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On a Necessary and Sufficient Condition of Metrizability

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It is well known that the second countability axiom is sufficient for a regular space to be metrizable, but it is not necessary. In this paper we shall show that some extention of the second countability axiom is necessary and sufficient for a regular space to be metrizable, and shall study some applications of this result.

Definition. Let R be a topological space and $\mathfrak{U}_\alpha=\{U_{\alpha\beta}\mid\beta\in B\}$ $(\alpha\in A)$ be open coverings of R. We call $\{\mathfrak{U}_\alpha\mid\alpha\in A\}$ an *open basis* of R, when each open set N of R can he represented in the form

$$N=\sum_{U_{\alpha\beta}\subset N}U_{\alpha\beta}.$$

β-Countability Axiom. We say that R satisfies *β-countability axiom*, when there exists an open basis of R consisting of an enumerable number of nbd (=neighbourhood) finite open coverings $\mathfrak{U}_n$.

It is an extention of the second countability axiom.

α-Countability Axiom, We say that R satisfies *α-countability axiom,* when there exists a collection of an enumerable number of nbd finite coverings $\{\mathfrak{U}_n\mid n=1, 2, \ldots\}$, $(\mathfrak{U}_n=\{U_{n\beta}\mid\beta\in B)$ such that for each pair of points $a, b\in R$, $a\neq b$, there exists $U_{n\beta}\in\mathfrak{U}_n$: $a\in U_{n\beta}$, $b\notin\bar{U}_{n\beta}$ for some n.

We shall say that $\{\mathfrak{U}_n\}$ satisfies the condition of α-countability, when $\{\mathfrak{U}_n\}$ has the above property.

Remark. The fact that $\mathfrak{U}_n$ covers R is not essential in α, β-countabilities. For when $\mathfrak{U}_n$ does not cover R, we may consider the covering $\{R, \mathfrak{U}_n\}$ in the place of $\mathfrak{U}_n$.

Theorem 1. *In order that a regular space R is metrizable, it is necessary and sufficient that R satisfies the β-countability axiom.*

Proof. Since the necessity is obvious from the theorem of A. H. Stone [1], we prove only the sufficiency.

1. Let R be a regular space satisfying the β-countability axiom.

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1) A. H. Stone. On Paracompactness and Product Space, Bull. of Amer. Math. 54 (1948) No. 10.

Then, in the first place we see that R is normal.

For let $\{\mathfrak{U}_n \mid n=1, 2, \ldots\}$ $(\mathfrak{U}_n=\{U_{n\beta} \mid \beta \in B\})$ be an open basis of R, and let F, G be disjoint closed sets of R; then for each point $a \in F$ we can choose $U_{n\beta} \in \mathfrak{U}_n$ for some n such that $\bar{U}_{n\beta} \cdot B=\phi$, $a \in U_{n\beta}$. We denote by $U_{n\beta}(a)$ such $U_{n\beta}$ for a.

Put $U_k = \sum_{n=k} U_{n\beta}(a)$; then it is easy to see that from the nbd finiteness of $\mathfrak{U}_k$, $U_k = \sum_{n=k} \bar{U}_{n\beta}(a)$ holds. And then $\bar{U}_k \subset G^c$ [2], $\sum_{k=1}^{\infty} U_k \supset F$ are obvious.

In the same manner we get open sets V_k such that $V_k \subset F^c$, $\sum_{k=1}^{\infty} V_k \supset G$.

Putting
$$M_i=U_i \cdot (\sum_{k=1}^{i} \bar{V}_k)^c, \quad M=\sum_{i=1}^{\infty} M_i;$$
$$N_i=V_i \cdot (\sum_{k=1}^{i} \bar{U}_k)^c, \quad N=\sum_{i=1}^{\infty} N_i,$$
we see that M, N are open sets such that $M \supset F$, $N \supset G$; $M \cdot N=\phi$.

2. Next we can show that for any nbd finite covering $\mathfrak{U}$ of R there exists a sequence of coverings $\mathfrak{U}_1$, $\mathfrak{U}_2$, ... such that $\mathfrak{U}_1^{\Delta} < \mathfrak{U}$, $\mathfrak{U}_2^{\Delta} < \mathfrak{U}_1$, ...

