Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 2, No. 1, Series A

On Whitney's Extension Theorem

By Minoru NAKAOKA

(Received May 31, 1951)

§ 1. Introduction

Let Y be an arcwise connected and simply connected topological space whose second homotopy group has a finite number of generators. Let K be a 4-dimensional finite complex. H. Whitney [10] gave an algebraic criterion for that a mapping f of the 2-section K^2 into Y be extendable over K. On the other hand, J. H. C. Whitehead gave many useful theorems on the investigation of homotopy type, and also defined the Pontrjagin squares [7,8]. We shall restate the Whitney's result in terms of the Pontrjagin squares, and prove this result by using Whitehead's theorems. The method is analogous to the Steenrod's [4]. We shall also give another definition of the Pontrjagin squares. I offer here my sincere thanks to Prof. A. Komatu, Messrs. T. Kudo and H. Uehara who gave me many valuable suggestions.

§ 2. Notations

(2. 1) Let K be a finite cell complex, whose cells are oriented. Let $C^r(K, A)$, $Z^r(K,A)$, $B^r(K, A)$ and $H^r(K, A)$ denote respectively the groups of r-A-cochains, r-A-cocycles, r-A-coboundaries and r-A-cohomology classes. We shall denote by $\{u\}$ the cohomology class containing u, an element of $Z^r(K, A)$. When cocycles u and v are cohomologous, we write $u \sim v$. When K_1 and K_2 are finite cell complexes and f is a cellular map of K_1 into K_2, we denote by f^* the homomorphism induced by f of $C^r(K_2, A), \dots, H^r(K_2, A)$ into $C^r(K_1, A), \dots, H^r(K_1, A)$, respectively. Let I_0 be the additive group of integers. A latin small letter attached with the symbol -- such as $\bar{u}$, means an integral cochain.

(2. 2) Throughout this paper, we suppose that a topological space Y is an arcwise connected, simply connected, Hausdorff space whose 2-dimensional homotopy group $\pi_2(Y)$ has a finite number of generators. Let y_* be a fixed point in Y. Maps always mean continuous maps. Let X, X', Y, Y' be topological spaces such that $X' \subset X$ and $Y' \subset Y$. The notation $f: (X, X') \to (Y, Y')$ means that f is a map of X into Y satisfying the condition $f(X') \subset Y'$. When maps $f, g: X \to Y$ are homotopic, we write $f \simeq g$.

§ 3. The Pontrjagin squares

(3. 1) Let A and A' be abelian groups and γ a map of A into A' satisfying the following conditions:

1) $\gamma(a)=\gamma(-a)$,

2) $\gamma(a+b)=\gamma(a)+\gamma(b)+[a, b], \quad a, b \in A$,

where $[a, b]$ is a bilinear function defined on $A\times A$ with values in A'. Then, in the similar way as in [7, p. 61], we get the following properties: i) $\gamma(na)=n^2\gamma(a)$, where n is an integer and $a\in A$. ii) $\gamma(a_1+a_2+\cdots+a_p)=\sum_i\gamma(a_i)+\sum_{i<j}[a_i, a_j]$, where $a_i\in A(i=1, 2, \dots, p)$. iii) $2\gamma(a)=[a, a]$. iv) If $ma=0$, then $(m^2, 2m)\gamma(a)=0$, where (p, q) denotes the greatest common measure of integers p and q.

(3. 2) We shall here recall the definition and some properties of the Pontrjagin squares which were given by J. H. C. Whitehead [8].

