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A Characterization of the Lattice of Lower Semi-Continuous Functions on T_1-Space

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I. Kaplansky characterized some sublattice of the lattice K of all continuous functions on topological space by the axiom of "translation lattice", lattice in which translations were defined. It seems that his purpose is to characterize K itself by some axiom of lattice. However the author showed that the lattice of semi-continuous functions characterized the topology of the completely regular (not necessarily compact) topological space, so the characterization of such a lattice is important in the effort to treat non-compact topological spaces with methods of lattice theory.

In this paper we shall characterize the lattice of all lower semi-continuous, non-negative and bounded functions on T_1-space.

We concern ourselves with the lattice L with the operation of all non-negative numbers which satisfies the following axioms.

Axioms. 1) *L is a complete distributive lattice with the least element* 0.

2) $\alpha \geqq \beta$ *implies* $\alpha f \geqq \beta f$. $f \geqq g$ *implies* $\alpha f \geqq \alpha g$. $\alpha(\beta f)=(\alpha\beta)f$.

3) $1 \cdot f = f$.

4) $\inf_{\alpha>0} \alpha f = 0$ *for every* f.

5) $(\sup_\alpha f_\alpha) \cap f = \sup_\alpha (f_\alpha \cap f)$.

(*S*) (*Separation axiom*) *If* $g \not\leqq f$, *then for all Ar. elements* φ, *there exist* α, β *and some max element* m *such that* $\alpha > \beta$, $f \cup \gamma m \geqq \alpha\varphi \geqq \beta\varphi$, $g \cup \gamma m \not\geqq \beta\varphi$ $(\gamma \geqq \gamma_0)$ *for some* γ_0.

(*E*) *There exists Ar. element* φ *such that*

$$\inf_\alpha (f_\alpha \cup \beta\varphi) = \beta\varphi \cup \inf_\alpha f_\alpha \text{ for any } f_\alpha, \beta,$$

$$\inf_\alpha (f \cup \beta_\alpha \varphi) = f \cup \inf_\alpha \beta_\alpha \varphi \text{ for any } f, \beta_\alpha.$$

(*We denote one of such Ar. elements by e.*)

Definitions. In the axioms above, we mean by an *Ar. element* an element φ such that for every element f of L $\alpha\varphi \geqq f$ for some α, and we mean by a *max element* a non-Ar. element m such that there exists some Ar. element $\varphi_m \geqq m$ such that if $m < f \leqq \varphi_m$, then f is Ar. We call this φ_m an *upper element of m.*

Remark. Small Latin letters $f, g, \ldots$ and Greak letters φ, ψ are used for elements of L. Greek letters $\alpha, \beta, \ldots$ except φ and ψ are used for oprators, i. e. positive numbers.

Lemma 1. *If m is a max element, then $m \cup f \geqq \alpha e$ and $\alpha > \beta$ implies $m \cup \beta e \ngeqq f$.*

Proof. From now forth we denote by m a max element and by φ_m one of upper elements of m.

1. We can reduce our problem to case of $\varphi_m \geqq \alpha e$. For, if this lemma is established in case of $\varphi_m \geqq \alpha e$, and for a general max element m, $m \cup f \geqq \alpha e$. $\alpha > \beta$ and $m \cup \beta e \geqq f$ hold, then also for a number γ such that $\gamma \varphi_m \geqq \alpha e$, $\gamma \geqq 1$ we get $\gamma m \cup f \geqq \alpha e$, $\gamma m \cup \beta e \geqq f$. However, since γm is obviously a max element having $\gamma \varphi_m$ as an upper element, this is a contradiction.

2. Firstly we prove that $m \cap \alpha e$ $(\varphi_m \geqq \alpha e, \alpha > 0)$ *is a max element having αe as an upper element.*

For let $m \cap \alpha e < f \leqq \alpha e \leqq \varphi_m$, then if $m = f \cup m$, from $\varphi_m \geqq \alpha e$ we get $m \cap \alpha e \geqq f$, which contradicts $m \cap \alpha e < f$. Hence it must be $m < f \cup m \leqq \varphi_m$; hence $f \cup m$ is Ar, i. e. there exists $\beta > 0$ such that $f \cup m \geqq \beta e$. Therefore $f \geqq (f \cap \alpha e) \cup (m \cap \alpha e) = (f \cup m) \cap \alpha e \geqq \beta e \cap \alpha e$, i. e. f is Ar.

