Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 1, No, 2, Series A

Note on Riemann's ξ-function

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(Received May 20, 1951)

A property of Riemann's ξ-function will be stated. The distribution of zero points of a function similar to the former has been studied. This is an extension of the result of Pólya.[1] §1. Certain property of ξ-function and the definition of a function $Y(z)$. §2. Asymptotic formulae for Bessel functions. §3. Distribution of zero points of $Y(z)$.

§1. Riemann defined the following ξ-function[2]:

$$\xi(t)=\frac{1}{2}-\left(t^2+\frac{1}{4}\right)\int_1^{\infty}\psi(x)x^{-\frac{3}{4}}\cos\left(\frac{t}{2}\log x\right)dx\,, \tag{1}$$

where $\psi(x)=\sum_1^{\infty}\exp(-n^2\pi x)$, and it satisfies the relation

$$2\psi(x)+1=\frac{1}{\sqrt{x}}\left(2\psi\left(\frac{1}{x}\right)+1\right). \tag{2}$$

In the following, $\Re f(z)$ and $\Im f(z)$ denote the real and imaginary part of a function $f(z)$ respectively. Let us put as $z=x+iy$.

When $|x|<\frac{1}{4}$, it is

$$\frac{1}{4z^2-\frac{1}{4}}=-\frac{1}{2}\left(\int_0^{\infty}\exp\left(\left(z-\frac{1}{4}\right)u\right)du+\int_0^{\infty}\exp\left(-\left(z+\frac{1}{4}\right)u\right)du\right).$$

So that we have, for $|x|<\frac{1}{4}$,

$$\xi(2iz)=\frac{1}{2}\left(4z^2-\frac{1}{4}\right)\int_0^{\infty}\Psi(u)(e^{zu}+e^{-zu})du, \tag{3}$$

where

$$\Psi(u)=\psi(e^u)e^{\frac{u}{4}}-\frac{1}{2}e^{-\frac{u}{4}}.$$

It can easily be proved that $\Psi(-u)=\Psi(u)$, when we use the relation (2).

The distribution of the zero points of ξ-function will be explained, if the distribution of zero points of a function

$$\Xi(z)=\int_{-\infty}^{\infty}\Psi(u)e^{zu}du$$

were explained.

Let us now consider a suitable sequence of functions

$$\Psi_1(u),\ \Psi_2(u),\ \Psi_3(u),\ \ldots\ldots\ldots, \tag{4}$$

where it is assumed that these functions are all even and they satisfy a convergence condition

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}(e^{\frac{u}{4}}+e^{-\frac{u}{4}})^2|\Psi(u)-\Psi_n(u)|^2du=0.$$

Then, by using Schwarz's inequality, we can prove that the functions

$$\Xi_n(z)=\int_{-\infty}^{\infty}\Psi_n(u)e^{zu}du,\ n=1,2,3,\cdots\cdots$$

tend to the function $\Xi(z)$ uniformly in a strip $|x|\leqq\frac{1}{4}-\varepsilon$, where ε is an arbitrary small positive number.

Accordingly, if such a suitable sequence (4) were found and the distributions of zero points of the functions $\Xi_n(z)$ were explained, the distribution of the zero points of $\Xi(z)$ would be explained.

In this peper we take up a function

$$\Phi(u)=\Sigma'(e^{\frac{u}{4}}+e^{-\frac{u}{4}})\exp(-n^2\pi(e^u+e^{-u}))-\frac{1}{2}(e^{\frac{u}{4}}+e^{-\frac{u}{4}})^{-1},$$

where the summation Σ' means the same as that of $\prod_{p\leqq N}(1-p^{-z})^{-1}=\sum' n^{-z}$, N being a fixed positive number. And we define a function

$$Y(z)=\int_{-\infty}^{\infty}\Phi(u)e^{zu}du, \tag{5}$$

and, in this paper, let us consider the distribution of its zero points.

§2. In order to solve the problem given in the preceding paragraph, we shall state some necessary functions and some of their properties.