Let K be a finite simplicial complex, and $\{\bar{c}_1, \dots, \bar{c}_q\}$ a canonical basis for $C^r(K, I_0)$. Let $\{A, A', \gamma\}$ be a system given in (3. 1). Let $u\in Z^r(K, A)$. Then, using the canonical basis, we may represent u by $\sum_{i=1}^{q} u_i\bar{c}_i$, where $u_i\in A$. The Pontrjagin square of u is given by

$$\mathfrak{p}u=\sum_i\gamma(u_i)\mathfrak{p}\bar{c}_i+\sum_{i<j}[u_i, u_j]\bar{c}_i\cup\bar{c}_j,$$

where $\mathfrak{p}\bar{c}_i=\bar{c}_i\cup\bar{c}_i+\bar{c}_i\cup_1\delta\bar{c}_i$ and r is even. We have the following properties.

i) When we pass from cocycles to cohomology classes, $\mathfrak{p}$ induces a map $\mathfrak{P}$ of $H^r(K, A)$ into $H^{2r}(K, A')$ which is independent of the choice of an order of vertices in K and a canonical basis of $C^r(K, I_0)$.

ii) $\mathfrak{P}$ is natural, that is to say, when g denotes a simplicial map of finite simplicial complexes $g: K_1\rightarrow K_2$, the commutativity holds in the diagram:

$$\begin{array}{ccc} H^r(K_1, A) & \xrightarrow{\mathfrak{P}} & H^{2r}(K_1, A') \\ g^*\uparrow & & g^*\uparrow \\ H^r(K_2, A) & \xrightarrow{\mathfrak{P}} & H^{2r}(K_2, A'), \end{array}$$

i.e., $g^*\mathfrak{P}=\mathfrak{P}g^*$.

iii) If $u_1, u_2, \dots, u_t\in Z^r(K, A)$,

$$\mathfrak{p}(u_1+u_2+\cdots+u_t)\sim\sum_i\mathfrak{p}u_i+\sum_{i<j}u_i\cup u_j,$$

where $\cup$ is the cup product given by considering the function $[a, b]$ in the definition of γ as a group pairing of A with itself to A'.

iv) $$2\mathfrak{p}u\sim u\cup u, \text{ where } u\in Z^r(K, A).$$

When X is a topological space, using the Čech cohomology theory, we can define $\mathfrak{P}: H^r(X, A)\rightarrow H^{2r}(X, A')$ in the usual way. And we can see that the similar properties hold. $\mathfrak{P}$ is a topological invariant and is called *the Pontrjagin squares.*

(3. 3) When A has a finite number of generators, we shall give another definition of the Pontrjagin squares. Let $\{a_1 \dots, a_t\}$ be a system of independent generators of A. Using this system, any $u\in Z^r(K, A)$ may be written in the form of $\sum_{i=1}^{t}\bar{u}_i\, a_i$. It is easily seen that $\bar{u}_i$ is a cocycle mod. m_i, where m_i is the order of a_i. Now we define

$$\mathfrak{p}'u=\sum_i\gamma(a_i)\ \mathfrak{p}\bar{u}_i+\sum_{i<j}[a_i,\ a_j]\bar{u}_i\cup\bar{u}_j\ ,$$

where $\mathfrak{p}\bar{u}_i=\bar{u}_i\cup\bar{u}_i+\bar{u}_i\cup_1\delta\bar{u}_i$ and r is even. Then we have the

Proposition. If $u\in Z^r(K, A)$,

$$\mathfrak{p}'u\sim\mathfrak{p}u$$

in $Z^{2r}(K, A')$.

Proof. Since $u\in Z^r(K, A)$, $\bar{u}_i$ is a cocycle mod. m_i and $\bar{u}_ia_i\in Z^r(K, A)$. It follows from iii) of (3. 2) that

$$\mathfrak{p}u=\mathfrak{p}(\textstyle\sum_{i=1}^{t}\bar{u}_ia_i)\sim\sum_i\mathfrak{p}(\bar{u}_ia_i)+\sum_{i<j}(\bar{u}_ia_i)\cup(\bar{u}_ja_j).$$

