Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 1, No. 2, Series A

On the Distribution of Zero Points of a Function which is related to Riemann's ξ-Function

By Osamu MIYATAKE

(Received May 31, 1951)

I wish to show that the distribution of zero points of a function which is related to Riemann's ξ-function, is in a state just as Riemann expected concerning ξ-function. §1. Introduction. §2. Definitions of functions necessary to the proofs, and relations between them. §3. Necessary Theorems 1, 2, 3, 4 and 5. §4. Results deduced from the Theorems 1, 2, 3, 4 and 5. §5. Proof of the conclusion. §6. Proofs of the Theorems 1 and 2.

§1. Riemann's ξ-function[1] is defined as

$$\xi(iz)=\frac{1}{2}\left(z^2-\frac{1}{4}\right)\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\left(\frac{1}{2}z+\frac{1}{4}\right)\zeta\left(z+\frac{1}{2}\right).$$

This is also represented as

$$\xi(z)=2\int_0^\infty \phi(t)\cos zt\,dt,$$

where $\phi(t)$ is even with respect to t and is given by the following series:

$$\phi(t)=2\pi e^{\frac{5}{2}t}\sum_{n=1}^{\infty}(2\pi e^{2t}n^2-3)n^2\exp(-n^2\pi e^{2t}).$$

Prof. Pólya[2] brought forth a function

$$\xi^*(z)=2\int_0^\infty \phi^*(t)\cos zt\,dt,$$

where

$$\phi^*(t)=4\pi^2(e^{\frac{9}{2}t}+e^{-\frac{9}{2}t})\exp(-\pi(e^{2t}+e^{-2t})),$$

and proved that $\xi^*(z)$ has infinitely many zero points and they are all on the real axis.

In order to develop his idea, let us consider as follows. By using an even function $\psi(t)$, we define a function $G(z)$ as

$$G(z)=\int_{-\infty}^{\infty}\psi(t)\,e^{zt}\,dt, \tag{1}$$

then $G(z)$ is even with respect to z. Here we assume that $G(z)$ and $\xi(z)$ are in the following relation:

$$\xi(z)=2\pi^2\left\{G\left(\frac{1}{2}iz-\frac{9}{4}\right)+G\left(\frac{1}{2}iz+\frac{9}{4}\right)\right\}. \tag{2}$$

In order that this relation may be satisfied, the function $\psi(t)$ which has not yet been determined, must satisfy the relation

$$4\pi^2\psi(2t)(e^{-\frac{9}{2}t}+e^{\frac{9}{2}t})=\phi(t).$$

From this, we obtain

$$\psi(t)=\sum_{n=1}^{\infty}\left\{\left(1-\frac{3}{2\pi n^2}e^{-t}\right)\Big/\left(1+e^{-\frac{9}{2}t}\right)\right\}n^4\exp(-n^2\pi e^t), \tag{3}$$

which can be written as

$$\psi(t)=\sum_{n=1}^{\infty}n^4\exp(-n^2\pi(e^t+e^{-t}))(1-R(n^2\pi,t)), \tag{4}$$

where $R(n^2\pi, t)$'s are even and

$$R(n^2\pi,t)=1-\left\{\left(1-\frac{3}{2\pi n^2}\right)\Big/\left(1+e^{-\frac{9}{2}t}\right)\right\}\exp(n^2\pi e^{-t}),$$

when t is positive. These functions $R(n^2\pi, t)$'s tend to zero when t tends to plus or minus infinity, but the tendency of going to zero is not uniform with respect to n.

Now we define functions

$$\psi_1(t)=\sum_{n=1}^{\infty}n^4\exp(-n^2\pi(e^t+e^{-t})),$$
$$\psi_2(t)=\sum{}' n^4\exp(-n^2\pi(e^t+e^{-t})),$$
$$\psi_3(t)=\sum_{n=1}^{\infty}n^3\exp(-n^2\pi(e^t+e^{-t})),$$

where the summation $\sum'$ of $\psi_2(t)$ is identical with that of

$$\zeta_2(z)\equiv\prod_{p\leq N}(1-p^{-z})^{-1}=\sum{}' n^{-z},$$

N being a certain fixed positive constant, and p's are prime numbers.

In the same way as the functions $G(z)$ and $\xi(z)$ were constructed from $\psi(t)$, functions $G_1(z)$, $\xi_1(t)$, $G_2(z)$, $\xi_2(t)$ and $G_3(z)$, $\xi_3(t)$ are constructed from $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ respectively. Hereafter, we always put as $z=x+iy$, and let A be a fixed positive constant and δ be a positive constant smaller than 1/5. $\Re f$ and $\Im f$ denote the real and imaginary parts of a function f respectively. Then we obtain the following results.

Theorem I. *The function $\xi_2(z)$ has infinitely many zero points in a strip $-A\leq y\leq A$. And these zero points are all on the real axis in a domain, where the absolute value of x is large enough. And the number of the zero points which are in $0\leq x\leq T$ is*

$$T/2\pi\cdot\log T/2\pi-T/2\pi+0(1).$$

Theorem II. *The function $\xi_3(z)$ has infinitely many zero points in a strip $-A\leq y\leq A$. And these zero points are all on the real axis in a domain, where the absolute value of x is large enough. And the number of the zero points which* *are in* $0 \leqq x \leqq T$ *is*

$$T/2\pi \cdot \log T/2\pi - T/2\pi + 0\,(1).$$

In the present paper, we shall prove the Theorem I. With respect to $\xi_1(z)$, we obtain no result. The Theorem II can be proved on the whole as the Theorem I is done.

§2. The following functions and the relations between them are often necessary on the way to the proofs. For details, see, for example, Watson [3].

1. $$J_\nu(z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{1}{2}z\right)^{\nu+2m} / m!\,\Gamma(\nu+m+1).$$

When k is any constant

$$J_\nu(z\,e^{k\pi i}) = e^{k\nu\pi i} J_\nu(z),$$
$$J_{-\nu}(z\,e^{k\pi i}) = e^{-k\nu\pi i} J_{-\nu}(z).$$

2. $$I_\nu(z) = \sum_{m=0}^{\infty} \left(\frac{1}{2}z\right)^{\nu+2m} / m!\,\Gamma(\nu+m+1).$$

When k is any constant,

$$I_\nu(z\,e^{k\pi i}) = e^{k\nu\pi i} I_\nu(z).$$

$$I_\nu(z) = \begin{cases} e^{-\frac{1}{2}\nu\pi i} J_\nu(z\,e^{\frac{\pi i}{2}}) & \left(-\pi < \arg z \leqq \frac{1}{2}\pi\right) \\ e^{\frac{3}{2}\nu\pi i} J_\nu(z\,e^{-\frac{3}{2}\pi i}) & \left(\frac{1}{2}\pi < \arg z \leqq \pi\right). \end{cases}$$

3. $$H_\nu^{(1)}(z) = \left\{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)\right\} / i \sin \nu\pi,$$
$$H_\nu^{(2)}(z) = \left\{J_{-\nu}(z) - e^{\nu\pi i} J_\nu(z)\right\} / -i \sin \nu\pi.$$

When $|z|$ is sufficiently small and ν is real and positive,

$$H_\nu^{(1)}(z) \sim 1/\Gamma(-\nu+1)\, i \sin \nu\pi \cdot \left(\frac{1}{2}z\right)^{-\nu}. \tag{5}$$

When ν is finite and $|z|$ increases,

$$H_\nu^{(1)}(z) \sim (2/\pi z)^{\frac{1}{2}} \exp\left(i\left(z - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right)\right). \quad (-\pi < \arg z < 2\pi) \tag{6}$$

4. $$K_\nu(z) = \frac{1}{2}\pi \left\{I_{-\nu}(z) - I_\nu(z)\right\} / \sin \nu\pi$$
$$= \frac{1}{2}\pi i\, e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(iz)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp(-z \cosh t - \nu t)\, dt. \tag{7}$$

When k is any constant,

$$K_\nu(z\,e^{k\pi i}) = e^{-k\nu\pi i} K_\nu(z). \tag{8}$$

When ν is finite and $|z|$ increases,

$$K_\nu(z) \sim (\pi/2z)^{\frac{1}{2}}\, e^{-z}. \quad \left(|\arg z| < \frac{3}{2}\pi\right) \tag{9}$$

This function occupies a central position in the proofs hereafter.

5. When $z\neq 0$ and μ_1, μ_2 are such arbitrary real numbers as $-\frac{1}{2}\pi<\mu_1, \mu_2<\frac{1}{2}\pi$,

$$H_\nu^{(1)}(z)=\frac{1}{\pi i}\int_{-\infty+i(\arg z+\mu_1)}^{\infty+i(\pi-\arg z+\mu_2)} \exp(z\sinh t-\nu t)\,dt.$$

§3. We are going to state the Theorems 1, 2, 3, 4 and 5 below. The proofs of them can be obtained by using the method of the steepest descent and will be given in §6. These results have many similar points to Watson [3], pp. 235–268.