$J_\nu(z)$ is the ordinary Bessel function of the first kind of order ν and argument z. When k is any constant

$$J_\nu(ze^{k\pi i})=e^{k\nu\pi i}J_\nu(z),\quad J_{-\nu}(ze^{k\pi i})=e^{-k\nu\pi i}J_{-\nu}(z).$$

Hankel function of the first kind is defined as

$$H_\nu^{(1)}(z)=\{J_{-\nu}(z)-e^{-\nu\pi i}J_\nu(z)\}/i\sin\nu\pi,$$

and it has the following integral representation:

$$H_\nu^{(1)}(z)=\frac{1}{\pi i}\int_{-\infty+i(\arg z+\mu_1)}^{\infty+i(\pi-\arg z+\mu_2)}\exp(z\sinh t-\nu t)dt, \tag{6}$$

where $z\neq0$ and $-\frac{\pi}{2}<\mu_1,\ \mu_2<\frac{\pi}{2}$.

By using this function, the function

$$K_\nu(z)=\frac{\pi i}{2}e^{\frac{1}{2}\nu\pi i}H_\nu^{(1)}(iz)$$

is defined. When z is real and positive, we can see from (6) that this function has the integral representation

$$K_\nu(z)=\frac{1}{2}\int_{-\infty}^{\infty}\exp(-z\cosh t-\nu t)dt. \tag{7}$$

Accordingly, from (5), we obtain

$$Y(z)=2\sum'\{K_{z+\frac{1}{4}}(2n^2\pi)+K_{z-\frac{1}{4}}(2n^2\pi)\}-R(z), \tag{8}$$

where

$$R(z)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{zu}}{e^{\frac{u}{4}}+e^{-\frac{u}{4}}}du.$$

In the next place, we deduce the asymptotic formulae of $H_z^{(1)}(ia)$, when a is real and positive, the value of x is finite and y tends to plus infinity. In the following, δ means a fixed positive number. For example $\delta=1/40$. And we separate the interval $(0, \infty)$ of a into five parts as

$$\begin{aligned} I_1:&\quad 0\leqq a<y-y^{\frac{1}{3}+\delta},\\ I_2:&\quad y-y^{\frac{1}{3}+\delta}\leqq a<y-y^{\frac{1}{4}},\\ I_3:&\quad y-y^{\frac{1}{4}}\leqq a<y+y^{\frac{1}{4}},\\ I_4:&\quad y+y^{\frac{1}{4}}\leqq a<y+y^{\frac{1}{3}+\delta},\\ I_5:&\quad y+y^{\frac{1}{3}+\delta}\leqq a<\infty.\end{aligned}$$

Then the following Theorems hold valid in each of these intervals. We can prove these Theorems, using the method of the steepest descent as in Watson [3] pp. 235-270.

Theorem 1. *In the interval I_1, put as $z=ia\cosh\gamma$, then the asymptotic formula*

$$H_z^{(1)}(ia)\sim\frac{\exp\left(z(\tanh\gamma-\gamma)-\frac{i\pi}{4}\right)}{\sqrt{-\frac{\pi iz}{2}\tanh\gamma}}\sum_0^\infty\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{A_m}{\left(-\frac{1}{2}z\tanh\gamma\right)^m}$$
$$+\frac{\exp\left(-z(\tanh\gamma-\gamma)-\frac{i\pi}{4}\right)}{\sqrt{\frac{\pi iz}{2}\tanh\gamma}}\sum_0^\infty\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{A_m}{\left(\frac{1}{2}z\tanh\gamma\right)^m}$$

holds valid. If we put as $\gamma=\alpha+i\beta$, then the position of γ is decided as

$$\alpha\leqq 0,\quad 0\leqq\beta<\frac{\pi}{2}\quad\text{for}\quad x\geqq 0,$$

$$\alpha\leqq 0,\quad 0\geqq\beta>-\frac{\pi}{2}\quad\text{for}\quad x\leqq 0.$$

And $A_0=1$, $A_1=\frac{1}{8}-\frac{5}{24}\coth^2\gamma$, ...

From this result, we obtain the formula

$$2K_z(a)=\sqrt{\frac{2\pi}{y}}e^{\frac{\pi iz}{2}}\left\{\left(\frac{y}{\pi}\right)^x e^{i\phi}\left(\frac{2\pi}{a}\right)^z\exp\left(f(a)\right)\right.$$
$$\left.+\left(\frac{y}{\pi}\right)^{-x}e^{-i\phi}\left(\frac{2\pi}{a}\right)^{-z}\exp\left(-f(a)\right)\right\}B(z)\left(1+O(y^{-\frac{3}{2}\delta})\right),\quad (9)$$

where $\phi=y\log\frac{y}{\pi}-y-\frac{\pi}{4}$. $B(z)=\sqrt{z}:\sqrt[4]{z^2+a^2}$ and

$$f(a)=f(a, z)=z-\sqrt{z^2+a^2}+z\log\frac{1}{i}(z+\sqrt{z^2+a^2})-z\log\frac{2z}{i}.$$