From the definition of the cup products,

$$\bar{u}_ia_i\cup\bar{u}_ja_j=[a_i\ ,\ a_j]\bar{u}_i\cup\bar{u}_j\ .$$

Therefore it remains to prove that $\mathfrak{p}(\bar{u}_ia_i)\sim\gamma(a_i)\mathfrak{p}\bar{u}_i$ for any i. Let $\delta\bar{c}_i=n_i\bar{d}_i$ $(i=1,\dots, q;\ n_i|n_{i+1})$, where $n_i\bar{d}_i=0$ if $i>p$, and $(\bar{d}_1,\dots,\bar{d}_p)$ is part of a canonical basis for $C^{r+1}(K, I_0)$. Let $\bar{u}_i=\sum_j\Omega_{ij}\bar{c}_j$ where Ω_{ij} is an integer. Then

$$\delta\bar{u}_ia_i=\textstyle\sum_i\Omega_{ij}n_j\bar{d}_ja_i=0.$$

It follows that for any j, $\Omega_{ij}n_ia_i=0$, hence $m_i\,|\,\Omega_{ij}n_j$.
Thus $\Omega_{ij}\bar{c}_j$ is a cocycle mod. m_i. Therefore, by [7,(4. 7), p. 61],

$$\mathfrak{p}\bar{u}_i=\mathfrak{p}(\textstyle\sum_j\Omega_{ij}\bar{c}_j)\sim\sum_j\mathfrak{p}(\Omega_{ij}\bar{c}_j)+\sum_{j<k}(\Omega_{ij}\bar{c}_j\cup\Omega_{ik}\bar{c}_k)\quad \text{mod. } (m_i^2,\ 2m_i).$$

Since $(m_i^2\ ,\ 2m_i)\gamma(a_i)=0$, $\mathfrak{p}(\Omega_{ij}\bar{c}_j)=\Omega_{ij}^2\mathfrak{p}\bar{c}_j$, $\Omega_{ij}^2\gamma(a_i)=\gamma(\Omega_{ij}a_i)$ and $2\gamma(a_i)=[a_i\ ,\ a_i]$, it turns out that

$$\begin{aligned}\gamma(a_i)\mathfrak{p}\bar{u}_i&\sim\textstyle\sum_j\gamma(a_i)\mathfrak{p}(\Omega_{ij}\bar{c}_j)+\sum_{j<k}\gamma(a_i)(\Omega_{ij}\bar{c}_j\cup\Omega_{ik}\bar{c}_k)\\&=\textstyle\sum_j\gamma(\Omega_{ij}a_i)\mathfrak{p}\bar{c}_i+\sum_{j<k}[\Omega_{ij}a_i\ ,\ \Omega_{ik}a_i]\bar{c}_j\cup\bar{c}_k.\end{aligned}$$

By the definition of $\mathfrak{p}$, the right hand is

$$\mathfrak{p}(\textstyle\sum_j\Omega_{ij}a_i\bar{c}_i)=\mathfrak{p}(\bar{u}_ia_i).$$

This completes the proof.
From this proposition it is seen that the operation $\mathfrak{P}'$ in cohomology classes induced by $\mathfrak{p}'$ coincides with the Pontrjagin square $\mathfrak{P}$. Since $\mathfrak{P}$ is independent on the choice of independent generators of A, so is $\mathfrak{P}'$.

§4. Relations between $\pi_2(Y)$ and $\pi_3(Y)$

(4. 1) Let S_i^2 be an oriented 2-sphere, and $S_1^2\vee S_2^2\vee\dots\vee S_t^2$ a space consisting of the collection of S_1^2, S_2^2, …, S_t^2 intersecting in the unique point e^0. Let E^2 be an oriented closed 2-cube and β_i: $(E^2, \dot{E}^2)\rightarrow(S_i^2\ , e^0)$ be a map of degree $+1$ such that $\beta_i|(E^2-\dot{E}^2)$ is a homeomorphism of $E^2-\dot{E}^2$ onto $S_i^2-e^0$. We define a map ω_{ij} of the boundary $\dot{\mathcal{E}}_{ij}^4=E_i^2\times\dot{E}_j^2+\dot{E}_i^2\times E_j^2$ of $\mathcal{E}_{ij}^4=E_i^2\times E_j^2$ onto $S_i^2\vee S_j^2$ as follows:

$$\omega_{ij}(x,\ y)=\begin{cases}\beta_i(x) & (x,\ y)\in E_i^2\times\dot{E}_j^2\ ,\\ \beta_j(y) & (x,\ y)\in\dot{E}_i^2\times E_j^2\ .\end{cases}$$

Let

$$\rho: (S_i^2 \vee S_j^2,\ e^0) \to (Y,\ y_*)$$

be a map such that $\rho|S_i^2$, $\rho|S_j^2$ represent the elements a_i, a_j of $\pi_2(Y)$, respectively. Then the element of $\pi_3(Y)$ represented $\rho\omega_{ij}$: $\mathcal{E}_{ij}^4 \to Y$ is the Whitehead product $a_i \circ a_j$ [5,6]. The Whitehead product is bilinear.