Theorem 1. *Put as $z=ia\cosh\gamma$, then the following asymptotic expansion can be obtained when y tends to plus infinity*:

$$H_z^{(1)}(ia)\sim\left(\exp\left(z(\tanh\gamma-\gamma)-\frac{1}{4}\pi i\right)\Big/\sqrt{-\frac{1}{2}\pi iz\tanh\gamma}\,\sum_{m=0}^{\infty}A_m\Gamma\left(m+\frac{1}{2}\right)\Big/\Gamma\left(\frac{1}{2}\right)\right.$$
$$\cdot\left(-\frac{1}{2}z\tanh\gamma\right)^m$$
$$+\left(\exp\left(-z(\tanh\gamma-\gamma)-\frac{\pi i}{4}\right)\Big/\sqrt{\frac{1}{2}\pi iz\tanh\gamma}\,\sum_{m=0}^{\infty}A_m\Gamma\left(m+\frac{1}{2}\right)\Big/\Gamma\left(\frac{1}{2}\right)\right.$$
$$\cdot\left(\frac{1}{2}z\tanh\gamma\right)^m.$$

This expansion holds valid uniformly for x, y and a such as $0<a\leqq y-y^{\frac{1}{3}+\delta}$, $-A\leqq x\leqq A$. $\gamma=\alpha+i\beta$ satisfies the following condition

$$\alpha\leqq 0,\quad 0\leqq\beta<\frac{1}{2}\pi\qquad\text{for}\quad x\geqq 0,$$
$$\alpha\leqq 0,\quad 0\geqq\beta>-\frac{1}{2}\pi\qquad\text{for}\quad x\leqq 0.$$

And
$$A_0=1,\quad A_1=\frac{1}{8}-\frac{5}{24}\coth^2\gamma,\quad A_2=\frac{3}{128}-\frac{77}{576}\coth^2\gamma-\frac{385}{3456}\coth^4\gamma,\ldots$$

Theorem 2. *Put as $z=ia\cosh\gamma$, then the asymptotic formula*

$$H_z^{(1)}(ia)\sim-\frac{2}{\sqrt{3}\,\pi}\exp(z(\tanh\gamma-\gamma))\tanh\gamma\exp\left(\frac{z}{3}\tanh^3\gamma\right)K_{\frac{1}{3}}\left(\frac{z}{3}\tanh^3\gamma\, e^{-i\pi}\right)$$
$$+\frac{2}{\sqrt{3}\,\pi}\exp(-z(\tanh\gamma-\gamma))\tanh\gamma\exp\left(-\frac{z}{3}\tanh^3\gamma\right)K_{\frac{1}{3}}\left(-\frac{z}{3}\tanh^3\gamma\right)$$

holds valid when y tends to infinity, where γ is in the same domain as in the Theorem 1. *And $y-y^{\frac{1}{3}+\delta}\leqq a\leqq y-y^{\frac{1}{4}}$.*

Theorem 3. *Put as $z=ia(1-\varepsilon)$, then the asymptotic formula*

$$H_z^{(1)}(ia)\sim-\frac{2}{3\pi}\sum_{m=0}^{\infty}e^{\frac{2}{3}(m+1)\pi i}B_m(\varepsilon ia)\sin\frac{1}{3}(m+1)\pi\Gamma\left(\frac{1}{3}m+\frac{1}{3}\right)\Big/\left(\frac{1}{6}ia\right)^{\frac{1}{3}\cdot(m+1)}$$

holds valid when y tends to infinity. Here $B_0(w)=1$, $B_1(w)=w$, $B_2(w)=\frac{1}{2}w^2-\frac{1}{20}$, ..., and $y-y^{\frac{1}{4}}\leqq a<y+y^{\frac{1}{4}}$.

Theorem 4. *In the interval $y+y^{\frac{1}{4}}\leqq a<y+y^{\frac{1}{3}+\delta}$, put as $z=ia\cos\gamma$, then the*

asymptotic formula

$$H_z^{(1)}(ia)\sim\frac{e^{\frac{\pi i}{6}}}{\sqrt{3}}\exp(iz(\tan\gamma-\gamma))\tan\gamma\exp\left(-\frac{1}{3}iz\tan^3\gamma\right)H_{\frac{1}{3}}^{(1)}\left(\frac{1}{3}z\tan^3\gamma\right)$$

holds valid when y tends to infinity. If we put as $\gamma=\alpha+i\beta$, *the position of* γ *is decided as*

$$0\leqq\alpha,\ 0\leqq\beta<\frac{1}{2}\pi \qquad \text{for} \qquad x\geqq0,$$

$$0\leqq\alpha,\ 0\geqq\beta>-\frac{1}{2}\pi \qquad \text{for} \qquad x\leqq0.$$

Theorem 5. *In the interval* $y+y^{\frac{1}{3}+\delta}\leqq a<\infty$, *put as* $z=ia\cos\gamma$, *then the asymptotic formula*

$$H_z^{(1)}(ia)\sim\exp\left(iz(\tan\gamma-\gamma)-\frac{1}{4}\pi i\right)\Big/\sqrt{\frac{1}{2}\pi z\tan\gamma}\cdot\sum_{m=0}^{\infty}A_m\Gamma\left(m+\frac{1}{2}\right)\Big/\Gamma\left(\frac{1}{2}\right)\times\left(\frac{1}{2}iz\tan\gamma\right)^m$$

holds valid when y tends to infinity. Here, $A_0=1$, $A_1=\frac{1}{8}+\frac{5}{24}\cot^2\gamma,\ldots$.

§4. *From the Theorem* 1, *we obtain the formula*

$$\left.\begin{aligned}2K_z(a)\sim\sqrt{\frac{2\pi}{y}}\,e^{\frac{\pi iz}{2}}\Big\{&\left(\frac{y}{\pi}\right)^x e^{i\phi}\left(\frac{2\pi}{a}\right)^z\exp(f(a))\\&+\left(\frac{y}{\pi}\right)^{-x}e^{-i\phi}\left(\frac{2\pi}{a}\right)^{-z}\exp(-f(a))\Big\}B(z)(1+0(y^{-\frac{3}{2}\delta})),\end{aligned}\right\} \qquad (10)$$

where $\phi=y\log y/\pi-y-\frac{1}{4}\pi$, $B(z)=\sqrt{z}:\sqrt[4]{z^2+a^2}$, *and*

$$f(a)=z-\sqrt{z^2+a^2}+z\log\frac{1}{i}(z+\sqrt{z^2+a^2})-z\log\frac{2z}{i}.$$

Proof. From the definition of $K_z(a)$ and the Theorem 1, we obtain

$$\begin{aligned}2K_z(a)\sim\pi i\,e^{\frac{\pi iz}{2}}\Big\{&\exp\left(z(\tanh\gamma-\gamma)-\frac{1}{4}\pi i\right)\Big/\sqrt{-\frac{1}{2}\pi iz\tanh\gamma}\cdot(1+Q)\\&+\exp\left(-z(\tanh\gamma-\gamma)-\frac{1}{4}\pi i\right)\Big/\sqrt{\frac{1}{2}\pi iz\tanh\gamma}\cdot(1-Q),\end{aligned}$$

where $\qquad Q=-1/8\,z\cdot\coth\gamma-5/24\,z\cdot\coth^3\gamma.$

And from (5), we obtain

$$\tanh\gamma=-\sqrt{z^2+a^2}/z, \qquad \gamma=-\log\frac{1}{ia}(z+\sqrt{z^2+a^2}). \qquad (11)$$

On the other hand,

$$\begin{aligned}&\exp\left(-\sqrt{z^2+a^2}+z\log\frac{1}{ia}(z+\sqrt{z^2+a^2})-\frac{1}{4}\pi i\right)\\&=\left(\frac{2\pi}{a}\right)^z\exp\left(-z+z\log\frac{z}{i}-z\log\pi-\frac{1}{4}\pi i+f(a)\right),\end{aligned}$$

$$\exp\left(\sqrt{z^2+a^2}-z\log\frac{1}{ia}(z+\sqrt{z^2+a^2})-\frac{1}{4}\pi i\right)$$
$$=\left(\frac{2\pi}{a}\right)^{-z}\exp\left(z-z\log\frac{z}{i}+z\log\pi-\frac{1}{4}\pi i-f(a)\right).$$

And

$$\pi i\exp\left(-z+z\log\frac{z}{i}-z\log\pi-\frac{1}{4}\pi i\right)\Big/\sqrt{\frac{1}{2}\pi iz}=\sqrt{\frac{2\pi}{y}}\left(\frac{y}{\pi}\right)^{x}e^{i\phi}\left(1+0\left(\frac{1}{y}\right)\right),$$
$$\pi i\exp\left(z-z\log\frac{z}{i}+z\log\pi-\frac{1}{4}\pi i\right)\Big/\sqrt{-\frac{1}{2}\pi iz}=\sqrt{\frac{2\pi}{y}}\left(\frac{y}{\pi}\right)^{-x}e^{-i\phi}\left(1+0\left(\frac{1}{y}\right)\right), \quad (12)$$

$\tanh\gamma=0(y^{-\frac{1}{3}+\frac{\delta}{2}})$, $\quad z\tanh^3\gamma=0(y^{\frac{3}{2}\delta})$.

Using the above ones, we directly obtain the result (10).

2. *When a is finite and y tends to infinity,*

$$2K_z(a)\sim\sqrt{\frac{2\pi}{y}}e^{\frac{\pi iz}{2}}\left\{\left(\frac{y}{\pi}\right)^{x}e^{i\phi}\left(\frac{2\pi}{a}\right)^{z}+\left(\frac{y}{\pi}\right)^{-x}e^{-i\phi}\left(\frac{2\pi}{a}\right)^{-z}\right\}(1+0(y^{-1})). \quad (13)$$

Proof. When a is finite and y tends to infinity,

$$\sqrt{z^2+a^2}=z+0(z^{-1}),$$
$$\gamma=\log a-\log 2y-i\tan x/y+0(y^{-2}),$$
$$\tan^{-1}x/y=x/y+0(y^{-3}),\quad \pi iz/2\cdot\tanh\gamma=-\pi iz/2+0(z^{-2}).$$

Using the above, we obtain

$$\pi i\exp\left(z(\tanh\gamma-\gamma)-\frac{1}{4}\pi i\right)\Big/\sqrt{-\frac{1}{2}\pi iz\tanh\gamma}=\sqrt{\frac{2\pi}{y}}\left(\frac{y}{\pi}\right)^{x}e^{i\phi}\left(\frac{2\pi}{a}\right)^{z}(1+0(y^{-1})),$$
$$\pi i\exp\left(-z(\tanh\gamma-\gamma)-\frac{1}{4}\pi i\right)\Big/\sqrt{\frac{1}{2}\pi iz\tanh\gamma}=\sqrt{\frac{2\pi}{y}}\left(\frac{y}{\pi}\right)^{-x}e^{-i\phi}\left(\frac{2\pi}{a}\right)^{-z}(1+0(y^{-1})).$$

Thus the result (13) is obtained.