The absolute value of $\exp(f(a))$ decreases from 1 to 2^{-x}, when a increaes from 2π to $y-y^{\frac{1}{3}}$ and x is positive, and increases from 1 to 2^{-x} when x is negative. Put as $a=ry$, then we obtain

$$B(z)=\frac{1}{\sqrt[4]{1-r^2}}\left(1+O\left(y^{-\frac{1}{6}}\right)\right).$$

Especially, for the bounded a, we have

$$2K_z(a)\sim\sqrt{\frac{2\pi}{y}}e^{\frac{\pi i z}{2}}\left\{\left(\frac{y}{\pi}\right)^x e^{i\phi}\left(\frac{2\pi}{a}\right)^z+\left(\frac{y}{\pi}\right)^{-x}e^{-i\phi}\left(\frac{2\pi}{a}\right)^{-z}\right\}. \qquad (10)$$

Theorem 2. *In the interval I_2, put as $z=ia\cosh\gamma$, then the asymptotic formula*

$$H_z^{(1)}(ia)\sim-\frac{2}{\sqrt{3}\pi}\exp\left(z\left(\tanh\gamma-\gamma\right)\right)\tanh\gamma\exp\left(\frac{z}{3}\tanh^3\gamma\right)K_{\frac{1}{3}}\left(\frac{z}{3}\tanh^3\gamma e^{-i\pi}\right)$$
$$+\frac{2}{\sqrt{3}\pi}\exp\left(-z\left(\tanh\gamma-\gamma\right)\right)\tanh\gamma\exp\left(-\frac{z}{3}\tanh^3\gamma\right)K_{\frac{1}{3}}\left(\frac{z}{3}\tanh^3\gamma\right)$$

hold valid, where γ is in the same domain as in the Theorem 1.

From this result, we obtain the formula

$$2K_z(a)=\sqrt{\frac{2\pi}{y}}e^{\frac{\pi i z}{2}}\left\{\left(\frac{y}{\pi}\right)^x e^{i\phi}\left(\frac{2\pi}{a}\right)^z\exp\left(f(a)\right)O\left(y^{\frac{1}{6}}\right)\right.$$
$$\left.+\left(\frac{y}{\pi}\right)^{-x}e^{-i\phi}\left(\frac{2\pi}{a}\right)^{-z}\exp\left(-f(a)\right)O\left(y^{\frac{1}{6}}\right)\right\}. \qquad (11)$$

Here, Landau's notation O is uniformly bounded with respect to all a in I_2.

Theorem 3. *In the interval I_3, put as $z=ia(1-\varepsilon)$, then the asymptotic formula*

$$H_z^{(1)}(ia)\sim-\frac{2}{3\pi}\sum_0^\infty e^{\frac{2}{3}(m+1)\pi i}B_m(\varepsilon ia)\sin\frac{1}{3}(m+1)\pi\frac{\Gamma\left(\frac{m}{3}+\frac{1}{3}\right)}{\left(\frac{1}{6}ia\right)^{\frac{1}{3}(m+1)}}$$

holds valid. Here $B_0(w)=1$, $B_1(w)=w$, $B_2(w)=\frac{w^2}{2}-\frac{1}{20}$, ...

From this result, we obtain the formula

$$2K_z(a)=\frac{\sqrt[3]{6}\Gamma\left(\frac{1}{3}\right)}{2\sqrt{3}\pi}e^{\frac{\pi i z}{2}}\frac{1}{\sqrt[3]{a}}(1+O(y^{-\frac{1}{3}})). \qquad (12)$$

Theorem 4. *In the interval I_4, put as $z=ia\cos\gamma$, then the asymptotic formula*

$$H_z^{(1)}(ia)\sim\frac{e^{\frac{\pi i}{6}}}{\sqrt{3}}\exp\left(iz(\tan\gamma-\gamma)\right)\tan\gamma\exp\left(-\frac{iz}{3}\tan^3\gamma\right)H_{\frac{1}{3}}^{(1)}\left(\frac{z}{3}\tan^3\gamma\right)$$

holds valid. If we put as $\gamma=\alpha+i\beta$, then the position of γ is decided as

$$0\leqq\alpha,\quad 0\leqq\beta<\frac{\pi}{2}\qquad\text{for } x\leqq0,$$

$$0\leqq\alpha,\quad 0\geqq\beta>-\frac{\pi}{2}\qquad\text{for } x\leqq0.$$

From this result, we obtain the formula

$$2K_z(a)=\sqrt{\frac{2\pi}{y}}e^{\frac{1}{2}\pi i z}\left(\frac{y}{\pi}\right)^x e^{i\phi}\left(\frac{2\pi}{a}\right)^z\exp\left(f(a)\right)O\left(y^{\frac{1}{6}}\right). \qquad (13)$$

