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ON DEFORMATIONS OF GENERALIZED CALABI–YAU AND GENERALIZED SU(n)-STRUCTURES

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Abstract

In this paper we will introduce the new notion of generalized geometric structures defined by systems of closed differential forms. From a cohomological point of view, we develop a unified approach to deformation problems and establish a criterion for unobstructed deformations of the generalized geometric structures. We construct the moduli spaces of the structures with the action of d-closed b-fields and show that the period map of the moduli space is locally injective under the certain cohomological condition (the local Torelli type theorem). We apply our approach to generalized Calabi–Yau structures and generalized SU(n)-structures and obtain unobstructed deformations of generalized Calabi–Yau structures if the $dd^{\mathcal{J}}$-property is satisfied. We also have unobstructed deformations of generalized SU(n)-structures and show that the period map of the moduli space of generalized SU(n)-structures is locally injective.

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Introduction

In the paper [6], the author introduced the notion of geometric structures defined by systems of closed differential forms which are based on the action of the gauge group of the tangent bundle of a manifold. This approach provides a systematic construction of smooth moduli spaces of Calabi–Yau, hyperKähler, G_2 and Spin(7)-structures. In the

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paper [14] Hitchin presented generalized complex and generalized Calabi–Yau geometries, which depend on the idea of replacing the tangent bundle by the direct sum of the tangent bundle T and the cotangent bundle T^* of a manifold X. The generalized complex geometry unifies different structures such as complex structures and symplectic structures. There are many articles already written on generalized complex, generalized Calabi–Yau and generalized Kähler geometries [14], [15], [11], [7], [8]. In this paper, however, we will develop the generalized geometry from a wide view point as in [6] which is of general nature with some new applications. Since there is an indefinite metric on the direct sum $T \oplus T^*$, the bundle of the Clifford algebra $\mathrm{CL}(X)$ of $T \oplus T^*$ naturally appears and we obtain various fibre bundles with fibres the Lie groups such as the spin group and the Clifford group $\mathrm{G_{cl}}$ which act on the differential forms on X by the spin representation at each point of X. We consider an orbit $\mathcal{B}(V)$ of the action of the Clifford group $\mathrm{G_{cl}}$. Then we define a $\mathcal{B}(V)$-structure Φ on X to be a system of closed differential forms in the orbit $\mathcal{B}(V)$ at each $x \in X$ (see Section 3.1). We develop the deformation problem of the $\mathcal{B}(V)$-structures. We establish a criterion for unobstructed deformations of the $\mathcal{B}(V)$-structures and we show that the local Torelli type theorem holds under the certain cohomological condition (Theorems 3.2.5, 3.2.6 and 3.2.7). Then we apply our approach to generalized Calabi–Yau structures and generalized $\mathrm{SU}(n)$-structures. A generalized Calabi–Yau structure ϕ is a non-degenerate, pure spinor which is a d-closed differential form on a manifold [14]. A generalized Calabi–Yau structure ϕ induces the generalized complex structure $\mathcal{J}_\phi$, which is regarded as a generalization of both complex structures with trivial canonical line bundle and symplectic structures[1]. Since the set of non-degenerate, pure spinors is an orbit $\mathcal{B}_{\mathrm{SL}}(V)$ of the action of $\mathrm{G_{cl}}$, generalized Calabi–Yau structures introduced by Hitchin [14] are considered as $\mathcal{B}_{\mathrm{SL}}(V)$-structures. Then our criterion is applied to the generalized Calabi–Yau structures.

Theorem 4.1.6. *Let ϕ be a generalized Calabi–Yau structure on a compact manifold X with the induced generalized complex structure $\mathcal{J}_\phi$. If the generalized complex structure $\mathcal{J}_\phi$ satisfies the $dd^{\mathcal{J}}$-property, we have unobstructed deformations of ϕ as generalized Calabi–Yau structures which are parametrized by an open set of the cohomology group $H^1(\#_{\mathcal{B}_{\mathrm{SL}}})$. Further the period map P from the space of deformations of ϕ to the de Rham cohomology group is locally injective, i.e., the local Torelli type theorem holds.*

(Note that $\#_{\mathcal{B}_{\mathrm{SL}}}$ is the deformations complex of generalized Calabi–Yau structures and $H^k(\#_{\mathcal{B}_{\mathrm{SL}}})$ is the cohomology group of the complex $\#_{\mathcal{B}_{\mathrm{SL}}}$, see Section 4.)

The $dd^{\mathcal{J}}$-property is a generalization of the ordinary $\partial\overline{\partial}$-lemma in Kähler geometry. Gualtieri showed that the $dd^{\mathcal{J}}$ property holds for generalized Kähler structures

[1]Note that generalized Calabi–Yau structures do not yield any metrical structure such as Ricci-flat Kähler metrics.

[12]. The generalized $\mathrm{SU}(n)$-structure[1] in [11] is a pair consisting of d-closed non-degenerate, pure spinors ϕ_0 and ϕ_1 on a $2n$ dimensional manifold such that the corresponding pair of generalized complex structures $(\mathcal{J}_{\phi_0}, \mathcal{J}_{\phi_1})$ yields a generalized Kähler structure. The generalized $\mathrm{SU}(n)$-structure is regarded as a natural generalization of the ordinary Ricci-flat Kähler metrics. Since the set of generalized $\mathrm{SU}(n)$-structures is an orbit $\mathcal{B}_{\mathrm{SU}}(V)$ of the diagonal action of $\mathrm{G_{cl}}$ on pairs of differential forms, generalized $\mathrm{SU}(n)$-structures are also considered as $\mathcal{B}_{\mathrm{SU}}(V)$-structures. Deformations of generalized $\mathrm{SU}(n)$-structures seem to be complicated. Our systematic approach, however, can be adapted to obtain unobstructed deformations of generalized $\mathrm{SU}(n)$-structures and the local Torelli type theorem,

Theorem 4.2.4. *Let $\Phi = (\phi_0, \phi_1)$ be a generalized $\mathrm{SU}(n)$-structure on a compact manifold X of dimension $2n$. Then we obtain unobstructed deformations of Φ as generalized $\mathrm{SU}(n)$-structures which are parametrized by an open set of the cohomology group $H^1(\#_{\mathcal{B}_{\mathrm{SU}}})$. Further the period map of the moduli space $\mathfrak{M}_{\mathrm{SU}}(X)$ is locally injective, i.e., the local Torelli type theorem holds.*

(Note that $\#_{\mathcal{B}_{\mathrm{SU}}}$ is the deformation complex of generalized $\mathrm{SU}(n)$-structures and $H^k(\#_{\mathcal{B}_{\mathrm{SU}}})$ is the cohomology group of the complex $\#_{\mathcal{B}_{\mathrm{SU}}}$.) In Section 1, we give an exposition of the Clifford algebra of the direct sum $V \oplus V^*$ and introduce various groups such as spin, pin and the Clifford group $\mathrm{G_{cl}}$. It is important that the exponential e^b (resp. e^β) for a 2-form $b \in \bigwedge^2 V^*$ (resp. a 2-vector $\beta \in \bigwedge^2 V$) gives an element of the spin group. The materials in this section are already explained in [21], [13] and [14]. The bundle of the Clifford algebra $\mathrm{CL}(X)$ is decomposed into the direct sum of the even Clifford bundle and the odd Clifford bundle. In Section 2, we introduce subbundles CL^k of $\mathrm{CL}(X)$ which carry a filtration of the even Clifford bundle and a filtration of the odd Clifford bundle:

$$\mathrm{CL}^0 \subset \mathrm{CL}^2 \subset \mathrm{CL}^4 \subset \cdots,$$
$$\mathrm{CL}^1 \subset \mathrm{CL}^3 \subset \mathrm{CL}^5 \subset \cdots.$$

Further we discuss differential operators acting on differential forms on X which arise as commutators between the exterior derivative d and the action of the bundle of the Clifford algebra $\mathrm{CL}(X)$. The Clifford–Lie operators of order 3 are introduced in Definition 2.1.2 The exterior derivative d is a Clifford–Lie operator of order 3 and the adjoint $e^{-a} \circ d \circ e^a$ for $a \in \mathrm{CL}^2$ is also a Clifford–Lie operator of order 3 (Proposition 2.1.8), which play a significant role in studying the deformation problem. In Section 3, the notion of $\mathcal{B}(V)$-structures is introduced. The Clifford group $\mathrm{G_{cl}}$ of $V \oplus V^*$ diagonally acts on the direct sum of l skew-symmetric tensors $\bigoplus^l \bigwedge^* V^*$. Let $\Phi =$

[1]The generalized $\mathrm{SU}(n)$-structure is called a generalized Calabi–Yau metrical structure in [11] and in order to avoid notational confusion, we use the terminology generalized $\mathrm{SU}(n)$-structures on which the special unitary group $\mathrm{SU}(n)$ arise as the isotropy group.

$(\phi_1, \dots, \phi_l)$ be an element of the direct sum $\bigoplus^l \bigwedge^* V^*$ and $\mathcal{B}(V)$ the orbit of G_{cl} through Φ. We fix the orbit $\mathcal{B}(V)$. Let X be a compact manifold X of dimension n. (Note that we always consider a real manifold in this paper.) The orbit $\mathcal{B}(V)$ yields the orbit $\mathcal{B}(T_x X)$ in $\bigoplus^l \bigwedge^* T_x^* X$ for each point $x \in X$ and we have a fibre bundle $\mathcal{B}(X)$ by

$$\mathcal{B}(X) := \bigcup_{x \in X} \mathcal{B}(T_x X) \to X.$$

The set of C^∞ global sections of $\mathcal{B}(X)$ is denoted by $\mathcal{E}_{\mathcal{B}}(X)$ and then we define a $\mathcal{B}(V)$-structure $\Phi = (\phi_1, \dots, \phi_l)$ on X to be a C^∞ global section of $\mathcal{B}(X)$ with $d\phi_i = 0$ for all $i = 1, \dots, l$. (For simplicity, we write it by $d\Phi = 0$.) We denote by $\tilde{\mathfrak{M}}_{\mathcal{B}}(X)$ the set of $\mathcal{B}(V)$-structures on X:

$$\tilde{\mathfrak{M}}_{\mathcal{B}}(X) = \{\Phi \in \mathcal{E}_{\mathcal{B}}(X) \mid d\Phi = 0\}.$$