3. *The absolute value of* $\exp(f(a))$ *is decreasing or increasing with respect to a in the interval* $(0, y-y^{\frac{1}{4}})$ *according as x is positive or negative respectively, and*

$$|\exp(f(a))|\sim 2^{-x}\qquad\text{for}\qquad a=y-y^{\frac{1}{4}}. \quad (14)$$

Especially, when a is fixed and y tends to infinity,

$$\exp(f(a))=1+0(z^{-1}).$$

Proof. From $\quad df/da=-a(z+\sqrt{z^2+a^2})^{-1},$

we can easily see that the real part of df/da is negative or positive according as x is positive or negative respectively. Accordingly, the absolute value of $\exp(f(a))$ is decreasing or increasing with respect to a according as x is positive or negative respectively. And from the expansion

$$f(a)=z-\sqrt{z^2+a^2}+z\log(1+\sqrt{z^2+a^2}/z)-z\log 2$$
$$=z-\frac{1}{2}(z^2+a^2)/z-\ \dots\ -z\log 2,$$

we obtain

$$\Re(f(a))_{a=y-y^{\frac{1}{4}}} \sim -x \log 2.$$

When a is fixed and y tends to infinity, $f(a)=0(1/z)$, and we obtain the desired result

$$\exp(f(a))=1+0(1/z).$$

4. *From the Theorem 2, we obtain the following asymptotic formula.*

$$\left.\begin{aligned} 2K_z(a) \sim \sqrt{\frac{2\pi}{y}}\, e^{\frac{\pi i z}{2}} \Big\{ \Big(\frac{y}{\pi}\Big)^x e^{i\phi}(2\pi/a)^z \exp(f(a))\, 0(y^{\frac{1}{6}}) \\ + \Big(\frac{y}{\pi}\Big)^{-x} e^{-i\phi}(2\pi/a)^{-z} \exp(-f(a))\, 0(y^{\frac{1}{6}}) \Big\}, \end{aligned}\right\} \quad (15)$$

where Landau's notation 0 *is uniformly bounded with respect to all values of* a *in* $(y-y^{\frac{1}{3}+\delta},\ y-y^{\frac{1}{4}})$.

Proof. Using the relation (11), we obtain

$$\left.\begin{aligned} H_z^{(1)}(ia) \sim (2\pi/a)^z \exp\Big(-z+z\log z/i - z\log\pi - \frac{1}{4}\pi i + f(a)\Big)\Big/\sqrt{\pi i z/2} \\ \times(-2i)\sqrt{z/6\pi}\tanh\gamma \exp\Big(\frac{z}{3}\tanh^3\gamma\Big) K_{\frac{1}{3}}\Big(\frac{z}{3}\tanh^3\gamma\, e^{-i\pi}\Big) \\ +(2\pi/a)^{-z}\exp\Big(z - z\log z/i + z\log\pi - \frac{1}{4}\pi i - f(a)\Big)\Big/\sqrt{-\pi i z/2} \\ \times 2i\sqrt{-z/6\pi}\tanh\gamma \exp\Big(-\frac{z}{3}\tanh^3\gamma\Big) K_{\frac{1}{3}}\Big(-\frac{z}{3}\tanh^3\gamma\Big). \end{aligned}\right\} \quad (16)$$

When $a=y-y^{\alpha}$, $\tanh\gamma=0(y^{-\frac{1}{2}+\frac{\alpha}{2}})$. So that $z\tanh^3\gamma/3$ tends to infinity or to zero when y tends to infinity, according as $a=y-y^{\frac{1}{3}+\delta}$ or $a=y-y^{\frac{1}{4}}$ respectively. That is, the absolute value of $z\tanh^3\gamma/3$ varies from a sufficiently large value to a sufficiently small value, when a moves from $y-y^{\frac{1}{3}+\delta}$ to $y-y^{\frac{1}{4}}$. When it is sufficiently large, we use the formula (8) and (9):

$$K_\nu\Big(\frac{z}{3}\tanh^3\gamma\, e^{-i\pi}\Big) \sim \text{const}\cdot\exp\Big(-\frac{z}{3}\tanh^3\gamma\Big)\Big/\sqrt{z}\tanh^{\frac{3}{2}}\gamma.$$

Thus

$$(-2i)\sqrt{z/6\pi}\tanh\gamma \exp(z/3\cdot\tanh^3\gamma)\, K_{\frac{1}{3}}(z/3\cdot\tanh^3\gamma\, e^{-i\pi}) = 0(y^{\frac{1}{6}-\frac{\delta}{4}}). \quad (17)$$

When the absolute value is sufficiently small, we use the formula (5):

$$K_\nu(z/3\cdot\tanh^3\gamma\, e^{-i\pi}) \sim \text{const.}\ 1/\sqrt[3]{z}\tanh\gamma,$$

so that, the left hand side of (17) is of $0(y^{\frac{1}{6}})$.

In the same way we can prove that the fourth term of (16) is of $0(y^{\frac{1}{6}})$. Using these results and (12), we obtain the formula (15). In this formula Landau's notation 0 is uniformly bounded with respect to all values of a.

5. *When* $y+y^{\frac{1}{4}} \leqq a < y+y^{\frac{1}{3}+\delta}$, *we obtain the following formula*

$$2K_z(a)\sim\sqrt{\frac{2\pi}{y}}\,e^{\frac{\pi i z}{2}}\left(\frac{y}{\pi}\right)^x e^{i\phi}\left(\frac{2\pi}{a}\right)^z \exp(f(a))\,0(y^{\frac{1}{6}}), \tag{18}$$

where Landau's notation 0 *is uniformly bounded with respect to all values of a.*
Proof. From the relation $z=ia\cos\gamma$, we obtain

$$\tan\gamma=i\sqrt{z^2+a^2}/z, \qquad \gamma=i\log\frac{1}{ia}(z+\sqrt{z^2+a^2}), \tag{19}$$

and

$$2K_z(a)\sim\pi i\,e^{\frac{\pi i z}{2}}(2\pi/a)^z\exp\left(-z+z\log z/i-z\log\pi-\frac{1}{4}\pi i\right)\exp(f(a))/\sqrt{\pi i z/2}$$
$$\times\sqrt{\pi i z/6}\,e^{\frac{5\pi i}{12}}\tan\gamma\exp(-iz/3\cdot\tan^3\gamma)\,H^{(1)}_{\frac{1}{3}}(z/3\cdot\tan^3\gamma).$$

When $a=y+y^\alpha$, $\tan\gamma=\sqrt{2}\,y^{-\frac{1}{2}+\frac{\alpha}{2}}$. Accordingly, $z\tan^3\gamma/3$ tends to infinity or to zero when y tends to infinity, according as $a=y+y^{\frac{1}{3}+\delta}$ or $a=y+y^{\frac{1}{4}}$ respectively. Using the formulae (5) and (6), we obtain

$$\sqrt{\pi i z/6}\,e^{\frac{5\pi i}{12}}\tan\gamma\exp(-iz/3\cdot\tan^3\gamma)\,H^{(1)}_{\frac{1}{3}}(z/3\cdot\tan^3\gamma)=0(y^{\frac{1}{6}}).$$

Here Landau's notation 0 is uniformly bounded with respect to all values of a.

6. *The absolute value of* $\exp(f(a))$ *is decreasing with respect to a in the interval* $(y+y^{\frac{1}{4}},\ y+y^{\frac{1}{3}+\delta})$, *and* $|\exp(f(a))|\sim 2^{-x}$, *when* $a=y+y^{\frac{1}{4}}$.

Proof. When $a=y+y^\alpha$, $z+\sqrt{z^2+a^2}\sim y^{\frac{1+\alpha}{2}}+iy$, so that $\Re(df/da)$ is always negative irrespective of x being positive or negative. And

$$\Big[\Re f(a)\Big]_{a=y+y^{\frac{1}{4}}}=\Big[\Re(z-(z^2+a^2)/2z-z\log 2+\dots)\Big]_{a=y+y^{\frac{1}{4}}}=-x\log 2.$$

7. *When* $y+y^{\frac{1}{3}+\alpha}\leqq a<\infty$, *we obtain the following asymptotic formula when y tends to infinity*:

$$2K_z(a)\sim\sqrt{\frac{2\pi}{y}}\,e^{\frac{\pi i z}{2}}\left(\frac{y}{\pi}\right)^x e^{i\phi}(2\pi/a)^z\exp(f(a))\,B(z). \tag{20}$$

And the absolnte value of $\exp(f(a))$ *is decreasing, and its approximate value for* $a=y+y^{\frac{1}{3}+\delta}$ *is* 2^{-x}, *and is equal to* $\exp\{(-\sqrt{k^2-1}+\tan^{-1}\sqrt{k^2-1})y\}$ *for* $a=ky(k>1)$.

The proof is similar to that of the formula (10).

§5. Before we prove the Theorem I, we rewrite the relation (2) as follows.

Put $\xi(iz)/2\pi^2$ as $\eta(z)$, then the relation (2) becomes

$$\eta(z)=G(z/2-9/4)+G(z/2+9/4), \tag{21}$$

or, as $G(z)$ is even,

$$\eta(z)=G(9/4-z/2)+G(9/4+z/2), \tag{22}$$

and we obtain

$$\eta(iy)=2\Re G(9/4+iy/2), \tag{23}$$

From this relation, the purely imaginary zero points of $\eta(z)$ are obtained as the ordinates which make $G(9/4+iy/2)$ be zero, i. e., make $\arg G(9/4+iy/2)$ be odd number times of $\pi/2$.

