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Homological Properties of Fibre Bundles

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Introduction

Let A be a fibre bundle over B and with fibre F. We assume that B is a finite complex and F is an orientable manifold, although the latter assumption is unnecessary for almost all cases. The structure group which acts on F is always denoted by G.

Our main concern is then to determine the homological characters of A assuming that we already know the homological characters of B, F and certain invariants characteristic to the fibre bundle [1].

In §§ 2-5 our main tool is established. It is given in an improved form of the one originated by J. Leray [9].

In § 6 a theorem is proved which may be regarded as a *theoretical* answer to our main problem as long as we are concerned with Betti numbers, showing that certain subgroups of the Kronecker product of $H(B)$ and $H(F)$ are characteristic to the fibre bundle.

In § 7 relations between our groups and intuition are given by examples.

In § 8 Gysin's theorems are proved as an application of our method.

In § 9-10 two Samelson's theorems [2] are utterly generalized. According to one of these generalized theorems a homology sphere of an odd dimension has a remarkable property: Every fibre bundle over an odd dimensional homology sphere, having a compact connected Lie group as its structure group, is a product bundle in the homological point of view. This has numerous applications as can be recognized by the Samelson's paper [3]. Moreover this leads us to conjecture that every Γ-manifold (in particular, every compact connected Lie group [4]) would have the same property.

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1) It is announced that this problem has been solved successfully by G. Hirsch in another form, [6].

2) Satz IV, and Satz VI of [13]. We refer to these theorems as the first and the second theorem respectively.

3) Note that in our case A need not be a Lie group any more and the theorem applies to Stiefel manifolds for example.

4) In this conjecture is included the following theorem: Let A be a compact connected Lie group and F its closed normal subgroup then $\mathfrak{P}_A(t)=\mathfrak{P}_B(t)\cdot\mathfrak{P}_F(t)$, wheer $B=A/F$.

In the appendix it is proved that an equivalence class of fibre bundles may be regarded as a fibre bundle over some mapping space [5]

§ 1. Let A be a fibre bundle over a complex B with fibre F. We shall denote by $B^{(q)}$ the q-section of B, and by $A^{(q)}$ the part of A over B^q. Then the (q, q', q'')-homology sequence (abreviated as: (q, q', q'')-H. S.) is the homology sequence of the triple [6] $(A^{(q)}, A^{(q')}, A^{(q'')})$, where $q \geq q' \geq q''$:

$$(1.1) \quad H^{p+1}(A^{(q)}, A^{(q')}) \xrightarrow{\alpha_p^{(q,q',q'')}} H^p(A^{(q')}, A^{(q'')}) \xrightarrow{\beta_p^{(q,q',q'')}} H^p(A^{(q)}, A^{(q'')}) \xrightarrow{\gamma_p^{(q,q',q'')}} H^p(A^{(q)}, A^{(q')})\ ^{7)}.$$

As is well known the above sequence is exact [6], in the sence that the kernel of each homomorphism is identical with the image of the preceeding one.

For simplicity we shall write $[q', q'')$, $(q, q']$, $[q']$, (q) instead of $(q'+1, q', q'')$, $(q, q', q'-1)$, $(q'+1, q', q'-1)$, $(q, q-1, -1)$ respectively.

We put $\mathfrak{E}^q(p-q)=H^p(A^{(q)}, A^{(q-1)})$, and define its subgroups $\mathfrak{Z}^q_k(p-q)$ $\mathfrak{B}^q_k(p-q)$ by [8]

$\mathfrak{Z}^q_k(p-q)=$[the kernel-image in $\mathfrak{E}^q(p-q)$ of the $[q-1, q-k-2)$–H. S.], $k=-1, 0, 1, 2, \dots$;

$\mathfrak{B}^q_k(p-q)=$[the kernel-image in $\mathfrak{E}^q(p-q)$ of the $(q+k+2, q)$–H. S.], $k=-2, -1, 0, 1, \dots$.

Clearly $\mathfrak{Z}^q_k=\mathfrak{Z}^q_{k+1}=\mathfrak{Z}^q_{k+2}=\dots$, $\mathfrak{B}^q_k=\mathfrak{B}^q_{k+1}=\mathfrak{B}^q_{k+2}=\dots$ for sufficiently large k: we shall denote them by $\mathfrak{Z}^q_*$, $\mathfrak{B}^q_*$ respectively. We see also that

$$(1.2) \quad \mathfrak{E}^q=\mathfrak{Z}^q_{-1}\supset\mathfrak{Z}^q_0\supset\mathfrak{Z}^q_1\supset\dots\supset\mathfrak{Z}^q_*\supset\mathfrak{B}^q_*\supset\dots\supset\mathfrak{B}^q_1\supset\mathfrak{B}^q_0\supset\mathfrak{B}^q_{-1}\supset\mathfrak{B}^q_{-2}=0.$$

Theorem 1. [9] $\mathfrak{B}^q_k(p)/\mathfrak{B}^c_{k-1}(p) \approx \mathfrak{Z}^{q+k+2}_k(p-k-1)/\mathfrak{Z}^{q+k+2}_{k+1}(p-k-1)$, $k=-1, 0, 1, \dots$.

Proof: We have only to examine the following diagram:

$$\begin{array}{ccccc}
 & & H^{q+p+1}(A^{(q+k+1)}, A^{(q)}) & & \\
 & \swarrow & \downarrow & & \\
\leftarrow H^{p+q}(A^{(q)}, A^{(q-1)}) & \leftarrow & H^{p+q+1}(A^{(q+k+2)}, A^{(q)}) & \leftarrow & H^{p+q+1}(A^{(p+k+2)}, A^{(q-1)}) \leftarrow \\
\swarrow & & \downarrow & \swarrow & \\
 & & H^{p+q+1}(A^{(q+k+2)}, A^{(q+k+1)}) & &
\end{array}$$

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5) In this paper $\cup$-and-$\cap$-product observations are scarcely given. They will appear later.

6) Cf. for instance, [14].

7) We put $H^p=0$ for $p<0$.

8) The meanings of these notations will be made clear later (§3).

9) This and the next theorems are analoguous to those obtained by J. Leray [9] for cohomology groups.

In the same way we can prove the following theorem. Before we state Theorem 2, we shall make the following definition:

$H^{p,q}=$[the image of the injection of $H^{p+q}(A^{(q)})$ into $H^{p+q}(A)$].