For let $\mathfrak{U}=\{U_\alpha \mid \alpha \in A\}$ be a nbd finite covering of R; then from the normality of R there exists an open covering $\mathfrak{V}=\{V_\alpha \mid \alpha \in A\}$ such that $\bar{V}_\alpha \subset U_\alpha$ for every α. We construct continuous functions f_α for every α such that
$$f_\alpha(a)=1 \quad (a \in \bar{V}_\alpha), \qquad 0 \leq f_\alpha(a) \leq 1.$$
$$f_\alpha(a)=0 \quad (a \ni U_\alpha^c),$$

Let $N_i(a)=\{x \mid (\forall\alpha) \mid f_\alpha(a)-f_\alpha(x) \mid < 1/2^i,\ x \in R\}$ $(i=0, 1, 2, \ldots)$ and let b be an arbitrary point of $N_i(a)$; then $|f_\alpha(a)-f_\alpha(b)| < 1/2^i$ for all α. We denote by $P(b)$ a nbd of b such that $P(b) \cdot U_{\alpha_j} \neq \phi$ $(j=1\ldots k)$, $P(b) \cdot U_\alpha=\phi$ $(\alpha \neq \alpha_j)$. Since $|f_{\alpha_j}(a)-f_{\alpha_j}(b)| < 1/2^i$ $(j=1\ldots k)$, there exists a nbd $Q(b)$ of b such that $x \in Q(b)$ implies $|f_{\alpha_j}(a)-f_{\alpha_j}(x)| < 1/2^i$ $(j=1\ldots k)$.

Let $x \in P(b) \cdot Q(b)$; then from $f_\alpha(b)=f_\alpha(x)=0$ $(\alpha \neq \alpha_j)$, we get $|f_\alpha(a) - f_\alpha(x)| < 1/2^i$ for all α, i. e. $P(b) \cdot Q(b) \subset N_i(a)$. Hence $N_i(a)$ is an open set. Putting $\mathfrak{N}_i=\{N_i(a) \mid a \in R\}$, we see that $\mathfrak{N}_{i+1}^{\Delta} < \mathfrak{N}_i$ $(i=0, 1, \ldots)$ and $\mathfrak{N}_0 < \mathfrak{U}$.

3. Next we shall show that R is perfectly normal.

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2) We denote by G^c the complement of G.

Let N be an arbitrary open set of R. For each point $a \in N$ there exists $U_{n\beta}(a)=U_{n\beta} \in \mathfrak{U}_n$ for some n such that $a \in U_{n\beta} \subset \bar{U}_{n\beta} \subset N$. From the nbd finiteness of $\mathfrak{U}_n$, $F_k=\sum_{n=k} \bar{U}_{n\beta}(a)$ is closed. Since $N=\sum_{k=1}^{\infty} F_k$ is obvious, G is an F_σ set, i. e. R is perfectly normal.

4. Now we shall show that R admits a countable uniformity, i. e. R is metrizable.

From the above result we can represent each $U_{n\beta}^c$ in the form $U_{n\beta}^c=\prod_{k=1}^{\infty} V_{n\beta}^{(k)}$, where $\{\mathfrak{U}_n\}(\mathfrak{U}_n=\{U_{n\beta} \mid \beta \in B\})$ is the open basis of R, and $V_{n\beta}^{(k)}$ are open sets.

Let Δ be a finite subset of β: $\Delta=\{\beta_1, \ldots, \beta_i\}$. Then from the nbd finiteness of $\mathfrak{U}_n$, $N_{n\Delta}^{(k)}=\prod_{\beta \in \Delta} U_{n\beta} \cdot \prod_{\beta \notin \Delta} V_{n\beta}^{(k)}$ is obviously an open set. Accordingly $\mathfrak{N}_n^{(k)}=\{N_{n\Delta}^{(k)} \mid \Delta$ running thru all finite subsetes of $\beta\}$ is a nbd finite covering of R.