We consider a sufficiently long section $\zeta_2^*(z)$ consisting of first N' terms of $\zeta_2(z)$, then we can find a certain fixed positive constant σ such as $|\zeta_2^*(z)|\geqq\sigma>0$ in $\frac{1}{2}-\varepsilon\leqq x\leqq A$, here ε is a fixed positive constant smaller than $\frac{1}{2}$. Because, the absolute value of $\zeta_2(z)$ is greater than a certain positive constant in $\frac{1}{2}-\varepsilon\leqq x\leqq A$. And we can assume that the absolute value of R is sufficiently small compared with that of $\zeta_2^*(z)$, when we put as

$$\zeta_2(z)=\zeta_2^*(z)+R.$$

$\xi_2(iz)/2\pi^2$ is written as $\eta_2(z)$, and we construct functions $\psi_2^*(t)$, $G_2^*(z)$, $\xi_2^*(z)$ and $\eta_2^*(z)$ corresponding to $\psi_2(t)$, $G_2(z)$, $\xi_2(z)$ and $\eta_2(z)$ respectively. Then we obtain the following Lemma.

Lemma. *The function $\xi_2^*(z)$ has infinitely many zero points in the strip $-A\leqq y\leqq A$, and these zero points are all on the real axis in a domain, where the absolute value of x is sufficiently large. And the number of zero points which are in $0\leqq x\leqq T$ is*

$$T/2\pi\cdot\log T/2\pi-T/2\pi+0\,(1).$$

Proof. From the definition,

$$G_2^*(z)=\int_{-\infty}^{\infty}\psi_2^*(t)\,e^{zt}\,dt=2\sum{}^*\,n^4K_z\,(2n^2\pi),$$

where the summation $\sum^*$ denotes the sum with respect to n's which are in the series of $\zeta_2^*(z)$. Hence, using the formula (13), we obtain

$$\eta_2^*(z)=G_2^*(z/2+9/4)+G_2^*(z/2\ -9/4)\sim$$

$$\left.\begin{aligned}&\sqrt{\frac{4\pi}{y}}\exp\left(-\pi y/4+\pi i/2\cdot(x/2+9/4)\right)\left\{\left(\frac{y}{2\pi}\right)^{\frac{x}{2}+\frac{9}{4}}e^{i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{-z-\frac{1}{2}}+\left(\frac{y}{2\pi}\right)^{-\frac{x}{2}-\frac{9}{4}}\right.\\&\qquad\left.\times e^{-i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{z+\frac{17}{2}}\right\}\\&+\sqrt{\frac{4\pi}{y}}\exp\left(-\pi y/4+\pi i/2\cdot(x/2-9/4)\right)\left\{\left(\frac{y}{2\pi}\right)^{\frac{x}{2}-\frac{9}{4}}e^{i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{-z+\frac{17}{2}}+\left(\frac{y}{2\pi}\right)^{-\frac{x}{2}+\frac{9}{4}}\right.\\&\qquad\left.\times e^{-i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{z-\frac{1}{2}}\right\}.\end{aligned}\right\}\quad(24)$$

Let ε' be any small positive number and y be sufficiently large, then, in $\varepsilon'\leqq x\leqq A$, the first term in the former braces is predominant compared with the remaining three terms, and, in $-A\leqq x\leqq-\varepsilon'$, the second term in the latter braces is predominant compared with the other terms. Accordingly, in $\varepsilon'\leqq x\leqq A$ or $-A\leqq x\leqq-\varepsilon'$, we obtain the following asymptotic formula when y tends to infinity.

$$\left.\begin{aligned}\eta_2^*(z):&\left\{\sqrt{\frac{4\pi}{y}}\exp\left(-\pi y/2+\pi i/2\cdot(x/2+9/4)\right)\left(\frac{y}{2\pi}\right)^{\frac{x}{2}+\frac{9}{4}}e^{i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{-z-\frac{1}{2}}\right.\\&\left.+\sqrt{\frac{4\pi}{y}}\exp\left(-\pi y/2+\pi i/2\cdot(x/2-9/4)\right)\left(\frac{y}{2\pi}\right)^{-\frac{x}{2}+\frac{9}{4}}e^{-i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{z-\frac{1}{2}}\right\}\sim1\end{aligned}\right\}\quad(25)$$

Especially, in $\varepsilon' \leqq x \leqq A$, we obtain

$$\eta_2^*(z): \left\{\sqrt{\frac{4\pi}{y}} \exp(-\pi y/2+\pi i/2\cdot(x/2+9/4)) \left(\frac{y}{2\pi}\right)^{\frac{x}{2}+\frac{9}{4}} e^{i\phi\left(\frac{y}{2}\right)} \sum{}^* n^{-z-\frac{1}{2}}\right\} \sim 1. \quad (26)$$

And the absolute value of $\sum{}^* n^{-z-\frac{1}{2}}$ is greater than σ in $0 \leqq x \leqq A$, so that we can conclude as: However small the positive number ε' may be, $\eta_2^*(z)$ has no zero point in the strips $\varepsilon' \leqq x \leqq A$ and $-A \leqq x \leqq -\varepsilon'$, when y is sufficiently large.

In the next place, we examine whether the formula (25) holds valid in $-\varepsilon' \leqq x \leqq \varepsilon'$ or not.

We consider a rectangle whose vertices are $A_0(A, y_0)$, $A_n(A, y_n)$, $B_n(-A, y_n)$ and $B_0(-A, y_0)$, where y_0 is smaller than y_n, and assume that y_0 is sufficiently large and fixed, so that the asymptotic formula (24) is valid sufficiently strictly on the boundary of the rectangle. Under these assumptions, y_0 and y_n are determined as follows: Let ι be a positive number, so as $\iota(N'+2) \leqq \pi/4$, and both y_0 and y_n satisfy the simultaneous inequalities

$$\begin{aligned} |y \log n| &< \iota \pmod{2\pi} \\ |\phi(y/2)| &< \iota \pmod{2\pi} \end{aligned} \quad (27)$$

where n's are those which are in the series of $\zeta_2^*(z)$. The fact that such y_0 and y_n are found infinitly, can be proved by using Dirichlet's Theorem. Dirichlet's Theorem: Let $x_1, x_2, x_3, \dots, x_N$ be N given positive numbers and y be an arbitrary positive integer. Then we can find an integer $t(\leqq y^N)$ and N integers $\beta_1, \beta_2, \dots, \beta_N$ such that inequalities

$$|t x_q - \beta_q| < 1/y, \quad q=1, 2, 3, \dots, N.$$

hold valid.

In this Theorem, we use $\log n$, $y/2\pi$ and $\iota/2\pi$ in place of x_q, t and $1/y$ respectively, then it can be concluded that we can find an integer $y/2\pi$ $(\leqq (2\pi/\iota)^N)$ and N integers β_q such as

$$|y/2\pi \cdot \log n - \beta_q| < \iota/2\pi, \quad q=1, 2, 3, \dots, N.$$

These inequalities are written as

$$|y \log n| < \iota \pmod{2\pi}. \quad (28)$$

On the other hand, $\phi(y/2)$ varies very quickly when a sufficiently large y varies. Accordingly we can find such y's that the inequality $|\phi(y/2)| < \iota \pmod{2\pi}$ holds valid within the limit of validity of the inequalities (28). These y's are found infinitly and they tend to fnfinity.

If y_0 and y_n are both y's which satisfy the inequalities (27), the asymptotic formula (25) holds good on the boundary of the rectangle $A_0 A_n B_n B_0$. In order to prove this, we rearrange the right hand side of (24) as

$$\eta_2^*(z)\sim\sqrt{\frac{4\pi}{y}}\exp(-\pi y/4+\pi ix/4)\left(\frac{y}{2\pi}\right)^{\frac{9}{4}}\left\{e^{\frac{9\pi i}{8}}\left(\frac{y}{2\pi}\right)^{\frac{x}{2}}e^{i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{-\frac{1}{2}-z}+e^{-\frac{9\pi i}{8}}\left(\frac{y}{2\pi}\right)^{-\frac{x}{2}}\right.$$
$$\left.\times e^{-i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{-\frac{1}{2}+z}\right\}$$
$$+\sqrt{\frac{4\pi}{y}}\exp(-\pi y/4+\pi ix/4)\left(\frac{y}{2\pi}\right)^{-\frac{9}{4}}\left\{e^{\frac{9\pi i}{8}}\left(\frac{y}{2\pi}\right)^{-\frac{x}{2}}e^{-i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{\frac{17}{2}+z}+e^{-\frac{9\pi i}{8}}\left(\frac{y}{2\pi}\right)^{\frac{x}{2}}\right.$$
$$\left.\times e^{i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{\frac{17}{2}-z}\right\},\qquad(29)$$

and write the former braces as

$$e^{\frac{9\pi i}{8}}\left(\frac{y}{2\pi}\right)^{\frac{x}{2}}e^{i\phi\left(\frac{y}{2}\right)}\left\{\sum{}^*n^{-\frac{1}{2}-z}+e^{-\frac{9\pi i}{4}}\left(\frac{y}{2\pi}\right)^{-x}e^{-2i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{-\frac{1}{2}+z}\right\}.$$

As $(N'+2)\iota\leqq\pi/4$, on A_0B_0 and A_nB_n,

$$-\pi/4\leqq\text{are}\, e^{-2i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{-\frac{1}{2}+z}\leqq\pi/4.$$

So that the absolute value of the sum in the last braces is greater than 1. Accordingly, the former term on the right hand side of (29) is predominant compared with the latter term on A_0B_0 and A_nB_n, i. e., the approximate formula (25) holds valid along the boundary of the rectangle $A_0A_nB_nB_0$. Of course, $\eta_2^*(z)$ has no zero point on its boundary.