Clearly we have

(1.4) $H^{p+q}(A)=H^{0,p+q}\supset H^{1,p+q-1}\supset\ldots\supset H^{p,q}\supset\ldots\supset H^{p+q,0}\supset H^{p+q+1,-1}=0.$

Theorem 2.

$$\mathfrak{Z}_*^q(p)/\mathfrak{B}_*^q(p)\approx H^{p,q}/H^{p+1,q-1}.$$

§ 2. Similar treatment is possible for the cohomology groups $\bar{H}$. Moreover, if we use suitable coefficient groups, for instance [10] group $\mathfrak{R}$ of rationals, for both homology and cohomology, the corresponding groups $\bar{\mathfrak{C}}^q(p)$, $\bar{H}^{p+q}(A)$ are dual to $\mathfrak{C}^q(p)$, $H^{p+q}(A)$ respectively, and the corresponding subgroups $\bar{\mathfrak{B}}_k^q(p)$, $\bar{\mathfrak{Z}}_k^q(p)$, $\bar{H}^{p,q}$ are annihilators of $\mathfrak{Z}_k^q(p)$, $\mathfrak{B}_k^q(p)$, $H^{p,q}$ respectively.[11]

§ 3. Let $\mathfrak{A}$ be the principal fibre bundle [12] of A, and let $\mathfrak{A}^{(q)}$ the part of $\mathfrak{A}$ over B^q. For any simplex σ_i^q of B let $\varphi_i^q: \sigma_i^q\to\mathfrak{A}$ be an arbitrarily chosen slicing map of σ_i^q into $\mathfrak{A}$, the existence of which is assured by the well-known Felbau's theorem.

Since for complexes various homology theories coincide, and since $\tilde{\sigma}$ [13] is a product bundle, we have

(3.1) $H^p(\tilde{\sigma}, \dot{\tilde{\sigma}})\approx H^{p-q}(\tilde{\sigma})\approx H^{p-q}(\tilde{\xi})\approx H^{p-q}(F)$, where ξ is a point of σ.

Let us consider these isomorphisms more precisely.

$\omega_1: H^{p-q}(\tilde{\xi})\to H^{p-q}(\tilde{\sigma})$ is induced by the injection mapping $\tilde{\xi}\to\tilde{\sigma}$. $\omega_2: H^{p-q}(F)\to H^{p-q}(\tilde{\xi})$ is induced by the homeomorphic mapping $\varphi(\xi)$: $F\to\tilde{\xi}$. $\omega_3: H^{p-q}(F)\to H^p(\tilde{\sigma}, \dot{\tilde{\sigma}})$ is induced by the mapping $\theta: F\times\sigma\to\tilde{\sigma}$, defined by

$$\theta(y, x)=\varphi(x)y,\ x\in\sigma,\ y\in F.$$

The image of $a^{p-q}\in H^{p-q}(F)$ under ω_3 is denote by $a^{p-q}\circ\sigma_i^q$. Similarly denoting by z^{p-q} a cycle of F, we denote by $z^{p-q}\circ\varphi_i^q$ the cycle $\theta(z^{p-q}, \sigma^q)$ of $\tilde{\sigma}$ mod $\dot{\tilde{\sigma}}$. Further the cycle $\theta(z^{p-q}, \dot{\tilde{\sigma}})$ of $\dot{\tilde{\sigma}}$ is denoted by $z^{p-q}\circ\dot{\sigma}$.

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10) We shall assume that the coefficient group of $\bar{H}^p$ and H^p is always $\mathfrak{R}$ unless otherwise mentioned, although in most cases any other cofficient groups will equally do.

11) To avoid repetition we omit the precicse definitions of the groups for cohomology, but the last statements enable us to understand what their precise definitions should be.

12) [1], [2].

13) Generally we shall denote by $\tilde{\Phi}$ the part of A over Φ, where Φ is any subset of B.

14) In th3s case ξ is assumed to lie in the interior of σ.

$\omega_4: H^p(\tilde{\sigma}, \dot{\tilde{\sigma}}) \to H^{p-q}(\tilde{\xi})$ [14] is as follows. Since any relative cycle c^p of $\tilde{\sigma}$ mod $\dot{\tilde{\sigma}}$ is a chain of $\tilde{\sigma}$ with boundary disjoint from $\tilde{\xi}$ we may consider the intersecticn $(c^p \cdot \tilde{\xi})_\sigma = (c^p \cdot \tilde{\xi})$ of c^p with $\tilde{\xi}$ in $\tilde{\sigma}$. It has dimension $p-q$ and may be regarded as a cycle of $\tilde{\xi}$. ω_4 can be obtained by assigning c^p to $(c^p \cdot \tilde{\xi})$.

§ 4. **J. Leray's homology theory.**[15] Let us consider linear forms $\sum_i \hat{a}_i \sigma_i^q$, where σ_i^q are the q-simplexes of B and $\hat{a}_i \in H^p(\tilde{\sigma}_i^q)$. The totality of them clearly forms an additive group which we call Leray's q-chain group and is denoted by $L^q(p)$.

We then define Leray's boundary homomorphism $l_{q,p}: L^q(p) \to L^{q-1}(p)$ by $l_{q,p}(\sum_i \hat{a}_i \sigma_i^q) = \sum_i (\sum_j [\sigma_j^{q-1} : \sigma_i^q] (\hat{a}_i)_{\sigma_j^{q-1}}) \sigma_j^{q-1}$, where $(\hat{a}_i)_{\sigma_j^{q-1}}$ is the inverse image of $\hat{a}_i$ of the injection homomorphism $H^p(\tilde{\sigma}_j^{q-1}) \to H^p(\tilde{\sigma}_i^p)$ (which is an isomorphism in this case). Cleary $l_{q-1,p} l_{q,p} = 0$, and this enables us to develope a homology theory with respect to $L^q(p)$ and $l_{q,p}$. The resulting homology theory is a special case of the one introduced by J. Leray and coincides to Steenrod's homology theory with local coefficients [16].

