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On the Growth of Subharmonic Functions and its Applications to a Study of the Minimum Modulus of Integral Functions

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I. Growth of subharmonic functions.

1. Let E be a measurable set on the positive x-axis and let $E(a, b)$ denote the part of E contained in (a, b). We put

$$\overline{\mu}E=\overline{\lim_{r\to\infty}}\frac{mE(1, r)}{\log r} \quad \text{and} \quad \underline{\mu}E=\underline{\lim_{r\to\infty}}\frac{mE(1, r)}{\log r},$$

where $mE(1, r)$ is the logarithmic measure of $E(1, r)$, namely

$$mE\,(1,r)=\int_{E(1,r)}\frac{dt}{t}.$$

Then clearly

$$(1)\qquad \overline{\mu}E+\underline{\mu}CE=1,$$

CE denoting the complementary set of E.

We consider an infinite domain D on the z-plane such that $z=\infty$ belongs to its boundary B. Let D_r denote a connected part of D contained in $|z|<r$ and B_r its boundary lying on $|z|=r$. Then there exists a bounded harmonic function in D_r which assumes 1 on B_r and 0 on the boundary lying in $|z|<r$, except at points of a set of logarithmic capacity zero, namely the harmonic measure $\omega_r(z)$ of B_r with respect to D_r. The following majoration of $\omega_r(z)$ is due to A. Beurling[1]:

$$(2)\qquad \omega_r(z)<2e^{-\frac{1}{2}mE(|z|,\, r)},\quad |z|<r.$$

Let $f(r)$ be a positive increasing function in $(0, \infty)$ such that $f(r)\to\infty$ as $r\to\infty$, and let $U(z)$ be a positive subharmonic function in D which satisfies the condition:

$$(3)\qquad \overline{\lim_{z\to z'}}U(z)\leq f(|z'|)$$

for every boundary point z', except at points of a set of logarithmic capacity zero.[2] Put

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1) A. Beurling, Études sur un problème de majoration, Thesis, Upsala, 1933.

2) We say that an infinite set E is of logarithmic capacity zero, whenever any finite subset of E is so.

$$\mathfrak{M}(r)=\underset{|z|=r}{\text{l.u.b.}}\ U(z)$$

and

(4) $$\sigma=\lim_{r\to\infty}\frac{\log f(r)}{\log r}$$

and suppose that there exists a positive increasing function $\varphi(r)$ in $(0, \infty)$ such that

(5) $$\mathfrak{M}(r)\leq\varphi(r)<\infty$$

for all $r(0\leq r<\infty)$.

Let Ω be the whole z-plane and E the set of absolute values of z which run over Ω-D. We can then prove

Theorem 1. *If* $\overline{\lim_{r\to\infty}}\frac{\mathfrak{M}(r)}{r^{\rho}}<\infty$ *and* $\overline{\mu}\equiv\overline{\mu}E>2(\rho-\sigma)>0$,

(6) $$\underline{\lim_{r\to\infty}}\frac{\log\mathfrak{M}(r)}{\log r}\leq\frac{\sigma\overline{\mu}}{\overline{\mu}-2(\rho-\sigma)},$$

and moreover if $\underline{\mu}\equiv\underline{\mu}E>2(\rho-\sigma)>0$,

(7) $$\overline{\lim_{r\to\infty}}\frac{\log\mathfrak{M}(r)}{\log r}\leq\frac{\sigma\underline{\mu}}{\underline{\mu}-2(\rho-\sigma)}.$$

If $\underline{\lim_{r\to\infty}}\frac{\mathfrak{M}(r)}{r^{\rho}}<\infty$ *and* $\underline{\mu}>2(\rho-\sigma)>0$,

(8) $$\underline{\lim_{r\to\infty}}\frac{\log\mathfrak{M}(r)}{\log r}\leq\frac{\sigma\overline{\mu}}{\underline{\mu}-2(\rho-\sigma)}.$$

Proof. Suppose $\overline{\lim_{r\to\infty}}\frac{\mathfrak{M}(r)}{r^{\rho}}<\infty(\rho>\sigma)$. Then, if R is large enough,

$$f(R)<KR^{\rho}\quad\text{and}\quad\mathfrak{M}(R)<KR^{\rho}$$

for a suitable constant K. Using (2), (4) and (5), we obtain

(9) $$U(z)<f(R)+\text{const.}\ \{KR^{\rho}-f(R)\}e^{-\frac{1}{2}mE(|z|,R)}$$
$$<f(R)\{1+\text{const.}\ e^{(\rho-\sigma+\varepsilon)\log R-\frac{1}{2}mE(|z|,R)}\}$$

for every $R(>|z|>0)$ and for any $\varepsilon>0$.