Let us now consider the variation of arg $\eta_2^*(z)$ for one round of z along the boundary of the rectangle.

On the abscissa $x=A$, the former term of the denominator of (25) is predominant compared with the second term, so that the variation of arg $\eta_2^*(z)$ is nearly equal to that of the argument of the former term, i. e., to

$$2\pi\theta_n\equiv\text{variation of }\left\{\phi(y/2)+\arg\sum{}^*n^{-\frac{1}{2}-z}\right\}.$$

Let us put the variation of arg $\eta_2^*(z)$ as $2\pi\theta_n+\varepsilon_n$. The variation of arg $\eta_2^*(z)$ along B_nB_0 is also $2\pi\theta_n+\varepsilon_n$. The variation of arg $\eta_2^*(z)$ along A_nB_n can be written as $-\frac{\pi iA}{2}+\varepsilon_n'$, where ε_n' is sufficiently small positive number. Because, the factors of the denominator of (25) which contribute to the variation of the argument are $e^{\frac{\pi ix}{4}}$, $e^{i\phi\left(\frac{y}{2}\right)}$, $\sum{}^*n^{-z-\frac{1}{2}}$, $e^{-i\phi\left(\frac{y}{2}\right)}$ and $\sum{}^*n^{z-\frac{1}{2}}$, and the variations of the arguments of the latter four terms are nearly equal to zero when ι in (27) is sufficiently small. In the same way, the variation of the argment along B_0A_0 can be written as $\pi iA/2+\varepsilon_0$.

Accordingly, the total variation of arg $\eta_2^*(z)$ along the boundary of the rectangle is

$$4\pi\theta_n+2\varepsilon_n+\varepsilon_n'+\varepsilon_0.$$

Consequently, the total number of the zero points contained in this rectangle is

$$1/2\pi\cdot(4\pi\theta_n+2\varepsilon_n'+\varepsilon_n'+\varepsilon_0).\qquad(30)$$

On the other hand, it can be seen from (23) that the purely imaginary zero points of $\eta_2^*(z)$ are given as iy's which satisfy the condition

$$\arg G_2^*(9/4+iy/2)=\pi/2\times \text{odd numbers}. \tag{31}$$

And

$$G_2^*(9/4+iy/2)\sim\sqrt{\frac{4\pi}{y}}\exp(-\pi y/4+9\pi i/8)\left\{\left(\frac{y}{2\pi}\right)^{\frac{9}{4}}e^{i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{-iy-\frac{1}{2}}+\left(\frac{y}{2\pi}\right)^{-\frac{9}{4}}\times e^{-i\phi\left(\frac{y}{2}\right)}\sum{}^*n^{\frac{17}{2}+iy}\right\},$$

so that the variation of $\arg G_2^*(9/4+iy/2)$ is determined by the former term in the braces. That is, if we put as

$$\pi\vartheta_n\equiv\text{variation of }\left\{\phi(y/2)+\arg\sum n^{-\frac{1}{2}-iy}\right\},$$

then the variation of $\arg G_2^*(9/4+iy/2)$ is equal to $\pi\vartheta_n+\varepsilon_n''$, where the absolute value of ε_n'' is sufficiently small.

Accordingly, the total number of non purely imaginary zero points of $\eta_2^*(z)$ in the rectangle is equal to

$$2\theta_n+\varepsilon_n/\pi+1/2\pi\cdot\varepsilon_n'-\vartheta_n-\varepsilon_n''/\pi,$$

i. e.,

$$1/\pi\left\{\text{variation of }(\arg\sum{}^*n^{-\frac{1}{2}-A-iy}-\arg\sum{}^*n^{-\frac{1}{2}-iy})\right\}+\varepsilon_n/\pi+\varepsilon_n'/2\pi-\varepsilon_n''/\pi. \tag{32}$$

And $\sum{}^*n^{-z}$ has no zero point in a rectangle $A_0(A, y_0)$, $A_n(A, y_n)$, $C_n\left(\frac{1}{2}, y_n\right)$, $C_0\left(\frac{1}{2}, y_0\right)$, so that the variation of $\arg\sum{}^*n^{-\frac{1}{2}-iy}$ is equal to the sum of $\arg_{C_0\to A_0}\sum{}^*n^{-z}$, $\arg_{A_0\to A_n}\sum{}^*n^{-z}$ and $\arg_{A_n\to C_n}\sum{}^*n^{-z}$. According to the first inequalities of (27), the sum of the variations of $\arg_{C_0\to A_0}\sum{}^*n^{-z}$ and $\arg_{A_n\to C_n}\sum{}^*n^{-z}$ is sufficiently small.

So that the value of (32) is sufficiently small, and it is a non negative integer. Accordingly it must be zero.

That is, $\eta_2^*(z)$ has only purely imaginary zero points in a domain, where the imaginary part of z is sufficiently large.

The variation of $\arg\sum{}^*n^{-\frac{1}{2}-iy}$ is nearly equal to that of $\arg\sum{}^*n^{-\frac{1}{2}-A-iy}$, and the latter is finite for all values of y, when $A=3/2$. Accordingly, the variation of $\arg\sum{}^*n^{-\frac{1}{2}-iy}$ is $O(1)$, and the total number of the zero points which lie between y_0 and y_n is $\phi(y_n/2)+O(1)$, where y_0 is fixed and y_n increases. Thus we have proved that the total number of the zero points of $\eta_2^*(z)$, which lie in $-A\leqq x\leqq A$, $0\leqq y\leqq T$, is

$$T/2\pi\cdot\log T/2\pi-T/2\pi+O(1),$$

and the number of non purely imaginary zero points is at most finite. Q.E.D.

Proof of the Theorem 1.

In the summations

$$G_2(z)=2\sum K_z(2n^2\pi),$$
$$\eta_2(z)=2\sum n^4K_{\frac{z}{2}+\frac{9}{4}}(2n^2\pi)+2\sum n^4K_{\frac{z}{2}-\frac{9}{4}}(2n^2\pi), \tag{33}$$

n runs all n's which are in the series of $\zeta_2(z)$. We partition these n's into five parts as: $1\leq n\leq[(y-y^{\frac{1}{3}+\delta}/2\pi)^{\frac{1}{2}}]$, $[(y-y^{\frac{1}{3}+\delta}/2\pi)^{\frac{1}{2}}]+1\leq n\leq[(y-y^{\frac{1}{4}}/2\pi)^{\frac{1}{2}}]$, $[(y-y^{\frac{1}{4}}/2\pi)^{\frac{1}{2}}]+1\leq n\leq[(y+y^{\frac{1}{4}}/2\pi)^{\frac{1}{2}}]$, $[(y+y^{\frac{1}{4}}/2\pi)^{\frac{1}{2}}]+1\leq n\leq[(y+y^{\frac{1}{3}+\delta}/2\pi)^{\frac{1}{2}}]$, $[(y+y^{\frac{1}{3}+\delta}/2\pi)^{\frac{1}{2}}]+1\leq n<\infty$, where $[a]$ is the integral part of a number a. Corresponding to these ranges of n, we have the following asymptotic formulae:

$$G_2(z)\sim\sqrt{\frac{4\pi}{y}}e^{\frac{\pi i z}{2}}(g_1+g_2+g_3+g_4+g_5),$$
$$\eta_2(z)\sim\sqrt{\frac{4\pi}{y}}\exp(\pi i/2(z/2+9/4))(h_1+h_2+h_3+h_4+h_5)$$
$$+\sqrt{\frac{4\pi}{y}}\exp(\pi i/2(z/2-9/4))(h_1'+h_2'+h_3'+h_4'+h_5'),$$

where

$$g_1=\{(y/\pi)^x e^{i\phi(y)}\sum n^{-2z+4}\exp(f(2n^2\pi))+(y/\pi)^{-x}e^{-i\phi(y)}\sum n^{2z+4}\exp(-f(2n^2\pi)\}B(z),$$
$$g_2=(y/\pi)e^{i\phi}\sum n^{-2z+4}\exp(f(2n^2\pi))O(y^{\frac{1}{6}})+(y/\pi)^{-x}e^{-i\phi(y)}\sum n^{2z+4}\exp(-f(2n^2\pi))O(y^{\frac{1}{6}}),$$
$$g_3=k\sqrt{y}\sum n^{\frac{10}{3}},\qquad k=-i/\sqrt[6]{3}\,\Gamma(1/3)\pi^{-\frac{4}{3}}$$
$$g_4=(y/\pi)^x e^{i\phi}\sum n^{-2z+4}\exp(f(2n^2\pi))\,O(y^{\frac{1}{6}}),$$
$$g_5=(y/\pi)^x e^{i\phi}\sum n^{-2z+4}\exp(f(2n^2\pi))\,B(z), \tag{34}$$