3. Next we show that *for a max element m, $\varphi_m \geqq \alpha e$ and $\alpha \geqq \beta$ imply $\frac{\beta}{\alpha}(m \cap \alpha e) = m \cap \beta e$.*

Since $\frac{\beta}{\alpha}(m \cap \alpha e) \leqq m \cap \beta e$ is obvious, we assume that $\frac{\beta}{\alpha}(m \cap \alpha e) < m \cap \beta e$. Then, since $m \cap \alpha e < \frac{\alpha}{\beta}(m \cap \beta e) \leqq \alpha e$, $\frac{\alpha}{\beta}(m \cap \beta e)$ is Ar. from $\varphi_m \geqq \alpha e$ and 2, i. e. $\frac{\alpha}{\beta}(m \cap \beta e) \geqq \gamma e$ for some $\gamma > 0$. Hence $m \geqq \frac{\beta\gamma}{\alpha} e$, which contradicts the fact that m is non-Ar.

4. We remark that $\alpha(f \cup g) = \alpha f \cup \alpha g$ holds generally. For $\geqq$ is obvious, and $\leqq$ can be taken from $f \cup g = \frac{1}{\alpha}(\alpha f) \cup \frac{1}{\alpha}(\alpha g) \leqq \frac{1}{\alpha}(\alpha f \cup \alpha g)$.

Next we show that *$n < e$ and $\gamma < 1$ imply $n \cup \gamma e \neq e$.*

For assume that $n \cup \gamma e = e$. If we assume $n \cup \gamma^k e = e$ for a positive integer k, then $\gamma n \cup \gamma^{k+1} e = \gamma e$; hence from $n \cap \gamma e \geqq \gamma n$ we get $\gamma e = (n \cap \gamma e) \cup \gamma^{k+1} e = (n \cup \gamma^{k+1} e) \cap \gamma e$, i. e. $n \cap \gamma^{k+1} e \geqq \gamma e$. Hence $n \cup \gamma^{k+1} e = n \cup (n \cup \gamma^{k+1} e) \geqq n \cup \gamma e = e$; hence $n \cup \gamma^{k+1} e = e$. Therefore we get $n \cup \gamma^k e = e$ for any positive integer k. Hence from Axiom (E) $e = \inf_k (n \cup \gamma^k e) = n \cup \inf \gamma^k e = n \cup 0 = n$, which contradicts $n < e$.

5. Now we prove Lemma 1. Assume that this proposition is false, i. e. $\varphi_m \geqq \alpha e$, $m \cup f \geqq \alpha e$, $\alpha > \beta$, $m \cup \beta e \geqq f$, then $m \cup \beta e \geqq m \cup f \geqq m \cup \alpha e$; hence from 3 $\alpha e = (m \cup \beta e) \cap \alpha e = (m \cap \alpha e) \cup \beta e = \frac{\alpha}{\beta}(m \cap \beta e) \cup \beta e$. Hence $\frac{1}{\alpha}\left\{\frac{\alpha}{\beta}(m \cap \beta e) \cup \beta e\right\} = \frac{1}{\alpha}\alpha e$, i. e. $\frac{1}{\beta}(m \cap \beta e) \cup \frac{\beta}{\alpha} e = e$. Since $\frac{1}{\beta}(m \cap \beta e) < e$ and $\alpha > \beta$, this formula contradicts 4. Thus this lemma is established.

