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On the Resolution of $\Delta U=cU+\varphi$ by the Iteration of Averaging Process

By Masao INOUE

1. Let D be a bounded domain in the 3-dimensional euclidean space, B its boundary, f a real continuous function defined on B and F a continuous extension of f onto D. Let $\rho(M)$ denote the distance from a point M of D to B and consider a continuous function $r(M)$ defined in D satisfying: $0<r(M)\leqq\rho(M),\ M\in D$.

Given a sequence of positive numbers λ_n such that $1\geqq\lambda_1\geqq\lambda_2\geqq \dots$, we put $\lambda_n r(M)\equiv r_n(M)$ and

$$U_1(M)=\frac{3}{4\pi r_1^3(M)}\iiint_{MP<r_1} F(P)d\omega_P=A_M^{(1)}(F).$$

Let now

$$U_n(M)=\frac{3}{4\pi r_n^3(M)}\iiint_{MP<r_n} U_{n-1}(P)d\omega_P \tag{1}$$

$$\equiv A_M^{(n)}(U_{n-1})\equiv A_M^{(n,1)}(F)$$

$$(n=1,\ 2,\ \dots,\ U_0=F).$$

If $\lambda_i=1(i=1,\ 2,\ \dots)$ and $\rho(M)\equiv r(M)$ in D, the sequence of $U_n(M)$ converges to the generalized solution $H_M^D(f)$ of the Dirichlet Problem for D corresponding to the boundary distribution f[1]. But if $\lambda_i\to 0$ very fast, it may be possible that $U_n(M)$ does not converge to $H_M^D(f)$. A question then arises naturally: Under what condition the sequence of $U_n(M)$ converges to $H_M^D(f)$ for any f on B?

This question has been solved in my preceding paper (N)[2]. The answer is as follows.

Theorem 1. *In order that the sequence of $U_n(M)$ defined by* (1) *converge to $H_M^D(f)$ in D for any f, it is necessary and sufficient that the series $\sum^{\infty}\lambda_i^2$ should be divergent. If $\sum^{\infty}\lambda_i^2$ diverges, the convergence*

1) F. W. Perkins, Sur la résolution du problème de Dirichlet par des médiations réitérées, C. R. Paris, t. 184 (1927).

2) M. Inoue, Sur la méthode des médiations réitérées dans le problème de Dirichlet, Memoirs of the Fac. of Sci., Kyûsyû Univ., V. 5, No. 1 (1950).

of U_n *is uniform over any closed subset of* D.

The proof of this theorem is based upon the following fact: Let V be the bounded function (univocally determined) satisfying $\Delta V=-1$ in D and vanishing on B, except at points of a set of capacity zero. Then

(2) $$A_M^{[n,1]}(V)\underset{u}{\rightarrow}0$$

provided $\sum^{\infty}\lambda_i^2$ is divergent, where $\underset{u}{\rightarrow}$ means the uniform convergence over any closed subset of D. Using this notation we can state the Theorem 1 as follows: If $\sum^{\infty}\lambda_i^2$ diverges,

(3) $$A_M^{[n,1]}(F)\underset{u}{\rightarrow}H_M^D(f)$$

for any continuous function f on B.

Remark. Given a function W in D and a function g on B. If $W(P)\to g(Q)$ as $P\to Q$, $P\in D$, $Q\in B$, we shall say that W takes g at Q and write $W(Q)=g(Q)$. If W takes f on B, except at points of a set of capacity zero, we shall say that W takes f almost everywhere on B.