Here, Landau's notation O is uniformly bounded for all a in I_4, and the absolute value of $\exp(f(a))$ is decreasing with respect to a whether x is positive or negative and its approximated value for $a=y+y^{\frac{1}{4}}$ is 2^{-x}.

Theorem 5. *In the interal* I_5, *put as* $z=ia\cos\gamma$, *then the asymptotic formula*

$$H_z^{(1)}(ia) \sim \frac{\exp\left(iz(\tan\gamma-\gamma)-\frac{\pi i}{4}\right)}{\sqrt{\frac{\pi z}{2}\tan\gamma}} \sum_0^\infty \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{A_m}{\left(\frac{iz}{2}\tan\gamma\right)^m}$$

holds valid. Here, $A_0=1$, $A_1=\frac{1}{8}+\frac{5}{24}\cot^2\gamma$, ...

From this result, we obtain the formula

$$2K_z(a) = \sqrt{\frac{2\pi}{y}}e^{\frac{1}{2}\pi iz}\left(\frac{y}{\pi}\right)^x e^{i\phi}\left(\frac{2\pi}{a}\right)^z \exp\left(f(a)\right)B\left(z\right)\left(1+O\left(y^{-\frac{3}{2}\delta}\right)\right). \quad (14)$$

The absolute value of $\exp(f(a))$ is decreasing whether x is positive or negative and its approximated value for $a=y+y^{\frac{1}{4}}$ is 2^{-x}, and is

$$\exp\{(-\sqrt{r^2-1}+\tan^{-1}\sqrt{r^2-1})y\}$$

for $a=ry$ $(r>1)$. And in the inteval I_5

$$B(z) = \frac{\sqrt{i}}{\sqrt[4]{r^2-1}}(1+O(y^{-\frac{1}{6}})).$$

In the last place, we obtain, by expanding the integrand in series,

$$2R(z) = \int_{-\infty}^{\infty} \frac{e^{-zu}}{e^{\frac{u}{4}}+e^{-\frac{u}{4}}}\, du = 2\pi \sec 2\pi z .$$

Accordingly

$$R(z) = O(e^{-2\pi y}).$$

§3. The absolute value of $\Pi_{p\leqq N}(1-p^{-z})^{-1}=\sum' n^{-z}$ is greater than a fixed positive number in $x\geqq\frac{1}{2}$. Then we divide the series $\sum' n^{-z}$ into two parts as

$$S \equiv \sum' n^{-z} = \sum'_{n\leqq k} + \sum'_{k<n} \equiv S_1+S_2 ,$$

and can make k so great that the absolute value of S_1 is sufficiently near to that of S and the absolute value of S_2 is sufficiently near to zero. Corresponding to this division of S, the function $Y(z)$ can be divided as

$$Y(z) = Y_1(z) + Y_2(z) - R(z).$$

Moreover, if we put as

$$2\sum' K_z(2n^2\pi) = G(z) ,$$

this function is also divided as

$$G(z) = G_1(z)+G_2(z) = 2\sum'_{n\leqq k}K_z(2n^2\pi)+ 2\sum'_{n>k}K_z(2n^2\pi) .$$

And the relations

$$Y(z) = G\left(z+\frac{1}{4}\right)+G\left(z-\frac{1}{4}\right)-R(z) , \quad (15)$$

$$Y_1(z)=G_1\left(z+\frac{1}{4}\right)+G_1\left(z-\frac{1}{4}\right) \quad (16)$$

hold valid. And, as the function $G(z)$ is even, we can see that

$$Y(iy) = 2\Re G\left(\frac{1}{4}+iy\right) - R(iy)\,, \tag{17}$$

$$Y_1(iy) = 2\Re G_1\left(\frac{1}{4}+iy\right). \tag{18}$$

Lemma 1. *The function $Y_1(z)$ has infinitely many zero points on the imaginary axis. And they do not accumulate in the finite region.*