Writing

(10) $$2(\rho-\sigma+\varepsilon)-\frac{mE(|z|,R)}{\log R}=\Theta(|z|, R, \varepsilon),$$

(9) becomes

(11) $$U(z)<f(R)\{1+\text{const.}\ e^{\frac{1}{2}\Theta(|z|, R, \varepsilon)\log R}\}.$$

Putting $R=|z|^{\alpha}(|z|=r>1, \alpha>1)$, $\Theta(|z|, R, \varepsilon)$ is written as follows:

$$\Theta(r, r^{\alpha}, \varepsilon)=2(\rho-\sigma+\varepsilon)-\frac{mE(1, r^{\alpha})}{\log r^{\alpha}}+\frac{mE(1, r)}{\alpha\log r}.$$

For any $\eta>0$, on the one hand,

$$\underline{\mu}-\eta<\frac{mE(1,\ r^{\alpha})}{\log r^{\alpha}} \quad \text{and} \quad \frac{mE(1,\ r)}{\log r}<\bar{\mu}+\eta$$

for all r sufficiently large; on the other hand, there exists a sequence of values of $r_i\rightarrow\infty$ such that

$$\bar{\mu}-\eta<\frac{mE(1,\ r_i^{\alpha})}{\log r_i^{\alpha}} \quad \text{and} \quad \frac{mE(1,\ r_i)}{\log r_i}<\bar{\mu}+\eta.$$

Thereby

$$\Theta(r,\ r^{\alpha},\ \varepsilon)<2(\rho-\sigma+\varepsilon)-\underline{\mu}+\frac{\bar{\mu}}{\alpha}+\eta(1+\frac{1}{\alpha})$$

and

$$\Theta(r_i,\ r_i^{\alpha},\ \varepsilon)<2(\rho-\sigma+\varepsilon)-\bar{\mu}+\frac{\bar{\mu}}{\alpha}+\eta(1+\frac{1}{\alpha}).$$

If $\bar{\mu}>2(\rho-\sigma)>0$, we can take ε so small that $\bar{\mu}>2(\rho-\sigma+\varepsilon)>0$. Then we can choose α such that $2(\rho-\sigma+\varepsilon)-\bar{\mu}+\bar{\mu}/\alpha<0$, that is,

$$\alpha>\frac{\bar{\mu}}{\bar{\mu}-2(\rho-\sigma+\varepsilon)}.$$

For such a α we can also find η and a positive constant δ so small that $\Theta(r_i,\ r_i^{\alpha},\ \varepsilon)<-\delta$. For α and η thus obtained,

$$\frac{1}{2}\Theta(r_i,\ r_i^{\alpha},\ \varepsilon)\log r_i\rightarrow 0$$

as $r_i\rightarrow\infty$. Hence, from (11),

$$U(z_i)<f(r_i^{\alpha})\{1+\varepsilon(r_i)\} \quad (|z_i|=r_i),$$

where $\varepsilon(r_i)\rightarrow 0$ as $r_i\rightarrow\infty$, and so

$$\varliminf_{r\rightarrow\infty}\frac{\log \mathfrak{M}(r)}{\log r}\leqq\sigma\alpha.$$

However α may be taken arbitrarily close to $\dfrac{\bar{\mu}}{\bar{\mu}-2(\rho-\sigma+\varepsilon)}$ and, since ε is arbitrary, to $\dfrac{\bar{\mu}}{\bar{\mu}-2(\rho-\sigma)}$. Thus we obtain the desired result:

$$\varliminf_{r\rightarrow\infty}\frac{\log \mathfrak{M}(r)}{\log r}\leqq\frac{\sigma\bar{\mu}}{\bar{\mu}-2(\rho-\sigma)}. \tag{6}$$