§ 5. We can now explain the meanings of $\mathfrak{C}^q(p-q)$, $\mathfrak{Z}_k^q$, $\mathfrak{B}_k^q$ etc. Let us start from the obvious isomorphisms:

$$(5.1) \qquad H^q(A^{(q)}, A^{(q-1)}) \approx \sum_i H^p(\tilde{\sigma}_i^q, \dot{\tilde{\sigma}}_i^q) \approx \sum_i H^{p-q}(\tilde{\sigma}_i^q) \approx \sum_i H^{p-q} \quad (F).$$

The first term is $\mathfrak{C}^q(p-q)$ by definition; the second may be identified with $L^q(p-q)$; the last may be identified with $C^q(B, H^{p-q}(F))$.

The isomorphism $\mathfrak{x}_1: C^q(B, H^{p-q}(F)) \to \mathfrak{C}^q(p-q)$ is given by

$$(5.2) \qquad \mathfrak{x}_1 (\sum_i a_i \sigma_i^q) = \sum_i a_i \circ \sigma_i^q, \text{ where } a_i \in H^{p-q}(F).$$

The isomorphism $\mathfrak{x}_2: C^q(B, H^{p-q}(F)) \to L^q(p-q)$ is given by

$$(5.3) \qquad \mathfrak{x}_2 (\sum_i a_i \sigma_i^q) = \sum_i (\omega_1 \omega_2 a_i) \sigma_i^q.$$

Between $\alpha_{p-1}^{(q-1)}$ and $l_{q,p-q}$ there exists a relation given by

Theorem 3. (5.3) $\quad \mathfrak{x}_1^{-1} \alpha_{p-1}^{(q-1)} \mathfrak{x}_1 = \mathfrak{x}_2^{-1} l_{q,p-q} \mathfrak{x}_2$.

Proof: For any $\sum a_i \sigma_i^q$, $a_i \in H^{p-q}(F)$, the coefficient on σ_j^{q-1} of $\mathfrak{x}_1^{-1} \alpha_{p-1}^{(q-1)} \mathfrak{x}_1 (\sum a_i \sigma_i^q) = \mathfrak{x}_1^{-1} \gamma_{p-1}^{(q-1)} \alpha_{p-1}^{(q)} (\sum_i a_i \circ \varphi_i^q) = \mathfrak{x}_1^{-1} \gamma_{p-1}^{(q-1)} \sum_i a_i \circ \dot{\varphi}_i^q$ is easily verified to be $b_j = \sum_i [\sigma_i^q : \sigma_j^{q-1}] \{\varphi_j^{q-1}(\xi_j^{q-1})\}^{-1} \varphi_i^q(\xi_j^{q-1}) a_i$, while the coefficient cn σ_j^{q-1} of $\mathfrak{x}_2^{-1} l_{q,p-q} \mathfrak{x}_2 (\sum_i a_i \sigma_i^q) = \mathfrak{x}_2^{-1} l_{q,q-1} (\sum_i (\omega_1 \omega_2 a_i) \sigma_i^q) = \sum_j \mathfrak{x}_2^{-1} (\sum_i (\omega_1 \omega_2 a_i)_{\sigma_j^{q-1}}) \sigma_j^{q-1}$

15) Cf. [9].
16) Cf. [15], [16].

is also b_j, since $\omega_1\omega_2$ is independent of the choice of ξ and $(\omega_1\omega_2 a_i)_{\sigma_j^{q-1}}$ is represented by the cyle $\varphi_i^q(\xi_j^{q-1})a_i$.

Theorem 3 can also be stated as

Theorem 3'. *$\alpha_{p-1}^{[q-1]}: \mathfrak{C}^q(p-q)\to\mathfrak{C}^{q-1}(p-q)$ is (when properly meant) identical with $l_{q,p-q}: L^q(p-q)\to L^{q-1}(p-q)$.*

From now on we shall assume that the structure group G is a connected Lie group. Then, in the proof of Theorem 3 $b_j=\sum[\sigma_i^q: \sigma_j^{q-1}]a_i$, and hence we have

Theorem 4. *(If G is connected), both $\alpha_{p-1}^{[q-1]}$ and $l_{q,p-q}$ are identical with the ordinary boundary homomorphism*

$$C^q(B, H^{p-q}(F))\to C^{q-1}(B, H^{p-q}(F)).$$

Thus we may identify $\mathfrak{C}^q(p-q)$ with $C^q(B, H^{p-q}(F))$ and may regard $\alpha_{p-1}^{[q-1]}$ as the ordinary boundary homomorphism operating on the chains of B.

Theorem 5. *$\mathfrak{Z}_\cdot^q(p-q)=\mathfrak{Z}^q(p-q)$, $\mathfrak{B}_{-1}^q(p-q)=\mathfrak{B}^q(p-q)$ are the q-cycle group, the q-boundary group respectively of B with coefficients in $H^{p-q}(F)$.*

Corollary 1. *If the q-th Betti number of B vanishes: $b^q(B)=0$, then $\mathfrak{Z}^q=\mathfrak{Z}_1^q=\ldots=\mathfrak{Z}_*^q=\mathfrak{B}_*^q\ldots=\mathfrak{B}_{-1}^q=\mathfrak{B}^q$.*

§ 6. The grups $\mathfrak{H}_k^q$, $\mathfrak{R}_k^q$, and their invariance. As we have seen in the preceding paragraphs, the groups $\mathfrak{Z}^q(p-q)$ and $\mathfrak{B}^q(p-q)$ are ordinary q-cycle gyoup and q-boundary group respectively of B. The factor group $H^q(p-q)=\mathfrak{Z}^q(p-q)/\mathfrak{B}^q(p-q)$ is therefore the q-homology group of B and is independent of the manner of subdividing B. We are now going to show that our groups $\mathfrak{Z}_k^q$ and $\mathfrak{B}_k^q$ are also invariant under subdivision if they are considered modulo $\mathfrak{B}^q$.

We define

$$\mathfrak{H}_k^q(p-q)=\mathfrak{Z}_k^q(p-q)/\mathfrak{B}^q(p-q),\ \mathfrak{H}_*^q=\mathfrak{Z}_*^q/\mathfrak{B}^q;$$

$$\mathfrak{R}_k^q(p-q)=\mathfrak{B}_k^q(p-q)/\mathfrak{B}^q(p-q),\ \mathfrak{R}_k^q=\mathfrak{B}_*^q/\mathfrak{B}^q.$$

Let dim $B=n$ and B' a subdivision of B. Various groups for B' will be denoted by $'\mathfrak{B}_k^q$, $'\mathfrak{H}_k^q$, etc. Further we use the following denoting way: for example, we mean by $A^{(n-1)'(q)}$ the part of A over the q-section of the subdivision (induced by $B\to B'$) of the $(n-1)$-section B^{n-1}.