(4.2) Let η: $\dot{E}^4 \to S^2$ be the Hopf map (i.e., a map with the Hopf invarient $+1$), and α: $S^2 \to Y$ a map representing a given element $a \in \pi_2(Y)$. The correspondence $\alpha \to \alpha\eta$ induces a map η_* of $\pi_2(Y)$ into $\pi_3(Y)$. Then η_* has the following properties [5,8,9]:

$$\eta_*(-a) = \eta_*(a) \qquad a \in \pi_2(Y),$$
$$\eta_*(a+b) = \eta_*(a) + \eta_*(b) + a \circ b \qquad a, b \in \pi_2(Y).$$

§5. The extension theorem

Let $K = K^4$ be a 4-dimensional finite cell complex whose cells σ_i^p are oriented, K^p the p-section of K and f a map of K^2 into Y. Since $\pi_1(Y)=0$, there exists a normal map f' such that $f' \simeq f$. For a map f', the difference cochain $d^2(f') = d^2(f', *) \in C^2(K,\ \pi_2(Y))$ is defined as usual, where $*$ denotes a constant map. Since $d^2(f')$ is independent on the choice of f', we shall define $d^2(f) = d^2(f')$. Clearly $\delta d^2(f)=0$ if and only if f is extendable over K^3.

Let us assume that f is extendable over K^3, and $\bar{f}$: $K^3 \to Y$ be an arbitrary extension of f over K^3. Then the 4-dimensional obstruction cocycle $c^4(\bar{f}) \in Z^4(K, \pi_3(Y))$ is to be defined. It is well known that the cohomology class $\{c^4(\bar{f})\}$ does not depend on the choice of an extension $\bar{f}$, but only on f[2]. Therefore we may denote this class by $\{z^4(f)\}$. Then the necessary and sufficient conditions for that f can be extended over K^4 are $\delta d^2(f)=0$ and $\{z^4(f)\}=0$.

(5. 2) By (4. 2), it is seen that $\{\pi_2(Y), \pi_3(Y), \eta_*\}$ is a system satisfying the conditions of (3. 1). And since $\pi_2(Y)$ has a finite number of generators, we obtain from (3. 2) or (3. 3) the Pontrjagin square of this system:

$$\mathfrak{P} = \mathfrak{P}' : H^2(K,\ \pi_2(Y)) \to H^4(K,\ \pi_3(Y)).$$

Then our main theorem is stated as follows.

Theorem 1. *If f: $K^2 \to Y$ is extendable over K^3,*

$$\{z^4(f)\} = \mathfrak{P}\{d^2(f)\}\ .$$

The proof will be given in §6 and §7.

From this theorem, we get

Theorem 2. (Extension theorem) The necessary and sufficient conditions for that a map f: $K^2 \to Y$ be extendable over K are $\delta d^2(f)=0$ and $\mathfrak{p}d^2(f) \sim 0$

Remark: Since $\mathfrak{P} = \mathfrak{P}'$, our result is different from the Whitney's [10] only in the respect that the latter contains some terms including $\cup_2$. However, it is easily seen by the following proposition, that these terms are coboundaries which vanish away in cohomology classes.

Proposition. If $u \in Z^p(K, I_0)$ and $p-i$ is odd, $2u \cup_i u \sim 0$.

Proof. See [4, p. 299].

Thus our result coincides with the Whitney's.