Then we define the moduli space $\mathfrak{M}_{\mathcal{B}}(X)$ of $\mathcal{B}(V)$-structures on X by the quotient space:

$$\mathfrak{M}_{\mathcal{B}}(X) = \tilde{\mathfrak{M}}_{\mathcal{B}}(X) / \widetilde{\mathrm{Diff}}_0(X),$$

where $\widetilde{\mathrm{Diff}}_0(X)$ is an extension of the diffeomorphisms of X by the action of d-exact b-fields (see Definition 3.1.2). Since the de Rham cohomology class $[\phi_i]$ of each component ϕ_i of $\Phi \in \tilde{\mathfrak{M}}_{\mathcal{B}}(X)$ is invariant under the action of $\widetilde{\mathrm{Diff}}_0(X)$, we have the period map:

$$P_{\mathcal{B}} \colon \mathfrak{M}_{\mathcal{B}}(X) \to \bigoplus^l H^*_{\mathrm{dR}}(X).$$

In order to discuss deformations of a $\mathcal{B}(V)$-structure Φ, we introduce a suitable deformation complex $\#_{\mathcal{B}}$ (Proposition 3.2.1):

$$0 \to \mathbf{E}^{-1}(X) \xrightarrow{d_{-1}} \mathbf{E}^0(X) \xrightarrow{d_0} \mathbf{E}^1(X) \xrightarrow{d_1} \mathbf{E}^2(X) \xrightarrow{d_2} \cdots.$$

Each vector bundle $\mathbf{E}^{k-1}(X)$ is defined by the action of the Clifford subbundle CL^k on Φ, that is, $\mathbf{E}^{k-1}(X) = \mathrm{CL}^k \cdot \Phi$ and the differential operator d_k is the restriction of d to the bundle $\mathbf{E}^k(X)$. An orbit $\mathcal{B}(V)$ is *an elliptic orbit* if the deformation complex $\#_{\mathcal{B}}$ is an elliptic complex in degree $k = 1, 2$. We denote by S the direct sum $\bigoplus_{p=0}^n \bigwedge^p T^*$. Then we obtain the full de Rham complex: $\cdots \xrightarrow{d} S \xrightarrow{d} S \xrightarrow{d} \cdots$. The cohomology group of the full de Rham complex is given by the full de Rham cohomology group $H^*_{\mathrm{dR}}(X) := \bigoplus_{p=0}^n H^p(X)$. Since the complex $\#_{\mathcal{B}}$ is a subcomplex of the direct sum of the full de Rham complex, we have the map $p^k_{\mathcal{B}}$ from the cohomology groups $H^k(\#_{\mathcal{B}})$ of the complex $\#_{\mathcal{B}}$ to the direct sum of the full de Rham cohomology groups $\bigoplus^l H^*_{\mathrm{dR}}(X)$. We say a $\mathcal{B}(V)$-structure Φ is *a topological structure* if the map $p^k_{\mathcal{B}}$ is injective for $k = 1, 2$ (Definition 3.2.3). Our criterion for unobstructed deformations and the local Torelli type theorem is shown in Theorem 3.2.5:

Theorem 3.2.5. *Let $\mathcal{B}(V)$ be an elliptic orbit and Φ a $\mathcal{B}(V)$-structure on a compact manifold X of dimension n. If Φ is a topological structure, then Φ is unobstructed and there exists a neighborhood U of Φ in the moduli space $\mathfrak{M}_{\mathcal{B}}(X)$ such that the restriction of the period map $P_{\mathcal{B}}|_U \colon U \to \bigoplus^l H^*_{\mathrm{dR}}(X)$ is injective. Further, if an orbit $\mathcal{B}(V)$ is an elliptic and topological orbit on X, the period map $P_{\mathcal{B}} \colon \mathfrak{M}_{\mathcal{B}}(X) \to \bigoplus^l H^*_{\mathrm{dR}}(X)$ is locally injective at each point, that is, the local Torelli theorem holds.*

In Section 4 we apply our approach to generalized Calabi–Yau structures and generalized SU(n)-structures. A generalized Calabi–Yau structure ϕ gives rise to a generalized complex structure $\mathcal{J}_\phi$, where d-closeness of ϕ implies the integrability of the structure $\mathcal{J}_\phi$. Deformations of generalized complex structures were discussed from the viewpoint of the Dirac structure and the Courant algebroid [11], [23]. The relations of deformations of generalized complex structures $\mathcal{J}_\phi$ and deformations of generalized Calabi–Yau structure ϕ is given in Proposition 4.1.8. We can obtain generalized hyper-Kähler, G_2 and Spin(7)-structures as special $\mathcal{B}(V)$-structures [10]. It must be noted that the generalized exceptional structures (G_2 and Spin(7)-structures) are discussed by Witt [27] from a different point of view. Our approach can be adapted in these interesting cases. We will discuss the deformation problems of other special structures in a forthcoming paper.

1. Clifford algebra and spin representation

1.1. The Clifford algebra preliminaries. Let V be an n dimensional real vector space and V^* the dual space of V. We denote by $\eta(v)$ by the natural pairing between $v \in V$ and $\eta \in V^*$. Then there is a symmetric bilinear form $\langle\ ,\ \rangle$ on the direct sum $V \oplus V^*$ which is defined by

$$\langle E_1, E_2\rangle = \frac{1}{2}\eta_1(v_2) + \frac{1}{2}\eta_2(v_1), \tag{1.1.1}$$

where $E_i = v_i + \eta_i \in V \oplus V^*$ for $i = 1, 2$. We denote by $\bigotimes^k(V \oplus V^*)$ the tensor product of k-copies of $V \oplus V^*$. Then the tensor algebra $\bigotimes(V \oplus V^*)$ of $V \oplus V^*$ is given by

$$\bigotimes(V \oplus V^*) := \sum_{i=0}^{\infty} \bigotimes^k(V \oplus V^*), \tag{1.1.2}$$

where $\bigotimes^0(V \oplus V^*) = \mathbb{R}$. We define $\mathcal{I}$ to be the two-sided ideal in $\bigotimes(V \oplus V^*)$ generated by all elements of the form $E \otimes E - \|E\|^2 1$ for $E \in V \oplus V^*$, where $\|E\|^2 = \langle E, E\rangle$. Then the Clifford algebra $\mathrm{CL}(V \oplus V^*)$ is defined to be the quotient algebra:

$$\mathrm{CL}(V \oplus V^*) = \bigotimes(V \oplus V^*)/\mathcal{I}. \tag{1.1.3}$$

The tensor product yields the product of the Clifford algebra which is called the Clifford product. We denoted by $\alpha \cdot \beta$ the Clifford product of α and β for $\alpha, \beta \in \mathrm{CL}(V \oplus V^*)$. Then it turns out that the following relation holds

$$E \cdot F + F \cdot E = 2\langle E, F\rangle 1, \quad E, F \in V \oplus V^*. \tag{1.1.4}$$

Since the ideal $\mathcal{I}$ is generated by tensors of degree 2, the Clifford algebra $\mathrm{CL}(V \oplus V^*)$ is decomposed into the even part and the odd part:

$$\mathrm{CL}(V \oplus V^*) = \mathrm{CL}^{\mathrm{even}} \oplus \mathrm{CL}^{\mathrm{odd}}, \tag{1.1.5}$$

where $\mathrm{CL}^{\mathrm{even}} = \sum_{i=0}^{\infty} \bigotimes^{2i}(V \oplus V^*)/\mathcal{I}$ and $\mathrm{CL}^{\mathrm{odd}} = \sum_{i=0}^{\infty} \bigotimes^{2i+1}(V \oplus V^*)/\mathcal{I}$. There are two involutions of $\mathrm{CL}(V \oplus V^*)$. The first one is the parity involution which is defined by

$$\tilde{\alpha} := \begin{cases} +\alpha, & (\alpha \in \mathrm{CL}^{\mathrm{even}}), \\ -\alpha, & (\alpha \in \mathrm{CL}^{\mathrm{odd}}), \end{cases} \tag{1.1.6}$$

for $\alpha \in \mathrm{CL}(V \oplus V^*)$. If we reverse the order in a simple product $\alpha = E_1 \cdot E_2 \cdots E_k \in \mathrm{CL}(V \oplus V^*)$ of $E_1, \dots, E_k \in V \oplus V^*$, we obtain the second involution σ of $\mathrm{CL}(V \oplus V^*)$:

$$\sigma(\alpha) = E_k \cdots E_2 \cdot E_1. \tag{1.1.7}$$

Since there is the natural isomorphism between the skew-symmetric tensors $\bigwedge^*(V \oplus V^*)$ and $\mathrm{CL}(V \oplus V^*)$ as $\mathbb{R}$-module, there is the metric $\langle\ ,\ \rangle$ on $\mathrm{CL}(V \oplus V^*)$ which is written as

$$\langle \alpha, \beta\rangle = \langle 1, \sigma(\alpha)\beta\rangle, \tag{1.1.8}$$

for $\alpha, \beta \in \mathrm{CL}(V \oplus V^*)$ (cf. [13]). The Clifford norm $\langle \alpha, \alpha\rangle$ of α is given by

$$\langle \alpha, \alpha\rangle = \langle 1, \sigma(\alpha)\alpha\rangle. \tag{1.1.9}$$

Let $\bigwedge^p V^*$ be the space of skew-symmetric tensor of degree p and S the direct sum of the spaces of skew-symmetric tensors:

$$S := \bigoplus_{p=0}^{\infty} \bigwedge^{p} V^*. \tag{1.1.10}$$

Then $E = v + \eta \in V \oplus V^*$ acts on S by the interior and the exterior product:

$$E \cdot \phi = i_v\phi + \eta \wedge \phi. \tag{1.1.11}$$

Since we have the identity:

$$E \cdot (E \cdot \phi) = i_v(\eta \wedge \phi) + \eta \wedge (i_v \phi) = \|E\|^2 \phi, \tag{1.1.12}$$

we have the action of $\mathrm{CL}(V \oplus V^*)$ on S which is called the spin representation. Let $\mathrm{CL}(V \oplus V^*)^\times$ be the group which consists of invertible elements of $\mathrm{CL}(V \oplus V^*)$. For each $g \in \mathrm{CL}(V \oplus V^*)^\times$, the twisted adjoint $\widetilde{\mathrm{Ad}}_g \colon \mathrm{CL}(V \oplus V^*) \to \mathrm{CL}(V \oplus V^*)$ is given by

$$\widetilde{\mathrm{Ad}}_g(\alpha) := \tilde{g}^{-1} \alpha g, \tag{1.1.13}$$

where $\alpha \in \mathrm{CL}(V \oplus V^*)$ and $\tilde{g}$ is the parity involution of g as in (1.1.6), (cf. [21]). The image $\widetilde{\mathrm{Ad}}_g(V \oplus V^*)$ is not contained in $V \oplus V^*$ for a general $g \in \mathrm{CL}(V \oplus V^*)^\times$. The Clifford group G_{cl} ($= \mathrm{G}_{\mathrm{cl}}(V \oplus V^*)$) is a subgroup of $\mathrm{CL}(V \oplus V^*)^\times$ defined by

$$\mathrm{G}_{\mathrm{cl}} := \{ g \in \mathrm{CL}(V \oplus V^*)^\times \mid \widetilde{\mathrm{Ad}}_g(V \oplus V^*) \subset V \oplus V^* \}. \tag{1.1.14}$$