2. Let φ be a bounded continuous function in D. Then there exists one and only one function U bounded and continuous in D, such that U takes f almost everywhere on B and satisfies

$$\tilde{\Delta}\, U(M)\equiv\lim_{h\to 0}\frac{10}{h^2}\Big[A_M^{(h)}(U)-U(M)\Big]=\varphi(M)$$

for every M of D, where

$$A_M^{(h)}(U)=\frac{3}{4\pi h^3}\iiint\limits_{MP<h}U(P)d\omega_P.$$

As is well known,

(4) $$U(M)=H_M^D(f)-\frac{1}{4\pi}\iiint\limits_D G(P,\ M)\varphi(P)d\omega_P,$$

where $G(P,\ M)$ is Green's function for D with pole at M. If φ satisfies the Hölder's condition at M, $\Delta U=\varphi$ holds at M. Hence if φ has continuous partial derivatives of the first order in D, $\tilde{\Delta}U=\Delta U=\varphi$ holds everywhere in D.

We now define the sequence of functions $U_n(M)$ by

(5) $$U_n(M)=A_M^{[n]}(U_{n-1})-\frac{1}{10}\varphi(M)r_n^2(M)$$

$$(n=1,\ 2,\ \ldots,\ U_0\equiv F).$$

Then $U_n(M)$ is written in the form

$$U_n(M) = A_M^{[n,1]}(F) - \frac{1}{10}\Big[\varphi(M) r_n^2(M) + A_M^{[n]}(\varphi r_{n-1}^2) + A_M^{[n,n-1]}(\varphi r_{n-2}^2) + \ldots + A_M^{[n,2]}(\varphi r_1^2)\Big]$$

$$= A_M^{[n,1]}(F) - S_M^n(\varphi),$$

say.

The main theorem obtained in (N) is stated as follows.

Theorem 2. *If the series $\sum^{\infty} \lambda_i^2$ is divergent and $\lambda_i \to 0$, the sequence of U_n defined by (5) converges uniformly over any closed subset of D to the function U for any f on B.*

Hence if this condition is satisfied, we see from (3) and (4) that

$$S_M^n(\varphi) \underset{u}{\rightarrow} \frac{1}{4\pi} \iiint_D G(P, M)\, \varphi(P) d\omega_P. \tag{6}$$

The purpose of this paper is to establish an analogous theorem for a more general equation $\Delta U = cU + \varphi$ (or $\tilde{\Delta} U = cU + \varphi$).

Recall here a lemma proved in (N) of which we shall make use later.

Lemma 1. *If U is regular*[3] *in D, it holds*

$$|A_M^{(h)}(U) - U(M)| \leq \frac{h^2}{5} \operatorname*{Max}_{\overline{MP} \leq h} |\Delta U(P)|, \quad h < \rho(M).$$

3. Let $c(\geq 0)$ and φ be bounded continuous functions in D and suppose that there exists a bounded regular function U in D, satisfying the following conditions:

a) U takes f almost everywhere on B;

b) $\Delta U = cU + \varphi$ in D.

For example, if c and φ have continuous partial derivatives of the first order in D, such a function exists[4].

Then

$$\tilde{\Delta} U(M) = \lim_{h \to 0} \frac{10}{h^2}\Big[A_M^{(h)}(U) - U(M)\Big]$$

$$= \Delta U(M) = c(M) U(M) + \varphi(M), \quad M \in D.$$

3) We say that U is regular, if U has continuous partial derivatives of the second order.

4) W. Füschel, Die erste Randwertaufgabe der allgemeinen selbstadjungierte elliptischen Differentialgleichung zweiter Ordnung in Raum für beliebige Gebiete, Math. Zeit. 34 (1932).

G. Tautz, Reguläre Randpunkte beim verallgemeinerten Dirichletschen Problems, Math. Zeit. 39 (1935).

Hence putting

$$\frac{10}{h^2}\left[A_M^{(h)}(U)-U(M)\right]=c(M)U(M)+\varphi(M)+\alpha(M,\ h),$$

we see that $\alpha(M,\ h)\to 0$ as $h\to 0$ for every M of D. Further by virtue of Lemma 1,

$$|\alpha(M,\ h)|\leqq 3\ \underset{P\in D}{\text{l.u.b.}}\,|c(P)U(P)+\varphi(P)|.$$

Therefore $\alpha(M,\ h)$ is bounded and continuous in $(M,\ h)$.

