$, we have:

$$\mathfrak{H}^0_{p-1}=\mathfrak{H}^0_\infty(0)+\cdots\mathfrak{H}^0_\infty(p)+\mathfrak{H}^0_\#(p+1)+\cdots\cdots,$$

$$\mathfrak{H}^0_\infty=\mathfrak{H}^0_\infty(0)+\cdots\mathfrak{H}^0_\infty(p)+\mathfrak{H}^0_\infty(p+1)+\cdots\cdots.$$

Since $\mathfrak{H}^0_\infty$ is an algebra, an element of $\mathfrak{H}^0_{p-1}(p+1)$, which may be generated by by multiplication and addition from lower dimensional elements of $\mathfrak{H}^0_{p-1}$, is an element of $\mathfrak{H}^0_\infty$. But according to Theorem 27, an element of $\mathfrak{H}^0_{p-1}(p+1)$ is either a $\circ$-minimal element or composable from lower dimensional elements. That is to say, an element of $\mathfrak{H}^0_\#$ which is not an element of $\mathfrak{H}^0_\#$ which is not an element of $\mathfrak{H}^0_\#$ which is not an element of $\mathfrak{H}^0_\infty$ is $\circ$-minimal. On the other hand there is no even dimensional $\circ$-minimal element since G is a compact connected Lie group. Hence $\mathfrak{H}^0_\infty(2\nu)=\mathfrak{H}^0_\#(2\nu)$.

References

1) Bräadli, E. R. & Eckmann, B, Cartetisches und Alexandersches Produkt in der Cohomologietheorie, Comment. Math. Helv. 24 (1949).
2) Chern, S. S. & Sun, Y. F., The imbedding Theorem for Fibre Bundles, Trans. Amer. Math. Soc., 67 (1949).
3) Chern, S. S. & Spanier, E., The homological structure of sphere bundles, Proc. Nat. Acad. Sci. U. S. A., 36 (1950).
4) Eilenberg, S. &. Steenrod, N. E., Axiomatic approach to homology theory, Proc. Nat. Acad. Sci. U. S. A., 31 (1945).
5) Gysin, W., Zur Homologietheorie der Abbildungen und Faserungen von Mannigfaltigkeiten, Comment. Math. Helv., 14 (1941).
6) Hirsch, G., Un isomorphism attaché aux structures fibrés, C. R., 227, 1948.
7) Hopf. H., Uber die Topologie der Gruppenmannigfaltigkeiten und ihrer Varallgemeinerungen, Ann. of Math., 42 (1941).
8) Koszul. J. L., Homologie des algèbre de Lie, Bull. Soc. math. de France,
9) Kudo, T., Homological Properties of Fibre Bundles, J. of the Inst. of Polytechn. Osaka City Univ., 1 (1959).
10) Leray, J., L'anneau d'homologie d'une représentation, C. R. 222, 1946. Structure de l'anneau d'homologie d'une représentation, ibid.
11) ——, Propriété de l'aneau d'homologie de la projection d'un espace fibré sur sa base, C. R., 223, 1946. Sur l'anneau d'homologie de l'espace homogène, quotient d'un groupe clos par un sous groupe abélien. conexe, maximum, ibid.
12) Morse, M., The calculus of Variations in the large, Amer. Math. Soc. Coll. publ., 1934.
13) Pontrjagin, L., On Homologies in compact Lie groups, Recueil math. de Moscou, 6 (1939).
14) Pontrjagin, L., Characteristic cycles on manifolds, C. R, Acad. Sci. USSR 35 (1942).
15) Samelson, H., Beiträge zur Topologie der Gruppenmannigfaltigkeiten, Ann. of Math, 42 (1941).
16) Spanier, E., Borsuk's cohomotopy groups, Ann. of Math., 50 (1949).
17) Steenrod, N. E., Topological method for the construction of tensor functions, Ann. of Math., 43 (1942).
18) ——, Homology with local coefficients, Ann. of Math., 44 (1943).
19) ——, Classification of Sphere Bundles, Ann. of Math., 45.
20) ——, On the cohomology invariants of mappings, Ann. of Math.,
21) ——, The topology of fibre bundles, Princeton Univ. Press,
22) Stifel, E., Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten, Comment. Math. Helv., 8 (1936).
23) Thom, R., Classes caracteristiques et i-carrés, C. R., 230, 1950.
24) Wang, C. H., The homology groups of the fibre bundles over a sphere, Duke Math. J., 1949.
25) Whitehead, G. W., The (n+2) homotopy group of the n-sphere, Ann. of Math., 52 (1950).
26) Whiehead, J. H. C., Combinatorial homotopy. I, Bull. Amer. Math. Soc. 55 (1949).
27) Whitney, H., Topological properties of differentiable manifolds, Bull. Amer. Math., Soc., 43 (1937).
28) ——, On the theory of sphere-bundles, Proc. Nat. Acad. Sci. U. S. A., 26 (1940).
29) ——, On the topology of differentiable manifolds, Lectures in topology, Michigan, 1941.
30) Colloque international de topologie algébrique, Paris, 1947.

Added in proof: The author obtained a proof of the following proposition (Cf. Remark 1, p. 137): *$\mathfrak{H}^0_{\#}$ generates $\mathfrak{H}^0$ if $\mathfrak{F}$ is a principal fibre bundle.* This may be proved by the infinitesimal method of J. L. Koszul [8] if $\mathfrak{F}$ is the universal principal fibre bundle given by N. E. Steenrod (Cf. p. 104).* If $\mathfrak{F}$ is general, it may be proved by making use of Theorem 22. The above proposition has the following consequences:

i) If B is homologous to zero in $'B$, and if $'\mathfrak{F}$ is a principal fibre bundle over $'B$, then the $\mathfrak{F}='\mathfrak{F}|B$ is homologically trivial.

ii) Every even dimensional homological sphere bundle is homologically trivial (Cf. Theorem 19, p. 133).

* In this case our proposition is contained in a theorem of A. Borel — La transgression dans les espaces fibrés principaux, C. R. Paris 232 (1951).

After this paper was presented to the editor, the author could read the following papers:

Séminaire de Topologie algébrique, ENS, III, 1950–1951.

J. P. Serre: Homologie singulière des espaces fibrés, Ann. of Math., vol. 54, 3 (1951).

The following papers of J. Leray are not available to the author, but he could know their outline through Math. Reviews:

L'aneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue, J. Math. Pures Appl. 29 (1950).

L'homologie d'un espace fibré dont la fibre est connexe, J. Math. Pures Appl. 29 (1950).