Remark. From Axiom (S) we see easily that if $f \ngeqq g$, there exist m and α such that $f \cup \beta m \geqq \alpha e$, $g \cup \beta m \ngeqq \alpha e$ $(\beta \geqq \beta_0)$ for some β_0. For $f \ngeqq g$ implies $f \cap g < f$, hence there exist m and α such that $(f\cap g)\cup\beta m\not\geqq\alpha e$. $f\cup\beta m\geqq\alpha e$ $(\beta\geqq\beta_0)$. Since $g\cup\beta m\geqq\alpha e$ implies $(f\cap g)\cup\beta m=(f\cup\beta m)\cap(g\cup\beta m)\geqq\alpha e$, it must be $g\cup\beta m\not\geqq\alpha e$ for these m, α and β.

Lemma 2. *If $\varphi_m\geqq f, \gamma e$; $f\cup m\not\geqq\gamma e$ for a max element m, then $m\cup\gamma e\geqq f$.*

Proof. 1. Firstly we show that *$f\cup m\not\geqq g$ and $\varphi_m\geqq f, g$ imply $f\cup\alpha m\not\geqq g$ for any $\alpha\geqq1$.*

We can see easily that $\alpha m\cap\varphi_m=m$, for if $\alpha m\cap\varphi_m>m$, then from $\varphi_m\geqq\alpha m\cap\varphi_m>m$, $\alpha m\cap\varphi_m$ must be Ar. But this is impossible; hence $m=\alpha m\cap\varphi_m$. From this fact $f\cup\alpha m\geqq g$ and $\alpha\geqq1$ are impossible. For then $g\leqq(f\cup\alpha m)\cap\varphi_m=(f\cap\varphi_m)\cup(\alpha m\cap\varphi_m)\leqq f\cup m$, which is a contradiction.

2. We see easily that *$\alpha m\cup\gamma e\geqq f$ and $\varphi_m\geqq f$ imply $m\cup\gamma e\geqq f$.* For from the proof of 1 $f\leqq(\alpha m\cup\gamma e)\cap\varphi_m=(\alpha m\cap\varphi_m)\cup(\gamma e\cap\varphi_m)\leqq m\cup\gamma e$.

3. Now we prove Lemma 2. From 1 and 2 we may assume that $\varphi_m\geqq\beta e\geqq f$ for some β. If $m\cup\gamma e\not\geqq f$, then from the remark above we can choose some max element n and α such that $n\cup f\geqq\alpha e$, $n\cup(m\cup\gamma e)\not\geqq\alpha e$ and $\varphi_n\geqq\beta e\geqq f$.

(a) In case that $n\cup m$ is non-Ar.

Take a number $\lambda\geqq1$ such that $\lambda\varphi_m\geqq\varphi_n$. Then, if $\lambda m\not\geqq n$, $\lambda m<n\cup\lambda m\leqq\lambda\varphi_m$; hence $n\cup\lambda m$ must be Ar, which is impossible. Therefore it must be $\lambda m\geqq n$. Hence $\alpha e\leqq n\cup f\leqq\lambda m\cup f$. Since $n\cup(m\cup\gamma e)\not\geqq\alpha e$; it must be $\alpha e>\gamma e$; hence $\lambda m\cup f>\gamma e$. Therefore from 1 we get $f\cup m\geqq\gamma e$, which is a contradiction.

(b) In case that $n\cup m$ is Ar.

There exists α_0 such that $n\cup m\geqq\alpha_0 e$, $\beta\geqq\alpha_0>0$. For this α_0 $\alpha_0 e=\alpha_0 e\cap(n\cup m)=(n\cap\alpha_0 e)\cup(m\cap\alpha_0 e)=\frac{\alpha_0}{\beta}(n\cap\beta e)\cup\frac{\alpha_0}{\beta}(m\cap\beta e)\leqq\frac{\alpha_0}{\beta}n\cup\frac{\alpha_0}{\beta}m$ holds from 3 of the proof of Lemma 1, $\varphi_n\geqq\beta e$ and $\varphi_m\geqq\beta e$. Hence $n\cup m\geqq\beta e\geqq f$; hence $n\cup m\cup\gamma e\geqq f\cup n\geqq\alpha e$, which is a contradiction. Thus the proof of Lemma 2 is complete.

Lemma 3. $f_\beta'=\inf\{m|m:\text{max}; \varphi_m\geqq\beta e, f; m\cup\beta e\geqq f\}\cap\beta e\leqq f$.

