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Note on Locally Free Groups

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1. Let G be a group. If any finite number of elements from G generate a free subgroup in G, then G is called to be *locally free.* Some properties of locally free groups were given by A. Kurosch.[1] For instance: No element except the identity is of finite order. Any subgroup of a locally free group is locally free. Free products of locally free groups are also locally free. The set-theoretical sum of an increasing sequence of locally free groups is also locally free. Conversely, in case of countable groups, we have easily:

Any countable locally free group is either a free group with a finite number of generators or the set-theoretical sum of an infinite increasing sequence of free groups with finite number of generators.

To this class of groups, belong free groups, the additive group of all rational numbers, all its subgroups and some others.

In this note we shall study some necessary and sufficient conditions for countable locally free groups to be exactly free. Section 2 is devoted to some preliminary remarks on properties of free groups and to the proof of a maximum condition which holds in free groups, in section 3 we define *$*$-subgroups* of locally free groups and give a condition in terms of $*$-subgroups, and finally in section 4 another condition is given which is in a form of maximum condition.

2. In this section let F be a free group with an arbitrary number of free generators. The number of free generators of F is called the *rank* of F and denoted by $\rho(F)$. As is well known, the rank is determined uniquely only by F itself.

Provided that F is of finite rank, F. Levi[2] has proved that a genertor system S of F is a free generator system if, and only if, S consists of the same number of elements as the rank of F. More

1) A. Kurosch, *Lakal freie Gruppen*, Comptes Rendus (Doklady) de l'Academie des Science de l'URSS 24 (1939) pp. 99-101.

2) F. Levi, *Über die Untergruppen der freien Gruppen*, Math. Zeitschr. Bd. 32 (1930) S. 315-317; Bd. 37 (1933) S. 90-97.

precisely, if a free group F is generated by n elements and r is the minimal number of defining relalions between those elements, then F is of rank n-r. Thererfore if a free group is generated by n elements and if there exists at least one non-trivial relation between them, then the rank of the group is finite and less than n. Obviously there holds

Lemma 1. *If F is the free product of two subgroups A and B, the rank of F is the sum of those of A and B:*

$$\rho(F) = \rho(A) + \rho(B).$$

As an immediate consequence of the subgroup-theorem [3] in free products of groups, we have

Lemma 2. *If K is a free factor of F and U is any subgroup of F, then $K \frown U$ is a free factor of U, where $K \frown U$ may be a non proper factor of U.*

Now we prove

Theorem 1. *In a free group F, for any positive integer r, there holds the maximum condition (C_r) for subgroups of the same finite rank r. That is, there exists no infinite properly increasing sequence of subgroups $U_1 \subset U_2 \subset \dots \subset U_n \subset \dots$, where every U_n is of rank r.*

Proof. Assume that the infinite sequence $U_1 \subset U_2 \subset \dots \subset U_n \subset \dots$ is properly increasing. Set $U = \smile U_n$, the set-theoretical sum of all U_n. The subgroup U of the free group F is also free and is of infinite rank, because the sequence is infinite and properly increasing. Any $r+1$ elements from a free generator system of U generats a free subgroup V of rank $r+1$. But these $r+1$ elements are contained in some suitably chosen U_m. Then, by lemma 2, $U = V * W$ and $V \leq U_m$ imply $U_m = V * W_m$ and, by lemma 1, $r = \rho(U_m) \geq \rho(V) = r+1$. This is a contradiction.

3. Let G be a *countable locally free* group. G is either a free group of finite rank or is represented as the set-thoretical sum of a countably infinite properly increasing sequence of free subgroups of finite rank:

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3) A. Kurosch, *Die Untergruppen der freien Produkte von beliebigen Gruppen*, Math. Ann. 109 (1934) S. 647. *Über freie Produkte von Gruppen*, Math. Ann. 108 (1933). S. 26.