§ 6. Reduced complex

(6. 1) We assumed that $\pi_2(Y)$ has a finite number of generators. Let $a_1, a_2, \dots, a_t$ be its independent generators, where the order of a_i is m_i. We may suppose that $m_i>1$ or $m_i=0$ according as $i \leqq s$ or $i>s$, and $m_i | m_{i+1}$. We note that the integers t, s and the system $\{m_1, m_2, \dots, m_t\}$ are invariants of $\pi_2(Y)$, hence of topological space Y. We shall construct the special 4-dimensional cell complex R which is called *the reduced complex R for Y* [7].

Let $\mathcal{E}_i^p$ be a p-dimensional oriented closed cube, and e_i^p its interior, when $p>0$. $R=R^4$ is the cell complex, which satisfies the following conditions:

(i) $R^1=R^0=$a single point e^0.

(ii) $R^2=R^1+e_1^2+\cdots+e_t^2$, where $e_i^2 (i=1, 2, \dots, t)$ is attached to R^1 by a map $\dot{\mathcal{E}}_i^2 \to e^0$. Thus $e_i^2+e^0$ is a 2-sphere S_i^2, and $R^2=S_1^2 \vee S_2^2 \vee \dots \vee S_t^2$.

(iii) $R^3=R^2+e_1^3+e_2^3+\cdots+e_s^3$, where $e_i^3 (i=1, 2, \dots, s)$ is attached to R^2 by a map $\dot{\mathcal{E}}_i^3 \to S_i^2$ of degree $m_i (>1)$.

(iv) $R^4=R^3+e_1^4+e_2^4+\cdots+e_t^4+e_{12}^4+e_{13}^4+\cdots+e_{1t}^4+e_{23}^4+\cdots+e_{t-1\,t}^4$, where $e_i^4 (i=1, 2, \dots, t)$ is attached to R^2 by the Hopf map η_i: $\dot{\mathcal{E}}_i^4 \to S_i^2$, and e_{ij}^4 $(i<j,\ i, j=1, 2, \dots, t)$ is attached to R^2 by the map ω_{ij}: $\mathcal{E}_{ij}^4 \to S_i^2 \vee S_j^2$ defined in (4.1).

(6.2) Let $h: R^2 \to Y$ be a map such that $h|S_i^2 : S_i^2 \to Y$ represents $a_i \in \pi_2(Y)$. Then since $c^3(h)=0$, h has an extension $\bar{h}$: $R^3 \to Y$. We shall prove Theorem 1 for R^4 and h. Let $\bar{e}_i^2$, $\bar{e}_i^4$, $\bar{e}_{ij}^4$ be integral cochains which take 1 as coefficient on e_i^2, e_i^4, e_{ij}^4 respectively. Since $\bar{e}_j^2 (j \geqq i)$ is a cocycle mod. m_i, $\bar{e}_i^2 \cup \bar{e}_j^2$ is a cocycle mod. m_i and $\mathfrak{p}\bar{e}_i^2$ is a cocycle mod. $(m_i^2, 2m_i)$. And, from [7, Theorem 5, p. 78], we have

$$\bar{e}_i^2 \cup \bar{e}_j^2 \sim \bar{e}_{ij}^4 \quad \text{mod. } m_i \quad (i<j),$$
$$\mathfrak{p}\,\bar{e}_i^2 \sim \bar{e}_i^4 \quad \text{mod. } (m_i^2, 2m_i).$$

Since $d^2(h)=\sum_i a_i \bar{e}_i^2$ and $a_i \bar{e}_i^2 \in Z^2(R, \pi_2(Y))$, it follows from (3.2) that

$$\begin{aligned}\mathfrak{p}d^2(h) &\sim \textstyle\sum_i \mathfrak{p}(a_i \bar{e}_i^2)+\sum_{i<j} a_i \bar{e}_i^2 \cup a_j \bar{e}_j^2 \\ &= \textstyle\sum_i \eta_*(a_i)\mathfrak{p}\bar{e}_i^2+\sum_{i<j}(a_i \circ a_j)\bar{e}_i^2 \cup \bar{e}_j^2.\end{aligned}$$