Proof. Assume that $f_\beta'\not\leqq f$, then there exist a max element n and γ such that $f_\beta'\cup n\geqq\gamma e$, $f\cup n\not\geqq\gamma e$, $\varphi_n\geqq f, \gamma e, \beta e$ from the remark. Since $\beta e\cup n\geqq((\inf m)\cap\beta e)\cup n\geqq\gamma e$ we get $\gamma\leqq\beta$ from Lemma 1. Hence from Lemma 2 $n\cup\beta e\geqq n\cup\gamma e\geqq f$ holds. Since $\varphi_n\geqq\beta e$, $n\cup\beta e\geqq f$, it must be $n\geqq\inf m$; hence $\gamma e\leqq f_\beta'\cup n=((\inf m)\cap\beta e)\cup n=n$, which contradicts the fact that n is non-Ar. Thus it must be $f_\beta'\leqq f$.

Lemma 4. *If $m\cup f\geqq\alpha e$, $\alpha>\beta$, $\varphi_m\geqq f$, then $m\cup f_\beta'\geqq\beta e$ and $\frac{\beta}{\gamma}(f_\beta'\cap\gamma e)\leqq f$ hold for every $\gamma\leqq\beta$, where f_β' is the one in Lemma* 3.

Proof. $\frac{\beta}{\gamma}(f_\beta'\cap\gamma e)\leqq\frac{\beta}{\gamma}((\inf m)\cap\gamma e)=\frac{\beta}{\gamma}\inf(m\cap\gamma e)=\inf\frac{\beta}{\gamma}(m\cap\gamma e)=\inf(m\cap\beta e)=\beta e\cap\inf m\leqq\beta e\cap f$ from 3 of the proof of Lemma 1 and Lemma 3.

Next we show that $\varphi_m\cap\inf m\not\leqq m$. For if we assume that $\varphi_m\cap\inf m\leqq m$, then from Axiom (E) $m\cup\beta e\geqq(\varphi_m\cap\inf m)\cup\beta e=\inf(m\cup\beta e)\cap(\varphi_m\cup\beta e)\geqq f$, which contradicts $m\cup f\geqq\alpha e$, $\alpha>\beta$ from Lemma 1. Hence $\varphi_m\cap\inf m\not\leqq m$; hence $m<m\cup(\varphi_m\cap\inf m)\leqq\varphi_m$. Therefore $m\cup(\varphi_m\cap\inf m)$ is Ar, i. e. there exists a number

γ_0 such that $m \cup \inf m \geqq \gamma_0 e$, where we may take γ_0 so that $\gamma_0 \leqq \beta$. Hence

$$\beta e \leqq \frac{\beta}{\gamma_0}(m \cup \inf m) \cap \beta e = \left(\frac{\beta}{\gamma_0} m \cap \beta e\right) \cup \left(\beta e \cap \frac{\beta}{\gamma_0} \inf m\right)$$
$$= \frac{\beta}{\gamma_0}(m \cap \gamma_0 e) \cup \inf \frac{\beta}{\gamma_0}(m \cap \gamma_0 e) = (m \cap \beta e) \cup \inf (m \cap \beta e) = \beta e \cap (m \cup \inf m) \leqq m \cup f_\beta'$$

from 3 of the proof of Lemma 1. Thus this lemma is proved.

Definitions. 1) $L \supset L_e = \left\{ f \mid f \leqq e,\ f = \frac{1}{\alpha}(f \cap \alpha e) \text{ for all } \alpha \leqq 1 \right\}$.

2) When $\varphi_m \geqq e$, we denote by m_e the max element $m \cap e$.

From 3 of the proof of Lemma 1, $m_e \in L_e$ is obvious.

3) We mean by a *unit ideal* a subset I of L such that

$$I = \{ u \mid u \in L_e,\ u \leqq m_e \} = I(m).$$

4) We denote by $\mathfrak{L}$ the set of all unit ideals of L.

5) $\mathfrak{L} \supset F(u) = \{ I \mid u \in I \}\ (u \in L_e)$.