For let $a \in R$ be an arbitrary point of R and let $a \in U_{n\beta}$ $(\beta \in \Delta)$, $a \notin U_{n\beta}(\beta \notin \Delta)$; then $a \in N_{n\Delta}^{(k)}$, i. e. $\mathfrak{N}_n^{(k)}$ covers R. The nbd finitess of $\mathfrak{N}_n^{(k)}$ is obvious.

Next let us show that for each open set N and each point $a \in N$ there exist $\mathfrak{N}_{n\Delta}^{(k)}$ such that $S(a, \mathfrak{N}_n^{(k)}) \subset N$[3].

Choose n such that for some β_0 $a \in U_{n\beta_0} \subset N$. Let $a \in U_{n\beta}$ $(\beta \in \Delta_0)$, $a \notin U_{n\beta}$ $(\beta \notin \Delta_0)$, and accordingly $\beta_0 \in \Delta_0$; then there exists k such that $a \notin V_{n\beta}^{(k)}$ for all $\beta_0 \in \Delta_0$. For these n, k, let us consider $\mathfrak{N}_n^{(k)}$. If $a \in N_{n\Delta}^{(k)}$, then from $a \notin V_{n\beta}^{(k)}$ $(\beta \in \Delta_0)$ it must be $\Delta_0 \subset \Delta$. Hence $N_{n\Delta}^{(k)} \subset U_{n\beta_0}$, i. e. $S(a, \mathfrak{N}_n^{(k)}) \subset U_{n\beta_0} \subset N$.

As we saw in 2, for every $\mathfrak{N}_n^{(k)}$, there exist sequences of open coverings $\{\mathfrak{N}_{n,i}^{(k)} \mid i=1, 2, \ldots\}$ such that $\mathfrak{N}_{n,i+1}^{(k)\Delta} < \mathfrak{N}_{n,i}^{(k)}$. Therefore we can construct a countable uiformity containing $\{\mathfrak{N}_n^{(k)} \mid n, k=1, 2, 3, \ldots\}$. Since this unifomity agrees with the topology of R, R is metrizable.

Theorem 2. *If a T-space R can be represented in the form* $R=\sum_{\alpha \in A} S_\alpha$, *where* $\{S_\alpha \mid \alpha \in A\}$ *is nbd finite and* S_α *are closed metrizable subspaces, then R is metrizable.*

Proof. 1. We shall prove first that R is regular.

Let N be an arbitrary open set of R and let $a \in N$. Suppose that

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3) $S(a, \mathfrak{N})=\sum_{a \in N \in \mathfrak{N}} N$

$a \in S_{\alpha i}(i=1\ldots k)$, $a \notin S_{\alpha}(\alpha \neq \alpha_i)$; then from the nbd finiteness of $\{S_{\alpha}\}$, $V(a)=\prod_{\alpha \neq \alpha_i} S^c_{\alpha}$ is an open nbd of a. From the regularity of S_{α_i}, there exists some open set $K_i(a)$ in S_{α_i} such that $a \in K_i(a) \subset \overline{K_i(a)} \subset N$.

Put $K_i(a)+S^c_i=U_i(a)$. Then, since $U_i(a)$ are open, $U(a)=V(a)\cdot\prod_{i=1}^{k} U_i(a)$ is an open nbd of a.

Since
$$U(a)=\prod_{\alpha\neq\alpha_i} S^c_{\alpha}\cdot\prod_{i=1}^{k}(K_i(a)+S^c_i)\subset\sum_{i=1}^{k}K_i(a),$$

we have $U(a)\subset\sum_{i=1}^{k}\overline{K_i(a)}\subset N$. Hence R is regular.