R. Baer und F. Levi, *Freie Produkte und ihre Untergruppen*, Compositio Math. 3 (1036) S. 391.

M. Takahasi, *Bemerkungen über den Untergruppenatz in freien Produkte*, Proc, Imp. Acad. Tokyo 20 (1944) pp. 589-594.

(1) $$G=\bigcup F_n,\quad F_1\subset F_2\subset\cdots\subset F_n\subset\cdots,$$

where F_n is a free subgroup of finite rank.

If every term F_n of the sequence (1) can be chosen as a free subgroup of the same finite rank r, that is, if any finite subset of G can be embeded in a free subgroup of rank r, the least such integer r is called, by A. Kurosch, the *rank* of the *locally free* group G.

Theorem 1 shows immediately that if a locally free group G of finite rank r is represented by the actually infinite and properly increasing sequence (1), then G can not be a free group.

Difinition. A subgroug K of a locally free group G, which is generated by a finite number of elements, that is, which is a free subgroup of finite rank, is called a *$*$-subgroup* of G, when any subgroup of G, which is a free subgroup of finite rank and contains K, has K as one of its free factors.

We can easily prove the following properties of $*$-subgroups.

(i) *If a $*$-subgroup K is of rank r, then K is a maximal subgroup of rank r,* that is, *there exists no free subgroup of rank r which contains K properly.*

Proof. If a free subgroup U of finite rank contains K, then by the definition of $*$-subgroups $U=K*V$, and by lemma 1,

$$\rho(U)=\rho(K)+\rho(V)=r+\rho(V).$$

Therefore, $U\neq K$ implies $V\neq 1$ and $\rho(V)>0$. hence $\rho(U)>r$.

(ii) *Any free factor of a $*$-sucgroup K is also a $*$-subgroup.*

Proof. Let $K=H*L$. By lemma 1 it is obvious that H is of finite rank. If U is a free subgroup of G which is of finite rank and contains H, then the join $K\bigcup U$ is also of finite rank and contains K. Therfore $K\bigcup U=K*P=H*L*P$. According to lemma 2, $U=(U\bigcap H)*V=H*V$. Hence H is a $*$-subgroup.

Lemma 2 shows immediately that

(iii) *Any free factor of G which is of finie rank is always a $*$-subgroup of G.*

(iv) *The meet $K\bigcap K'$ of two $*$-subgroups K and K' is also a $*$-subgroup.*

Proof. $K\bigcup K'$ is of finite rank and contains the $*$-subgroup K, therefore $K\bigcup K'=K*P$ and, by lemma 2, $K'=K'\bigcap K*Q$. According to what we have just proved above in (ii), $K\bigcap K'$ is a $*$-subgroup.

Quite analogously as in (ii), wh have also:

(v) *If K is a $*$-subgroup and U is any free subgroup of finite rank, which in general does not contain K, then $K \frown U$ is a free factor of U.*

(vi) *A free subgroup K of finite rank is a $*$-subgroup if and only if K is a common free factor of every F_m which contains K, for any one fixed sequence of subgroups F_n which defines G as (1).*

Proof. Necessity is obvious.

Let U be a free subgroup of finite rank which contains K. U and K are both contained in some one F_m. By lemma 2, $F_m=K*P\supseteq U$ implies $U=(U\frown K)*Q=K*Q$. Hence K is a $*$-snbgroup.

(vii) *Any $*$-subgroup of a free group F is always a free factor of F.*

Proof. If F is a free group and if K is a $*$-subgroup of F, then there exists always a free factor L of F, which is of finite rank and contains K. From $L=K*L'$ and $F=L*M$ follows $F=K*(L'*M)$.

Now we have a condition, in terms of $*$-subgroups, for a countable locally free group to be exactly free.

Theorem 2. *For a countable locally free group G to be exactly free, it is necessary and sufficient that any finite number of elements of G can be embeded in some one $*$-subgroup of G.*

Proof. Necessity follows from the above statement (iii).