6) $f_\alpha^* = \sup \{ u \mid u \in L_e,\ \alpha u \leqq f \cap \alpha e \}\ (\alpha \geqq 0)$

We can easily see that $f_\alpha^* \in L_e$. For $\frac{1}{\beta}(f_\alpha^* \cap \beta e) = \frac{1}{\beta}\{(\sup u) \cap \beta e\} = \frac{1}{\beta} \sup (u \cap \beta e) = \frac{1}{\beta} \sup \beta u = f_\alpha^*$ for $\beta \leqq 1$ from Axiom (5).

Next if $\beta \geqq \alpha$, $u \in L_e$ and $\beta u \leqq f \cap \beta e$, then $\alpha u \leqq \frac{\alpha}{\beta}(f \cap \beta e) \leqq f \cap \alpha e$, i. e. we get $F(f_\beta^*) \supseteqq F(f_\alpha^*)$ for $\beta \geqq \alpha$.

Theorem. *In order that a lattice L with the operation of all non-negative numbers is operation-isomorphic with the lattice of all lower semi-continuous, non-negative and bounded functions on some T_1-space, it is necessary and sufficient that L satisfies Axioms 1)–5), (S) and (E).*

Proof. Since it is easy to see the validity of the necessity, we shall show that an operation-lattice L satisfying 1)–5), (S) and (E) is operation-isomorphic with the lattice of all lower semi-continuous, non-negative and bounded functions on some T_1-space.

1. We introduce a topology into $\mathfrak{L}$ by the closed sets

$$F(u) = \{ I \mid u \in I \} \quad (u \in L_e)$$

$F(\cup u_\alpha) = \Pi F(u_\alpha)$ is obvious. We show that $F(u_1 \cap u_2) = F(u_1) \cup F(u_2)$. Since $\supseteqq$ is obvious, we prove $\subseteqq$. Assume that $u_1 \cap u_2 \in I$, $u_1 \notin I$, $u_2 \notin I$ and $I = I(m)$, then $u_1 \nleqq m_e$; hence $e \geqq u_1 \cup m_e > m_e$. Therefore $L_e \ni u_1 \cup m_e \geqq \beta e$ for some $0 < \beta \leqq 1$; hence $u_1 \cup m_e = \frac{1}{\beta}((u_1 \cup m_e) \cap \beta e) = e$. We get $u_2 \cup m_e = e$ on the same ground. Hence $m_e \cup (u_1 \cap u_2) = e$, which contradicts $u_1 \cap u_2 \leqq m_e$. It is easy to see that $F(m_e) = \{I(m)\}$. Hence L is a T_1-space by this topology.

We denote by $L(\mathfrak{L})$ the lattice of all lower semi-continuous, non-negative and bounded functions on $\mathfrak{L}$. We define a mapping from L into $L(\mathfrak{L})$ in the following manner

$$L \ni f \to F \in L(\mathfrak{L}),\ F(I)=\inf\left\{\alpha \,|\, I \in F(f_\alpha^*)\right\}.$$

Since $\{F(I)\leqq\alpha\}=\prod_{\alpha<\beta} F_\beta$ is a closed set in $\mathfrak{L}$, $F(I)$ is a lower semi-continuous function on $\mathfrak{L}$. (We denote by F_α $F(f_\alpha^*)$).

Next we see that $u\in L_e$, $\delta<1$ and $u\leqq\delta e$ imply $u=0$. For, then $u=\frac{1}{\delta}(u\frown\delta e)=\frac{1}{\delta}u$, i. e. $u=\delta u$; hence $u=\delta^k u$ for any positive integer k. Therefore $u=\inf\delta^k u=0$.

From this fact we see that $f\leqq\gamma e<\alpha e$ implies $F_\alpha=\mathfrak{L}$. For if $\alpha u\leqq f\frown\alpha e$, then $\alpha u\leqq\gamma e$, i. e. $u\leqq\frac{\gamma}{\alpha}e$; hence $u=0$. Therefore it must be $f_\alpha^*=0$, i. e. $F_\alpha=\mathfrak{L}$. This fact shows that $f\leqq\gamma e$ implies $F(I)\leqq\gamma$. Therefore $F(I)\in L(\mathfrak{L})$.