Noting that $(m_i^2, 2m_i)\eta_*(a_i)=0$ and $m_i(a_i \circ a_j)=0$, we have

$$\mathfrak{p}d^2(h) \sim \textstyle\sum_i \eta_*(a_i)\bar{e}_i^4+\sum_{i<j}(a_i \circ a_j)\bar{e}_{ij}^4.$$

On the other hand, by the difinition,

$$c^4(\bar{h})=\textstyle\sum_i \eta_*(a_i)\bar{e}_i^4+\sum_{i<j}(a_i \circ a_j)\bar{e}_{ij}^4.$$

Thus
$$\mathfrak{p}d^2(h) \sim c^4(\bar{h}),$$
i. e.,
$$\mathfrak{P}\{d^2(h)\}=\{z^4(h)\}. \qquad \text{Q. E. D.}$$

Remark. The above-mentioned reduced complex is a special case of reduced complexes which were given by J. H. C. Whitehead [7].

(6.3) Here we shall note some properties of homotopy groups of R which will be needed in the following parts. Let β_i be a map which was defined in (4.1), and b_i, b_i' be respectively an element of $\pi_2(R^2)$, an element of $\pi_2(R^3)$, both of which are represented by β_i. Let $i: R^2 \to R^3$ be an identity map, i_* the homomorphism of $\pi_3(R^2)$ into $\pi_3(R^3)$ induced by i. Then i_* is onto [5. Lemma 3]. On the other hand, $\pi_3(R^2)$ is a free abelian group which is generated by $\eta_*(b_i)$ $(i=1, 2, \dots, t)$ and $b_i \circ b_j (i<j,\ i, j=1, 2, \dots, t)$ [1, 7]. Therefore $\pi_3(R^3)$ is generated by $\eta_*(b_i')$ $(i=1, 2, \dots, t)$ and $b_i' \circ b_j'$ $(i<j,\ i, j=1, 2, \dots, t)$.

§7. Proof of Theorem I

(7.1) Let $f: K^2 \to Y$ be an arbitrary map. Since $\pi_1(Y)=0$, there exists a normal map f' such that $f' \simeq f$. Let σ_k^2 be an arbitrary oriented 2-cell. We shall denote by $\sum_{i=1}^t c_{ki} a_i$ an element of $\pi_2(Y)$ which is represented by $f' | \dot{\sigma}_k^2$, where c_{ki} is an integer. In each σ_k^2, choose t disjoint closed 2-cubes ${}^kE_1^2, {}^kE_2^2, \dots, {}^kE_t^2$, oriented in agreement with σ_k^2. Let $g: K^2 \to R^2$ be a map such that $g | {}^kE_i^2 : ({}^kE_i^2, {}^k\dot{E}_i^2) \to (S_i^2, e^0)$ is a map of degree c_{ki} for any k, i and $g(K^2 - \bigcup_{k,i} {}^kE_i^2) = e^0$. It is clear that $hg \simeq f'$, hence $hg \simeq f$. Therefore $d^2(f) = d^2(hg)$.

Now suppose that f can be extended over K^3. It follows from the homotopy extension property that hg has an extension. Since $\pi_1(Y)=0$, it is seen [2] that

$$\{z^4(f)\} = \{z^4(hg)\}.$$

From these arguments, without loss of generality, we may suppose that f is hg for the purpose of proving Theorem 1.

(7.2) Suppose that $f = hg: K^2 \to Y$ has an extension over K^3. Let σ_j^3 be an arbitrary oriented 3-cell of K, and $\dot{\sigma}_j^3 = \sum_k \varepsilon_k^j \sigma_k^2$, where ε_k^j is the incidence number between σ_j^3 and σ_k^2. Then $h | \dot{\sigma}_j^3$, $f | \dot{\sigma}_j^3$ represent $\sum_{k,i} \varepsilon_k^j c_{ki} b_i \in \pi_2(R^2)$, $\sum_{k,i} \varepsilon_k^j c_{ki} a_i \in \pi_2(Y)$ respectively. Since f is extendable over K^3,

$$\sum_k \varepsilon_k^j c_{ki} \equiv 0 \mod. m_i$$

for any $i = 1, 2, \dots, t$. Therefore, $\sum_k \varepsilon_k^j c_{ki} = 0$ if $i > s$, and $\sum_k \varepsilon_k^j c_{ki} = n_i^j m_i$ (n_i^j : integer) if $i \leqq s$.