Proof. By using the asymptotic formula (10), we obtain

$$G_1\left(\frac{1}{4}+iy\right) \sim \sqrt{\frac{2\pi}{y}}\, e^{-\frac{1}{2}\pi y+\frac{1}{8}\pi i}\left(\frac{y}{\pi}\right)^{\frac{1}{4}} e^{i\phi} {\sum_{n\leqq k}}' \left\{ n^{-\frac{1}{2}-2iy} + \left(\frac{y}{\pi}\right)^{-\frac{1}{2}} e^{-2i\phi} n^{\frac{1}{2}+2iy}\right\}. \tag{19}$$

The second series in the right hand side is negligible, compared with the first series when y is sufficiently large. So that we can say that the argument of $G_1\left(\frac{1}{4}+iy\right)$ increases infinitely when y tends to infinity. On the other hand, from (18), the purely imaginary zero points of $Y_1(z)$ are obtained as the points satisfying the condition

$$\arg G_1\left(\frac{1}{4}+iy\right) = \frac{\pi}{2} \times \text{ odd number.}$$

Accordingly the function $Y_1(z)$ has infinitely many purely imaginary zero points. These zero points do not accumulate in the finite region, because the function $Y_1(z)$ is an integral function. Q. E. D.

Lemma 2. *Let ε be any small positive number, then the function $Y_1(z)$ has no zero point in the strip $\varepsilon \leqq x \leqq \frac{1}{4}$, when y is sufficiently large.*

Proof. From (10) and (16), we obtain

$$Y_1(z) \sim \sqrt{\frac{2\pi}{y}}\, e^{\frac{\pi i z}{2}} \left(\frac{y}{\pi}\right)^{\frac{1}{4}} \sum{}' \frac{1}{\sqrt{n}} \left\{\left(\frac{y}{\pi}\right)^{x} e^{i(\phi+\frac{\pi}{8})} n^{-2z} + \left(\frac{y}{\pi}\right)^{-x} e^{-i(\phi+\frac{\pi}{8})} n^{2z}\right\},$$

$$|x| \leqq \frac{1}{4}\,. \tag{20}$$

Accordingly

$$Y_1(z) \sim \sqrt{\frac{2\pi}{y}}\, e^{\frac{\pi i z}{2}} \left(\frac{y}{\pi}\right)^{x+\frac{1}{4}} e^{i(\phi+\frac{\pi}{8})} \sum{}' n^{-2z-\frac{1}{2}}, \quad \varepsilon \leqq x \leqq \frac{1}{4}\,. \tag{21}$$

While the asymptotic formula (20) holds valid uniformly in the strip $|x| \leqq \frac{1}{4}$, the formula (21) does not hold valid exactly, unless the larger we make y according as the smaller ε is. And the right hand side of the formula (21) has no zero point in the strip $\varepsilon \leqq x \leqq \frac{1}{4}$. So that the function $Y_1(z)$ has no zero point in the same strip, when y is sufficiently large.

Lemma 3. *Let ε be any small positive number, then there are infinitely many y's, satisfying the simultaneous inequalies*

$$|y \log n| < \varepsilon \pmod{2\pi}, \quad \left|\phi(y) + \frac{\pi}{8}\right| < \varepsilon \pmod{2\pi}, \tag{22}$$

where the number of the integers n is finite. And there is such y that its magnitude is greater than an arbitrary positive number.

Proof. Let x_1, x_2, ..., x_N be N given positive numbers, y be an arbitrary positive integer. Then there are an integer t smaller that y^N and N integers β_1, β_2, ..., β_N, as satisfy the inequalities

$$|tx_q-\beta_q|<\frac{1}{y},\quad q=1,2,\ldots\ldots,N.$$

(Dirichlet's Theorem)

Put, in these inequalities, $y/2\pi$, $\log q$ and $2\pi/\varepsilon$ in place of t, x_q and y respectively, then we have the first inequalities of (22). And as $\phi\sim y\log y$, ϕ varies very quickly when the sufficiently large y varies. So it is possible that the inequalities (22) hold valid simultaneously by making y vary within the limit of validity of the first inequalities of (22). That is, at least one y that satisfies (22) exists. Next, let y_i's be solutions of

$$|y_i\log n|<\frac{\varepsilon}{2^{i+1}}(\mathrm{mod}\ 2\pi),\ i=1,2,3,\ldots\ldots \tag{23}$$

In the above Dirichlet's Theorem, t is an integer and non zero, so that the solutions y_i's of (23) are all greater than 1. Then the numbers

$$y=y_1,\ y_1+y_2,\ y_1+y_2+y_3,\ \ldots\ldots$$

all satisfy the first inequalities of (22), and this sequence tends to infinity. We can, here, make y vary slightly in order that the second of (22) also may hold valid. Q. E. D.