Lemma 1. *For $q\leq n-1$, any cycle c^p of $A'^{(q)}$ mod $A^{(n-1)'(q-k)}$ is* *homologous in* $A'^{(q+1)}$ *to a cycle of* $\dot{A}^{(n+1)'(q)}$ *mod* $\dot{A}^{(n-1)'(q-k)}$.

Proof: c^p can be decomposed as $c^p = c_0^p + c_1^p + \ldots + c_l^p$, where $c_0^p \subset A^{(n-1)}$, $c_i^p \subset \sigma_i^n$. The lemma follows from the obvious fact that the statement of the lemma is true for each c_i^q.

$$\begin{array}{ccccccc} H^p(A^{(q)}, A^{(q-1)}) & \overset{\gamma}{\leftarrow} & H^p(A^{(q)}, A^{(q-k-2)}) & \overset{\gamma}{\leftarrow} & H^p(A^{(q)}) & \overset{\alpha}{\rightarrow} & H^{p+1}(A^{(q+k+2)}, A^{(q)}) \\ \downarrow\theta & & \downarrow\theta & & \downarrow\theta & & \downarrow\theta \\ H^p(A'^{(q)}, A'^{(q-1)}) & \overset{\gamma}{\leftarrow} & H^p(A'^{(q)}, A'^{(q-k-2)}) & \overset{\gamma}{\rightarrow} & H^p(A'^{(q)}) & \overset{\alpha}{\rightarrow} & H^{p+1}(A'^{(q+k+2)}, A'^{(q)}) \end{array}$$

$$\begin{array}{c} \overset{\beta}{\rightarrow} H^{p+1}(A^{(q+1)}, A^{(q)}) \\ \downarrow\theta \\ \overset{\beta}{\rightarrow} H^{p+}(A'^{(q+1)}, A'^{(q)}), \end{array}$$

where γ, β, θ are induced by the identity mappings, and α is the boundary homorphism. Clearly commutativity relations hold in the diagram, and hence θ induces homorphisms

$$\mathfrak{H}_k^q(p-q) \rightarrow {}'\mathfrak{H}_k^q(p-q), \tag{6.1}$$

$$\mathfrak{R}_k^q(p-q) \rightarrow {}'\mathfrak{R}_k^q(p-q). \tag{6.2}$$

Lemma 2. *The homomorphism (6.1) is onto.*

Proof: Assuming that the lemma is already proved for B with dim $B \leq n-1$, we shall prove it for B with dim $B=n$.

Let $c^p \in H^p(A'^{(q)}, A'^{(q-k-2)})$, where $k \geqslant 0$. By Lemma 1 there exists a chain $d^p \subset A'^{(q-k-1)}$ such that $\dot{d}^p = -\dot{c}^p - c^{p-1}$ for some $c^{p-1} \subset A^{(n-1)'(n-k-2)}$. The chain $c^p + d^p$ clearly belongs to $H^p(A'^{(q)}, A^{(n-1)'(q-k-2)})$ and the image $\gamma(c^p+d^p)$ in $H^p(A'^{(q)}, A'^{(q-1)})$ is identical with $\gamma(c^p)$, since $d^d \subset A'^{(q-1)}$.

(i) The case $q \leq n-1$. By Lemma 1 there exists a chain $e^{p+1} \subset A'^{(q+1)}$ such that $c_1^p = c^p + d^p - \dot{e}^{p+1} \subset A^{(n-1)'(q)}$. The element c_1^p, when regarded as belonging to $H^p(A^{(n-1)'(q)}, A^{(n-1)'(q-k-2)})$, is mapped by γ into $\gamma c_1^p \in {}_{n-1}'\mathfrak{Z}_k^q(p-q)$, ${}_{n-1}'\mathfrak{Z}_k^q(p-q)$ is the group $\mathfrak{Z}_k^q$ for $(B^{n-1})'$.

Now let $\bar{\theta}$: ${}_{n-1}'\mathfrak{H}_k^q \rightarrow {}'\mathfrak{H}_k^q$ be induced by $\bar{\theta}$: $A^{(n-1)'(q)} \rightarrow A'^{(q)}$, and let θ: $\mathfrak{H}_k^q \rightarrow {}_{n-1}'\mathfrak{H}_k^q$ be induced by $\bar{\bar{\theta}}$: $A^{(q)} \rightarrow A^{(n-1)'(q)}$. $\bar{\theta}$ is onto, because $\bar{\theta}(\gamma c_1^p) = \gamma\bar{\theta} c_1^p = \gamma(c^p + d^p - \dot{e}^{p+1}) = \gamma(c^p) - \gamma(\dot{e}^{p+1}) \equiv \gamma(c^p)$ mod ${}'\mathfrak{B}^q(p-q)$, and γc^p may be any element of ${}'\mathfrak{Z}^q(p-q)$. On the other haod $\bar{\bar{\theta}}$ is onto since B^{n-1} is at most $(n-1)$-dimensional. Thus $\theta = \bar{\theta}\bar{\bar{\theta}}$ is proved to be onto.

(ii) The case $q=n$. In this case $A^{(n)} = A'^{(n)}$. Since $(c^p + d^p) = c^{p-1} \subset A^{(n-1)'(n-k-1)}$, by Lemma 1 there exists a chain $e^p \subset A^{(n-1)'(n-k-1)}$ such that $\dot{e}^p - c^{p-1} \subset A^{(n-2)'(n-k-1)}$. Repeating the same argument we see that for a suitable $f^p \subset A'^{(n-k-1)}$, $f^p - c^p \subset A^{(n-k-2)}$. Then $c_1^p = c^p + d^p - f^p$ is an element of $H^p(A^{(n)}, A^{(n-k-2)})$ and hence $\gamma c_1^p \in \mathfrak{Z}_k^n(p-n)$ and $\theta(\gamma c_1^p) = \gamma\theta c_1^p = \gamma c_1^p = \gamma c^p$. Thus $\theta: \mathfrak{H}_k^n \to '\mathfrak{H}_k^n$ is onto in this case also. Quite similarly we can prove

Lemma 3. $\mathfrak{K}_k^q(p-q) \to '\mathfrak{K}_k^q(p-q)$ is onto.