Let $G=\bigvee F_n,\quad F_1\subset F_2\subset \dots \subset F_n\subset \dots$

be such a representation of G by an infinite increasing sequence of free grous of finite rank as (1).

According to the condition, there exists a $*$-subgroup K_n which contains F_n, moreover there exists a $*$-subgroup K'_{n+1} which coutains both K_n and F_{n+1}, because K_n and F_{n+1} are both of finite rank. Hence we have an increasing sequence of $*$-subgroups such that

$$G=\bigvee K'_{n+1} \text{ and } K'_2\subseteqq K'_3\subseteqq K'_4\subseteqq \dots \subseteqq K'_{n+1}\subseteqq \dots .$$

Since every K'_{n+1} is a $*$-subgroup, $K'_{n+2}=K'_{n+1}*P_{n+1}$, therfore we have $G=K'_2*P_2*P_3*\dots$, hence G is a free group.

4. According to Theorem 1 and the statements in the previous section, there arises the question whether the condition (C_r) in Theorem 1 is also sufficient or not for a countable locally free group G to be exactly free. But the auther has not been able to answer it.

As one of stronger conditions than (C_r) we take the following condition:

(D) *There exists no infinite properly increasing sequence of free subgroups of finite rank of G,* $V_1 \subset V_2 \subset \dots \subset V_n \subset \dots$, *where* V_1 *is not contained in any proper free factor of any* V_n.

Then we have

Theorem 3. *For a countable locally free froup G to be free, it is necessary and sufficient that the condition (D) holds.*

Proof. Necessity: Let $V_1 \subset V_2 \subset \dots \subset V_n \subset \dots$ be an infinite properly increasing sequence of subgroups of finite rank in a *free* group F.

Then the set-theoretical sum V of all V_n is a *free* group of a countably infinite rank.

According to Theorem 2, V_1 can be embeded in a $*$-subgroup K of V. Since $V = \bigcup V_n \supset K$, K is contained in every V_m for sufficiently large m, hence it is a common free factor of them. Therefore V_1 can be embeded in a *proper* free factor of some V_n.

Sufficiency: Let G be a countable locally free group, and, as in section 3, take an increasing sequence of free subgroups of finite rank such that

$$(1) \qquad G = \bigcup F_n, \quad F_1 \subseteqq F_2 \subseteqq \dots \subseteqq F_n \subseteqq \dots .$$

Assume that this sequence is infinite and properly increasing.

For any fixed F_n, accoding to the condition (D), we can chose an infinite subsequence of (1) such that every term has a *proper* free factor containing F_n.

Let
$$F_n \subset F_{m_1} \subset F_{m_2} \subset \dots \subset F_{m_i} \subset \dots$$
be such a subsequence.

Denote by K_{m_i} the least proper free factor of F_{m_i} which contains F_n. Actually K_{m_i} is determined uniquely, because the meet of two distinct proper free factors of F_{m_i} containing F_n is also a proper free factor of F_{m_i} contaning F_n, and its rank is less than those of two.

Now we have an infinite sequence of these K_{m_i}:

$$F_n \subseteqq K_{m_1} \subseteqq K_{m_2} \subseteqq \dots .$$

If this sequence is infinite and properly increasing, according to (D) again, there must exist some m_i such that K_{m_i} has a *proper* free factor L_{m_i} containing F_n. Then L_{m_i} is a free factor of F_{m_i} which contains F_n and $L_{m_i} \subset K_{m_i}$. This contradicts to the definition of the least free factor K_{m_i} of F_{m_i} which contains F_n.

Therefore we can conclude that, for a sufficiently large m, m_j . m implies $K_{m_j}=K_m$. Then, according to the statement (vi) in section 3, K is a $*$-subgroup of G.

Hence any finite number of elements of G can be embeded in a $*$-subgroup and, by Theorem 2, it can be proved that G is exactly free.

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