2. Let us supbose that $\{\mathfrak{U}^{(\alpha)'}_n \mid n=1, 2, \ldots\}$ is a countable open basis consisting of nbd finite coverings $\mathfrak{U}^{(\alpha)'}_n=\{U^{(\alpha)}_{n,\beta} \mid \beta\in B\}$. We denote by $\{U_{\gamma} \mid \gamma\in\Gamma\}$ an open covering of R such that $\bar{U}_{\gamma}$ meets a finite number of $U^{(\alpha)'}_{1,\beta}(\alpha\in A,\ \beta\in B)$ only, and by $\mathfrak{U}_{\alpha}$ a nbd finite refinement of $\{U_{\gamma}\cdot S_{\alpha} \mid \gamma\in\Gamma\}$ in S_{α}. Considering $\mathfrak{U}^{(\alpha)}_2=\mathfrak{U}^{(\alpha)'}_2\wedge\mathfrak{U}_{\alpha}=\{U^{(\alpha)}_{2,\beta} \mid \beta\in B\}$, we see that for each α, β, $\bar{U}^{(\alpha)}_{2,\beta}$ meets a finite number of $U^{(\alpha)'}_{1,\beta}$ $(\alpha\in A,\ B\in B)$ only.

Assume that for a fixed n we get $\{U^{(\alpha)}_{n\beta} \mid \alpha\in A,\ \beta\in B\}$ such that for each α, β, $\bar{U}^{(\alpha)}_{n\beta}$ meets a finite number of $\bar{U}^{(\alpha)}_{n-1,\beta}$ only, and $\mathfrak{U}^{(\alpha)}_n=\{U^{(\alpha)}_{n\beta} \mid \beta\}$ is a nbd finite open covering of S_{α} having the form $\mathfrak{U}^{(\alpha)}_n=\mathfrak{U}^{(\alpha)'}_n\wedge\mathfrak{U}_{\alpha}$; then for $n+1$ we get $\{U^{(\alpha)}_{n+1,\beta} \mid \alpha\in A,\ \beta\in B\}$ having the same property in the above manner. Thus we get sequence of coverings $\{U^{(\alpha)}_{1,\beta}\}$, $\{U^{(\alpha)}_{2,\beta}\}$, $\{U^{(\alpha)}_{3,\beta}\}$, ... such that $\bar{U}^{(\alpha)}_{n+1,\beta}$ meets a finite number $\bar{U}^{(\alpha)}_{n\beta}$ only, and $\{\mathfrak{U}^{(\alpha)}_n \mid n=1, 2, \ldots\}$ is an open basis of S_{α} consisting of nbd finite coverings $\mathfrak{U}^{(\alpha)}_n=\{U^{(\alpha)}_{n\beta} \mid \beta\in B\}$.

3. Put
$$V^{(\alpha_0)}_{n\beta_0}=(U^{(\alpha_0)}_{n\beta_0}+S^c_{\alpha_0})\cdot\prod_{(\bar{U}^{(\alpha)}_{n+1,\beta})^c\supset U^{(\alpha_0)}_{n\beta_0}}(\bar{U}^{(\alpha)}_{n+1,\beta})^c;$$

then $V^{(\alpha)}_{n,\beta}$ is an open set of R from the nbd finiteness of $\{\bar{U}^{(\alpha)}_{n+1,\beta}\}$ and $V^{(\alpha)}_{n\beta}\cdot S_{\alpha}=U^{(\alpha)}_{n\beta}$.

Put $W_{n\Delta}=\prod_{i=1}^{k} V^{(\alpha_i)}_{n\beta(\alpha_i)}\cdot\prod_{\alpha\neq\alpha_i} S^c_{\alpha}$ $(\Delta=\{(\alpha_i\beta(\alpha_i)) \mid i=1\ldots k\}$, $\alpha_i\neq\alpha_j$ for $i\neq j)$; then $\mathfrak{W}_n=\{W_{n\Delta} \mid \Delta$ running thru all sets having the form of the above $\Delta\}$ is obviously an open covering of R. Let us show that $\mathfrak{W}_n$ is nbd finite.