Since $\widetilde{\mathrm{Ad}}_g$ is an orthogonal endmorphism of $V \oplus V^*$, we have the short exact sequence:

$$1 \to \mathbb{R}^\times \to \mathrm{G}_{\mathrm{cl}} \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(V \oplus V^*) \to \mathrm{id}. \tag{1.1.15}$$

Since every element g of the Clifford group G_{cl} is written as a simple product $E_1 \cdot \cdots \cdot E_k$ for $E_1, \ldots, E_k \in V \oplus V^*$, it follows that the Clifford norm of $g \in \mathrm{G}_{\mathrm{cl}}$ is given by $\sigma(g) \cdot g$. We define the pin group $\mathrm{Pin}(V \oplus V^*)$ by

$$\mathrm{Pin}(V \oplus V^*) = \{ g \in \mathrm{G}_{\mathrm{cl}} \mid \sigma(g) \cdot g = \pm 1 \}, \tag{1.1.16}$$

and the spin group $\mathrm{Spin}(V \oplus V^*)$ is defined by

$$\mathrm{Spin}(V \oplus V^*) := \mathrm{Pin}(V \oplus V^*) \cap \mathrm{CL}^{\mathrm{even}}. \tag{1.1.17}$$

Then we also have the short exact sequence using the adjoint map:

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(V \oplus V^*) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(V \oplus V^*) \to \mathrm{id}. \tag{1.1.18}$$

We denote by Spin_0 the identity component of $\mathrm{Spin}(V \oplus V^*)$. Then Spin_0 is given by

$$\mathrm{Spin}_0 = \{ g \in \mathrm{Spin}(V \oplus V^*) \mid \sigma(g) \cdot g = 1 \}. \tag{1.1.19}$$

1.2. Spin representation. The Lie algebra $\mathrm{so}(V \oplus V^*)$ of the Lie group $\mathrm{SO}(V \oplus V^*)$ is decomposed into three parts:

$$\mathrm{so}(V \oplus V^*) = \mathrm{End}(V) \oplus \bigwedge^2 V \oplus \bigwedge^2 V^*. \tag{1.2.1}$$

In fact each $a \in \mathrm{so}(V \oplus V^*)$ gives the endmorphism of $V \oplus V^*$ which is written as

$$\begin{pmatrix} A & \beta \\ b & -A^* \end{pmatrix}$$

with $A \in \mathrm{End}(V)$, $b \in \bigwedge^2 V^*$, $\beta \in \bigwedge^2 V$ and $A^* \in \mathrm{End}(V^*)$ is defined by $A^*(\eta)(v) = \eta(Av)$ for $v \in V$ and $\eta \in V^*$, where a 2-form b is regarded as the homomorphism from V to V^* and a 2-vector β is also considered as the one from V^* to V. We denote by q the embedding of $\mathrm{GL}(V)$ into $\mathrm{SO}(V \oplus V^*)$,

$$q\colon \mathrm{GL}(V) \to \mathrm{SO}(V \oplus V^*), \tag{1.2.2}$$

which is given by

$$q(g) = \begin{pmatrix} g & 0 \\ 0 & (g^*)^{-1} \end{pmatrix}, \tag{1.2.3}$$

where $g \in \mathrm{GL}(V)$. Let Ad: $\mathrm{Spin}(V \oplus V^*) \to \mathrm{SO}(V \oplus V^*)$ be the adjoint map as in Section 1.1. Since the kernel of the map Ad is $\mathbb{Z}_2$, the inverse image $\mathrm{Ad}^{-1}(q(g))$ consists of two elements g_{cl} and $-g_{\mathrm{cl}}$, where $q(g) = \mathrm{Ad}(g_{\mathrm{cl}}) = \mathrm{Ad}(-g_{\mathrm{cl}})$.

REMARK The exponential $e^b = 1 + b + (1/2!)b^2 + \cdots$ gives an element of Spin_0 for $b \in \bigwedge^2 V^*$ which gives the action of the b-filed. The exponential $e^\beta = 1 + \beta + (1/2!)\beta^2 + \cdots$ is also an element of Spin_0 for $\beta \in \bigwedge^2 V$.

Since $\mathrm{Spin}(V \oplus V^*)$ is the subgroup of $\mathrm{CL}(V \oplus V^*)$, the representation on $S = \bigwedge^\bullet V^*$ of the Clifford algebra $\mathrm{CL}(V \oplus V^*)$ yields the representation ρ_{spin} of $\mathrm{Spin}(V \oplus V^*)$,

$$\rho_{\mathrm{spin}}\colon \mathrm{Spin}(V \oplus V^*) \to \mathrm{GL}(S). \tag{1.2.4}$$

We also denote by ρ^*_{GL} the linear representation of $\mathrm{GL}(V)$ on $S = \bigwedge^* V^*$.

Lemma 1.2.1. *Let g_{cl} be an element of $\mathrm{Spin}(V \oplus V^*)$ such that $q(g) = \mathrm{Ad}(g_{\mathrm{cl}})$ for $g \in \mathrm{GL}(V)$. Then we have*

$$\rho_{\mathrm{spin}}(g_{\mathrm{cl}}) = \pm|\det g|^{1/2}(\rho^*_{\mathrm{GL}}(g))^{-1}, \tag{1.2.5}$$

where $|\det g|^{1/2}$ denotes the positive square root of the absolute value of the determinant of g.

Proof. We decompose $\mathrm{GL}(V)$ into the positive symmetric part and the orthogonal group with respect to a positive-definite metric g_V on V (the Cartan decomposition). Thus $g \in \mathrm{GL}(V)$ is uniquely written as $g = hk$ where h is positive symmetric and k is orthogonal. Then there is a unique symmetric endmorphism A such that $e^A = h$ and A is described as $A = \sum_{i,j} A^i_j e_i \otimes \theta^j$, where $\{e_i\}_{i=1}^n$ is an orthogonal basis and $\{\theta^j\}_{j=1}^n$ denotes the dual basis. We define $A_{\mathrm{cl}} \in \bigwedge^2(V \oplus V^*) \subset \mathrm{CL}$ by

$$A_{\mathrm{cl}} = \frac{1}{2}\sum_{i,j} A^i_j(e_i \cdot \theta^j - \theta^j \cdot e_i).$$

Then $e^{A_{\mathrm{cl}}} \in \mathrm{Spin}(V \oplus V^*)$ satisfies $\mathrm{Ad}(e^{A_{\mathrm{cl}}}) = q(e^A) = q(h)$ and we have the equation:

$$\rho_{\mathrm{spin}}(e^{A_{\mathrm{cl}}}) = |\det e^A|^{1/2}(\rho^*_{\mathrm{GL}}(e^A))^{-1}. \tag{1.2.6}$$

The orthogonal element k is decomposed into a finite product of reflections, that is,

$$k = R_{u_1} \circ \cdots \circ R_{u_r},$$

where R_{u_i} is the reflection with respect to $u_i \in V$ which is given by $R_{u_i}(u_i) = -u_i$ and $R_{u_i}(v) = v$ for all $v \in V$ with $g_V(u_i, v) = 0$. We denote by θ_{u_i} the dual 1-form of u_i with respect to the metric g_V for $i = 1, \dots, r$. Then it turns out that $(u_i - \theta_{u_i}) \cdot (u_i + \theta_{u_i}) \in \mathrm{Spin}(V \oplus V^*)$ gives $\mathrm{Ad}((u_i - \theta_{u_i}) \cdot (u_i + \theta_{u_i})) = q(R_{u_i})$ and we have $(u_i - \theta_{u_i}) \cdot (u_i + \theta_{u_i}) \cdot \phi = R^*_{u_i}\phi$ for $\phi \in \bigwedge^\bullet V^*$. We define k_{cl} by

$$k_{\mathrm{cl}} = \prod_{i=1}^r (u_i - \theta_{u_i}) \cdot (u_i + \theta_{u_i}). \tag{1.2.7}$$

Then it follows that $k_{\mathrm{cl}} \in \mathrm{Spin}(V \oplus V^*)$ satisfies $\mathrm{Ad}(k_{\mathrm{cl}}) = q(k)$ and we have the equation:

$$\rho_{\mathrm{spin}}(k_{\mathrm{cl}}) = |\det k|^{1/2}(\rho^*_{\mathrm{GL}}(k))^{-1}. \tag{1.2.8}$$

(Note that $|\det k| = |(-1)^r| = 1$.) We define g_{cl} by $e^{A_{\mathrm{cl}}} \cdot k_{\mathrm{cl}}$. Then it follows that $\mathrm{Ad}(g_{\mathrm{cl}}) = \mathrm{Ad}(e^{A_{\mathrm{cl}}}) \circ \mathrm{Ad}(k_{\mathrm{cl}}) = q(h)q(k) = q(hk) = q(g)$. We also have

$$\begin{aligned}
\rho_{\mathrm{spin}}(g_{\mathrm{cl}}) &= \rho_{\mathrm{spin}}(e^{A_{\mathrm{cl}}}) \circ \rho_{\mathrm{spin}}(k_{\mathrm{cl}}) \\
&= |\det e^A|^{1/2}(\rho^*_{\mathrm{GL}}(e^A))^{-1} \circ (\rho^*_{\mathrm{GL}}(k))^{-1} \\
&= |\det g|^{1/2}(\rho^*_{\mathrm{GL}}(g))^{-1}.
\end{aligned}$$

Then we also have $\rho_{\mathrm{spin}}(-g_{\mathrm{cl}}) = -|\det g|^{1/2}(\rho^*_{\mathrm{GL}}(g))^{-1}$. Hence we obtain the result. $\square$

A lift of the map q is a map $p\colon \mathrm{GL}(V) \to \mathrm{Spin}(V \oplus V^*)$ such that $\mathrm{Ad} \circ p = q$:

$$\begin{tikzcd}
 & \mathrm{Spin}(V \oplus V^*) \arrow[d, "\mathrm{Ad}"] \\
\mathrm{GL}(V) \arrow[ur, "p"] \arrow[r, "q"'] & \mathrm{SO}(V \oplus V^*).
\end{tikzcd}$$

The composition $\rho_{\text{spin}} \circ p$ gives rise to the representation of $\mathrm{GL}(V)$ on S. The following lemma implies that there is the canonical lift p of the map q:

Lemma 1.2.2. *There is the lift* $p\colon \mathrm{GL}(V) \to \mathrm{Spin}(V \oplus V^*)$ *such that the representation* $\rho_{\text{spin}} \circ p$ *is given by*

$$\rho_{\text{spin}} \circ p(g) = |\det g|^{1/2} (\rho^*_{\mathrm{GL}}(g))^{-1}. \tag{1.2.9}$$

Proof. From Lemma 1.2.1, it suffices to determine the sign of $\rho_{\text{spin}} \circ p(g)$. The inverse image $\mathrm{Ad}^{-1}(q(g))$ is $\{g_{\text{cl}}, -g_{\text{cl}}\}$. If g_{cl} satisfies $\rho_{\text{spin}}(g_{\text{cl}}) = |\det g|^{1/2}(\rho^*_{\mathrm{GL}}(g))^{-1}$, we define $p(g)$ to be g_{cl}. Otherwise, we choose $-g_{\text{cl}}$. Thus we can choose $p(g)$ which satisfies the equation (1.2.9). $\square$

REMARK 1.2.3. There is the another lift $\hat{p}\colon \mathrm{GL}(V) \to \mathrm{Spin}(V \oplus V^*)$ with $\mathrm{Ad} \circ \hat{p} = q$ which satisfies

$$\rho_{\text{spin}} \circ p(g) = \operatorname{sgn}(\det g) |\det g|^{1/2} (\rho^*_{\mathrm{GL}}(g))^{-1},$$

for $g \in \mathrm{GL}(V)$. Note that $p|_{\mathrm{GL}_0(V)} = \hat{p}|_{\mathrm{GL}_0(V)}$ for the identity component $\mathrm{GL}_0(V)$.