Theorem 10. *(Invariance of $\mathfrak{H}_k^q$ and $'\mathfrak{K}_k^q$ under subdivision). The homomorphisms (6.1) and (6.2) are isomorphisms onto.*

Proof: Let $\zeta \in \mathfrak{Z}_k^q$ and let $\theta\zeta \in '\mathfrak{B}^q(p-q)$. Since $\zeta \in \mathfrak{Z}_0^q$, and since the theorem is true for $k=0$, $\theta\zeta \in '\mathfrak{B}^q(p-q)$ implies $\zeta \in \mathfrak{B}^q(p-q)$. This means that $\theta: \mathfrak{H}_k^q \to '\mathfrak{H}_k^q$ is an isomorphism. But, since we already know that θ is onto, the theorem is proved.

It can be proved[17] that $\mathfrak{H}_k^q$ and $\mathfrak{K}_k^q$ (in particular $\mathfrak{H}_*^q$ and $\mathfrak{K}_*^q$) are invariants of the equivalence class of fibre bundles.

Theorem 11.

$$H^r(A) \approx \sum_{r=p+q} \mathfrak{H}_*^q(p)/\mathfrak{K}_*^q(p),$$

where $\mathfrak{H}_*^q(p)$, $\mathfrak{K}_k^q(p)$ are subgroups of $\mathfrak{H}^q(p) \equiv H^q(B, H^p(F)) \approx H^q(B) \otimes H^p(F)$[18], which are characteristic to the equivalence class of A.

§ 7. **Intuitive meanings of $\mathfrak{H}_k^q(p-q)$ and $\mathfrak{K}_k^q(p-q)$ in their special cases.** (i) *The case: $p=q$.* In this case $\mathfrak{H}_k^q$, $\mathfrak{B}_k^q$ are all subgroups of the homology group $H^q(B) = \mathfrak{H}^q(0)$ of the base space. $\mathfrak{H}_*^q(0)$ is the image of the projection $\pi: H^q(A) \to H^q(B)$.

(ii) *The case: $q=0$.* In this case $\mathfrak{H}_k^q$, $\mathfrak{B}_k^q$ are all subgroups of the homology group $H^p(F) = \mathfrak{H}^0(p)$ of the fibre. $\mathfrak{H}_*^0(p)$ is the kernel of the injection $\iota: H^p(F) \to H^p(A)$.

(iii) *The case: $p-q=d=dim\ F$.* In this case $\mathfrak{H}_k^q(d)$, $\mathfrak{K}_k^q(d)$ are all subgroups of the homology group $H^q(B) = \mathfrak{H}^q(d)$. $\mathfrak{K}_*^q(d)$ is the kernel of the Hopf's inverse homorphism $\pi': H^q(B) \to H^{q+d}(A)$.

§ 8. **Gysin's theorem.**

Lemma 4. *If F is a homology d-sphere,*

$$\mathfrak{B}^q(d) = \mathfrak{B}_{d-2}^q(d) \supset \mathfrak{B}_{d-1}^q(d) = \mathfrak{B}_*^q(d)\ ;$$

$$\mathfrak{Z}^q(0) = \mathfrak{Z}_{d-1}^q(0) \supset \mathfrak{Z}_d^q(0) = \mathfrak{Z}_*^q(0).$$

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17) By the standard method proving the invariance of Betti groups.

18) $M \otimes N$ is the Kronecker product of M and N.

Proof: By Theorem 1, we have

$$\mathfrak{B}^q_k(d)/\mathfrak{B}^q_{k-1}(d)\approx\mathfrak{Z}^{q+k+2}_k(d-k-1)/\mathfrak{Z}^{q+k+2}_k(d-k-1).$$

The left hand side of this isomorphism vanishes if $1\leq d-k-1\leq d-1$, i.e. $0\leq k\leq d-2$. In the same way we can prove the second relation.

Theorem 12. *If F is a homology d-sphese, we have*

$$\mathfrak{R}^q_*(d)\approx\mathfrak{H}^{q+d+1}(0)/\mathfrak{H}^{q+d+1}_*(0).$$

Proof: By Theorem 1, we have

$$\mathfrak{B}^q_{d-1}(d)/\mathfrak{B}^q_{d-2}(d)\approx\mathfrak{Z}^{q+d+1}_{q-1}(0)/\mathfrak{Z}^{q+d+1}_d(0).$$

By Lemma 4 it becomes

$$\mathfrak{R}^q_*(d)\approx\mathfrak{B}^q_*(d)/\mathfrak{B}^q(d)\approx\mathfrak{Z}^{q+d+1}(0)/\mathfrak{H}^{q+d+1}_*(0)\approx\mathfrak{H}^{q+d+1}(0)/\mathfrak{H}^{q+d+1}_*(0).$$

Lemma 5. $\mathfrak{Z}^q(d)=\mathfrak{Z}^q_*(d)$.

Lemma 6. *If $H^{p+q}(A)=0$, $\mathfrak{Z}^q_*(p)=\mathfrak{B}^q_*(p)$.*

Lemma 7. *If $H^{q+d}(A)=0$, $\mathfrak{R}^q_*(d)=H^q(d)$.*

The proof of Lemma 5 is similar to the one of Lemma 4, Lemma 6 can be proved by making use of Theorem 2 and Lemma 5.

Theorem 13. *If F and B are homology d-and n-sphere respectivesy, then for q with $0\leq q\leq n-1$,*

$$\mathfrak{H}^q(d)\approx\mathfrak{H}^{q+d+1}(0)/\mathfrak{H}^{q+d+1}_*(0).$$

Remark: By (i), (iii), § 7, we see that Theorem 12, 13 are identical with those given by W. Gysin [19].

§ 9. Generalization of Samelson's second theorem.

Theorem 14. *If the injection homomorphisms $H^p(F)\to H^p(A)$ are all isomorphisms ($p=0, 1, 2, \ldots$), then*

$$\mathfrak{Z}^q(p)=\mathfrak{Z}^q_*(p),\ \mathfrak{B}^q(p)=\mathfrak{B}^q_*(p).$$

This theorem is deduced by induction from the following **Theorem 14′**. Let $\dim B=n$, $'B=\sigma^n+B$, $\dot{\sigma}^n\subset B$. Furhter let the injection homomorphisms $H^p(F)\to H^p(A)$ are all isomorphisms. Then if the conditions: for any p, q

(i) $\mathfrak{Z}^q(p)=\mathfrak{Z}^q_*(p)$, $\mathfrak{B}^q(p)=\mathfrak{B}^q_*(p)$,

(ii) for any bounding cycle $Z^{q-1}\in B^{q-1}(B)$ having a slicing map into A, and for any $a^p\in H^p(F)$, the cycle $a^p\circ z^{q-1}$ [20] is homologous to a cycle of F in A, are satisfied for B, then the same conditions are satisfied for $'B$ also.