Let $a\in R$ be an arbitrary point of R. If $a\in S_{\alpha_i}(i=1\ldots k)$ $a\notin S_{\alpha}$ $(\alpha\neq\alpha_i)$ and $a\in U^{(\alpha_i)}_{n+1,\beta(\alpha_i)}(i=1\ldots k)$; then

$$a \in U(a) = \prod_{i=1}^{k} (U^{(\alpha_i)}_{n+1,\,\beta(\alpha_i)} + S^c_{\alpha_i}) \cdot \prod_{\alpha \neq \alpha_i} S^c_\alpha \subset \sum_{i=1}^{k} U^{(\alpha_i)}_{n+1,\,\beta(\alpha_i)}\,.$$

Since $\bar{U}^{(\alpha_i)}_{n+1,\,\beta(\alpha_i)}$ meets $V^{(\alpha)}_{n\beta}$ only when $U^{(\alpha)}_{n\beta} \cdot \bar{U}^{(\alpha_i)}_{n+1,\,\beta(\alpha_i)} \neq \phi$, $\bar{U}^{(\alpha_i)}_{n+1,\,\beta(\alpha_i)}$ meets a finite number of $V^{(\alpha)}_{n\beta}$ for a fixed n. we denote by $V^{(\alpha_i)}_{n\beta(\alpha_i)}(i=1\dots l)$ all $V^{(\alpha)}_{n\beta}$ which meets $U^{(\alpha_i)}_{n+1,\,\beta(\alpha_i)}$. Hence $W_{n\Delta} \cdot U^{(\alpha_i)}_{n+1,\,\beta(\alpha_i)} \neq \phi$ only when $\Delta \subset \{(\alpha^i, \beta(\alpha^i)) \mid i=1\dots l\}$, i. e. $U^{(\alpha_i)}_{n+1,\,\beta(\alpha_i)}$ meets a finite number of $W_{n\Delta}$ only. Therefore $\sum_{i=1}^{k} U^{(\alpha_i)}_{n+1,\,\beta(\alpha_i)}$ or $U(a)$ meets only a finite number of $W_{n\Delta}$, i. e. $\mathfrak{W}_n$ is nbd finite.

4. Finally, we can show that $\{\mathfrak{W}_n \mid n=1, 2, \dots\}$ is an open basis of R. For assume that $a \in N$. N is an arbitrary open set of R and that $a \in S_{\alpha_i}(i=1, \dots, k)$, $a \notin S_\alpha(\alpha \neq \alpha_i)$; $a \in U^{(\alpha_i)}_{n\beta(\alpha_i)} \subset N$. $(i=1, \dots, k)$. For $\Delta = \{(\alpha_i, \beta(\alpha_i)) \mid i=1\dots k\}$, we get $a \in W_{n\Delta}$ and

$$W_{n\Delta} = W_{n\Delta} \cdot \sum_{i=1}^{k} S_{\alpha_i} \subset \prod_{i=1}^{k} V^{(\alpha_i)}_{n\beta(\alpha_i)} \cdot \sum_{i=1}^{k} S_{\alpha_i} = \sum_{i=1}^{k} (S_{\alpha_i} \cdot \prod_{j=1}^{k} V^{(\alpha_j)}_{n\beta(\alpha_j)}) \subset \sum_{i=1}^{k} S_{\alpha_i} \cdot V^{(\alpha_i)}_{n\beta(\alpha_i)}$$
$= \sum_{i=1}^{k} U^{(\alpha_i)}_{n\beta(\alpha_i)} \subset N$, i. e. $\{\mathfrak{W}_n\}$ is an open basis of R.

Therefore from theorem 1 R is metrizable.

Theorem 3. *In order that a fully normal space R is completely metrizable, it is necessary and sufficient that R is topologically complete and satisfies the α-countability axiom.*

Proof. Since the necessity of the condition is obvious, we shall prove only the sufficiency.

Let R be a fully normal and topologically complete space satisfying the α-countability axiom. Since R is topologically complete, from N. A. Shanin's theorem[4)] there exists a countable collection $\{\mathfrak{U}_n \mid n=1, 2, \dots\}$ of open coverings $\mathfrak{U}_n$, which has the following property: If a maximum filter $\mathfrak{F} = \{F_\alpha\}$ of closed sets F_α has no convergent then point, for the open covering $\mathfrak{G} = \{F^c_\alpha\}$, there exists some element $\mathfrak{U}_n = \{U_\beta\}$ of $\{\mathfrak{O}_n\}$ such that for every β and some α, $\bar{U}_\beta \subset F^c_\alpha$.