Moreover, if $\underline{\mu}>2(\rho-\sigma)>0$, we can choose ε, α and η such that

$$\underline{\mu}>2(\rho-\sigma+\varepsilon)>0,\ \alpha>\frac{\bar{\mu}}{\underline{\mu}-2(\rho-\sigma+\varepsilon)}$$

and

$$\Theta(r,\ r^{\alpha},\ \varepsilon)<-\delta$$

for all r sufficiently large, δ being a suitable positive constant. Then

$$U(z)<f(r^{\alpha})\{1+\varepsilon(r)\} \quad (|z|=r),$$

where $\varepsilon(r)\to 0$ as $r\to\infty$. From this it follows just as before that

(7) $$\varliminf_{r\to\infty}\frac{\log \mathfrak{M}(r)}{\log r}\leq\frac{\sigma\bar{\mu}}{\underline{\mu}-2(\rho-\sigma)}.$$

Next suppose $\varliminf_{r\to\infty}\frac{\mathfrak{M}(r)}{r^{\rho}}<\infty$ and $\underline{\mu}>2(\rho-\sigma)>0$. Then there exists a sequence of values of $r_i\to\infty$ such that

$$U(z_i)<f(r_i^{\alpha})\{1+\varepsilon(r_i)\} \quad (|z_i|=r_i)$$

for $\alpha>\frac{\bar{\mu}}{\underline{\mu}-2(\rho-\sigma)}$, where $\varepsilon(r_i)\to 0$ as $r_i\to\infty$. From this it follows as before that

(8) $$\varliminf_{r\to\infty}\frac{\log \mathfrak{M}(r)}{\log r}\leq\frac{\sigma\bar{\mu}}{\underline{\mu}-2(\rho-\sigma)}.$$

The theorem is thus proved.

For later use we quote the following

Theorem 2. *Let $U(z)$ be a subharmonic function in D. If $\varlimsup_{z\to z'} U(z)\leq K<\infty$ for every finite boundary point z', and if $\varlimsup_{r\to\infty}\frac{\mathfrak{M}(r)}{r^{\rho}}<\infty$ $(\rho>0)$ and $\bar{\mu}E>2\rho$, then*

$$U(z)\leq K \quad in \quad D,$$

where K is a constant[3].

2. M. Tsuji first introduced the upper and lower strong logarithmic densities of E as follows[4]:

$$\bar{\lambda}E=\varlimsup_{r,\,s\to\infty}\frac{mE(r,\,sr)}{\log s} \quad \text{and} \quad \underline{\lambda}E=\varliminf_{r,\,s\to\infty}\frac{mE(r,\,sr)}{\log s}.$$

Then

(12) $$0\leq\underline{\lambda}E\leq\underline{\mu}E\leq\bar{\mu}E\leq\bar{\lambda}E\leq 1$$

and

(13) $$\bar{\lambda}E+\underline{\lambda}CE=1.$$

By means of the strong lower logarithmic density, we can state the following

Theorem 3. *If $\varliminf_{r\to\infty}\frac{\mathfrak{M}(r)}{r^{\rho}}<\infty$ and $\underline{\lambda}\equiv\underline{\lambda}E>2(\rho-\sigma)>0$,*

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3) M. Inoue, Une étude sur les fonctions sousharmoniques et ses applications aux fonctions holomorphes, Memoirs of the Fac. of Sci., Kyūsyū Univ., Vol. 3, No. 1 (1943).

4) M. Tsuji, Wiman's theorem on integral functions of order $<\frac{1}{2}$, to appear in the Proc. Jap. Acad., where he uses the notations $\bar{\lambda}$, $\underline{\lambda}$, $\bar{\lambda}^*$ and $\underline{\lambda}^*$, instead of $\bar{\mu}$, $\underline{\mu}$, $\bar{\lambda}$ and $\underline{\lambda}$, respectively.