19) W. Gysin [4].

Remark: If the condition of Theorem 14 is satisfied for B, the same is satisfied for any subcomplex B_1 of B.

Proof of Theorem 14'. We shall first show that under the assumptions of the theorem the relation $\mathfrak{Z}^n(p)=\mathfrak{Z}^n_*(p)$ holds.

Let $\zeta=\sum_i a_i\sigma^n_i+a\sigma^n$ be an element of $'\mathfrak{Z}^n(p-n)$, then $\sum a_i\dot\sigma^n_i+a\dot\sigma^n=0$, and hence $a\dot\sigma^n=-\sum a_i\dot\sigma^n_i\in\mathfrak{B}^{n-1}(p-n)=\mathfrak{B}^{n-1}_*(p-n)$. But, since $H^{p-n}(F)$ is a finite dimensional vector space over the field of rational numbers, the last relation implies that $\dot\sigma^n\sim 0$ in B when $a\neq 0$ (the case $a=0$ is trivial!).

By assumption (ii) it follows that $a\circ\dot\sigma^n\sim b^{p-1}$ in A, where b^{p-1} is a cycle of F. This b^{p-1} can not $\not\sim 0$ in F, since otherwise $a\circ\dot\sigma^n\sim 0$ in $'A$, which contradicts the relation $(a\circ\sigma^n)$ in $'A=a\circ\dot\sigma^n\not\sim 0$ in $'A$. Thus b^{p-1} must ~ 0 in F, and eence $a\circ\dot\sigma^n\sim 0$ in A, hence (9.1) $a\circ\dot\sigma^n+\sum a_i\circ\dot\sigma^n_i\sim 0$ in A.

Now consider the following diagram:

$$\begin{array}{ccc}
H^p('A,\ A^{(n-1)}) & \xleftarrow{\ \theta\ } & H^p(A,\ A^{(n-1)}) \\
{\scriptstyle '\alpha^{(n)}_{p-1}}\searrow & & \swarrow{\scriptstyle \alpha^{(n)}_{p-1}} \\
 & H^{p-1}(A^{(n-1)}) & \\
{\scriptstyle \beta^{(n)}_{p-1}}\swarrow & & \searrow{\scriptstyle \gamma^{(n-1)}_{p-1}} \\
H^{p-1}(A) & & H^{p-1}(A^{(n-1)},\ A^{(n-2)}).
\end{array}$$

Commutativity relations hold in this diagram, and chain $a\circ\dot\sigma^n+\sum a_i\circ\dot\sigma^n_i$ represents the element $'\alpha^{(n)}_{p-1}\zeta$ of $H^{p-1}(A^{(n-1)})$. Relation (9.1) implies that $\beta^{(n)}_{p-1}{}'\alpha^{(n)}_{p-1}\zeta=0$, and hence $'\alpha^{(n)}_{p-1}\zeta=\alpha^{(n)}_{p-1}\zeta_1$ for some $\zeta_1\in H^p(A, A^{(n-1)})$. Since [boundary in B of ζ_1]$=\gamma^{(n-1)}_{p-1}\alpha^{(n)}_{p-1}\zeta_1=\gamma^{(n-1)}_{p-1}{}'\alpha^{(n)}_{p-1}\zeta=$[boundary in $'B$ of ζ]$=0$, from the assumption $\mathfrak{B}^{n-1}(p-n)=\mathfrak{B}^{n-1}_*(p-n)$ we may conclude $'\alpha^{(n)}_{p-1}\zeta=\alpha^{(n)}_{p-1}\zeta_1=0$.

Thus $\zeta\in\mathfrak{Z}^n_*(p-n)$, which proves the relation

$$\mathfrak{Z}^n_*(p-n)=\mathfrak{Z}^n(p-n).$$

Now since $'\mathfrak{Z}^q_k=\mathfrak{Z}^q_k$ for $q<n$, $'\mathfrak{Z}^q=\mathfrak{Z}^q=\mathfrak{Z}^q_*={}'\mathfrak{Z}^q_*$, and, since $'\mathfrak{B}^q_k/'\mathfrak{B}^q_{k-1}\approx{}'\mathfrak{Z}^{q+k+2}_k/'\mathfrak{Z}^{q+k+2}_{k+1}$, from the fact just proved we may conclude that $'\mathfrak{B}^q_*(p-n)={}'\mathfrak{B}^q(p-n)$. Let us prove the property (ii) for $'B$. Let $z^{q-1}\sim 0$ in $'B$. Since the case $z^{q-1}\sim 0$ in B (in particular the case $q<n$) is clear, we suppose $q=n$ and $z^{n-1}\not\sim 0$ in B, $z^{n-1}\sim 0$ in $'B$. Then for some rational number $m\neq 0$ we have $z^{n-1}-m\dot\sigma^n\sim 0$ in B. Since $a\circ z^{n-1}-ma\circ\dot\sigma^n=a\circ(z^{n-1}-m\dot\sigma^n)\sim\subset F$ [21] in A by the assumption (ii) and

$a \circ \dot{\sigma}^n \sim F$ in $'A$, $a \circ z^n$ [1]$\subset F$ in $'A$, which proves the condition (ii) for $'B$. Thus Theorem 14′ is proved.

§ **10. Generalization of Samelson's first theorem.** Samelson's first theorem can be generalized as follows.