$$h_1=(y/2\pi)^{\frac{x}{2}+\frac{9}{4}}e^{i\phi(\frac{y}{2})}\sum n^{-\frac{1}{2}-z}\exp(f(2n^2\pi))B_n(z/2+9/4),\ B_n(z)\equiv\sqrt{z}:\sqrt[4]{z^2+4n^4\pi^2}$$
$$+(y/2\pi)^{-\frac{x}{2}-\frac{9}{4}}e^{-i\phi(\frac{y}{2})}\sum n^{\frac{17}{2}+z}\exp(-f(2n^2\pi))\,B_n(z/2+9/4)=h_1^{(1)}+h_1^{(2)}\text{ say},$$
$$h_2=(y/2\pi)^{\frac{x}{2}+\frac{9}{4}}e^{i\phi(\frac{y}{2})}\sum n^{-\frac{1}{2}-z}\exp(f(2n^2\pi))\,O(y^{\frac{1}{6}})$$
$$+(y/2\pi)^{-\frac{x}{2}-\frac{9}{4}}e^{-i\phi}\sum n^{\frac{17}{2}+z}\exp(f(2n^2\pi))\,O(y^{\frac{1}{6}})=h_2^{(1)}+h_2^{(2)}\text{ say},$$
$$h_3=k\sqrt{y/2}\sum n^{\frac{10}{3}},$$
$$h_4=(y/2\pi)^{\frac{x}{2}+\frac{9}{4}}e^{i\phi}\sum n^{-\frac{1}{2}-z}\exp(f(2n^2\pi))\,O(y^{\frac{1}{6}}),$$
$$h_5=(y/2\pi)^{\frac{x}{2}+\frac{9}{4}}e^{i\phi}\sum n^{-\frac{1}{2}-z}\exp(f(2n^2\pi))\,B_n(z/2+9/4),$$
$$h_1'=(y/2\pi)^{\frac{x}{2}-\frac{9}{4}}e^{i\phi}\sum n^{\frac{17}{2}-z}\exp(f(2n^2\pi))\,B_n(z/2-9/4)$$
$$+(y/2\pi)^{-\frac{x}{2}+\frac{9}{4}}e^{-i\phi}\sum n^{-\frac{1}{2}+z}\exp(-f(2n^2\pi))\ B_n(z/2-9/4)\equiv h'^{(1)}_1+h'^{(2)}_1,$$
$$h_2'=(y/2\pi)^{\frac{x}{2}-\frac{9}{4}}e^{i\phi}\sum n^{\frac{17}{2}-z}\exp(f(2n^2\pi))\,O(y^{\frac{1}{6}})$$
$$+(y/2\pi)^{-\frac{x}{2}+\frac{9}{4}}e^{-i\phi}\sum n^{-\frac{1}{2}+z}\exp(-f(2n^2\pi))\,O(y^{\frac{1}{6}})\equiv h'^{(1)}_2+h'^{(2)}_2,$$
$$h_3'=k\sqrt{y/2}\sum n^{\frac{10}{3}},$$

$h_4'=(y/2\pi)^{\frac{x}{2}-\frac{9}{4}}e^{i\phi}\sum n^{\frac{17}{2}-z}\exp(f(2n^2\pi))0(y^{\frac{1}{6}})$,

$$h_5'=(y/2\pi)^{\frac{x}{2}-\frac{9}{4}}e^{i\phi}\sum n^{\frac{17}{2}-z}\exp(f(2n^2\pi))B_n(z/2-9/4). \tag{35}$$

Under the last ten series, h_1 is predominant in $x\geqq 0$ compared with the other eight series other than h_1', and h_1' is predominant in $x\leqq 0$ compared with the other eight series other than h_1. We shall briefly examine these conclusions below. Here we must notice that $\delta\leqq 1/5$.

(1) $h_1^{(1)}$ is predominant compared with $h_1^{(2)}$.

We compare the absolute values of

$$\sum n^{-\frac{1}{2}-z}\exp(f(2n^2\pi))B_n(z/2+9/4) \tag{36}$$

and

$$(y/2\pi)^{-x-\frac{9}{2}}\sum n^{\frac{17}{2}+z}\exp(-f(2n^2\pi))B_n(z/2+9/4). \tag{37}$$

The absolute value of $B_n(z)$ is greatest for $a=y-y^{\frac{1}{3}+\delta}$ when $0\leqq a\leqq y-y^{\frac{1}{3}+\delta}$ and its order is $0(y^{\frac{1}{6}-\frac{\delta}{4}})$. Hence the order of (37) is $0(y^{-x-\frac{13}{3}-\frac{\delta}{4}}\sum n^{\frac{17}{2}+x})$. And, as the maximum of $2n^2\pi$ is $y-y^{\frac{1}{3}+\delta}$, the order of $\sum n^{\frac{17}{2}+x}$ is $0(y^{\frac{17}{4}+\frac{x}{2}})$. So that (37) tends to zero, when y tends to infinity. On the other hand, the series of (36) tends to $\sum n^{-\frac{1}{2}-z}$, as we shall see later. Accordingly, $h_1^{(1)}$ is predominant compared with $h_1^{(2)}$.

(2) $h_1^{(1)}$ is predominant compared with $h_2^{(1)}$.

The function which is to be compaed with (36), is $\sum n^{-\frac{1}{2}-z}\exp(f(2n^2\pi))0(y^{\frac{1}{6}})$. The number of terms of this series does not surpass $0(y^{-\frac{1}{6}+\delta})$. So that the order of the above function is $0(y^{-\frac{1}{4}-x+\delta})$, and it tends to zero when y tends to infinity.

(3) $h_1^{(1)}$ is predominant compared with $h_2^{(2)}$.

Instead of (37), we may consider the series

$$(y/2\pi)^{-x-\frac{9}{2}}\sum n^{\frac{17}{2}+z}\exp(-f(2n^2\pi))0(y^{\frac{1}{6}}).$$

The order of this series is $0(y^{-\frac{1}{4}-\frac{x}{2}+\delta})$.

(4) $h_1^{(1)}$ is predominant compared with h_3.

In this case we obtain $0(y^{-\frac{1}{12}-\frac{x}{2}})$.

(5) $h_1^{(1)}$ is predominant compared with h_4.

This case can be treated in the same way as in (2).

(6) $h_1^{(1)}$ is predominant compared with h_5.

In this case, we may consider the series $\sum n^{-\frac{1}{2}-z}\exp(f(2n^2\pi))B_n(z/2+9/4)$ instead of (36). And we obtain $0(y^{-\frac{1}{4}+\frac{3}{4}\delta-\frac{x}{2}})$.

(7) $h_1^{(1)}$ is predominant compared with $h'^{(1)}_1$.

In this case, we obtain $0(y^{-\frac{1}{12}-\frac{x}{2}-\frac{\delta}{4}})$.

(8) $h_1^{(1)}$ is predominant compared with $h'^{(1)}_2$.

In this case, we obtain $0(y^{-\frac{1}{12}-\frac{x}{2}})$.

(9) $h_1^{(1)}$ is predominant compared with $h'^{(2)}_2$.

In this case, we obtain $0(y^{-\frac{1}{12}-\frac{x}{2}})$.

(10) $h_1^{(1)}$ is predominant compared with h_4' and h_5'.

Accordingly, we obtain the following asymptotic expansion in $-A\leqq x\leqq A$:

$$\left.\begin{aligned}\eta_2(z)\sim&\sqrt{4\pi/y}\exp(\pi i/2\cdot(z/2+9/4))(y/2\pi)^{\frac{x}{2}+\frac{9}{4}}e^{i\phi(\frac{y}{2})}\\&\times\sum n^{-\frac{1}{2}-z}\exp(f(2n^2\pi))B_n(z/2+9/4)\\&+\sqrt{4\pi/y}\exp(\pi i/2\cdot(z/2-9/4))(y/2\pi)^{-\frac{x}{2}+\frac{9}{4}}e^{-i\phi(\frac{y}{2})}\\&\times\sum n^{-\frac{1}{2}+z}\exp(-f(2n^2\pi))B_n(z/2-9/4),\end{aligned}\right\}\qquad(38)$$

where

$$2\pi\leqq 2n^2\pi\leqq(y/2)-(y/2)^{\frac{1}{3}+\delta}.$$

We divide the above summation into two parts: $\sum^*+\sum'$, where $\sum^*$ consists of first N' terms, and $\sum'$ consists of the remaining terms.

When y tends to infinity, we can see, from (13), that

$$\sum^*\sim\sum^* n^{-\frac{1}{2}-z}.$$

In the next place, we consider the remaining part $\sum'$. The absolute value of this part is small compared with that of $\sum^*$, when the number of terms of $\sum^*$ is sufficiently large.

Eventually, joining together these results, we can conclude that the absolute value of $\eta_2(z)$ is nearly equal to that of $\eta_2^*(z)$ on the boundary of the rectangle given in the Lemma, when y_0 is sufficiently large. Then, according to Rouché's Theorem, the number of zero points of $\eta_2(z)$ in the rectangle, is equal to that of $\eta_2^*(z)$ in the same rectangle.

On the other hand, $\eta_2(iy)$ becomes zero for y's which satisfy the relation $\arg G_2(9/4+iy/2)=\pi/2\times$odd number. And, from the fact that $h_1^{(1)}$ in (35) is predominant compared with $h_1^{(2)}$, h_2, h_3, h_4 and h_5, we obtain

$$\begin{aligned}G_2(9/4+iy/2)\sim&\sqrt{4\pi/y}\exp(-\pi y/4+9\pi i/8)(y/2\pi)^{\frac{9}{4}}e^{i\phi(\frac{y}{2})}\\&\times\sum n^{-\frac{1}{2}-iy}\exp(f(2n^2\pi))B_n(z/2+9/4).\end{aligned}\qquad(39)$$

As we have already seen, the summation in the right hand side is written as $\sum^* n^{-\frac{1}{2}-iy}+R$, and the absolute value of R is sufficiently small compared with that of the former term. And the variation of

$$\arg \sum n^{-\frac{1}{2}-iy} \exp(f(2n^2\pi)) B_n(z/2+9/4)$$

is nearly equal to that of $\arg \sum^* n^{-\frac{1}{2}-iy}$.

So that the number of the purely imaginary zero points of $\eta_2(z)$ is equal to that of $\eta_2^*(z)$. Consequently, all zero points of $\eta_2(z)$ are all on the imaginary axis, when y is sufficiently large. And the number of the zero points which lie in $0 \leqq y \leqq T$, is

$$T/2\pi \cdot \log T/2\pi - T/2\pi + 0(1).$$

§6. Proofs of the Theorems 1 and 2.