Now $U(M)$ may be written in the form

$$U(M)=\frac{10}{10+c(M)h^2}A_M^{(h)}(U)-\frac{\varphi(M)h^2}{10+c(M)h^2}-\frac{\alpha(M,\ h)h^2}{10+c(M)h^2}.$$

Applying this relation in succession for $h=r_n,\ r_{n-1},\ \dots,\ r_1$, we have

$$U(M)=A_M^{[n,\ 1]}(U,\ c)-S_M^n(\varphi,\ c)-S_M^n(\alpha,\ c), \tag{7}$$

where

$$A_M^{[n]}(U,\ c)\equiv\frac{10}{10+c(M)r_n^2(M)}A_M^{[n]}(U),$$

$$A_M^{[n,\ m]}(U,\ c)\equiv A_M^{[n]}(A^{[n-1]}(\dots(A^{[m]}(U,\ c),\ c)\dots c),\ c)$$

and

$$S_M^n(\alpha,\ c)\equiv\frac{\alpha(M,\ r_n)r_n^2(M)}{10+c(M)r_n^2(M)}+\sum_{i=1}^{n-1}A_M^{[n,\ i+1]}\left(\frac{\alpha(P,\ r_i)r_i^2}{10+c(P)r_i^2(P)},\ c\right).$$

Suppose that $\sum^{\infty}\lambda_i^2$ is divergent and $\lambda_i\to 0$. Since $c\geqq 0$,

$$|S_M^n(\alpha,\ c)|\leqq I_n(M)+J_n(M),$$

where

$$I_n(M)=\frac{1}{10}\Big[|\alpha(M,\ r_n)|r_n^2(M)+A_M^{[n]}(|\alpha(P,\ r_{n-1})|r_{n-1}^2)$$
$$+\dots+A_M^{[n,\ s+1]}(|\alpha(P,\ r_s)|r_s^2)\Big]$$

and

$$J_n(M)=\frac{1}{10}\Big[A_M^{[n,\ s]}(|\alpha(P,\ r_{s-1})|r_{s-1}^2)+\dots+A_M^{[n,\ 2]}(|\alpha(P,\ r_1)|r_1^2)\Big],$$

s being an integer such that $n>s>1$.

It is readily seen by Theorem 1 that $J_n(M)\vec{u}0$ as $n\to\infty$. Putting

$$\delta=\underset{M\in D}{\text{Max}}\,r(M)\quad\text{and}\quad\underset{h\leq\lambda_s\delta}{\text{Max}}|\alpha(M,\ h)|=\alpha_s(M),$$

α_s becomes a bounded continuous function in D. So referring to (6), we have

$$I_n(M)\leqq\frac{1}{10}\Big[\alpha_s(M)r_n^2(M)+A_M^{[n]}(\alpha_s r_{n-1}^2)+\dots+A_M^{[n,\ s+1]}(\alpha_s r_s^2)\Big]$$

$$\overrightarrow{u}\ u_s(M)=\frac{1}{4\pi}\iiint_D G(P,\ M)\,\alpha_s(P)\,d\omega_P.$$

But $\alpha_n \geqq \alpha_{n+1}(n\geqq s)$ and $\alpha_n \to 0$ as $n\to\infty$. Hence $\alpha_n(M)\overrightarrow{u}0$ in D. Therefore $u_s\overrightarrow{u}0$ as $s\to\infty$. Thereby it follows that