(14) $$\overline{\lim_{r\to\infty}}\frac{\log \mathfrak{M}(r)}{\log r}\leqq\frac{\sigma\lambda}{\lambda-2(\rho-\sigma)}.$$

If $\underline{\lim}_{r\to\infty}\frac{\mathfrak{M}(r)}{r^\rho}<\infty$ *and* $\lambda>2(\rho-\sigma)>0$,

(15) $$\underline{\lim}_{r\to\infty}\frac{\log \mathfrak{M}(r)}{\log r}\leqq\frac{\sigma\lambda}{\lambda-2(\rho-\sigma)}.$$

Proof. First suppose $\overline{\lim}_{r\to\infty}\frac{\mathfrak{M}(r)}{r^\rho}<\infty$ and $\lambda>2(\rho-\sigma)>0$. Here we obtain as before

(11) $$U(z)<f(R)\{1+\text{const. } e^{\frac{1}{2}\Theta(|z|,\,R,\,\varepsilon)\log R}\}$$

for every $R(>|z|>0)$, where ε is chosen so small that $\lambda>2(\rho-\sigma+\varepsilon)>0$.

Let β be a finite positive value greater than $2(\rho-\sigma+\varepsilon)/\lambda$, and put $s(r)=r^{\frac{\beta}{1-\beta}}$, $R=rs(r)$, $|z|=r$. Then, if r is large enough,

$$\Theta(r,\ rs(r),\ \varepsilon)<2(\rho-\sigma+\varepsilon)-(\lambda-\varepsilon')\frac{\log s(r)}{\log rs(r)}$$

for any $\varepsilon'>0$. Since $\frac{\log s}{\log rs}=\beta>2(\rho-\sigma+\varepsilon)/\lambda$, we find, if ε' is small enough,

$$\Theta(r,\ rs(r),\ \varepsilon)<-\delta$$

for r sufficiently large, δ being a suitable positive constant. Hence, from (11),

$$U(z)<f(rs(r))\{1+\varepsilon(r)\},$$

where $\varepsilon(r)\to 0$ as $r\to\infty$. From this it follows that

$$\overline{\lim_{r\to\infty}}\frac{\log \mathfrak{M}(r)}{\log r}\leqq\frac{\sigma}{1-\beta}.$$

However β may be taken arbitrarily close to $2(\rho-\sigma+\varepsilon)/\lambda$ and, since ε is arbitrary, to $2(\rho-\sigma)/\lambda$. Thus we obtain

(14) $$\overline{\lim_{r\to\infty}}\frac{\log \mathfrak{M}(r)}{\log r}\leqq\frac{\sigma\lambda}{\lambda-2(\rho-\sigma)}.$$

Secondly suppose $\underline{\lim}_{r\to\infty}\frac{\mathfrak{M}(r)}{r^\rho}<\infty$ and $\lambda>2(\rho-\sigma)>0$. We can then see by a similar manner as in the proof of Theorem 1 that there exists a sequence of values of $r_i\to\infty$ such that

$$U(z_i)<f(r_is(r_i))\{1+\varepsilon(r_i)\}(|z_i|=r_i),$$

where $\varepsilon(r_i)\to 0$ as $r_i\to\infty$. From this it follows as before that

$$\text{(15)}\qquad \varliminf_{r\to\infty}\frac{\log \mathfrak{M}(r)}{\log r}\leq\frac{\sigma\lambda}{\lambda-2(\rho-\sigma)}.$$

The theorem is thus proved.

II. Unbounded Dirichlet Problem [5].

3. Let E be an unbounded closed set on the positive x-axis and D be the entire z-plane outside E. Let $f(r)$ be a positive and continuous increasing function in $(0, \infty)$ such that $f(r)\to\infty$ as $r\to\infty$, and put

$$\sigma=\varlimsup_{r\to\infty}\frac{\log f(r)}{\log r}.$$

We will then prove

Theorem 4. *If $\underline{\mu}\equiv\underline{\mu}E>2\sigma$, there exists a harmonic function $U(z)$ in D, such that*

$$U(z)=f(|z|)$$

almost everywhere [6] *on E and*

$$U(z)<\text{const.}\ |z|^{\frac{1-\mu}{2}+\sigma+\varepsilon}$$

in D for any $\varepsilon>0$.