2. Firstly we show that if $f\to F_f$, $g\to G_g$ by this mapping, and $g<f$, then $G_g\leqq\neq F_f$.

When $g<f$, there exist $\alpha>0$ and a max element m such that $g\smile m\ngeqq\alpha e$, $f\smile m\geqq\alpha e$, $\varphi_m\geqq f$, e, αe.

a) Let us prove that $F_f(I(m))\geqq\alpha$.

For any positive number $\beta<\alpha$, from Lemma 4 there exists u such that $u\smile m\geqq\beta e$, $\frac{\beta}{\gamma}(u\frown\gamma e)\leqq f$ for all $\gamma\leqq\beta$, where

$$u=f_\beta'=\inf\left\{m\,|\,m:\ \max,\ \varphi_m\geqq\beta e,\ f;\ m\smile\beta e\geqq f\right\}\frown\beta e.$$

Let $\gamma\leqq\beta$, then from 2 and 3 of the proof of Lemma 1,

$$\frac{\beta}{\gamma}(u\frown\gamma e)=\frac{\beta}{\gamma}(\inf(m\frown\beta e)\frown\gamma e)=\inf\frac{\beta}{\gamma}((m\frown\beta e)\frown\gamma e)=\inf(m\frown\beta e)=u.$$

Since $\gamma\beta\leqq\beta$ holds for $\gamma\leqq1$, we get $\frac{\beta}{\gamma\beta}(u\frown\gamma\beta e)=u$ from the above mentioned fact. Hence $\frac{1}{\gamma}\left(\frac{1}{\beta}u\frown\gamma e\right)=\frac{1}{\beta}u$, i. e. $\frac{1}{\beta}u\in L_e$.

Since $u\leqq f$ from Lemma 3, we get $\beta\cdot\frac{1}{\beta}u\leqq f\frown\beta e$.

Next, assume that $\frac{1}{\beta}u\leqq m_e=m\frown e$, then $\beta e\leqq u\smile m\leqq\beta m_e\smile m$, which is impossible. Hence it must be $\frac{1}{\beta}u\nleqq m_e$. Therefore $f_\beta^*\nleqq m_e$ for all $\beta<\alpha$. Thus $F_f(I(m))\geqq\alpha$ is proved.

b) Next we prove that for some $\beta<\alpha$, $G_g(I(m))\leqq\beta$ holds.

If $g\frown\alpha e\geqq\alpha u$, $u\in L_e$, then $g\smile m\geqq\alpha u\smile m$. Since $\alpha u\smile m\geqq\alpha e$ implies $g\smile m\geqq\alpha e$, and this is a contradiction, $\alpha u\smile m\ngeqq\alpha e$ holds.

If $u\smile m_e=e$, then $\alpha e=\alpha u\smile\alpha m_e=\alpha u\smile\alpha(m\frown e)=\alpha u\smile(m\frown\alpha e)\leqq\alpha u\smile m$ from 3 of the proof of Lemma 1; hence it must be $u\smile m_e\leqq\neq e$. Therefore from 1 we get $u\leqq m_e$; hence $g_\alpha^*\leqq m_e$.

Now we prove $\sup_{\beta<\alpha}\beta e=\alpha e$ generally. If $\sup_{\gamma<1}\gamma e=e$ is established, then $\frac{1}{\alpha}\sup_{\beta<\alpha}\beta e=\sup_{\beta<\alpha}\frac{\beta}{\alpha}e=e$, i. e. $\sup_{\beta<\alpha}\beta e=\alpha e$, so we may prove $\sup_{\gamma<1}\gamma e=e$.

Assume that $\sup_{\gamma<1}\gamma e<e$, then from Axiom (S) there exist m, $\alpha>\beta$ such that

$e\cup m\geqq \alpha e>\beta e$, $m\cup\sup_{\gamma<1}\gamma e\not\geqq\beta e$. Therefore if $\alpha>1$, it must be $e\cup m\not\geqq e$ from Lemma 1, which is impossible; hence $\beta<1$. But this contradicts $m\cup\sup_{\gamma<1}\gamma e\not\geqq\beta e$. Therefore $\sup_{\gamma<1}\gamma e=e$.