In σ_j^3, choose s closed 3-cubes ${}^jE_1^3, {}^jE_2^3, \dots, {}^jE_s^3$ oriented in agreement with σ_j^3, having only a single point p_j in common. Let ${}^jE_*^3 = {}^jE_1^3 \vee {}^jE_2^3 \vee \dots \vee {}^jE_s^3$ if $s>0$ and ${}^jE_*^3 = p_j$ if $s=0$. Let $\psi_j : ({}^jE_*^3, p_j) \to (R^3, e^0)$ be a map such that $\psi_j | {}^jE_i^3 : ({}^jE_i^3, {}^j\dot{E}_i^3) \to (\mathcal{E}_i^3, \dot{\mathcal{E}}_i^3)$ is a map of degree n_i^j. Let ${}^jQ^3$ be a 3-cube such that ${}^jE_*^3 \subset {}^jQ^3 \subset \operatorname{Int.} \sigma_j^3$. ${}^jQ^3$ is oriented in agreement with σ_j^3. Let $q(p)$ be a point which a straight line $\overline{p_j\, p}$ interesects with ${}^jQ^3$, where p is an arbitrary point of ${}^jE_*^3$. Map all points of $\overline{pq(p)}$ to $\psi_j(p) \in R^2$, and ${}^jQ^3 - {}^jE_* - \bigcup_{p \in {}^j\dot{E}_*^3} \overline{pq(p)}$ on e^0. Then we get an extension of ψ_j, $\bar{\psi}_j : {}^jQ^3 \to R^3$, such that $\bar{\psi}_j | {}^j\dot{Q}^3 : {}^j\dot{Q}^3 \to R^2$ represents an element $\sum_i n_i^j m_i b_i$ of $\pi_2(R^2)$. On the other hand, $g | \dot{\sigma}_j^3$ also represents $\sum_{i,k} \varepsilon_k^j c_{ki} b_i$

$=\sum_i n_i^j m_i b_i$. Therefore $g|\dot{\sigma}_j^3$ and $\psi_j|{}^j\dot{Q}^3$ are homotopic in R^2. Using this homotopy, we obtain a mapping $\bar{g}_j : \sigma_j^3 \to R^3$, which is an extension of $\bar{\psi}_j$ and $g|\dot{\sigma}_j^3$. Let $\bar{g}$: $K^3 \to R^3$ be a map such that $\bar{g}|\sigma_j^3 = \bar{g}_j$, then $\bar{g}$ is an extension of g, and $\bar{f} = \bar{h}\bar{g}$ is an extension of f.

(7. 3) Let σ_l^4 be an arbitrary oriented 4-cell. Then, from (6. 3), an element of $\pi_3(R^3)$ which is represented by a map $\bar{g}|\dot{\sigma}_l^4$ have a form

$$\sum_i \Gamma_i^l \eta_*(b_i') + \sum_{i<j} \Gamma_{ij}^l b_i' \circ b_j',$$

where Γ_i^l, Γ_{ij}^l are integers.