Proof. Let $\omega(z, r)$ denote a bounded harmonic function in D which assumes 1 a. e. on $E(r, \infty)$ and vanishes a. e. on $E(0, r)$, that is the harmonic measure of $E(r, \infty)$ with respect to D. If we put

$$\text{(16)}\qquad U_r(z)=-\int_0^r f(t)d\omega(z, t),$$

$U_r(z)$ is bounded and harmonic in D, assumes f a. e. on $E(0, r)$ and vanishes a. e. on $E(r, \infty)$.

$U_r(z)$ is written:

$$U_r(z)=\Big[f(t)\omega(z, t)\Big]_r^0+\int_0^r\omega(z, t)df(t).$$

For fixed z and for any $\varepsilon'>0$ such that $\underline{\mu}>2\varepsilon'$, we can find a positive constant t_0 such that, for every $t(>t_0>|z|)$,

$$\omega(z, t)<\omega_t(z)<2e^{-\frac{1}{2}mE(|z|, t)}<\frac{2}{t^{\underline{\mu}/2-\varepsilon'}}.$$

Then, for $r>t_0>|z|$,

5) The initiative of this problem was taken by M. Tsuji, see his paper quoted. The main reasoning which follows is due to him.

6) "almost everywhere" means "except at points of a set of logarithmic capacity zero" Hereafter we write for simplicity "a. e." instead of "almost everywhere".

$$U_r(z) < f(t_0) + 2\int_{t_0}^{r} \frac{df(t)}{t^{\mu/2-\varepsilon'}}$$

$$< f(t_0) + \frac{2f(r)}{r^{\mu/2-\varepsilon'}} + \int_{t_0}^{r} \frac{f(t)\,dt}{t^{1+\mu/2-\varepsilon'}}.$$

Since $\mu > 2\sigma$, it holds

$$f(t) < \text{const.}\, t^{\mu/2-\varepsilon}$$

for every $t > 0$ and for any ε such that $\mu/2 - \sigma > \varepsilon > 0$. Hence

$$U_r(z) < f(t_0) + \frac{\text{const.}}{r^{\varepsilon-\varepsilon'}} + \text{const.} \int_1^{r} \frac{dt}{t^{1+\varepsilon-\varepsilon'}},$$

and so

$$\lim_{r\to\infty} U_r(z) < \infty,$$

provided $\varepsilon > \varepsilon'$.

It is clear that $U_r(z)$ is increasing function of r for fixed z. Hence $\lim\limits_{r\to\infty} U_r(z)$ exists at each point z of D. Denoting the limit by $U(z)$, we see that $U(z)$ is expressible in the form

$$U(z) = -\int_0^{\infty} f(t)\,d\omega(z,\ t). \tag{17}$$

Evidently $U(z)$ is harmonic in D and assumes f a. e. (at regular points) on E. We call $U(z)$ (constructed in this way) the solution of the Dirichlet Problem for D and f.

Now consider the expression (17). For every $|z|(>1)$ and for any $\varepsilon'(\mu > 2\varepsilon' > 0)$, we can choose a constant c sufficiently large so that

$$U(z) < f(c|z|) + 2\int_{c|z|}^{\infty} e^{-\frac{1}{2}mE(|z|,\ t)}\,df(t)$$

$$< f(c|z|) + 2|z|^{\frac{1}{2}} \int_{c|z|}^{\infty} \frac{df(t)}{t^{\mu/2-\varepsilon'}}.$$

From this we deduce for any $\varepsilon > 0$ and for any η such that $\mu/2 - \sigma > \eta > 0$,

$$U(z) < \text{const.}\ |z|^{\sigma+\varepsilon} + \text{const.}\ |z|^{\frac{1}{2}} \int_{c|z|}^{\infty} \frac{dt}{t^{1+\eta-\varepsilon'}},$$

if ε' is small enough, for instance, $\varepsilon' < \varepsilon$. Putting $\eta = \mu/2 - \sigma - \varepsilon/2$ and choosing ε' so small that $2\varepsilon' < \varepsilon$, it follows that

$$U(z) < \text{const.}\,|z|^{\sigma+\varepsilon} + \text{const.}\,|z|^{\frac{1-\mu}{2}+\sigma+\varepsilon} \int_1^{\infty} \frac{dt}{t^{1+\varepsilon/2-\varepsilon'}}$$

$$< \text{const.}\,|z|^{\frac{1-\mu}{2}+\sigma+\varepsilon}.$$

This is the desired result. The theorem is thus completely proved.