Since R is fully normal, each $\mathfrak{U}_n$ has a nbd refinement, which we denote by $\mathfrak{U}'_n$.

Denote by $\{\mathfrak{V}_n\}$ the countable collection of nbd finite open coverings satisfying the condition of α-countability. Then $\mathfrak{W}_n = \mathfrak{U}'_n \wedge \mathfrak{V}_n$ are nbd finite and $\{\mathfrak{W}_n\}$ satisfies the condition of α-countability as well

4) N. A. Shanin, On the Theory of Bicompact Extension of Topological Spaces, C. R. URSS, 38 (1943) No. 5-6.

as the above mentioned condition satisfied by $\{\mathfrak{U}_n\}$.

Let N be an open set of R. Then we can show that for an arbitrary point a of N there exist some $\mathfrak{W}_{n_i}$ and some W_{n_i} $(i=1\dots k)$ such that $a\in\prod_{i=1}^{k} W_{n_i}\subset N$, $W_{n_i}\in\mathfrak{W}_{n_i}$.

To show this, assume the contrary. If we denote by $W_1, W_2, \dots$ all the elements of some $\mathfrak{W}_n$ which contain a, then since the condition of α-countability is satisfied by $\{\mathfrak{W}_n\}$, $\{G^c\cdot\overline{W}_n|n=1, 2, \dots\}$ is a closed filter having no cluster point. We denote by $\mathfrak{H}=\{H_\alpha|A\}$ a maximum closed filter containing $\{G^c\cdot\overline{W}_n\}$ and by $\mathfrak{H}'$ the open covering $\{H_\alpha^c|A\}$. Since $\mathfrak{H}$ has no convergent point, there exists an element $\mathfrak{W}_n=\{W_{n\beta}|B\}$ of $\{\mathfrak{W}_n\}$ such that for every elements $W_{n\beta}$ of $\mathfrak{W}_n$ and some $H_\alpha \overline{W}_{n\beta}\subset H_\alpha^c\in\mathfrak{H}'$ holds. Let $a\in W_m\in\mathfrak{W}_n$, $\overline{W}_m\subset H_\alpha^c\in\mathfrak{H}'$; then $\overline{W}_m$, $H_\alpha\in\mathfrak{H}'$ and $\overline{W}_m\cdot H_2=\phi$ hold, which is a contradiction. Thus we have shown the existence of W_{n_i} such that $a\in\prod_{i=1}^{k} W_{n_i}\subset N$.

From $\mathfrak{W}_n=\{W_{n_\beta}|B\}$ we construct nbd open coverings $\mathfrak{W}(n_1\dots n_k)=\bigwedge_{i=1}^{k}\mathfrak{W}_{n_i}$. The enumerable collection $\{\mathfrak{W}(n_1\dots n_k)|i=1, 2, \dots; n_i=1, 2, \dots\}$ satisfies the condition of α-countability; hence R is metrizable by theorem 1. Since R is topologically complete, by Čech's theorem[5] R is completely metrizable.

Remark. When R is regular, β-countability axiom contains α-countability axiom, but α-countability axiom does not contain the lst countability axiom. The direct product of an enumerable infinite number of unit intervals $[0, 1]$ satisfies the α-countacility axiom but it does not the lst countability axiom, when its topology is the strong topology. This fact shows that theorem 3 is essentially different from theorem 1.

Corollary 1. *Let a topological space R be the sum of S_α: $R=\sum_{\alpha\in A} S_\alpha$, where $\{S_\alpha|\alpha\in A\}$ is nbd finite in R, and S_α are fully normal closed subspaces being at most of cardinal number $\mathfrak{n}$: $|S_\alpha|\leq\mathfrak{n}$. Then in order that R admits some complete uniformity being at most of cardinal number $\mathfrak{n}$, it is necessary and sufficient that R can be a meet of at most $\mathfrak{n}$ number of open sets in some bicompact T_2-space.*

Proof. Since theorem 1, 2, 3 holds obviously about uniformity of

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5) E. Čech. On Bicompact Spaces, Anns. of Math. 38 (1939).

at most power $\mathfrak{n}$, too, the validity of this corollary is almost obvious from theorems 1, 2, 3, the property of cardinal number, $\mathfrak{n}^2=\mathfrak{n}$ and the extension of Čech's theorem by the author [6].