$$S_M^n(\alpha,\ c)\overrightarrow{u}0.$$

We thus obtain from (7),

(8) $$A_M^{[n,1]}(U,\ c)-S_M^n(\varphi,\ c)\overrightarrow{u}U(M).$$

4. Let now the function $U_n(M)$ be defined by

(9) $$U_n(M)=A_M^{[n]}(U_{n-1},\ c)-\frac{\varphi(M)r_n^2(M)}{10+c(M)r_n^2(M)}$$

$$(n=1,\ 2,\ \ldots,\ U_0\equiv F).$$

Then $U_n(M)$ may be written

$$U_n(M)=A_M^{[n,1]}(F,\ c)-S_M^n(\varphi,\ c).$$

Our purpose is to prove that $U_n\overrightarrow{u}U$. To prove this, it obviously suffices to show that $A_M^{[n,1]}(F-U,\ c)\overrightarrow{u}0$. To this end we will consider a sequence of functions $A_M^{[n,1]}(\omega,\ c)$, $\omega(M)$ being a bounded continuous function defined in D, vanishing almost everywhere on B. Put $\bar{\omega}(Q)=\limsup_{P\to Q}\omega(P)$, Q being a boundary point of D. Then the set E_ε of all boundary points Q at which $\bar{\omega}(Q)\geqq\varepsilon$ for an arbitrary positive constant ε, is closed and of capacity zero. Covering E_ε by a finite number of regular surfaces S_ε, we denote by $v_\varepsilon(M)$ the conductor potential of the finite body (not necessarily connected) bounded by S_ε. Since E_ε is of capacity zero, we can find S_ε such that $0<v_\varepsilon(M)<\varepsilon$ on a given closed subset R of D. Suppose that $|\omega|\leqq K$ in D. Then, for a suitable positive constant L_ε,

$$\omega(M)\leqq L_\varepsilon V(M)+Kv_\varepsilon(M)+\varepsilon,\ M\in D.$$

From this it follows that

$$A_M^{[n,1]}(\omega,\ c)\leqq L_\varepsilon A_M^{[n,1]}(V,\ c)+KA_M^{[n,1]}(v_\varepsilon,\ c)+\varepsilon.$$

On account of the superharmonicity of v_ε,

$$A_M^{[n,1]}(v_\varepsilon,\ c)\leqq A_M^{[n,1]}(v_\varepsilon)\leqq v_\varepsilon(M)<\varepsilon \quad \text{on} \quad R.$$

On the other hand, by virtue of (2),

$$A_M^{[n,1]}(V,\ c)\leqq A_M^{[n,1]}(V)\overrightarrow{u}0.$$

Therefore

$$A_M^{[n,1]}(\omega, c) < (K+2)\varepsilon \quad \text{on} \quad R,$$

for all sufficiently large n. A similar reasoning gives

$$-A_M^{[n,1]}(\omega, c) < (K+2)\varepsilon \quad \text{on} \quad R,$$

for all sufficiently large n. Consequently, since ε is arbitrary,

$$A_M^{[n,1]}(\omega, c) \underset{u}{\to} 0.$$

Observing this fact we know that

$$A_M^{[n,1]}(F-U, c) \underset{u}{\to} 0,$$

and so

(10) $$U_n(M) = A_M^{[n,1]}(F, c) - S_M^n(\varphi, c) \underset{u}{\to} U(M).$$ [5]

Thus we obtain

Theorem 3. *Let $c(\geqq 0)$ and φ be bounded continuous functions in D. If there exists a bounded regular function U (univocally determined if it exists) taking f almost everywhere on B and satisfying $\Delta U = cU + \varphi$ in D, and if the series $\sum^{\infty} \lambda_i^2$ is divergent and $\lambda_i \to 0$, then the sequence of U_n defined by* (9) *converges uniformly to U over any closed subset of D for any f.*

In the way we can state the following uniqueness theorem:

Theorem 4. *Let $c(\geqq 0)$ be a bounded continuous function in D. If a bounded regular function U vanishes almost everywhere on B and satisfies $\Delta U = cU$ in D, U must be identically zero.*

5. Now we return to the general case where $c(\geqq 0)$ and φ are quite arbitrary bounded and continuous functions. Then the existence of U which satisfies *a*) and *b*) is not necessarily admitted.