4. We will now prove

Theorem 5. *Let $V_r(z)$ be a bounded harmonic function in D_r which assumes $f(|z|)$ at each boundary point z, except at points of a set of logarithmic capacity zero. Then, if $\underline{\mu}E>2\sigma$, $V_r(z)$ converges as $r\to\infty$ towards the solution of the Dirichlet Problem for D and f.*

Proof. Let $\omega_t(z, r)$ denote a bounded harmonic function in D_r which vanishes a. e. on the boundary lying in $|z|<t(\leq r)$ and assumes 1 a. e. on the rest of boundary. Then clearly

$$V_r(z)=f(r)\omega_r(z)-\int_0^r f(t)d\omega_t(z, r)\ ^{7)}$$

and

$$U_r(z)\geq-\int_0^r f(t)d\omega_t(z, r).$$

Hence

$$U_r(z)\geq V_r(z)-f(r)\omega_r(z).$$

Since $\underline{\mu}>2\sigma$, $f(r)\omega_r(z)\to 0$ as $r\to\infty$. Therefore

$$U(z)=\lim_{r\to\infty} U_r(z)\geq \overline{\lim_{r\to\infty}}\, V_r(z).$$

On the other hand,

$$U_r(z)=\Big[f(t)\omega(z, t)\Big]_r^0+\int_0^r \omega(z, t)df(t)$$

$$<\Big[f(t)\omega(z, t)\Big]_r^0+\int_0^r \omega_t(z, r)df(t)$$

$$=V_r(z)-f(r)\omega(z, r).$$

Evidently $f(r)\omega(z, r)\to 0$ as $r\to\infty$. Therefore

$$U(z)\leq \underline{\lim_{r\to\infty}}\, V_r(z).$$

Consequently

$$U(z)=\lim_{r\to\infty} V_r(z), \quad \text{q. e. d.}$$

Theorem 6. *Let $f(r)$ be convex in $\log r$ and let $U(z)$ be the solution of the Dirichlet Problem for D and f. Then $U(z)\geq f(|z|)$.*

Proof. By the preceding theorem, $U(z)=\lim_{r\to\infty} V_r(z)$. If we put $\mathfrak{F}(z)=f(|z|)$, $\mathfrak{F}(z)$ is subharmonic in the whole z-plane. Hence $V_r(z)\geq \mathfrak{F}(z)=f(|z|)$ in D_r for any $r>0$. Consequently $U(z)\geq f(|z|)$.

As a special case of Theorem 4, we can state in view of Theorem 6,

Theorem 7. *There exists a harmonic function $U(z)$ in D such that*

7) One may see $\omega_r(z)=\omega_r(z, r)$.

$$U(z)=|z|^{\rho}$$

almost everywhere on E *and*

$$|z|^{\rho}\leqq U(z)<\text{const.}\,|z|^{\frac{1-\mu E}{2}+\rho+\varepsilon}$$

in D *for any* $\rho(0<\rho<\mu/2)$ *and for any* $\varepsilon>0$[8].

III. Applications.

Let $F(z)$ be a non-constant integral function of order ρ:

$$\rho=\varlimsup_{r\to\infty}\frac{\log\log M(r)}{\log r},$$

where $M(r)$ denotes the maximum of $|F(z)|$ on $|z|=r$. We define the lower order ρ' of F as follows:

$$\rho'=\varliminf_{r\to\infty}\frac{\log\log M(r)}{\log r}.$$

Let $f(r)$ be a positive and continuous increasing function of r in $(0, \infty)$ such that $f(r)\to\infty$ as $r\to\infty$, and put

$$\sigma=\lim_{r\to\infty}\frac{\log f(r)}{\log r}.$$

Denote by $m(r)$ the minimum of $|F(z)|$ on $|z|=r$.