2. Clifford–Lie operators

2.1. Clifford–Lie operators. We use the same notation as in Section 1. Let X be a real manifold of dimension n. Then we consider the direct sum $T \oplus T^*$ of the tangent bundle $T = TX$ and the cotangent bundle $T^* = T^*X$. Let $\mathrm{CL}(X) = \mathrm{CL}(T \oplus T^*)$ be the Clifford bundle on X:

$$\mathrm{CL}(X) := \bigcup_{x \in X} \mathrm{CL}(T_x X \oplus T_x^* X) \to X.$$

We also define the Clifford group bundle $\mathrm{G}_{\text{cl}}(X) = \mathrm{G}_{\text{cl}}(T \oplus T^*)$ by:

$$\mathrm{G}_{\text{cl}}(X) := \bigcup_{x \in X} \mathrm{G}_{\text{cl}}(T_x X \oplus T_x^* X) \to X.$$

Let π be the natural projection,

$$\pi : \bigotimes (T \oplus T^*) \to \mathrm{CL}(X) = \bigotimes (T \oplus T^*)/\mathcal{I}.$$

We define CL^{2i} by the image:

$$\mathrm{CL}^{2i} = \pi\left(\bigoplus_{l=0}^{i} \bigotimes^{2l} (T \oplus T^*) \right).$$

Then we have the filtration of $\mathrm{CL}^{\mathrm{even}}$:

$$\mathrm{CL}^0 \subset \mathrm{CL}^2 \subset \mathrm{CL}^4 \subset \cdots .$$

We also define CL^{2i+1} by the image:

$$\mathrm{CL}^{2i+1} = \pi\left(\bigoplus_{l=0}^{i} \bigotimes^{2l+1} (T \oplus T^*)\right),$$

which also gives the filtration of $\mathrm{CL}^{\mathrm{odd}}$,

$$\mathrm{CL}^1 \subset \mathrm{CL}^3 \subset \mathrm{CL}^5 \subset \cdots .$$

Let $S(X)$ be the bundle of differential forms $\bigwedge^* T^*X$ on a manifold X. By using the spin representation on each fibre as in Section 1, the bundle of the Clifford algebra $\mathrm{CL}(X)$ acts on $S(X)$. Let $\mathcal{L}_E$ be the anti-commutator $\{d, E\} = dE + Ed$ for a section E of the bundle $T \oplus T^*$. (For simplicity, we denote it by $E \in \mathrm{CL}^1 = T \oplus T^*$.) For $E = v + \theta \in T \oplus T^*$ we have $\mathcal{L}_E = \mathcal{L}_v + (d\theta)$, where $\mathcal{L}_v$ is the ordinary Lie derivative and $(d\theta)$ acts on $S(X)$ by the wedge product. Next we consider a bracket $[\mathcal{L}_E, F] = \mathcal{L}_E F - F\mathcal{L}_E$ for $E, F \in T \oplus T^*$.

Lemma 2.1.1. *The bracket $[\mathcal{L}_E, F]$ is a section of $T \oplus T^*$.*

Proof. When we write $E = v + \theta$, $F = w + \eta \in T \oplus T^*$, then we have

$$\begin{aligned}[\mathcal{L}_E, F] &= [\mathcal{L}_v + (d\theta), w + \eta]\\ &= [\mathcal{L}_v, w] + [\mathcal{L}_v, \eta] + [(d\theta), w] + [(d\theta), \eta].\end{aligned}$$

Since $[(d\theta), w] = -i_w(d\theta) \in (T \oplus T^*)$ and $[\mathcal{L}_v, w] \in T \oplus T^*$ is the ordinary bracket of vector fields v and w, we have the result. $\square$

In this paper Clifford algebra valued Lie derivatives play an significant role.

DEFINITION 2.1.2 (Clifford–Lie operators). *A Clifford–Lie operator of order* 3 on X is a differential operator acting on $S(X)$ which is locally written as

$$L = \sum_{i,j} a^{ij} E_i \mathcal{L}_{E_j} + K,$$

on every open set U on X for some $E_i \in \mathrm{CL}^1(TU \oplus T^*U)$, $a^{ij} \in C^\infty(U)$ and $K \in \mathrm{CL}^3(TU \oplus T^*U)$.

Note that the operator L acts on a differential form ϕ by $L\phi = \sum_{i,j} a^{ij} E_i \cdot \mathcal{L}_{E_j}\phi + K \cdot \phi$ on an open set U, where $K \cdot \phi$ denotes the spin representation of CL on the differential form. Let $\{x_1, \dots, x_n\}$ be a local coordinates of X. We denote by v_i the vector field $\partial/\partial x_i$ and $\theta^i = dx^i$. Then the exterior derivative d is locally written as

$$d = \sum_{i=0}^{n} \theta^i \wedge \mathcal{L}_{v_i}.$$

Hence d is the Clifford–Lie operator of order 3.

Lemma 2.1.3. *Let a be a section of* $\mathrm{CL}^2(T \oplus T^*)$ *which acts on $S(X)$ by the spin representation. If L is a Clifford–Lie operator of order* 3 *then the commutator $[L, a]$ is also a Clifford–Lie operator of order* 3.

Proof. Let f be a function on X and $E = v + \theta$ a section of $T \oplus T^*$. Since we have

$$\mathcal{L}_E(fa) = (\mathcal{L}_E f)a + f\mathcal{L}_E a,$$

where $\mathcal{L}_E f = \mathcal{L}_v f \in C^\infty(X)$. We have the following equality on an open set U on X:

$$\begin{aligned}
[L, fa] &= L(fa) - faL \\
&= \sum_{ij} a_{ij} E_i \mathcal{L}_{E_j}(fa) - faL + K(fa) \\
&= \sum_{ij} a_{ij} E_i (\mathcal{L}_{E_j} f)a + f[L, a].
\end{aligned}$$

Since $E_i(\mathcal{L}_{E_j} f)a \in \mathrm{CL}^3(T \oplus T^*)$, it is sufficient to show the lemma in the case $a = F_1F_2$ for $F_i \in T \oplus T^*$ $(i = 1, 2)$. The bracket $[\mathcal{L}_E, F_1F_2]$ is given by

$$\begin{aligned}
[\mathcal{L}_E, F_1F_2] &= \mathcal{L}_E(F_1F_2) - F_1F_2\mathcal{L}_E \\
&= [\mathcal{L}_E, F_1]F_2 + F_1\mathcal{L}_E F_2 - F_1F_2\mathcal{L}_E \\
&= [\mathcal{L}_E, F_1]F_2 + F_1[\mathcal{L}_E, F_2].
\end{aligned}$$

Hence it follows from Lemma 2.1.1 that $[\mathcal{L}_E, F_1F_2] \in \mathrm{CL}^2$. The bracket $[E_1\mathcal{L}_{E_2}, F_1F_2]$ is given by

$$\begin{aligned}
[E_1\mathcal{L}_{E_2}, F_1F_2] &= E_1\mathcal{L}_{E_2}F_1F_2 - F_1F_2E_1\mathcal{L}_{E_2} \\
&= E_1[\mathcal{L}_{E_2}, F_1F_2] + E_1F_1F_2\mathcal{L}_{E_2} - F_1F_2E_1\mathcal{L}_{E_2} \\
&= [E_1, F_1F_2]\mathcal{L}_{E_2} + E_1[\mathcal{L}_{E_2}, F_1F_2].
\end{aligned}$$

Since $[E_1, F_1F_2] = 2\langle E_1, F_1\rangle F_2 - 2\langle E_1, F_2\rangle F_1 \in \mathrm{CL}^1 = (T \oplus T^*)$, it follows that the bracket $[E_1\mathcal{L}_{E_2}, F_1F_1]$ is a Clifford–Lie operator of order 3. Since $[K, F_1F_2] \in \mathrm{CL}^3$ for $K \in \mathrm{CL}^3$, the result follows from the equation:

$$\begin{aligned}[L, F_1F_2] &= \left[\sum_{ij} a_{ij} E_i\mathcal{L}_{E_j}, F_1F_2\right] + [K, F_1F_2] \\ &= \sum_{ij} a_{ij}[E_i\mathcal{L}_{E_j}, F_1F_2] + [K, F_1F_2].\end{aligned}$$

□

Lemma 2.1.4. *The commutator* $[d, a]$ *is a Clifford–Lie operator of order* 3.

Proof. Since d is a Clifford–Lie operator of order 3, the result follows from Lemma 2.1.3. We shall give the following direct proof. We have $[d, fa] = dfa - fad = (df)a + f[d, a]$ for a function f. Hence it is sufficient to show the lemma in the case $a = E_1E_2$, where $E_i \in T \oplus T^*$ $(i = 1, 2)$. Then the bracket $[d, a]$ is written as

$$\begin{aligned}[d, a] &= dE_1E_2 - E_1E_2d \\ &= \mathcal{L}_{E_1}E_2 - E_1dE_2 - E_1E_2d \\ &= \mathcal{L}_{E_1}E_2 - E_1\mathcal{L}_{E_2} \\ &= E_2\mathcal{L}_{E_1} - E_1\mathcal{L}_{E_2} + [\mathcal{L}_{E_1}, E_2].\end{aligned}$$

Hence the result follows from $[\mathcal{L}_{E_1}, E_2] \in \mathrm{CL}^1 \subset \mathrm{CL}^3$. □

Proposition 2.1.5. *For* $a_1, a_2 \in \mathrm{CL}^2(T \oplus T^*)$, $[[d, a_1], a_2]$ *is a Clifford–Lie operator of order* 3. *Further we denote by* $\mathrm{Ad}_a\, L$ *the commutator* $[L, a]$. *Then the composition* $\mathrm{Ad}_{a_1}(\mathrm{Ad}_{a_2}(\cdots \mathrm{Ad}_{a_n}\, d)\cdots)$ *is a Clifford–Lie operator of order* 3 *for* $a_1, \ldots, a_n \in \mathrm{CL}^2$.