Theorem 6. *The function $Y_1(z)$ has only purely imaginary zero points in the strip $|x|\leqq\frac{1}{4}$, when the ordinate of y is sufficiently large. And the number of these zero points whose ordinates are smaller than y, is approximately*

$$\frac{y}{\pi}\log\frac{y}{\pi}+O(y)\ .$$

Proof. Consider a rectangle R whose vertices are $A_0\left(\frac{1}{8},\ y_0\right)$, $A_n\left(\frac{1}{8},\ y_n\right)$ $B_n\left(-\frac{1}{8},\ y_n\right)$ and $B_0\left(-\frac{1}{8},\ y_0\right)$, where $y_0<y_n$. And determine y_0 and y_n in the following way. First, let y_0 be sufficiently large so that the asymptotic formula (21) may hold valid sufficiently exactly on A_0A_n. Secondly, let y_0 and y_n be solutions of the inequalities (22) of the Lemma 3. y_0 is fixed and y_n tends to infinity. Here, the n's in (22) are these which appear in the series of $Y_1(z)$.

Along the boundary of the rectangle R, let z make one round in the positive sense and we examine the variation of the arg $Y_1(z)$. Let the variation of $\phi+\arg\sum' n^{-2z-\frac{1}{2}}$ which arises when z moves along A_0A_n, be θ_n. Then the variation of $\arg Y_1(z)$ is written as $\theta_n+\varepsilon_n$, where the absolute value of ε_n is sufficiently small. When ε in (22) is sufficiently small, the variation of the argument of the series of (20) along A_nB_n is $-\frac{\pi}{8}+\varepsilon_n'$. In the same manner, the variation of $\arg Y_1(z)$ along B_0A_0 is $\frac{\pi}{8}+\varepsilon_0'$, where the absolute value is ε_0' is also sufficiently small. Accordingly the variation of $\arg Y_1(z)$ along the boundary of the rectangle R is

$$2\theta_n+2\varepsilon_n+\varepsilon_n'+\varepsilon_0'. \tag{24}$$

Next, we enumerate the number of the zero points of $Y_1(z)$ on the imaginary axis. As has already been stated in the proof of the Lemma 1, the purely imaginary zero points of $Y_1(z)$ are given as the points which satisfy the condition $\arg G_1\left(\frac{1}{4}+iy\right)=\pi/2\times$odd number. On the other hand, from (10), we obtain

$$G_1\left(\frac{1}{4}+iy\right)\sim\sqrt{\frac{2\pi}{y}}\,e^{-\frac{\pi}{2}y}e^{\frac{\pi i}{8}}\left(\frac{y}{\pi}\right)^{\frac{1}{4}}e^{i\phi}\sum{}'\,n^{-\frac{1}{2}-2iy}. \tag{25}$$

So, if we put the variation of $\phi+\arg\sum{}'\,n^{-\frac{1}{2}-2iy}$ which arises when y varies from y_0 to y_n, as ϑ_n, then the variation of $\arg G_1\left(\frac{1}{4}+iy\right)$ is $\vartheta_n+\varepsilon_n''$, where the absolute value of ε_n'' is sufficiently small. Accordingly, the number of purely imaginary zero points of $Y_1(z)$ which are contained in the rectangle R, is not smaller than $\frac{1}{\pi}(\vartheta_n+\varepsilon_n'')-1$. From this result and (24), we can conclude that the number μ of zero points which are not purely imaginary is not greater than

$$\frac{1}{2\pi}(2\theta_n+2\varepsilon_n+\varepsilon_n'+\varepsilon_0')-\frac{1}{\pi}(\vartheta_n+\varepsilon_n'')+1,$$

i. e.,

$$\mu\leqq\frac{1}{\pi}\{v.\,o.\,\arg\sum{}'\,n^{-\frac{3}{4}-2iy}-v.\,o.\,\arg\sum{}'\,n^{-\frac{1}{2}-2iy}+\varepsilon_n-\varepsilon_n''+(\varepsilon_n'+\varepsilon_0')/2\}+1,$$

where the notation "$v.\,o.$" means "variation of".