5. We will now prove

Theorem 8. *If* $0\leqq\sigma<\rho<\frac{1}{2}$,

$$\overline{\mu}E(\log m(r)>f(r))\geqq(1-2\rho)\,(1-\frac{\sigma}{\rho}). \tag{18}$$

Proof. Since $0<\rho<\frac{1}{2}$, $F(z)$ may be represented in the form

$$F(z)=a_0 z^k\prod_1^{\infty}(1-\frac{z}{a_n}),\quad a_n\neq 0.$$

As far as we are concerned with a study of the set of r on which $\log m(r)>f(r)$, the most unfavourable case, fixing the moduli of zeros, will occur, for instance, when all $a_n>0$. Then $m(r)$ will evidently be as small as possible and $M(r)$ as large as possible. We will therefore consider without loss of generality,

$$F(z)=|a_0|z^k\prod_1^{\infty}(1-\frac{z}{|a_n|}),\quad a_n\neq 0,$$

of which the order is unchanged; $m(r)$ is always attained on the positive x-axis.

8) This theorem is obtained by M. Tsuji in a more general case. See his paper quoted.

We now suppose that

$$\underline{\mu}\equiv\underline{\mu}E(\log m(r)\leq f(r))>2(\rho-\sigma)+\frac{\sigma}{\rho}.$$

Let D be the entire z-plane outside E. Replacing U, ρ, φ with $\log^+|F(z)|$, $\rho+\varepsilon(\varepsilon>0)$, $\log^+ M(r)$ in Theorem 1, we have in virtue of (7),

$$\rho\leq\frac{\sigma}{\underline{\mu}-2(\rho-\sigma)},$$

since $\overline{\mu}E\leq1$ and ε is arbitrary. This gives contradictory

$$\underline{\mu}\leq2(\rho-\sigma)+\frac{\sigma}{\rho}.$$

Hence

$$\underline{\mu}E(\log m(r)\leq f(r))\leq2(\rho-\sigma)+\frac{\sigma}{\rho}$$

and so by (1),

$$\overline{\mu}E(\log m(r)>f(r))\geq(1-2\rho)\,(1-\frac{\sigma}{\rho}).$$

As a corollary we can state:

If $0<\rho<\frac{1}{2}$ *and* $\lim\limits_{r\to\infty}\frac{\log f(r)}{\log r}=0$,

(19) $$\overline{\mu}E(\log m(r)>f(r))\geq1-2\rho.$$

Especially if $0<k<\infty$,

(20) $$\overline{\mu}E(\log m(r)>(\log r)^k)\geq1-2\rho.$$

We have also:

If $0<\rho<\frac{1}{2}$ *and* $0<\alpha<1$,

(21) $$\overline{\mu}E(\log m(r)>r^{\alpha\rho})\geq(1-2\rho)\,(1-\alpha).$$

Applying (6) we will reach by a similar manner:

If $0\leq\sigma<\rho'\leq\rho<\frac{1}{2}$,

$$\mu E(\log m(r)>f(r))\geq1-2\rho-\frac{\sigma}{\rho'}\,(1-2\rho').$$

But we know at present a better result. In fact, in a preceding paper [9], the author obtained the following result:

If $0\leq\rho<\frac{1}{2}$,

(22) $$\underline{\mu}E(\log m(r)>r^{\rho'-\varepsilon}))\geq1-2\rho$$

for any $\varepsilon>0$.

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9) M. Inoue, Sur le module minimum des fonctions sousharmoniques et des fonctions entières d'ordre $<\frac{1}{2}$, Memoirs of the Fac. of Sci., Kyūsyū Univ., Vol. 4, No. 2 (1949).

Evidently this contains the above mentionned as a special case.

6. Applying Theorem 3 we obtain

Theorem 9. *If* $0 \leqq \sigma < \rho < \frac{1}{2}$,

(23) $$\bar{\lambda} E(\log m(r) > f(r)) \geqq 1 - 2\rho.$$

Proof. Suppose $\underline{\lambda} \equiv \underline{\lambda} E(\log m(r) \leqq f(r)) > 2\rho$. Then, proceeding as in the proof of Theorem 8, we conclude from (14) that

$$\rho \leqq \frac{\sigma \underline{\lambda}}{\underline{\lambda} - 2(\rho - \sigma)};$$

that is, $\rho \leqq \sigma$ since $\underline{\lambda} > 2\rho$. This is a contradiction. Hence

$$\underline{\lambda} E(\log m(r) \leqq f(r)) \leqq 2\rho,$$

and so by (12),

$$\bar{\lambda} E(\log m(r) > f(r)) \geqq 1 - 2\rho.$$

Corollary. *If* $0 < \rho < \frac{1}{2}$

(24) $$\bar{\lambda} E(\log m(r) > r^{\rho - \varepsilon}) \geqq 1 - 2\rho$$

for any $\varepsilon > 0$.