Theorem 15. *Ig $B=B^{2\nu+1}$ is an odd dimensional homology sphere, and A a fibre bundle over B with fibre a homogeneous space $F=G/U$, where G is a compact connected Lie group. Then the Künneth equality holds between the Poincaré polynomials of A, B and F:*

$$\mathfrak{P}_A(t) = \mathfrak{P}_B(t)\mathfrak{P}_F(t).$$

We shall first prove the

Lemma 7. *If $B=B^n$ is a homology sphere, then for some n-simplex σ_0^n of B, $'B=B-Int\ \sigma_0^n$ is acyclic.*

Proof: Let $z^n=\sum m_i\sigma_i^n$ be the basic n-cycle of B, i. e. the cycle belonging to the element 1 of $H^n(B, \mathfrak{R})$, and let σ_0^n be an n-simlex which appears in z^n with non-vanishing coefficient $m_0 \neq 0$. Now, let $\sum_{i\neq 0} r_i\sigma_i^n$ be any cycle of B- Int σ_0^n, then it is also a cycle of B and hence a multiple of $\sum m_i\sigma_i^n$:

$$\sum_{i\neq 0} r_i\sigma_i^n = s\sum_{i\neq 0} m_i\sigma_i^n + sm_0\sigma_0^n, \text{ hence clearly } \sum_{i\neq 0} r_i\sigma_i^n = 0.$$

Now let $\sum r_j\sigma_j^{n-1}$ be an $(n-1)$-cycle of B- Int σ_0^n. From $b_{n-1}(B)=0$ there exists a chain $\sum s_i\sigma_i^n$ with

$$(10.1) \qquad (\sum s_i\sigma_i^n)^{\cdot} = s_0\dot{\sigma}_0^n + \sum_{i\neq 0} s_i\sigma_i^n = \sum r_j\sigma_j^{n-1}$$

Since $m_0\dot{\sigma}_0^n + \sum_{i\neq 0} m_i\dot{\sigma}_i^n = 0$ and $m_0 \neq 0$, by (10.1) we have $\sum r_j\sigma_j^{n-1} = (s_0/m_0)$ $(-\sum_{i\neq 0} m_i\dot{\sigma}_i^n) - \sum_{i\neq 0} s_i\dot{\sigma}_i^n = \sum_{i\neq 0} (-(s_0/m_0)m_i + s_i)\dot{\sigma}_i^n$. Hence $\sum r_j\sigma_j^{n-1} \sim 0$ in B- Int σ_0^n. Since the lower dimensional case is trivial, we have for all q (>0)

$$b^q(B\text{-Int } \sigma_0^n) = 0.$$

Lemma 8, *Let $'B$ be an acyclic complex and $'A$ a fibre bundle over B with fibre F. Then each of the injection homomorphisms $H(F) \rightarrow H(A)$, $\overline{H}(F) \leftarrow \overline{H}(A)$ is an isomorphism onto. Moreover minimal elements [22] in $H(F)$ and $H('A)$ correspond under the above isomorphism.*

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21) "$\# \sim \supset F$" means "$\#$ is homologous to a cycle in F"

22) $Z^p \in H^p(K)$, where K is a complex, is called a minimal element of $H(K)$ if for any $\zeta^q \in \overline{H}(K)$ $\zeta^q \cap Z^q = 0$ unless $q=0$ or $=p$. This definition clearly coincide with the ordinary one if K is a manifold. As in the case of manifold any minimal element is mapped under a continuous mapping to a minimal element, in particular, every spherical cycle is minimal. Cf [5], [13].

Proof: Since for $q>0$, $\mathfrak{Z}^q(p)=\mathfrak{B}^q(p)$, the inclusions $\mathfrak{Z}^q(p)\supset\mathfrak{Z}^q_*(p)\supset\mathfrak{B}^q_*(p)\supset\mathfrak{B}^q(p)$ imply $\mathfrak{Z}^q_*(p)=\mathfrak{B}^q_*(p)$. It follows that $H^{p,q}/H^{p+1,q-1}\approx\mathfrak{Z}^q_*(p)/\mathfrak{B}^q_*(p)=0$ $(q>0)$. Hence $H^{p+q}(A)=H^{0,p+q}=H^{1,p+q-1}=\ldots=H^{p+q,0}$, which shows that the injection $\chi: H(F)\to H('A)$ is an isomorphism onto. Similarly for $\bar{\chi}: \bar{H}(F)\leftarrow H('A)$.

To prove the second statement it is sufficient to show that if $z^p\in H^p('A)$ is minimal then the element $\chi^{-1}z^p\in H^p(F)$ is also minimal, since the converse is trivial. Let ζ^p run around over all the elements of $\bar{H}^q('A)$, then $\bar{\chi}\zeta$ runs around over all the elements of $\bar{H}^q(F)$.

Since $\chi(\bar{\chi}\zeta^q\wedge\chi^{-1}z^p)=\zeta^q\wedge z^p$, and since χ is an isomorphim, $\chi\zeta^q\wedge\bar{\chi}^{-1}z^p=0$ provided that $q\neq 0$, p, which shows that $\chi^{-1}z^p$ is minimal in $H^q(F)$.

Proof of Theorem 15. Let $B=B^{2\nu+1}$ be as in the theorem, and $\sigma_0^{2\nu+1}$, $'B$ be as in Lemma 7. Let us consider the homology sequence of the pair $(A, 'A)$:

$$\begin{array}{ccccccccc}\to H^{p+1}(A, 'A) & \xrightarrow{\alpha_p} & H^p('A) & \xrightarrow{\beta_p} & H^p(A) & \xrightarrow{\gamma_p} & H^p(A, 'A) & \to \\ \cup & & \cup & & \cup & & \cup & \\ \nu_{p+1} & & \lambda_p & & \mu_p & & \nu_p & \end{array}$$

Let φ be a slicing map $\sigma_0^{2\nu+1}\to\mathfrak{A}$ and $\dot{\varphi}$ its boundary: $\dot{\varphi}=\varphi|\dot{\sigma}_0^{2\nu+1}$, $\dot{\varphi}(\dot{\sigma}^{2\nu+1})\subset{}'\mathfrak{A}$, where $\mathfrak{A}$ is the principal fibre bundle of A and $'\mathfrak{A}$ its part over $'B$. As in § 3 the elements of $H^{p+1}(A, 'A)$ may be written in the form $a\circ\varphi$, where $a\in H^{p-n-1}(F)$. We have also $\alpha_p(a\circ\varphi)=a\circ\dot{\varphi}$, where $\dot{\varphi}$ is a spherical (and hence a minimal) element of $H^{2\nu}('A)$. But, since every even dimensional minimal cycle in a compact connected Lie group G is homologous to 0 [23], by Lemma 8 $\dot{\varphi}\sim 0$ in $'A$. Hence α_p is a null homorphism for each p.