Proof. The result follows form Lemma 2.1.3 and Lemma 2.1.4. □

REMARK 2.1.6. In the case of $a_1, a_2 \in \mathrm{End}(TX)$, the bracket $[[d, a_1], a_2]$ is given in terms of the Nijenhuis tensor of a_1 and a_2. In the case $a_1, a_2 \in \bigwedge^2 T$, the bracket $[[d, a_1], a_2]$ is the Schouten bracket. In general the bracket $[[d, a_1], a_2]$ is not a tensor but a Clifford–Lie operator of order 3.

Let a be a section of CL^2 and L an operator acting on $S(X)$. We successively define an operator $(\mathrm{Ad}_a)^l L$ acting on $S(X)$ by

$$(\mathrm{Ad}_a)^l L = [(\mathrm{Ad}_a)^{l-1} L, a].$$

We also define a formal power series $(\exp(\mathrm{Ad}_a))L$ by

$$\begin{aligned}(\exp(\mathrm{Ad}_a))L &= \sum_{l=0}^{\infty} \frac{1}{l!}(\mathrm{Ad}_a)^l L \\ &= d + [L, a] + \frac{1}{2!}[[L, a], a] + \cdots .\end{aligned}$$

Lemma 2.1.7. *The power series* $(\exp(\mathrm{Ad}_a))L$ *is given by*

$$(\exp(\mathrm{Ad}_a))L = e^{-a} \circ L \circ e^a.$$

Proof. It follows from definition of $(\mathrm{Ad}_a)^l L$ that

$$(\mathrm{Ad}_a)^l L = \sum_{m=0}^{l} \frac{(-1)^m l!}{m!\,(l-m)!} a^m L a^{l-m}.$$

Then by a combinatorial calculation we have

$$La^k = \sum_{l=0}^{k} \frac{k!}{l!\,(k-l)!} a^{k-l} (\mathrm{Ad}_a)^l L.$$

Then we have

$$\begin{aligned}Le^a &= e^a\left(L + (\mathrm{Ad}_a)L + \frac{1}{2!}(\mathrm{Ad}_a)^2 L + \frac{1}{3!}(\mathrm{Ad}_a)^3 L + \cdots\right) \\ &= e^a (\exp(\mathrm{Ad}_a))L.\end{aligned}$$

Hence the result follows. □

Proposition 2.1.8. *If L is a Clifford–Lie operator of order* 3 *and* $a \in \mathrm{CL}^2$, *then* $e^{-a} \circ L \circ e^a = (\exp(\mathrm{Ad}_a))L$ *is also a Clifford–Lie operator of order* 3. *In particular* $(\exp(\mathrm{Ad}_a))d$ *is a Clifford–Lie operator of order* 3.

Proof. The result follows from Proposition 2.1.5 and Lemma 2.1.7. □

3. Deformations of generalized geometric structures

3.1. Generalized geometric structures ($\mathcal{B}(V)$-structures). Let V be an n dimensional real vector space and V^* the dual space of V. As in Section 1 the space of the skew-symmetric tensors $\mathrm{S} := \bigwedge^* V^*$ is regarded as the spin representation of CL ($= \mathrm{CL}(V \oplus V^*)$), which induces the representation of the Clifford group G_{cl} (=

$\mathrm{G}_{\mathrm{cl}}(V \oplus V^*))$. We consider the direct sum of the spin representations on which $\mathrm{G}_{\mathrm{cl}}(V \oplus V^*)$ acts diagonally:

$$\bigoplus^l S := \overbrace{\left(\bigwedge^* V^* \oplus \cdots \oplus \bigwedge^* V^*\right)}^{l \text{ times}}.$$

Let $\Phi_V = (\phi_1, \ldots, \phi_l)$ be an element of the direct sum $\bigoplus^l S$. Then we have the orbit $\mathcal{B}(V)$ of $\mathrm{G}_{\mathrm{cl}}(V \oplus V^*)$ through Φ_V:

$$\mathcal{B}(V) := \{g \cdot \Phi_V \mid g \in \mathrm{G}_{\mathrm{cl}}(V \oplus V^*)\}.$$

From now on we fix an orbit $\mathcal{B}(V)$. We also denote by $\mathcal{A}(V)$ the orbit of $\mathrm{GL}(V)$ through Φ_V.

As in Lemma 1.2.2 in Section 1, we have the lift p: $\mathrm{GL}(V) \to \mathrm{Spin}(V \oplus V^*)$ which satisfies $\mathrm{Ad} \circ p = q$ and (1.2.9). Thus we have,

$$\rho_{\mathrm{spin}} \circ p(g) = |\det g|^{1/2} (\rho^*_{\mathrm{GL}}(g))^{-1}, \quad \text{for} \quad g \in \mathrm{GL}(V).$$

Since the action on $\bigoplus^l S$ is diagonal, we have

$$(|\det g|^{-1/2} p(g)) \cdot \Phi_V = (\rho^*_{\mathrm{GL}(V)}(g))^{-1} \Phi_V,$$

for $\Phi_V \in \bigoplus^l S$. Since the Clifford group is the extension of pin by the $\mathbb{R}^\times$, it follows that $|\det g|^{-1/2} p(g) \in \mathrm{G}_{\mathrm{cl}}$. It implies that the $\mathrm{GL}(V)$-orbit $\mathcal{A}(V)$ is embedded into the $\mathrm{G}_{\mathrm{cl}}(V \oplus V^*)$-orbit $\mathcal{B}(V)$:

$$\mathcal{A}(V) \hookrightarrow \mathcal{B}(V). \tag{3.1.1}$$

The inclusion (3.1.1) shows that the group G_{cl} is suitable for our construction, rather than spin group. Let X be a compact manifold of dimension n. As in Section 2 we have the Clifford bundle $\mathrm{CL}(X)$ and the Clifford group bundle $\mathrm{G}_{\mathrm{cl}}(X)$ on X. For an identification $h \colon V \to T_x X$ for each $x \in X$, we define the set $\mathcal{B}(T_x X)$ by $\mathcal{B}(T_x X) = h_*(\mathcal{B}(V)) \in \bigoplus^l \bigwedge^* T^*_x X$. It follows from (3.1.1) that the orbit $\mathcal{B}(T_x X)$ does not depend on a choice of an identification h and thus $\mathcal{B}(T_x X)$ is canonically defined as the submanifold of the direct sum of forms $\bigoplus^l \bigwedge^* T^*_x X$, which is in fact a homogeneous space. Hence we have the fibre bundle $\mathcal{B}(X) \to X$:

$$\mathcal{B}(X) := \bigcup_{x \in X} \mathcal{B}(T_x X) \to X.$$

Let H be the isotropy group of the action of $\mathrm{G}_{\mathrm{cl}}(V \oplus V^*)$ at Φ_V:

$$H := \{g \in \mathrm{G}_{\mathrm{cl}}(V \oplus V^*) \mid g \cdot \Phi_V = \Phi_V\}.$$

Then $\mathcal{B}(X)$ is the fibre bundle with fibre $\mathrm{G_{cl}}(V \oplus V^*)/H$ and $\mathcal{B}(X)$ is embedded into the direct sum of differential forms $\bigoplus^l \bigwedge^* T^*X$. We denote by $\mathcal{E}_B(X)$ the set of C^∞-sections of the fibre bundle $\mathcal{B}(X)$:

$$\mathcal{E}_B(X) := C^\infty(X, \mathcal{B}(X)).$$

Each section $\Phi \in \mathcal{E}_B(X)$ consists of l differential forms on which the exterior derivative d acts. Let $\tilde{\mathfrak{M}}_B(X)$ be the set of d-closed sections of $\mathcal{B}(X)$:

$$\tilde{\mathfrak{M}}_B(X) := \{\Phi \in \mathcal{E}_B(X) \mid d\Phi = 0\}.$$

DEFINITION 3.1.1. *A generalized geometric structure on X associated with the orbit $\mathcal{B}(V)$* is a d-closed section $\Phi \in \tilde{\mathfrak{M}}_B(X)$. For simplicity, we call a d-closed section Φ a $\mathcal{B}(V)$-structure on X.

The diffeomorphism group $\mathrm{Diff}(X)$ naturally acts on $\tilde{\mathfrak{M}}_B(X)$ by the pull back, since $\mathrm{GL}(V)$-orbit $\mathcal{A}(V)$ is a subset of the Clifford group orbit $\mathcal{B}(V)$. We denote by $\mathrm{Diff}_0(X)$ the identity component of $\mathrm{Diff}(X)$. Since the exponential $e^{d\gamma}$ is a section of the bundle $\mathrm{Spin}_0(X)$ for a 1-form γ, we have the action of $e^{d\gamma}$ on $\mathcal{B}(V)$-structures $\tilde{\mathfrak{M}}_{\mathcal{B}}(X)$,

$$\Phi \mapsto e^{d\gamma} \wedge \Phi, \quad (\gamma \in T^*X).$$

Let $\widetilde{\mathrm{Diff}}_0(X)$ be the group generated by the composition of the action of $\mathrm{Diff}_0(X)$ and d-exact 2-forms:

$$\widetilde{\mathrm{Diff}}_0(X) := \{e^{d\gamma} \wedge f^* \mid \gamma \in T^*,\ f \in \mathrm{Diff}_0(X)\}.$$

Here the group $\widetilde{\mathrm{Diff}}_0(X)$ is regarded as a subgroup of the automorphisms of the bundle $\mathrm{Spin}_0(X)$:

$$\begin{array}{ccc} \mathrm{Spin}_0(X) & \longrightarrow & \mathrm{Spin}_0(X) \\ \downarrow & & \downarrow \\ X & \longrightarrow & X. \end{array}$$

Hence the group $\widetilde{\mathrm{Diff}}_0(X)$ is an extension of $\mathrm{Diff}_0(X)$ by d-exact 2-forms $d\left(\bigwedge^1 T^*\right)$:

$$0 \to d\left(\bigwedge^1 T^*\right) \to \widetilde{\mathrm{Diff}}_0(X) \to \mathrm{Diff}_0(X) \to 0.$$

DEFINITION 3.1.2. *A moduli space $\mathfrak{M}_B(X)$ of $\mathcal{B}(V)$-structures on X* is the quotient space of $\tilde{\mathfrak{M}}_B(X)$ by the action of $\widetilde{\mathrm{Diff}}_0(X)$:

$$\mathfrak{M}_B(X) := \tilde{\mathfrak{M}}_B(X)/\widetilde{\mathrm{Diff}}_0(X).$$

We denote by F the direct sum $\bigoplus^l \bigwedge^* T^*X$ of the bundle of differential forms. By using a Riemannian metric on X, we define the Sobolev space $L^2_s(X, F)$ consisting of square integrable section of F up to order s, where we take s sufficiently large. We also define $C^l(X, F)$ by sections of F of class C^l for $s > l + n/2$. Then it follows from the Sobolev embedding theorem that $L^2_s(X, F) \subset C^l(X, F)$. Since $\tilde{\mathfrak{M}}_{\mathcal{B}}(X)$ is a subset of $L^2_s(X, F)$, we have a completion $\tilde{\mathfrak{M}}^s_{\mathcal{B}}(X)$ of $\tilde{\mathfrak{M}}_{\mathcal{B}}(X)$ with respect to the Sobolev norm $\| \ \|_{L^2_s}$. We also have a completion $\mathrm{Diff}^{s+1}(X)$ of $\mathrm{Diff}(X)$ and a completion $\widetilde{\mathrm{Diff}}(X)^s$ of $\widetilde{\mathrm{Diff}}(X)$ (cf. [4], Section 3 in [6].)