The arguments of $\sum{}'\,n^{-\frac{3}{4}-2iy}$ and $\sum{}'\,n^{-\frac{1}{2}-2iy}$ are sufficiently near to those of $\Pi_{p\leqq N}(1-p^{-\frac{3}{4}-2iy})^{-1}$ and $\Pi_{p\leqq N}(1-p^{-\frac{1}{2}-2iy})^{-1}$ respectively. So the difference $v.\,o.$ $\arg\sum{}'\,n^{-\frac{3}{4}-2iy}-v.\,o.\,\arg\sum{}'\,n^{-\frac{1}{2}-2iy}$ is sufficiently near to

$$\arg\Pi_{p\leqq N}(p^{\frac{1}{2}+2iy}-1)(p^{\frac{3}{4}+2iy}-1)^{-1}. \tag{26}$$

As the radii $p^{\frac{1}{2}}$ and $p^{\frac{3}{4}}$ of two circles whose centers are both the point 1 are greater than 1, the absolute value of $\arg(p^{\frac{1}{2}+2iy}-1)(p^{\frac{3}{4}+2iy}-1)^{-1}$ does not surpass $\pi/2$ for all values of y. So the value of (26) does not surpass a finite value for all values of y. Accordingly the $v.\,o.\,\sum{}'\,n^{-\frac{3}{4}-2iy}-v.\,o.\,\arg\sum{}'\,n^{-\frac{1}{2}-2iy}$ is finite for all value of y. So that μ is finite whatever large value of y_n we may adopt. As has already been shown in the Lemma 1, the function $Y_1(z)$ has infinitely many zero points in the strip $|x|\leqq\frac{1}{4}$. So that the zero points of $Y_1(z)$ whose ordinates are sufficiently large, are all purely imaginary.

When y is a sufficiently large positive number, the number of zero points whose ordinates are greater than zero and smaller than y, is approximately

$$\frac{1}{\pi}(\phi(y)+v.\,o.\,\sum{}'\,n^{-\frac{1}{2}-2iy}),$$

i. e.,

$$\frac{y}{\pi}\log\frac{y}{\pi}+O(y).$$ Q. E. D.

Next, let us consider the whole function $Y(z)$.

Lemma 4. *On the boundary of the rectangle R, we have $|Y_2(z)/Y_1(z)|<1$.*

Proof. We divide the summation $Y_2(z)$ into five parts: $2\pi \leq a < y - y^{\frac{1}{3}+\delta}$, $y - y^{\frac{1}{3}+\delta} \leq a < y - y^{\frac{1}{4}}$, $y - y^{\frac{1}{4}} \leq a \leq y + y^{\frac{1}{4}}$, $y + y^{\frac{1}{4}} \leq a < y + y^{\frac{1}{3}+\delta}$ and $y + y^{\frac{1}{3}+\delta} \leq a < \infty$, where $a = 2n^2\pi$, and name the summations corresponding these parts as $\sum_1$, $\sum_2$, $\sum_3$, $\sum_4$ and $\sum_5$ respectively.

We obtain

$$\sum_2 = \sqrt{\frac{2\pi}{y}}\, e^{\frac{\pi i z}{2}} \sum{}' \left\{ \left(\frac{y}{\pi}\right)^x e^{i\phi} n^{-2z}\, O\left(y^{\frac{1}{6}}\right) + \left(\frac{y}{\pi}\right)^{-x} e^{-i\phi} n^{2z}\, O\left(y^{\frac{1}{6}}\right) \right\}$$

from the asymptotic formula which has already been given. Then, on A_0A_n, we obtain

$$\sum_2 : Y_1(z) = \sum{}' \left\{ n^{-2z} O\left(y^{\frac{1}{6}}\right) + \left(\frac{y}{\pi}\right)^{-2x} e^{-2i\phi} n^{2z} O\left(y^{\frac{1}{6}}\right) \right\}$$

$$: \left(\frac{y}{\pi}\right)^{\frac{1}{4}} e^{\frac{\pi i}{8}} \sum{}' n^{-2z-\frac{1}{2}}.$$