1. Proof of the Theorem 1.

$$\begin{aligned} ia \sinh t - zt &= z \operatorname{sech} \gamma \sinh t - zt \\ &= z \operatorname{sech} \gamma \sinh \gamma - \gamma z - izG(t, \gamma) \operatorname{sech} \gamma, \end{aligned}$$

where

$$G(t, \gamma) = i(\sinh t - t \cosh \gamma - \sinh \gamma + \gamma \cosh \gamma).$$

Accordingly

$$H_z^{(1)}(ia) = 1/\pi i \cdot \exp(z(\tanh \gamma - \gamma)) L(z, \gamma),$$

where

$$L(z, \gamma) = \int_{-\infty+\frac{\pi i}{2}}^{\infty+\frac{\pi i}{2}} \exp(-izG(t, \gamma) \operatorname{sech} \gamma)\, dt.$$

The stationary points of $G(t, \gamma)$ are $\pm\gamma + 2h\pi i$, $h = 0, \pm 1, \pm 2, \ldots$

When we put as $t = u + iv$, Debye's contour

$$\Im G(t, \gamma) = 0, \tag{40}$$

which passes through γ, is given by

$$\begin{aligned} \sinh u \cosh v = u \cosh \alpha \cosh \beta - v \sinh \alpha \sin \beta + \sinh \alpha \cos \beta - \alpha \cosh \alpha \cos \beta \\ + \beta \sinh \alpha \sin \beta. \end{aligned} \tag{41}$$

Let us trace this curve. We introduce a new real variable w, and consider the curve (41) as the projection of the intersection of a surface $w = \sinh u \cos v$ and a plane $w = u \cosh \alpha \cos \beta - v \sinh \alpha \sin \beta + \sinh \alpha \cos \beta - \alpha \cosh \alpha \cos \beta + \beta \sinh \alpha \sin \beta$. Let P_0 and Q be the points of intersection of the above plane with the u- and v-axes respectively. Then P_0 and Q are both on the negative sides of the origin. Because, as $\alpha \leqq 0$ and $0 \leqq \beta < \pi/2$ when $x \geqq 0$, the coefficients of u and v are both positive, and the constant term is also positive. Let $P_{n/2}$ be the point of intersection of the above plane with the axis $v = n\pi/2$, $w = 0$. Then the above curve (41) has two branches. One of them starts from $(-\infty, -\pi/2+0)$ and terminates at $(-\infty, 3\pi/2+0)$. We name this branch as Γ_0.

$$\Gamma_0: \ (-\infty, -\pi/2+0) \to \gamma \to P_{\frac{1}{2}} \to P_{\frac{3}{2}} \to (-\infty, 3\pi/2+0).$$

The other branch starts from $(-\infty, \pi/2-0)$ and passes the following several points and terminates at $(\infty, -(2n+1)\pi/2-0)$. We name this branch as Γ_n.

$$\Gamma_n: \ (-\infty, \pi/2-0) \to \gamma \to Q \to P_{-\frac{3}{2}} \to \cdots\cdots \to P_{-2n-\frac{3}{2}} \to (\infty, \ -(2n+1)\pi/2-0),$$

where n is a finite positive integer which is determined according to α, β.

Instead of (40), we consider an equation

$$\Im G(t, -\gamma)=0 .$$

The contour which corresponds to this equation is symmetric with that of (40) with respect to the origin. Let C_0 be a branch which is symmetric with Γ_0.

$$C_0: \ (+\infty, \pi/2-0) \to -\gamma \to (+\infty, \ -3\pi/2-0)$$

In general, a branch of the contour

$$\Im G(t, -\gamma-2h\pi i)=0 .$$

is given by

$$C_h: \ (+\infty, \ -2h\pi+\pi/2-0) \to -\gamma-2h\pi i \to (+\infty, \ -2h\pi-3\pi/2-0).$$

Accordingly

$$L(z,\gamma)=\left(\int_{\Gamma_n}+\int_{C_n}+\cdots\cdots+\int_{C_0}\right) \exp(-izG(t,\gamma) \operatorname{sech}\gamma)\, dt \tag{42}$$

In the first place, we consider the first integral in the parentheses. We put as $\zeta=-G(t,\gamma)$, then, as

$$\Re G(t,\gamma)=-\cosh u \sin v+u \sinh \alpha \sin \beta+v \cosh \alpha \cos \beta+\cosh \alpha \sin \beta$$
$$-\beta \cosh \alpha \cos \beta-\alpha \sinh \alpha \sin \beta,$$

ζ starts from the origin and tends to plus infinity, when t on Γ_n starts from γ and tends to $-\infty+\pi i/2$ or to $\infty+(-2n\pi-3\pi/2)i$. So that, two t's correspond to one real number ζ. Let t_1 be the one which is in $(\gamma, -\infty+(-2\pi-3\pi/2)i)$ and t_2 be the other one which is in $(-\infty+\pi i/2, \gamma)$. Then

$$\int_{\Gamma_n} \exp(-izG(t,\gamma) \operatorname{sech}\gamma) dt=\int_0^\infty \exp(iz\zeta \operatorname{sech}\gamma)(dt_1/d\zeta-dt_2/d\zeta)d\zeta.$$

Next we shall discuss the expansion of t_1 and t_2 in the descending powers of ζ. Since ζ and $d\zeta/dt$ vanish when $t=\gamma$, it follows that the expansion of ζ in powers of $t-\gamma$ begins with a term in $(t-\gamma)^2$; by reverting this expansion, we obtain expansions of the forms

$$t_1-\gamma=\sum_{m=0}^{\infty} a_m/m+1 \ \zeta^{\frac{1}{2}(m+1)}, \quad t_2-\gamma=\sum_{m=0}^{\infty}(-1)^{m+1} a_m/m+1 \ \zeta^{\frac{1}{2}(m+1)},$$

and these expansions are valid for sufficiently small values of $|\zeta|$.
Moreover

$$a_m=1/2\pi i \cdot \int^{(0+,0+)} dt_1/d\zeta \cdot \zeta^{-\frac{1}{2}(m+1)} d\zeta=1/2\pi i \cdot \int^{(\gamma+)} dt/\zeta^{\frac{1}{2}(m+1)}.$$

The double circuit in the ζ-plane is necessary in order to dispose of the fractional powers of ζ; and a single circuit round γ in the t-plane corresponds to the double circuit round the origin in the ζ-plane. From the last contour integral, it follows that a_m is the coefficicient of $1/t-\gamma$ in the expansion of $\zeta^{-\frac{1}{2}(m+1)}$ in ascending powers of $t-\gamma$; we are thus enabled to calculate the coefficients a_m's.

Write $t-\gamma=T$ and we obtain

$$\zeta=T^2(c_0+c_1T+c_2T^2+\cdots),$$

where $c_0=-i/2!\sinh\gamma,\ c_1=-i/3!\cosh\gamma,\ c_2=-i/4!\sinh\gamma, \dots$

Therefore a_m is the coefficient of T^m in the expansion of

$$\{c_0+c_1T+c_2T^2+\cdots\}^{-\frac{1}{2}(m+1)}.$$

The coefficient in this expansion will be called $a_0(m)$, $a_1(m)$, $a_2(m)$, ... , and so we obtain

$$a_0(m)=c_0^{-\frac{1}{2}(m+1)},\quad a_1(m)=c_0^{-\frac{1}{2}(m+1)}\{-(m+1)c_1/2\cdot 1!c_0\},\dots$$

By substitution we find that

$$a_0=a_0(0)=c_0^{-\frac{1}{2}},$$
$$a_1=a_1(1)=c_0^{-\frac{2}{2}}\left\{1/3\cdot\coth\gamma\right\},$$
$$a_2=a_2(1)=c_0^{-\frac{3}{2}}\left\{1/8-5/24\cdot\coth^2\gamma\right\},$$
$$a_3=a_3(1)=c_0^{-\frac{4}{2}}\left\{2/15\cdot\coth\gamma-4/27\cdot\coth^3\gamma\right\},$$
$$a_4=a_4(1)=c_0^{-\frac{5}{2}}\left\{3/128-77/576\cdot\coth^2\gamma-385/3456\cdot\coth^4\gamma\right\},$$
....

using these coefficients, we obtain

$$d(t_1-t_2)/d\zeta=\sum_{m=0}^{\infty}a_m\zeta^{m-\frac{1}{2}},$$

and so the asymptotic expansion

$$\int_{\Gamma_n}\exp(-izG(t,\gamma)\operatorname{sech}\gamma)\,dt\sim\sum_{m=0}^{\infty}a_{2m}\Gamma\left(m+\frac{1}{2}\right)\left(-iz\operatorname{sech}\gamma\right)^{-m-\frac{1}{2}} \tag{43}$$

is obtained when positive y is large. Here $a_0=(-i/2\cdot\sinh\gamma)^{-\frac{1}{2}}$. And, as $a_0=[t_1-\gamma/\zeta^{\frac{1}{2}}]_{\zeta\to+0}$, $\arg a_0$ is a negative acute angle, so that we have $0<\arg(-i\sinh\gamma)<\pi$. Accordingly we obtain

$$\int_{\Gamma_n}\exp(-izG(t,\gamma)\operatorname{sech}\gamma)dt\sim e^{-\frac{\pi i}{4}}\Big/\sqrt{-\pi iz/2\tanh\gamma}\sum_{m=0}^{\infty}\Gamma\left(m+\frac{1}{2}\right)\Big/\Gamma\left(\frac{1}{2}\right)$$
$$\times A_m/(-z/2\cdot\tanh\gamma)^m. \tag{44}$$

Here $\arg(-\pi iz/2\cdot\tanh\gamma)=\pi/2+\arg(-i\sinh\gamma)$, and $A_0=1, A_1=\frac{1}{2}\left(\frac{1}{8}-5/24\cdot\coth^2\gamma\right)$, $A_2=3/128-77/576\cdot\coth^2\gamma-385/3456\cdot\coth^4\gamma$,......In general, A_m is a polynomial of $\coth^2\gamma$ of the m-th degree. (Watson[3], pp. 242–243). Accordingly, $A_m/(-z/2\tanh\gamma)^m=0(y^{-\frac{3}{2}m\delta})$. Thus the formula (44) holds valid in $0\leqq a\leqq y-y^{\frac{1}{3}+\delta}$.