3.2. Main theorems (deformations of $\mathcal{B}(V)$-structures). Let $\mathcal{B}(V)$ be the fixed orbit of the action of the Clifford group $\mathrm{G_{cl}}(V \oplus V^*)$ as in Section 3.1 and Φ a $\mathcal{B}(V)$-structure on a manifold X. In order to consider deformations of $\mathcal{B}(V)$-structures of Φ, we introduce a deformation complex of the $\mathcal{B}(V)$-structure Φ. As in Section 2 there are the filtration of the even Clifford bundle $\mathrm{CL}^{\mathrm{even}}$ and the one of the odd Clifford bundle $\mathrm{CL}^{\mathrm{odd}}$:

$$\mathrm{CL}^0 \subset \mathrm{CL}^2 \subset \mathrm{CL}^4 \subset \cdots,$$
$$\mathrm{CL}^1 \subset \mathrm{CL}^3 \subset \mathrm{CL}^5 \subset \cdots.$$

Then the action of CL^k on Φ gives vector bundles $\mathbf{E}^{k-1}(X)$ on X:

$$\mathbf{E}^{k-1}(X) := \mathrm{CL}^k \cdot \Phi,$$

which also carry the corresponding filtrations:

$$\mathbf{E}^{-1}(X) \subset \mathbf{E}^1(X) \subset \mathbf{E}^3(X) \subset \cdots,$$
$$\mathbf{E}^0(X) \subset \mathbf{E}^2(X) \subset \mathbf{E}^4(X) \subset \cdots.$$

(Note that we shift the degree of vector bundles.) The vector bundle $\mathbf{E}^{-1}(X)$ is the line bundle generated by Φ. The vector bundle $\mathbf{E}^0(X)$ is generated by $E \cdot \Phi$ for all $E \in T \oplus T^*$ over $\mathrm{C}^\infty(X)$ and $\mathbf{E}^1(X)$ is generated by $E_1 \cdot E_2 \cdot \Phi$ for all $E_1, E_2 \in T \oplus T^*$. Each $\mathbf{E}^k(X)$ is a subbundle of the direct sum of the bundle of differential forms on which the exterior derivative d diagonally acts.

Proposition 3.2.1. *There is a differential complex $\#_{\mathcal{B},\Phi}$ for each $\Phi \in \tilde{\mathfrak{M}}_{\mathcal{B}}(X)$,*

$$(\#_{\mathcal{B},\Phi}) \qquad 0 \to \mathbf{E}^{-1}(X) \xrightarrow{d_{-1}} \mathbf{E}^0(X) \xrightarrow{d_0} \mathbf{E}^1(X) \xrightarrow{d_1} \mathbf{E}^2(X) \xrightarrow{d_2} \cdots,$$

where d_k is given by the restriction $d|_{\mathbf{E}^k(X)}$. The cohomology groups of the complex $\#_{\mathcal{B},\Phi}$ is denoted by $\mathrm{H}^k(\#_{\mathcal{B},\Phi})$,

$$H^k(\#_{\mathcal{B},\Phi}) := \frac{\ker d_k \colon \Gamma(\mathbf{E}^k(X)) \to \Gamma(\mathbf{E}^{k+1}(X))}{\mathrm{im} d_{k-1} \colon \Gamma(\mathbf{E}^{k-1}(X)) \to \Gamma(\mathbf{E}^k(X))}.$$

Then the first cohomology group $H^1(\#_{\mathcal{B},\Phi})$ is regarded as the infinitesimal tangent space of deformations of the $\mathcal{B}(V)$-structure Φ, where $\Gamma(\mathbf{E}^k(X))$ denotes smooth global sections of the bundle $\mathbf{E}^k(X)$.

(For simplicity, the complex $\#_{\mathcal{B},\Phi}$ is often denoted by $\#_{\mathcal{B}}$ and the cohomology group $H^k(\#_{\mathcal{B},\Phi})$ is also written as $H^k(\#_{\mathcal{B}})$.)

Proof. A section of $\mathbf{E}^{-1}(X)$ is written as $f\Phi$ for a function f. Hence $d(f\Phi) = df \wedge \Phi$ and we see that the image $d(\mathbf{E}^{-1}(X))$ is included in $\mathbf{E}^0(X)$. We denote by $\mathcal{L}_F$ the anti-commutator $dF + Fd$ acting on forms where $F \in T \oplus T^*$. When we write $F = v + \eta$ for $v \in T$ and $\eta \in T^*$, the anti-commutator $\mathcal{L}_F$ is given by

$$\mathcal{L}_F = \mathcal{L}_v + (d\eta)\wedge,$$

where $\mathcal{L}_v$ denotes the Lie derivative. Then we have

$$\begin{aligned}\mathcal{L}_F(f\Phi) &= \mathcal{L}_v(f\Phi) + (d\eta)\wedge(f\Phi)\\ &= (\mathcal{L}_v f)\Phi + f\mathcal{L}_v\Phi + f(d\eta)\wedge\Phi,\end{aligned}$$

where $\mathcal{L}_v f \in C^\infty(X)$. Since $\mathrm{GL}(TX)$ is the subbundle of $\mathrm{G}_{\mathrm{cl}}(X)$, diffeomorphisms of X act on $\mathcal{E}_{\mathcal{B}}(X)$. Hence we have

$$\mathcal{L}_v\Phi \in T_\Phi\mathcal{E}_{\mathcal{B}}(X).$$

A subset $\mathrm{G}_{\mathrm{cl0}}(X)$ with the identity of $\mathrm{G}_{\mathrm{cl}}(X)$ is given by the exponential of CL^2,

$$\mathrm{G}_{\mathrm{cl0}}(X) = \{e^a \mid a \in \mathrm{CL}^2\}.$$

Since the tangent space $T_\Phi\mathcal{E}_{\mathcal{B}}(X)$ is generated by the action of $\mathrm{G}_{\mathrm{cl0}}(T \oplus T^*)$, we have

$$T_\Phi\mathcal{E}_{\mathcal{B}}(X) \cong \mathrm{CL}^2 \cdot \Phi = \mathbf{E}^1(X).$$

Hence we have

$$\mathcal{L}_v\Phi \in \mathbf{E}^1(X).$$

Then it follows that $\mathcal{L}_F(\mathbf{E}^{-1}(X)) \subset \mathbf{E}^1(X)$. We also have

$$d(F\cdot\Phi) = \mathcal{L}_F\Phi - Fd\Phi = \mathcal{L}_F\Phi.$$

Hence we have $d(\mathbf{E}^0(X)) \subset \mathbf{E}^1(X)$. For $F_1, F_2 \in T \oplus T^*$ we have

$$\mathcal{L}_{F_1}(F_2\cdot\Phi) = [\mathcal{L}_{F_1}, F_2]\Phi + F_2\cdot\mathcal{L}_{F_1}\Phi.$$

It follows from Lemma 2.1.1 that $[\mathcal{L}_{F_1}, F_2] \in T \oplus T^*$. Hence from $\mathcal{L}_{F_1}\Phi \in \mathbf{E}^1(X)$ we have $\mathcal{L}_F(\mathbf{E}^0(X)) \subset \mathbf{E}^2(X)$. We shall show that $d\mathbf{E}^k(X) \subset \mathbf{E}^{k+1}(X)$ by induction on k.

We assume that $d\mathbf{E}^{k-2}(X) \subset \mathbf{E}^{k-1}(X)$ and $\mathcal{L}_F(\mathbf{E}^{k-2}(X)) \subset \mathbf{E}^k(X)$ for some $k \geq 1$ and for all $F \in T \oplus T^*$. Then for $F_1, F_2 \in T \oplus T^*$ and $s \in \mathbf{E}^{k-2}(X)$ we have

$$\begin{aligned} d(F_1 \cdot F_2 \cdot s) &= \mathcal{L}_{F_1}(F_2 \cdot s) - F_1 \cdot dF_2 \cdot s \\ &= [\mathcal{L}_{F_1}, F_2] \cdot s + F_2 \cdot \mathcal{L}_{F_1} s \\ &\quad - F_1 \cdot \mathcal{L}_{F_2} s + F_1 \cdot F_2 \cdot ds. \end{aligned}$$

It follows from our assumption ($ds \in \mathbf{E}^{k-1}(X)$ and $\mathcal{L}_F s \in \mathbf{E}^k(X)$) that $d(F_1 \cdot F_2 \cdot s) \in \mathbf{E}^{k+1}(X)$ since $[\mathcal{L}_{F_1}, F_2] \cdot s \in \mathbf{E}^{k-1}(X) \subset \mathbf{E}^{k+1}(X)$. Hence $d(\mathbf{E}^k(X)) \subset \mathbf{E}^{k+1}(X)$. For $F_3 \in T \oplus T^*$ we also have

$$\begin{aligned} \mathcal{L}_{F_3}(F_1 \cdot F_2 \cdot s) &= [\mathcal{L}_{F_3}, F_2] \cdot F_1 \cdot s + F_2 \cdot \mathcal{L}_{F_3}(F_1 \cdot s) \\ &= [\mathcal{L}_{F_3}, F_2] \cdot F_1 \cdot s + F_2 \cdot [\mathcal{L}_{F_3}, F_1] \cdot s \\ &\quad + F_2 \cdot F_1 \cdot \mathcal{L}_{F_3} s. \end{aligned}$$