The function U_n defined by (9) is the sum of $A_M^{[n,1]}(F, c)$ and $-S_M^n(\varphi, c)$. First we are concerned with $A_M^{[n,1]}(F, c)$. We can always find two bounded functions $\underline{c}_m(\geqq 0)$ and $\bar{c}_m(\geqq 0)$ having continuous partial derivatives of the first order, such that

$$c - \frac{1}{m} \leqq \underline{c}_m \leqq c \leqq \bar{c}_m \leqq c + \frac{1}{m}.$$

Suppose $F \geqq 0$. Then clearly

(11) $$A_M^{[n,1]}(F, \bar{c}_m) \leqq A_M^{[n,1]}(F, c) \leqq A_M^{[n,1]}(F, \underline{c}_m).$$

Also by Theorem 3,

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5) This shows that U is univocally determined if it exists.

(12) $$A_M^{[n,1]}(F, \underline{c}_m) \overrightarrow{u} \underline{W}_m(M) \quad \text{as } n\to\infty,$$

(13) $$A_M^{[n,1]}(F, \bar{c}_m) \overrightarrow{u} \overline{W}_m(M) \quad \text{as } n\to\infty,$$

where $\underline{W}_m$ is bounded, takes f almost everywhere on B and satisfies $\Delta \underline{W}_m = \underline{c}_m \underline{W}_m$ in D; $\overline{W}_m$ is bounded, takes f almost everywhere on B and satisfies $\Delta \overline{W}_m = \bar{c}_m \overline{W}_m$ in D.

Here we prove the following lemma:

Lemma 2. *There is a positive constant dependent only on c (independent of m and M), such that*

(14) $$|\underline{W}_m(M) - \overline{W}_m(M)| < C/m \quad \textit{in } D.$$

In fact,

$$\Delta(\overline{W}_m - \underline{W}_m) = \underline{c}_m(\overline{W}_m - \underline{W}_m) + \eta \overline{W}_m,$$

where $\eta = \bar{c}_m - \underline{c}_m$ and $|\eta| \leqq 2/m$, $\overline{W}_m - \underline{W}_m$ (bounded) vanishes almost everywhere on B. Therefore

$$\overline{W}_m(M) - \underline{W}_m(M) = \lim_{n\to\infty} S_M^n(\eta \overline{W}_m, \underline{c}_m)$$

for every M of D. However $\overline{W}_m(M)$ is bounded with respect to M and m; so that, for a suitable positive constant K, $|\overline{W}_m(M)| < K$ holds for all M and for all positive integers m. Hence

$$|\underline{W}_m(M) - \overline{W}_m(M)| < \frac{2K}{m} \lim_{n\to\infty} S_M^n(1) = \frac{2}{m} KV(M).$$

We thereby find a positive constaut C desired in (14).

On account of (11), (12), (13) and Lemma 2, we conclude that $\overline{W}_m$ and $\underline{W}_m$ converge uniformly as $m\to\infty$ towards a same continuous function, say $A_M^{[\infty]}(F, c)$, and so $A_M^{[n,1]}(F, c) \overrightarrow{u} A_M^{[\infty]}(F, c)$.

It now remains to find the properties of the function $A_M^{[\infty]}(F, c)$ thus obtanied. By Lemma 1,

$$\frac{10}{h^2}\left|\left[A_M^{(h)}(\overline{W}_m) - \overline{W}_m(M)\right] - \left[A_M^{h}(\overline{W}_{m+p}) - \overline{W}_{m+p}(M)\right]\right|$$
$$\leqq 2 \operatorname*{Max}_{MP\leq h} |\bar{c}_m(P)\overline{W}_m(P) - \bar{c}_{m+p}(P)\overline{W}_{m+p}(P)|$$
$$(m, p = 1, 2, \ldots\ldots).$$

Making $p\to\infty$, next $h\to 0$ and finally $m\to\infty$, we obtain

$$\tilde{\Delta} A_M^{[\infty]}(F, c) = c(M) A_M^{[\infty]}(F, c).$$

On the other hand, it is readily seen that $A_M^{[\infty]}(F, c)$ takes f almost everywhere on B.

When F is not $\geqq 0$, we shall put $F = F^+ - (-F)^+$, where $\operatorname{Max}(F, 0)$

$=F^+$. The above results hold for each F^+ and $(-F)^+$, hence certainly for F.