If follows that β_p is an isomorphism and γ_p is an onto-homomorphism. Hence $H^p(A)\approx\mu_p+\nu_p\approx H^p('A)+H^p(A, 'A)$. But $H^p('A)\approx H^p(F)$ by Lemma 8, and $H^p(A, 'A)\approx C^n(\sigma_0^{2\nu+1}, H^{p-n}(F))\approx H^{p-n}(F)$. Therefore $H^p(A)\approx H^p(F)+H^{p-n}(F)$. Comparing the ranks, we have

$$b^p(A)=b^p(F)+b^{p-n}(F),$$

i. e.
$$\mathfrak{P}_A(t)=(1+t^n)\mathfrak{P}_F(t)=\mathfrak{P}_B(t)\mathfrak{P}_F(t).$$

Thus the theorem is completely proved.

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23) [13].

Appendix

§ 11. Since the content of this appendix is closely related to the paper of Chern-Sun [1], we shall often use their notations and terminologies.

Let $\mathfrak{F}^*=\{F, G, X^*, B^*; \psi^*, \phi_U^*\}$ be a fibre bundle and let B be a complex. The mapping space $\{B^*\}^B$ becomes a metric space which is locally contractible[24] if B is assumed to be metric and locally contractible. Let Ω be a component of $\{B^*\}^B$ containing a given mapping f_0. According to [1] Theorem 2.1, to every $f\in\Omega$ corresponds a fibre bundle $B(f)$, the graph $\mathfrak{G}(\psi^{-1*}f, B)$ of the many valued function $\psi^{-1*}f : B\rightarrow X^*$ [25]. $B(f)$'s are all equivalent among them. This naturally leads us to the following investigation.

Let Δ be the graph of the many valued function $\Omega\ni f\rightarrow B(f)\subset B\times X^*$. We may identify Δ with a subset Δ' of $\Omega\times B\times X^*$ defined by

$$\{(f, b, x^*)\in\Delta'\}\leftrightarrow\{x^*\in\psi^{-1*}f(b)\}.$$

Further let Γ be the group of automophisms of $B(f_0)$ giving self-equivalences of $B(f_0)$.

Theorem A. *Let G be a ANR, and let B be a complex. Then Δ is a fibre bundle over Ω with fibre $B(f_0)$ and the structure group Γ.*

Proof: Let f_1 be any element of Ω. Since $f_1 : B\rightarrow B^*$ is uniformly continuous, we can subdivide B into cells $\{\sigma^q\}_{q=0,1,2,\ldots,n}$, such that the mesh of each cell σ being so small that $f_1(\sigma)$ is contained in a coordinate neighbournood U_σ^* of the fibre bundle $\mathfrak{F}^*$. Let V_0 be a neighbourhood of f_1 in Ω such that $f(\sigma)\subset U_\sigma^*$ for any $f\in V_0$ and any σ. We shall construct a sequence of neighbourhoods $V_0\supset V_1\supset V_2\supset\ldots\supset V_n$ and a family $\{\theta_f^{(q)}\}_{f\in V_q}$ of mappings such that $\theta_f^{(q)}$ is an isomorphic mapping $B^q(f_1)\rightarrow B^q(f)$ depending continuously on the parameter $f\in V_q$ and reducing to the identity for $f=f_1$. Let us proceed inductively and assume that we have already constructed V_{q-1} and $\theta_f^{(q-1)}$. We define $\lambda_f(b, x^*)=\lambda_{f,b}(x^*)$ by $\theta_f^{(q-1)}(b, x^*)=(b, \lambda_{f,b}(x^*))$, where $f\in V_{q-1}$, $b\in B^{q-1}$, and $x^*\in f(b)$. Let σ^q be a q-simplex and $\phi_{U_{\sigma^q}^*}$ be abbreviated as ϕ_{σ^q}.

Since $\theta_f^{(q)}$ is isomorphic, $\mu_{f,b}=\phi_{\sigma^q, f(b)}^{-1}\lambda_{f,b}\phi_{\sigma^q, f_1(b)}$ belongs to G for

24) [7].
25) [8].

$b\in\dot{\sigma}^q$, and reduces to this identity for $f=f_1$. Let us define $\mu_{f_1,b}=1$ for $b\in\sigma^q$. Since G is an ANR, the function $\mu_{f,b}$: $V_{q-1}\times\dot{\sigma}^q+(f_1)\times\sigma^q\rightarrow G$ may be extended to a function $'\mu_{f,b}$: $W\rightarrow G$, where W is a neighbourhood of $V_{q-1}\times\dot{\sigma}^q+(f_1)\times\sigma^q$ in $V_{q-1}\times\sigma^q$.

Let $V_{q,\sigma^q}\subset V_{q-1}$ be a neighbourhood of f_1 such that $V_{q,\sigma^q}\times\sigma^q\subset W$ [26]. We define $\theta_f^{(q)}(b, x^*)$ for $f\in V_{q,\sigma^q}$, $b\in\sigma^q$, $x^*\in f_1(b)$ by

$$\theta_f^{(q)}(b, x^*)=(b, \phi_{\sigma^q, f(b)}{}'\mu_{f,b}\overset{-1}{\phi}_{\sigma^q, f_1(b)}(x^*)).$$

If we put $V_q=\bigcap_{\sigma^q}V_{q,\sigma^q}$, $\theta_f^{(q)}(b, x^*)$ is defined for $f\in V_q$, $b\in B^q$, $x^*\in f_1(b)$ and gives an isomorphic mapping of $B^q(f_1)$ onto $B^q(f)$. Thus the q-th step of our induction is completed. Since the 0-th step is clear we may assure that we can construct V_n and $\theta_f^{(n)}$: $B(f_1)\rightarrow B(f)$. This being proved the theorem follows at once from the fact that Ω is path-wise connected.

Corollary. *For any f_1, $f_2\in\Omega$, $B(f_1)$ and $B(f_2)$ are equivalent.*[27]

Corollary. *$H^r(B(f))$ is a local group on Ω in the sence of N. E. Steenrod.*[28]

More generally

Theorem B. *Let $B\supset B_1\supset B_2\supset B_3$ be subcomplexes of B, and let $T(f)$ be any kernel-image of the homology sequence of the triple $(B_1(f), B_2(f), B_3(f))$. Then $T(f)$ is a local group on Ω; in particular it is independent of f as an abstract group.*

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