Hence it follows from our assumption $\mathcal{L}_F s \in \mathbf{E}^k(X)$ that $\mathcal{L}_{F_3}(F_1 \cdot F_2 \cdot s) \in \mathbf{E}^{k+2}(X)$. Hence $\mathcal{L}_F(\mathbf{E}^k(X)) \subset \mathbf{E}^{k+2}(X)$. We have already shown that our assumption holds for $k = 1, 2$. Therefore we have $d\mathbf{E}^k(X) \subset \mathbf{E}^{k+1}(X)$ for all k by induction. The tangent space of the orbit of $\widetilde{\mathrm{Diff}}_0(X)$ is given by the Lie derivative $\mathcal{L}_v \Phi$ and $d\gamma \wedge \Phi$ for $v \in T$ and $\gamma \in T^*$. Hence it follows that the image $d(\Gamma(E^0(X))$ is the tangent space of $\widetilde{\mathrm{Diff}}_0(X)$. As we see, the tangent space of $\mathcal{E}_{\mathcal{B}}(X)$ is global sections of $\mathbf{E}^1(X)$. Hence the infinitesimal tangent space of deformations of Φ is given by the first cohomology group $H^1(\#_{\mathcal{B}})$. $\square$

The direct sum $\bigoplus^l S$ $(= \bigoplus^l \bigwedge^* T^*)$ is invariant under the action of the exterior derivative d which yields the direct sum of the full de Rham complex. Then the complex $\#_{\mathcal{B},\Phi}$ is the subcomplex of the direct sum of the full de Rham complex $\bigoplus^l S = \bigoplus^l \bigwedge^* T^*$,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbf{E}^{-1}(X) & \xrightarrow{d_{-1}} & \mathbf{E}^0(X) & \xrightarrow{d_0} & \mathbf{E}^1(X) & \xrightarrow{d_1} & \mathbf{E}^2(X) & \longrightarrow & \cdots, \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \bigoplus^l \bigwedge^* T^* & \xrightarrow[d]{} & \bigoplus^l \bigwedge^* T^* & \xrightarrow[d]{} & \bigoplus^l \bigwedge^* T^* & \xrightarrow[d]{} & \bigoplus^l \bigwedge^* T^* & \longrightarrow & \cdots. \end{array}$$

We denote by $\bigoplus^l H^*_{\mathrm{dR}}(X)$ $(= \bigoplus^l \bigoplus_{p=0}^{\dim X} H^p(X, \mathbb{R}))$ the direct sum of the full de Rham cohomology group. Then we have the map $p^k_{\mathcal{B},\Phi}$:

$$p^k_{\mathcal{B},\Phi} \colon H^k(\#_{\mathcal{B},\Phi}) \to \bigoplus^l H^*_{\mathrm{dR}}(X).$$

Since the action of $\widetilde{\mathrm{Diff}}_0(X)$ on $\tilde{\mathfrak{M}}_{\mathcal{B}}(X)$ preserves the de Rham cohomology class $[\Phi] = ([\phi_1], \dots, [\phi_l])$ of a $\mathcal{B}(V)$-structure $\Phi = (\phi_1, \dots, \phi_l)$, we have the map $P_{\mathcal{B}}$:

$$P_{\mathcal{B}} \colon \mathfrak{M}_{\mathcal{B}}(X) \to \bigoplus^{l} H^*_{\mathrm{dR}}(X).$$

The map $P_{\mathcal{B}}$ is called *the period map*.

DEFINITION 3.2.2. An orbit $\mathcal{B}(V)$ is *elliptic* if the differential complex $\#_{\mathcal{B}}$ is exact in degree $k = 1, 2$, that is, the symbol complex of the differential complex $\#_{\mathcal{B}}$ is exact in degree $k = 1, 2$.

DEFINITION 3.2.3. Let $\mathcal{B}(V)$ be an orbit of $\mathrm{G}_{\mathrm{cl}}(V \oplus V^*)$ as before and X a compact manifold of dimension n. A $\mathcal{B}(V)$-structure Φ on X is *a topological structure* if the maps $p^k_{\mathcal{B},\Phi} \colon H^k(\#_{\mathcal{B},\Phi}) \to \bigoplus^l H^*_{\mathrm{dR}}(X)$ are injective for $k = 1, 2$. An orbit $\mathcal{B}(V)$ is *a topological orbit* on X if every $\mathcal{B}(V)$-structure on X is a topological structure.

Clearly the elliptic condition depends only on the choice of an orbit $\mathcal{B}(V)$. However the topological condition relies on the choice of a $\mathcal{B}(V)$-structure Φ on X.

DEFINITION 3.2.4. A $\mathcal{B}(V)$-structure Φ on X is *unobstructed* if for each representative α of the infinitesimal tangent space $\mathrm{H}^1(\#_{\mathcal{B}})$, there exists a smooth one parameter family of deformations $\Phi_t \in \tilde{\mathfrak{M}}_{\mathcal{B}}(X)$ with $\Phi_0 = \Phi$ such that

$$\frac{d}{dt}\Phi_t|_{t=0} = \alpha,$$

where $|t| < \varepsilon$ for sufficiently small constant $\varepsilon > 0$.

If Φ is unobstructed, each infinitesimal tangent generates actual deformations and the space of deformations of Φ is locally given by an open set of $H^1(\#_{\mathcal{B}})$. From the viewpoint as in [6], we have the following criterion for unobstructed deformations of $\mathcal{B}(V)$-structures and the local Torelli-type theorem:

Theorem 3.2.5. *Let $\mathcal{B}(V)$ be an elliptic orbit and Φ a $\mathcal{B}(V)$-structure on a compact manifold X of dimension n. If Φ is a topological structure, then Φ is unobstructed and there exists a neighborhood U of Φ in the moduli space $\mathfrak{M}_{\mathcal{B}}(X)$ such that the restriction of the period map $P_{\mathcal{B}}|_U \colon U \to \bigoplus^l H^*_{\mathrm{dR}}(X)$ is injective. Further, if an orbit $\mathcal{B}(V)$ is an elliptic and topological orbit on X, the period map $P_{\mathcal{B}} \colon \mathfrak{M}_{\mathcal{B}}(X) \to \bigoplus^l H^*_{\mathrm{dR}}(X)$ is locally injective at each point, that is, the local Torelli theorem holds.*

Theorem 3.2.5 is reduced to the following Theorems 3.2.6 and 3.2.7.

Theorem 3.2.6. *Let Φ be a $\mathcal{B}(V)$-structure on a compact manifold X of dimension n as in* Theorem 3.2.5. *If Φ is a topological structure, then there exists a neighborhood U of Φ in the moduli space $\mathfrak{M}_{\mathcal{B}}(X)$ such that the restriction of the period map $P_{\mathcal{B}}|_U : U \to \bigoplus^l H^*_{\mathrm{dR}}(X)$ is injective.*

(Note that Theorem 3.2.6 is regarded as a generalization of the Moser's stability theorem for symplectic structures and volume forms.)

Theorem 3.2.7. *Let Φ be a $\mathcal{B}(V)$-structure as in* Theorem 3.2.5. *If $p^2_{\mathcal{B},\Phi}$ is injective, then Φ is unobstructed.*

The proof of Theorem 3.2.7 is given in the next Section 3.3. The rest of this section is devoted to proof of Theorem 3.2.6.

Let $\tilde{U}$ be a neighborhood of Φ in $\tilde{\mathfrak{M}}_{\mathcal{B}}(X)$. For $\Psi \in \tilde{U}$, we have vector bundles $\mathbf{E}^k_\Psi = \mathrm{CL}^{k+1} \cdot \Psi$ and the differential complex $\#_{\mathcal{B},\Psi} = (\mathbf{E}^*_\Psi, d)$ which gives the cohomology groups $H^k(\#_{\mathcal{B},\Psi})$ and the maps $p^k_{\mathcal{B},\Psi} : H^k(\#_{\mathcal{B},\Psi}) \to \bigoplus^l H^*_{\mathrm{dR}}(X)$.

In order to obtain Theorem 3.2.6, we shall show the following lemma:

Lemma 3.2.8. *Let $\{\Phi_n\}_{n=1}^\infty$ be a sequence of $\mathcal{B}(V)$-structures which converges to a $\mathcal{B}(V)$ structure Φ, that is,*

$$\lim_{n\to\infty} \Phi_n = \Phi \in \tilde{\mathfrak{M}}_{\mathcal{B}}(X),$$

where we use the Sobolev norm $\|\ \|_{L^2_s}$. We denote by $\mathbf{E}^k_n(X)$ the vector bundle $\mathrm{CL}^{k+1} \cdot \Phi_n$ and by $\#_{\mathcal{B},n}$ the deformation complex $\{\mathbf{E}^*_n\}$ with cohomology groups $H^k(\#_{\mathcal{B},n})$. If the map $p^k_{\mathcal{B},n} : H^k(\#_{\mathcal{B},n}) \to H^*_{\mathrm{dR}}(X)$ is not injective for all n, then the map $p^k_{\mathcal{B},\Phi}$ is not injective also, where $k = 1, 2$.

Lemma 3.2.8 shows that the injectivity of the map $p^k_{\mathcal{B}}$ is an open condition, that is, if $p^k_{\mathcal{B},\Phi}$ is injective for $\Phi \in \tilde{\mathfrak{M}}_{\mathcal{B}}(X)$, then there exists an neighborhood $\tilde{U}$ of Φ such that $p^k_{\mathcal{B},\Psi}$ is also injective for all $\Psi \in \tilde{U}$.