Corollary 2. *In order that a regular space R is metrizable, it is necessary and sufficient that there exists a coollection $\{f_\alpha \mid \alpha \in A\}$ of continuous functions f_α such tnat*

$$f=sup \sum_{i=1}^{k} f_{\alpha_i} \text{ is a continuous function,}$$
$$0 \leqq f_\alpha(x) \leqq 1 \quad \text{for all } \alpha,$$

and for any open set N and any point $a \in N$, there exists an element f_α of $\{f_\alpha\}$:

$$f_\alpha(a)>0,$$
$$f_\alpha(x)=0 \quad (x \in N^c).$$

Proof. Necessity: We denote by $\mathfrak{U}_n=\{U_{n\alpha} \mid \alpha \in A_n\}$ a nbd finite refinement of $\mathfrak{S}_n=\{\{x \mid \rho(ax)>1/2^n\} \mid a \in R\}$, where we denote by ρ the metric of R. Since the non-negative function $\rho_{n\alpha}(x)=\rho(x, U^c_{n\alpha})$ is continuous, $f_{n\alpha}(x)=\frac{1}{2^n}\cdot\frac{\rho_{n\alpha}(x)}{\sum\limits_{x \in U_{n\alpha}} \rho_{n\alpha}(x)}$ is a continuos function such that function $0 \leqq f_{n\alpha}(x) \leqq 1/2^n$. Obviously $sup \sum_{i=1}^{k} f_{n_i\alpha_i}=1$, and for any open set N and any point $a \in N$, $f_{n\alpha}(a)>0$, $f_{n\alpha}(x)=0$ $(x \in N^c)$ for $U_{n\alpha}$ such that $a \in U_{n\alpha} \subset N$. Hence $\{f_{n\alpha} \mid n=1, 2, \ldots, \alpha \in A_n\}$ is the collection of continuous functions in the condition of this corollary.

Sufficiency: Let R be a regular space having such a family $\{f_\alpha \mid \alpha \in A\}$ of continuous functions. Let us show the nbd finiteness of the open covering $\mathfrak{U}_n=\{U_{n\alpha} \mid \alpha \in A\}$, where $U_{n\alpha}=\{x \mid f_\alpha(x)>1/2^n\}$.

Let a be an arbitrary point of R. Then, assume that $f_{\alpha_1}(a)+\ldots+f_{\alpha_k}(a)>f(a)-1/2^{n+2}$. We denote by $U(a)$ an open nbd of a such that $x \in U(a)$ implies $f(x)<f(a)+1/2^{n+1}$ and by $U_i(a)$ nbds of a such that $x \in U_i(a)$ implies $f_{\alpha_i}(x)>f_{\alpha_i}(a)-1/k\cdot 2^{n+2}$. Then the nbd $V(a)=U(a)\cdot\prod_{i=1}^{k} U_i(a)$ is disjoint from $U_{n\alpha}$, but $U_{n\alpha_i}$ $(i=1\ldots k)$. For, if there would be a point $b \in V(a)\cdot U_{n\alpha}$ $(\alpha\alpha \neq_i)$, then it would be $f(b)<f(a)+1/2^{n+1}$ and $f(b) \geqq \sum_{i=1}^{k} f_{\alpha_i}(b)+f_\alpha(b) > \sum_{i=1}^{k} f_{\alpha_i}(a)-1/2^{n+2}+1/2^n > f(a)-1/2^{n+1}+1/2^n = f(a)+1/2^{n+1}$ hold at the same time, which is a contradiction. Hench $\mathfrak{U}_n$ is nbd

6) On Topological Completeness, Sugaku, 2 (1949), in Japanes. The content of this paper is unpublished in foreign language.

finite.

Since $\{\mathfrak{U}_n \mid n=1, 2, \ldots\}$ is obviously an open basis of R, R satisfies the β-countability axiom. Therefore R is metrizable from theorm 1.

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