This is a result recently obtained by M. Tsuji [10].

7. Applying Theorems 4 and 6, we obtain

Theorem 10. *Let* $f(r)$ *be convex in* $\log r$. *If* $0 \leqq \sigma < \rho < \frac{1}{2}$,

(25) $$\bar{\mu} E(\log m(r) > f(r)) \geqq \min \{(1 - 2\rho),\ 2(\rho - \sigma)\}.$$

Proof. Suppose

$$\bar{\mu} E(\log m(r) > f(r)) < 1 - 2\rho,$$

so that

$$\underline{\mu} E(\log m(r) \leqq f(r)) > 2\rho.$$

Let $E \equiv E(\log m(r) \leqq f(r))$, $\underline{\mu} \equiv \underline{\mu} E$, and let D be the entire z-plane outside E. $U(z)$ denotes the solution of the Dirichlet Problem for D and f. Consider

$$W(z) = \log |F(z)| - U(z).$$

Then, by Theorem 6, $U(z) \geqq f(|z|)$ in D. Therefore $W(z) \leqq 0$ everywhere on E and

$$\varlimsup_{r \to \infty} \frac{\max_{|z|=r} W(z)}{r^{\rho + \varepsilon}} \leqq 1$$

for any $\varepsilon > 0$ such that $\underline{\mu} > 2(\rho + \varepsilon)$. According to Theorem 2, $W(z)$

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10) M. Tsuji, loc. cit.

$\leqq 0$ in D. Hence, by Theorem 4,

$$\log M(|z|) \leqq U(z) < \text{const.}\, |z|^{\frac{1-\mu}{2}+\sigma+\varepsilon}.$$

in D for any $\varepsilon > 0$. Since ε is arbitrary, this yields

$$\rho \leqq \frac{1-\mu}{2} + \sigma,$$

so that

$$\mu \leqq 1-2(\rho-\sigma).$$

Finally by (1),

$$\overline{\mu}E(\log m(r) > f(r)) \geqq 2(\rho-\sigma).$$

This proves the theorem.

Consequently

Theorem 11. *Let $f(r)$ be convex in $\log r$. If $\sigma \leqq 2\rho - \frac{1}{2}\left(<\frac{1}{2}\right)$,*

(26) $$\overline{\mu}E(\log m(r) > f(r)) \geqq 1-2\rho$$

and if $2\rho - \frac{1}{2} \leqq \sigma < \rho < \frac{1}{2}$,

(27) $$\mu E(\log m(r) > f(r)) \geqq 2(\rho-\sigma).$$

We thus obtain the following

Corollary. *If $0 < \frac{1}{2} - \rho < \varepsilon$,*

(28) $$\mu E(\log m(r) > r^{\rho-\varepsilon}) \geqq 1-2\rho$$

and if $0 < \varepsilon < \frac{1}{2} - \rho$,

(29) $$\overline{\mu}E(\log m(r) > r^{\rho-\varepsilon}) \geqq 2\varepsilon.$$

But, for $0 < \varepsilon < \rho(1-2\rho)$ $\left(< \frac{1}{2} - \rho \text{ evidently}\right)$, there exists by M. Tsuji [10] an integral function of order $\rho\left(0 < \rho < \frac{1}{2}\right)$ such that

(30) $$\overline{\mu}E(\log m(r) > r^{\rho-\varepsilon}) < 1-2\rho.$$

The relation between $\overline{\mu}E(\log m(r) > r^{\rho-\varepsilon})$ and $1-2\rho$ for $\frac{1}{2} - \rho > \varepsilon > \rho$ $(1-2\rho)$ remains unknown.

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