Similarly, we obtain

$$\int_{C_0} \exp(-izG(t,\gamma)\operatorname{sech}\gamma)dt = e^{\frac{\pi i}{4}} \Big/ \sqrt{\pi iz/2\cdot\tanh\gamma} \cdot \sum_{m=0}^{\infty} \Gamma\left(m+\frac{1}{2}\right) \Big/ \Gamma\left(\frac{1}{2}\right) \times A_m/(z/2\tanh\gamma)^m.$$

here, $\arg(\pi iz/2\cdot\tanh\gamma) = \pi/2 + \arg(i\sinh\gamma)$ and $0 > \arg(i\sinh\gamma) > -\pi$.

Concerning the contour C_k, we obtain

$$\int_{C_k} \exp(-izG(t,\gamma)\operatorname{sech}\gamma)dt = e^{2k\pi iz}\int_{C_0} \exp(-izG(t,\gamma)\operatorname{sech}\gamma)dt.$$

So that it can be neglected compared with that of C_0. Thus we obtain the Theorem 1.

2. Proof of the Theorem 2.

We consider the case $x \geqq 0$ only.

As has been proved above,

$$H_z^{(1)}(ia) \sim 1/\pi i \cdot \left(\int_{C_0} + \int_{\Gamma_n}\right) \exp(ia\sinh t - zt)\, dt.$$

In the first place, we consider the integral taken along Γ_n. We put as

$G(t,\gamma) = -\tau\cosh\gamma$, where $\tau = -i\tanh\gamma(\cosh w - 1) - i(\sinh w - w)$, $w = t - \gamma$. (45)

In the interval $y - y^{\frac{1}{3}+\delta} \leqq a \leqq y - y^{\frac{1}{4}}$, $\cosh\gamma = z/ia$ tends to 1, so that γ is in the neighbourhood of the origin. As the inequalities $d\alpha/da > 0$ and $d\beta/da > 0$ hold valid, γ starts from $-\infty + (\pi/2 + 0)i$ and moves towards upper right, when a increases from zero to $y - y^{\frac{1}{4}}$. When $a = y - y^{\frac{1}{4}}$, we obtain

$$\alpha = -\sqrt{2}\, y^{-\frac{3}{8}}, \quad \beta = xy^{-\frac{5}{8}}/\sqrt{2},$$

and so we obtain $\arg\gamma \to \pi (y \to +\infty)$.

Expanding the right hand side of (45) in powers of w, we obtain

$$\tau = -i\tanh\gamma\cdot w/2 - iw^3/3!,$$

and, neglecting the infinitesimal w^3, we obtain $u^2 - v^2 = 0$ from $\Im\tau = 0$. And a branch $u + v = 0$ is adopted.

Instead of the integral

$$\int e^{iz\tau} dw = \int_{\Im\tau=0} \exp(-a\tau\cosh\gamma)dw, \tag{46}$$

we consider an integral

$$\int \exp(-a\tau\cosh\gamma)dw, \tag{47}$$

where $\tau = -i/2\cdot W^2\tanh\gamma - i/6\cdot W^3$, and the path of integration is $\Im\tau = 0$. When we put as $W = U + iV$ and $\tanh\gamma = -h + ik$, the contour $\Im\tau = 0$ is given by

$$-h(U^2 - V^2) - 2kUV + 1/3\cdot(U^3 - 3UV^2) = 0,$$

Putting as $k=0$, we obtain

$$-h(U^2-V^2)+1/3\cdot(U^3-3UV^2)=0. \tag{48}$$

One of the branches of this curve starts from $\infty e^{\frac{5\pi i}{6}}$, passes the origin and then tends to $h-i\infty$. This branch contacts with $\Im\tau=0$ in the w-plane at the origin. So that we may consider the integral (47) instead of the integral (46). Thus we can carry out the following calculations.

$$\int \exp(-a\tau\cosh\gamma)dw \sim \int \exp(-a\tau\cosh\gamma)dW$$

$$\left(\int_{-\tan\gamma+\infty e^{\frac{5\pi i}{6}}}^{-\tanh\gamma} + \int_{-\tanh\gamma}^{-\tanh\gamma-i\infty}\right)\exp(-a\tau\cosh\gamma)dW$$

$$=-\exp(z/3\cdot\tanh^3\gamma)\left\{\int_0^\infty e^{\frac{5\pi i}{6}}\exp(iz\xi^3/6-e^{\frac{5\pi i}{6}}\xi z\tanh^2\gamma/2)d\xi + i\int_0^\infty \exp(iz\xi^3/6+iz\xi\tanh^2\gamma)d\xi\right\}$$

$$=-\exp(z/3\cdot\tanh^3\gamma)\left\{e^{\frac{5\pi i}{6}}\sum_{m=0}^{\infty}1/m!\,(-e^{\frac{5\pi i}{6}}z/2\cdot\tanh^2\gamma)^m\int_0^\infty \exp(iz/6\cdot\xi^3)\xi^m d\xi + i\sum_{m=0}^{\infty}1/m!\,(iz/2\cdot\tanh^2\gamma)^m\int_0^\infty \exp(iz/6\cdot\xi^3)\xi^m d\xi\right\}$$

$$=-\exp(z/3\cdot\tanh^3\gamma)\times$$

$$\Big[1/3\cdot e^{\frac{5\pi i}{6}}\Big\{\sum_{k=1}^{\infty}1/(3k-1)!\cdot(-e^{\frac{5\pi i}{6}})^{3k-1}\cdot(z/2)^{3k-1}(\tanh^2\gamma)^{3k-1}(6i/z)^k\Gamma(k)$$
$$+\sum_{k=0}^{\infty}1/(3k)!\cdot(-e^{\frac{5\pi i}{6}})^{3k}(z/2)^{3k}(\tanh^2\gamma)^{3k}(6i/z)^{k+\frac{1}{3}}\Gamma(k+1/3)$$
$$+\sum_{k=0}^{\infty}1/(3k+1)!\cdot(-e^{\frac{5\pi i}{6}})^{3k+1}(z/2)^{3k+1}(\tanh^2\gamma)^{3k+1}(6i/z)^{k+\frac{2}{3}}\Gamma(k+2/3)\Big\}$$
$$+i/3\Big\{\sum_{k=1}^{\infty}1/(3k-1)!\cdot i^{3k-1}\cdot(z/2)^{3k-1}(\tanh^2\gamma)^{3k-1}(6i/z)^k\Gamma(k)$$
$$+\sum_{k=0}^{\infty}1/(3k)!\,i^{3k}(z/2)^{3k}(\tanh^2\gamma)^{3k}(6i/z)^{k+\frac{1}{3}}\Gamma(k+1/3)$$
$$+\sum_{k=0}^{\infty}1/(3k+1)!\,i^{3k+1}(z/2)^{3k+1}(\tanh^2\gamma)^{3k+1}(6i/z)^{k+\frac{2}{3}}\Gamma(k+2/3)\Big\}\Big].$$

The coefficients of z^{3k-1}, z^{3k} and z^{3k+1} are zero, $\sqrt{3}\exp(3/2\cdot k\pi i+2/3\cdot\pi i)$ and $\sqrt{3}\exp(3/2\cdot k\pi i+5/6\cdot\pi i)$ respectively. Thus the above series becomes as

$$-\exp(z/3\cdot\tanh^3\gamma)2\pi/3\cdot\tanh\gamma e^{\frac{5\pi i}{6}}\left\{I_{-\frac{1}{3}}(z/3\cdot\tanh^3\gamma)-e^{-\frac{2\pi i}{3}}I_{\frac{1}{3}}(z/3\cdot\tanh^3\gamma)\right\}.$$

Using the formulae of §2, the above formula is

$$-\exp(z/3\cdot\tanh^3\gamma)2\pi i/3\cdot\tanh\gamma\left\{e^{\frac{\pi i}{3}}I_{-\frac{1}{3}}(z/3\cdot\tanh^3\gamma)-e^{-\frac{\pi i}{3}}I_{-\frac{1}{3}}(z/3\cdot\tanh^3\gamma)\right\}$$

$$=-\exp(z/3\cdot\tanh^3\gamma)\cdot 2i/\sqrt{3}\cdot\tanh\gamma K_{\frac{1}{3}}(z/3\cdot\tanh^3\gamma e^{-i\pi}).$$

Thus we have

$$1/\pi i \cdot \int_{\Gamma_n} \exp(ia \sinh t - zt)dt = -2/\sqrt{3}\,\pi \cdot \tanh\gamma \exp(z/3 \cdot \tanh^3\gamma + z(\tanh\gamma - \gamma))K_{\frac{1}{3}}(z/3 \cdot \tanh^3\gamma e^{-i\pi}).$$

Similarly, we obtain

$$1/\pi i \cdot \int_{C_0} \exp(ia \sinh t - zt)dt = -2/\sqrt{3}\,\pi \cdot \tanh\gamma \exp(-z/3 \tanh^3\gamma - z(\tanh\gamma - \gamma))K_{\frac{1}{3}}(-z/3 \cdot \tanh^3\gamma).$$

Q.E.D.

As the proofs of the Theorem 3, 4 and 5 have already been given in Watson [3], we may omit them.

References

1) Riemann, B: Werke.
2) Pólya, G: Acta Math., 1926.
3) Watson, G. N.: Theory of Bessel Functions, (Cambridge), 1922.