Next consider $S_M^n(\varphi, c)$. Suppose $\varphi \geqq 0$. Take $\underline{c}_m$, $\bar{c}_m$ as before and $\underline{\varphi}_m(\geqq 0)$, $\bar{\varphi}_m(\geqq 0)$, bounded functions having continuous partial derivatives of the first order, such that

$$\varphi - \frac{1}{m} \leqq \underline{\varphi}_m \leqq \varphi \leqq \bar{\varphi}_m + \frac{1}{m} \quad \text{in } D.$$

Then, for any m and n,

$$S_M^n(\bar{\varphi}_m, \underline{c}_m) \geqq S_M^n(\varphi, c) \geqq S_M^n(\underline{\varphi}_m, \bar{c}_m). \tag{15}$$

Also by Theorem 3,

$$S_M^n(\bar{\varphi}_m, \underline{c}_m) \underset{u}{\rightarrow} \bar{\Phi}_m(M) \quad \text{as } n \to \infty, \tag{16}$$

$$S_M^n(\underline{\varphi}_m, \bar{c}_m) \underset{u}{\rightarrow} \underline{\Phi}_m(M) \quad \text{as } n \to \infty, \tag{17}$$

where $\bar{\Phi}_m = 0$ almost everywhere on B and $\Delta\bar{\Phi}_m = \underline{c}_m \bar{\Phi}_m + \bar{\varphi}_m$ in D; $\underline{\Phi}_m = 0$ almost everywhere on B and $\Delta\underline{\Phi}_m = \bar{c}_m \underline{\Phi}_m + \underline{\varphi}_m$ in D. Proceeding as before we can find a positive constant C independent of M and m, such that

$$|\underline{\Phi}_m(M) - \bar{\Phi}_m(M)| < C/m \quad \text{in } D. \tag{18}$$

On account of (15), (16), (17) and (18), we conclude that $\bar{\Phi}_m$ and $\underline{\Phi}_m$ converge uniformly as $m \to \infty$ towards a same continuous function, say $S_M^\infty(\varphi, c)$, and so

$$S_M^n(\varphi, c) \underset{u}{\rightarrow} S_M^\infty(\varphi, c).$$

Furthermore with the help of Lemma 1,

$$\tilde{\Delta} S_M^\infty(\varphi, c) = c(M) S_M^\infty(\varphi, c) - \varphi(M)$$

for every M of D. It is easily seen that $S_M^\infty(\varphi, c)$ vanishes almost everywhere on B.

When φ is not $\geqq 0$, we can proceed as for F and prove that the above results hold in this general case.

Putting

$$A_M^{[\infty]}(F, c) - S_M^\infty(\varphi, c) = U(M),$$

$U(M)$ is bounded continuous and satisfies the following conditions:

a) U takes f almost everywhere on B;

b') $\tilde{\Delta} U = cU + \varphi$ in D.

Thus we obtain

Theorem 5. *If $c(\geqq 0)$ and φ are bounded continuous functions in D, and if the series $\sum^{\infty} \lambda_i^2$ is divergent and $\lambda_i \to 0$, then the sequence of U_n defined by* (9) *converges uniformly over any closed subset of D towards a bounded and continuous function U taking f almost everywhere on B and satisfying*

$$\tilde{\Delta} U = cU + \varphi \quad \textit{in } D.$$

If c and φ have continuous partial derivatives of the first order, $\Delta U = cU + \varphi$ holds everywhere in D.

Remark. The above reasonings and results are valid in the case where one takes the peripheral mean

$$L_M^{(h)}(F) = \frac{1}{4\pi h^2} \iint_{\overline{MP} = h} F(P)\, d\gamma_P$$

instead of the spatial mean $A_M^{(h)}(F)$. Here it is necessary to take the coefficient 6 instead of 10 in (5), (9), etc.

The whole results are valid similary in the plane as in the space for the circumferential mean or areal mean, where the coefficieht must be taken 4 or 8 respectively.

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