And when y becomes sufficiently large, the number of terms of $\sum_2$ decreases sufficiently. So the numerator in the right hand side can be considered as of order $O(y^{\frac{1}{24}})$. And the absolute value of $\sum' n^{-2z-\frac{1}{2}}$ in the denominator is greater than a fixed positive number. Accordingly the above ratio is of order $O(y^{-\frac{5}{24}})$. On A_0B_0 and A_nB_n, we obtain

$$\sum_2 : Y_1(z) = \sum{}' \left\{ \left(\frac{y}{\pi}\right)^x e^{i\phi} n^{-2z} O\left(y^{\frac{1}{6}}\right) + \left(\frac{y}{\pi}\right)^{-x} e^{-i\phi} n^{2z} O\left(y^{\frac{1}{6}}\right) \right\}$$

$$: \sum{}' \left(\frac{y}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{n}} \left\{ \left(\frac{y}{\pi}\right)^x e^{i(\phi+\frac{\pi}{8})} n^{-2z} + \left(\frac{y}{\pi}\right)^{-x} e^{-i(\phi+\frac{\pi}{8})} n^{2z} \right\}.$$

The denominator in the right hand side is sufficiently near to

$$\sum{}' \left(\frac{y}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{n}} \left\{ \left(\frac{y}{\pi}\right)^x n^{-2x} + \left(\frac{y}{\pi}\right)^{-x} n^{2x} \right\},$$

so that it is of order $O(y^{\frac{1}{4}+|x|})$. Accordingly, the above ratio is of order $O(y^{-\frac{1}{12}-|x|})$. Eventually, on the boundary of the rectangle R, the absolute value of $\sum_2$ is negligible, compared with that of $Y_1(z)$.

In the same manner, $\sum_3$ and $\sum_4$ are both of order $e^{-\frac{\pi y}{2}} O(y^{-\frac{1}{3}})$, so that they are also negligible, compared with $Y_1(z)$ on the boundary of the rectangle R.

In the next place, let us consider $\sum_1$ and $\sum_5$. From the asymptotic formula (9). We obtain

$$\sum_1 = \sqrt{\frac{2\pi}{y}}\, e^{\frac{\pi i z}{2}} \left(\frac{y}{\pi}\right)^{\frac{1}{4}} \sum{}' \frac{1}{\sqrt{n}} \left\{ \left(\frac{y}{\pi}\right)^x e^{i\phi} n^{-2z} O(1) + \left(\frac{y}{\pi}\right)^{-x} e^{-i\phi} n^{2z}\, O(1) \right\} \left(\sqrt[4]{1-r^2}\right)^{-1}.$$

The interval between terms of this series grow greater in the similar manner as a geometrical series does. So the absolute value of $\sum n^{-2z-\frac{1}{2}} / \sqrt[4]{1-r^2}$ is of $O\left(\frac{1}{\sqrt{k}}\right)$, when $x \geq 0$. In the same way $y^{-2x} \sum n^{2z-\frac{1}{2}} / \sqrt[4]{1-r^2} = O\left(\frac{1}{\sqrt{k}}\right)$. Accordingly, if k is sufficiently large, the absolute value of $\sum_1$ is very small, compared with that of $Y_1(z)$ on the boundary of R. In the same way, we can conclude that the absolute value of $\sum_5$ is also negligible, compared with that of $Y_1(z)$. $R(z)$ is also negligible, Consequently, from Rouche's Theorem, the number of zero points of $Y(z)$, contained in R is equal to that of $Y_1(z)$.

The purely imaginary zero points of $Y(z)$ are such y's that satisfy the condition

$$\arg G\left(\frac{1}{4}+iy\right)=\frac{\pi}{2}\times \text{odd number} + O\left(e^{-2\pi y}\alpha^{-1}\right) ,$$

where α is the absolute value of $G\left(\frac{1}{4}+iy\right)$ and its magnitue is $O(y^{-\frac{1}{4}}e^{-\frac{\pi y}{2}})$. And, on the abscissa $\frac{1}{4}$, the asymptotic formula of $G\left(\frac{1}{4}+iy\right)$ is sufficiently near to that of $G_1\left(\frac{1}{4}+iy\right)$. So we obtain the following Theorem.

Theorem 7. *The function $Y(z)$ has only purely imaginary zero points in the domain in which y is sufficiently large, and the number of the zero points in $0\leqq y\leqq Y$ is*

$$\frac{Y}{\pi}\log\frac{Y}{\pi}+O(Y).$$

References

1) Pólya, G: Acta Math., 1926.
2) Riemann, B: Werke.
3) Watson, G. N: Theory of Bessel Functions (Cambridge), 1922