Next we show the existence of $\beta<\alpha$ such that $m_e\geqq g_\beta^*$.

For if we assume that $m_e\not\geqq g_\beta^*=\sup\{u\,|\,u\in L_c,\ \beta u\leqq g\cap\beta e\}$ for all $\beta<\alpha$, then from 1 we get $e\leqq m_e\cup g_\beta^*$. Hence $\beta e\leqq\beta m_e\cup\beta g_\beta^*\leqq\beta m_e\cup(\beta e\cap g)\leqq\alpha m_e\cup g$. Therefore $\alpha e=\sup_{\beta<\alpha}\beta e\leqq\alpha m_e\cup g\leqq(\alpha m\cap\varphi_m)\cup g=m\cup g$ from the proof of 1 of Lemma 2, which is a contradiction. Therefore $m_e\geqq g_\beta^*$ for some $\beta<\alpha$, i.e. $G_g(I(m))\leqq\beta<\alpha$ holds. Hence $G_g(I(m))\lneqq F_f(I(m))$ is proved.

Now it is easy to see that the mapping $f\to F$ is one-to-one.

Let $f\to F_f$, $g\to G_g$ and $f\neq g$, then if $f\not\leqq g$, there exist m, α such that $m\cup f\geqq\alpha e$, $m\cup g\not\geqq\alpha e$, $\varphi_m\geqq f$, αe, e. Hence from the above mentioned fact $F_f(I)\not\leqq G_g(I)$.

Since it is obvious that $f\leqq g$ implies $F_f(I)\leqq G_g(I)$, this mapping $f\leftrightarrow F_f(I)$ is one-to-one.

3. We prove that for every element $F(I)\leqq\gamma$ of $L(\mathfrak{L})$ there exists some element of L corresponding to $F(I)$.

Let $\{I\,|F(I)\leqq\alpha\}=F(u_\alpha)=\{I\,|u_\alpha\in I\}$ $(u_\alpha\in L_c)$, then we can show that $f=\sup_{\alpha\leqq\gamma}\alpha u_\alpha$ corresponds to $F(I)$. Let us assume that f corresponds to $F_f(I)$, and prove $F_f(I)=F(I)$.

a) Firstly we prove that $\{I\,|u_\alpha\in I\}\supseteqq\{I\,|u_{\alpha_0}\in I\}$ implies $u_\alpha\leqq u_{\alpha_0}$.

If u_α, $u_{\alpha_0}\in L_c$ and $u_\alpha\not\leqq u_{\alpha_0}$, there exists m such that $m_e\cup u_\alpha=e$, $u_{\alpha_0}\leqq m_e$. For then there exist m, $\delta>0$ such that $m\cup u_\alpha\geqq\delta e$, $m\cup u_{\alpha_0}\not\geqq\delta e$, $\varphi_m\geqq e$. Since $\delta>1$ contradicts $m\cup e\geqq\delta e$ by Lemma 1, it must be $\delta\leqq 1$; hence $m\cup u_{\alpha_0}\not\geqq e$. If $u_{\alpha_0}\not\leqq m$, then from 1 we get $u_{\alpha_0}\cup m_e=e$, which contradicts $m\cup u_{\alpha_0}\not\geqq e$. Hence it must be $u_{\alpha_0}\leqq m_e$. Next from $m\cup u_\alpha\geqq\delta e$ and from 1 we get $m_e\cup u_\alpha=e$. Therefore $u_{\alpha_0}\in I(m)$, $u_\alpha\notin I(m)$, i.e. $\{u_\alpha\in I\}\not\supseteqq\{u_{\alpha_0}\in I\}$.

b) Now let α_0 be a fixed number, then for any number $\beta>\alpha_0$, $u\in L_c$ and $\beta u\leqq f\cap\beta e=\sup_{\alpha\leqq\gamma}\alpha u_\alpha\cap\beta e$ imply $u\leqq u_{\alpha_0}$.