Now choose in σ_l^4 $\dfrac{t(t+1)}{2}$ closed 4-cubes ${}^lE_1^4, \dots, {}^lE_t^4, {}^lE_{12}^4, {}^lE_{13}^4, \dots, {}^lE_{1t}^4, {}^lF_{23}^4, \dots, {}^lE_{t-1\,t}$, oriented in agreement with σ_l^4, with a single point p_l' in common. Let ${}^lE_*^4 = {}^lE_1^4 \vee \dots \vee {}^lE_t^4 \vee {}^lE_{12}^4 \vee {}^lE_{13}^4 \wedge \dots \vee {}^lE_{t-1\,t}^4$. Let $\varphi_l : ({}^lE_*^4, p_l') \to (R^4, e^0)$ be a map such that $\varphi_l|{}^lE_i^4 : ({}^lE_i^4, {}^l\dot{E}_i^4) \to (\dot{\mathcal{E}}_i^4, \mathcal{E}_i^4)$, $\varphi_l|{}^lE_{ij}^4 : ({}^lE_{ij}^4, {}^l\dot{E}_{ij}^4) \to (\mathcal{E}_{ij}^4, \dot{\mathcal{E}}_{ij}^4)$ are maps of degree Γ_i^l, Γ_{ij}^l respectively. Let ${}^lQ^4$ be an 4-cube which is oriented in agreement with σ_l^4 and satisfies the condition ${}^lE_*^4 \subset {}^lQ^4 \subset$ Int. σ_l^4. Using the similar argument as in (6. 2), we can construct an extension of φ_l, $\bar{\varphi}_l : {}^lQ^4 \to R$ such that ${}_l\bar{\varphi}|{}^l\dot{Q}^4$ represents $\sum_i \Gamma_i^l \eta_*(b_i') + \sum_{i<j} \Gamma_{ij}^l b_i' \circ b_j' \in \pi_3(R^3)$. On the other hand, since $\bar{g}|\dot{\sigma}^4$ also represents it, $\bar{\varphi}|{}^l\dot{E}_i^4$ and $\bar{g}|\dot{\sigma}_l^4$ are homotopic in R^3. Using this homotopy, we can get a map $\bar{\bar{g}}_l : \sigma_l^4 \to R^4$ which is an extention of $\bar{g}|\dot{\sigma}_l^4$ and $\bar{\varphi}_l$. Let $\bar{\bar{g}} : K^4 \to R^4$ be a map such that $\bar{\bar{g}}|\sigma_l^4 = \bar{\bar{g}}_l$, then $\bar{\bar{g}}$ is an extension of $\bar{g}$.

Now using the general theory of continuous extension by Hu[3], we can see that

$$\{c^4(\bar{h}\bar{g})\} = g^*\{c^4(h)\},$$
$$\{d^2(hg)\} = \bar{g}^*\{d^2(h)\}.$$

Since $\{z^4(f)\} = \{c^4(\bar{h}\bar{g})\}$, $\{c^4(\bar{h})\} = \mathfrak{P}\{d^2(h)\}$ and $\mathfrak{P}$ is natural, we have

$$\{z^4(f)\} = \{c^4(\bar{h}\bar{g})\} = \bar{\bar{g}}^*\{c^4(\bar{h})\} = \bar{\bar{g}}^*\mathfrak{P}\{d^2(h)\}$$
$$= \mathfrak{P}\bar{\bar{g}}^*\{d^2(h)\} = \mathfrak{P}\{d^2(hg)\} = \mathfrak{P}\{d^2(f)\}.$$

Thus we have completed the proof of Theorem 1.

Literatures

[1] A. L. Blakers and W. S. Massey: The homotopy groups of a triad, Ann. of Math. 52, (1951).

[2] S. Eilenberg: Cohomology and continuous mapping, Ann. of Math., 41, (1940).

[3] S. T. Hu: Extension and classification of maps. Osaka Math. Journ., 2, (1950).

[4] N. E. Steenrod: Products of cocycles and extensions of mappings, Ann. of Math. 48 (1947).

[5] G. W. Whitehead: A generalization of the Hopf invariants, Ann. of Math., 51, (1950).

[6] J. H. C. Whitehead: On adding relations to homotopy groups, Ann. of Math., 42, (1941).

[7] J. H. C. Whitehead: On simply connected, 4-dimensional polyhedra, Comm. Math. Helv., 22 (1949).

[8] J. H. C. Whitehead: A certain exact sequence, Ann. of Math. 52 (1951).

[9] H. Whitney: Relations between the second and third homotopy groups of a simply connected space, Ann. of Math., 50 (1949).

[10] H. Whitney: An extension theorem for mappings into simply connected spaces, Ann. of Math. 50 (1949).