For $u\not\leqq u_{\alpha_0}$ implies $u_{\alpha_0}\leqq m_e$, $u\cup m_e=e$ for some m from a). Since we can choose this m so that $\varphi_m\geqq\beta e$, e, we get

$$m\cup\sup\alpha u_\alpha\geqq m\cup\beta u\geqq\beta m_e\cup\beta u=\beta e.$$

In the other hand, from a) we get $m_e\geqq u_\alpha$ for any $\alpha\geqq\alpha_0$. Hence

$$\sup_{\alpha\leqq\gamma}\alpha u_\alpha=\sup_{\gamma\geqq\alpha\geqq\alpha_0}\alpha u_\alpha\cup\sup_{\alpha<\alpha_0}\alpha u_\alpha\leqq\lambda m_e\cup\alpha_0 e,$$

where λ is a number such that $\lambda\geqq\gamma$, $\lambda e\geqq\varphi_m$. Hence $\lambda m_e\cup\alpha_0 e\geqq\lambda m_e\cup\sup\alpha u_\alpha\geqq\beta e$ (for some $\beta>\alpha_0$) from the above mentioned fact. Hence from Lemma 1 we get $\lambda m_e\cup\alpha_0 e\not\geqq\alpha_0 e$, but this is impossible. Therefore $\beta>\alpha_0$, $u\in L_c$ and $\beta u\leqq\sup\alpha u_\alpha\cap\beta e$ imply $u\leqq u_{\alpha_0}$.

This proposition shows that $F(I)\leqq\alpha_0$ or $u_{\alpha_0}\in I$ implies $f_\beta^*\leqq u_{\alpha_0}$ for any $\beta>\alpha_0$, i. e. $f_\beta^*\in I$. Hence $F_f(I)\leqq\alpha_0$. Thus $F_f(I)\leqq F(I)$ is proved.

c) Next we prove $F(I)\leqq F_f(I)$.

Let $F_f(I)\leqq\alpha_0<\gamma$ and $I=I(m)$, then $f_\beta^*\in I(m)$ for all $\beta>\alpha_0$. For an arbitrary number β such that $\gamma\geqq\beta>\alpha_0$, $\beta u_\beta\leqq\sup\alpha u_\alpha\frown\beta e$ holds; hence from the above mentioned remark $u_\beta\leqq m_e$. Hence $F(I)\leqq\beta$. Therefore $F(I)\leqq\alpha_0$, i.e. $F(I)\leqq F_f(I)$ is proved. Thus $F_f(I)=F(I)$ is established.

4. It is easy to see that this isomorphism between L and $L(\mathfrak{L})$ is an operation-isomorphism.

Let $f\to F_f(I)$, $\lambda f\to F_{\lambda f}(I)$, and let us see $F_{\lambda f}(I)=\lambda F_f(I)$.

$$F_f(I)=\inf\{\alpha\,|\,f_\alpha^*\in I\},$$

$$F_{\lambda f}(I)=\inf\{\alpha\,|\,(\lambda f)_\alpha^*\in I\}\text{ and}$$

$$(\lambda f)_{\alpha\lambda}^*=\sup\{u\,|\,u\in L_c,\ \lambda\alpha u\leqq\lambda f\frown\lambda\alpha e\}=\sup\{u\,|\,u\in L_c,\ \alpha u\leqq f\frown\alpha e\}=f_\alpha^*$$

show the equivalence between $(\lambda f)_{\lambda\alpha}^*\in I$ and $f_\alpha^*\in I$ for all α; hence $F_{\lambda f}(I)=\lambda F_f(I)$. Thus the proof of this theorem is complete.

References

1) I. Kaplansky, Lattices of Continuous Functions II, Amer. J. of Math. Vol. 70, No. 3 (1948).

2) J. Nagata, On Lattices of Functions on Topological Spaces and of Functions on Uniform Spaces, Osaka Math. J. Vol. 1, No. 2 (1949).

3) We mean by "complete" the condition that if $f_\alpha\leqq f$ for some f and all α, then there exists $\sup_\alpha f_\alpha$.