Proof of Lemma 3.2.8. We take a Riemannian metric on the manifold X. Then we have the Laplacian $\triangle_{n,k} = d^*_k d_k + d_{k-1} d^*_{k-1}$ defined by the complex $\{\mathbf{E}^*_n\}$ acting on sections of $\mathbf{E}^k_n(X)$. We denote by $\mathbb{H}^k(\#_{\mathcal{B},n})$ the kernel of the Laplacian $\triangle_{n,k}$. Since the complex $\#_{\mathcal{B},n}$ is elliptic in degree $k = 1, 2$, the cohomology group $H^k(\#_{\mathcal{B},n})$ is isomorphic to $\mathbb{H}^k(\#_{\mathcal{B},n})$. We also have the ordinary Laplacian $\triangle$ which acts on $\bigoplus^l S$ and we denote by Π the L^2-projection to the harmonic forms with respect to $\triangle$. If $p^k_{\mathcal{B},n}$ is not injective, we have $a_n \in \mathrm{CL}^{k+1}$ such that $a_n \cdot \Phi_n$ is a non-zero element of $\mathbb{H}^k(\#_{\mathcal{B},n})$ satisfying $\Pi(a_n \cdot \Phi_n) = 0$. For each Φ_n we can take a section g_n of the fibre bundle

$\mathrm{G}_{\mathrm{cl}}(X)$ such that $g_n \cdot \Phi_n = \Phi$ and $g_n \to \mathrm{id}$ as $n \to \infty$. By the left multiplication L_{g_n} of g_n, we identify $\mathbf{E}^k_n(X)$ with $\mathbf{E}^k(X) = \mathrm{CL}^{k+1}(X) \cdot \Phi$,

$$\begin{aligned} L_{g_n} : \mathbf{E}^k_n(X) &\to \mathbf{E}^k(X), \\ a_n \cdot \Phi_n &\mapsto g_n \cdot a_n \cdot \Phi_n = (\mathrm{Ad}_{g_n} a_n) \cdot \Phi. \end{aligned}$$

Then the elliptic operator $\tilde{\triangle}_{\mathcal{B},n}$ on $\mathbf{E}^1(X)$ is defined by

$$\tilde{\triangle}_{\mathcal{B},n} = L_{g_n} \triangle_{\mathcal{B},n} L^{-1}_{g_n}.$$

We put $b_n = \mathrm{Ad}_{g_n} a_n$. Then we have

$$\tilde{\triangle}_{\mathcal{B},n} b_n \cdot \Phi = L_g \triangle_{\mathcal{B},n} (a_n \cdot \Phi_n) = 0.$$

We take a_n such that the Sobolev norm of $b_n \cdot \Phi$ is normalized,

$$\|b_n \cdot \Phi\|_{L^2_4} = 1.$$

Then from Rellich lemma there exists a subsequence $\{b_m \cdot \Phi\}_m$ which converges to $b \cdot \Phi \in \mathbf{E}^1(X)$ with respect to the norm L^2_2. Since $\tilde{\triangle}_{\mathcal{B},m} b_m \cdot \Phi = 0$, using the elliptic estimate, we have,

$$\|b_m \cdot \Phi\|_{L^2_4} \le C_1 \|b_m \cdot \Phi\|_{L^1} \le C_2 \|b_m \cdot \Phi\|_{L^2_2},$$

where $C_i \neq 0$ does not depend on m for $i = 1, 2$. Hence we have the bound,

$$0 \neq C_3 \le \|b \cdot \Phi\|_{L^2_2}.$$

The family of elliptic operator $\{\tilde{\triangle}_{\mathcal{B},m}\}_m$ also converges to the operator $\triangle_{\mathcal{B},\Phi}$ as $m \to \infty$, where $\triangle_{\mathcal{B},\Phi}$ denotes the Laplacian of the complex $\#_{\mathcal{B},\Phi}$ acting on $\mathbf{E}^k$.

Hence we have

$$\triangle_{\mathcal{B}}(b \cdot \Phi) = 0.$$

Since $g_m \to 1$ $(m \to \infty)$, the sequence $\{a_m \cdot \Phi_m\} = \{g_m^{-1} \cdot b_m \cdot \Phi\}_m$ converges to $b \cdot \Phi$ $(n \to \infty)$. Hence it follows from $\Pi(a_m \cdot \Phi_m) = 0$ that

$$\Pi(b \cdot \Phi) = 0.$$

Hence $b \cdot \Phi \neq 0$ is an element of $\ker p^1_{\mathcal{B},\Phi}$ and we have the result. $\square$

We shall show that the dimension $H^1(\#_{\mathcal{B},\Psi})$ is constant for all Ψ in a sufficiently small $\tilde{U}$:

Proposition 3.2.9. *Let Φ be a $\mathcal{B}(V)$-structure on a compact manifold X for an elliptic orbit $\mathcal{B}(V)$. If Φ is a topological structure, then there exists a neighborhood $\tilde{U}$ of Φ in $\tilde{\mathfrak{M}}_{\mathcal{B}}(X)$ such that* $\dim H^1(\#_{\mathcal{B},\Psi}) = \dim H^1(\#_{\mathcal{B},\Phi})$, *for all* $\Psi \in \tilde{U}$.

Proof. Taking the quotient $Q^k_\Psi = \bigoplus^l S/\mathbf{E}^k_\Psi$, we obtain the quotient complex $\{(Q_\Psi, d_Q)\}$ and the short exact sequence:

$$0 \to \mathbf{E}^*_\Psi \to \bigoplus^l S \to Q^*_\Psi \to 0.$$

Since $\mathbf{E}^*_\Psi$ and $\bigoplus^l S$ are elliptic, it follows that the quotient complex Q^*_Ψ is also elliptic. (This follows from the long exact sequence of the symbol complexes.) Then we have the long exact sequence,

$$\cdots \to H^1(\#_{\mathcal{B},\Psi}) \xrightarrow{p^1_{\mathcal{B},\Psi}} \bigoplus^l H^*_{\mathrm{dR}}(X) \to H^1(Q^*_\Psi) \to H^2(\#_{\mathcal{B},\Psi}) \xrightarrow{p^2_{\mathcal{B},\Psi}} \bigoplus^l H^*_{\mathrm{dR}}(X) \to \cdots.$$

It follows from Lemma 3.2.8 that the maps $p^1_{\mathcal{B},\Psi}$ and $p^2_{\mathcal{B},\Psi}$ are injective for all $\Psi \in \tilde{U}$. Thus we have the exact sequence,

$$0 \to H^1(\#_{\mathcal{B},\Psi}) \xrightarrow{p^1_{\mathcal{B},\Psi}} \bigoplus^l H^*_{\mathrm{dR}}(X) \to H^1(Q^*_\Psi) \to 0.$$

Then we have

$$\dim \bigoplus^l H^*_{\mathrm{dR}}(X) = \dim H^1(\#_{\mathcal{B},\Psi}) + \dim H^1(Q^*_\Psi).$$

Since $\dim H^1(\#_{\mathcal{B},\Psi})$ and $\dim H^1(Q^*_\Psi)$ are upper semi-continuous in Ψ (see [17]) and $\dim H^*_{\mathrm{dR}}(X)$ is invariant, it follows that $\dim H^1(\#_{\mathcal{B},\Psi}) = \dim H^1(\#_{\mathcal{B},\Phi})$ for all Ψ in a sufficiently small neighborhood $\tilde{U}$ of Φ. □

Proof of Theorem 3.2.6. Let $\tilde{U}$ be a neighborhood of Φ such that for every $\Psi \in \tilde{U}$, the map $p^1_{\mathcal{B},\Psi}$ is injective and $\dim H^1(\#_{\mathcal{B},\Psi}) = \dim H^1(\#_{\mathcal{B},\Phi})$ holds. Let $\{\Phi_t\}$ be a smooth family of $\mathcal{B}(V)$-structures in the neighborhood $\tilde{U}$ parametrized by $t \in [0, 1]$. We assume that the d-closed form Φ_t belongs to the same de Rham cohomology class as Φ_0 for all t, that is, there exists A_t such that

$$\Phi_t - \Phi_0 = dA_t. \tag{3.2.1}$$

Since the group $\widetilde{\mathrm{Diff}}_0(X)$ is generated by the action of $\mathrm{Diff}_0(X)$ and the action of d-exact b-fields, Theorem 3.2.6 is reduced to the following proposition:

Proposition 3.2.10. *Let Φ_t be as in the proof of* Theorem 3.2.6. *Then there exist a smooth family of diffeomorphisms $\{f_t\}$ and a smooth family of d-exact* 2*-forms $\{d\gamma_t\}$ such that*

$$e^{d\gamma_t} \wedge f_t^*\Phi_t = \Phi_0, \quad \textit{for all} \quad t \in [0,1]. \tag{3.2.2}$$

Proof. By differentiating the equation (3.2.2), we have

$$\frac{d}{dt}(e^{d\gamma_t} \wedge f_t^*\Phi_t) = 0, \quad \forall t \in [0,1], \tag{3.2.3}$$

which is equivalent to

$$e^{d\gamma_t} \wedge d\dot{\gamma}_t \wedge f_t^*\Phi_t + e^{d\gamma_t} \wedge \dot{f}_t^*\Phi_t + e^{d\gamma_t} \wedge f_t^*\dot{\Phi}_t = 0. \tag{3.2.4}$$

By the left action of $(f_t^{-1})^*(e^{-d\gamma_t})$, we have

$$(f_t^{-1})^*(d\dot{\gamma}_t \wedge f_t^*\Phi_t) + (f_t^{-1})^*\dot{f}_t^*\Phi_t + \dot{\Phi}_t = 0.$$

We set $(f_t^{-1})^*\dot{\gamma}_t = \tilde{\gamma}_t$. Since $(f_t^{-1})^*\dot{f}_t^*\Phi_t$ is given as the Lie derivative $\mathcal{L}_{v_t}\Phi_t$ for a vector field v_t, it follows from (3.2.1) that

$$(d\tilde{\gamma}_t) \wedge \Phi_t + \mathcal{L}_{v_t}\Phi_t + d\dot{A}_t = 0. \tag{3.2.5}$$

Since Φ_t is d-closed, we have

$$\mathcal{L}_{v_t}\Phi = di_{v_t}\Phi_t. \tag{3.2.6}$$

We substitute (3.2.6) in (3.2.5) and we have

$$(d\tilde{\gamma}_t) \wedge \Phi_t + di_{v_t}\Phi_t = d(\tilde{\gamma}_t + v_t)\cdot\Phi_t = -d\dot{A}_t, \tag{3.2.7}$$

where $(v_t + \tilde{\gamma}_t) \in T \oplus T^*$ acts on Φ_t by the Clifford multiplication. We denote by $\mathbf{E}_t^k(X)$ the vector bundle $\mathrm{CL}^{k+1}\cdot\Phi_t$ and $\#_{\mathcal{B},t}$ the complex $\{\mathbf{E}_t^*(X)\}$. Then $(\tilde{\gamma}_t + v_t)\cdot\Phi_t$ is a section of $\mathbf{E}_t^0(X)$ and $-\dot{\Phi}_t = -d\dot{A}_t$ is a section of $\mathbf{E}_t^1(X)$. Hence $-d\dot{A}_t$ yields the class $-[d\dot{A}_t] \in H^1(\#_{\mathcal{B},t})$ of the deformation complex $\#_{\mathcal{B},t}$:

$$\mathbf{E}_t^0 \xrightarrow{d_0} \mathbf{E}_t^1 \xrightarrow{d_1} \cdots.$$

Then we see that the class $[-d\dot{A}_t] \in H^1(\#_{\mathcal{B},t})$ vanishes since the class $-[d\dot{A}_t]$ is represented by the d-exact form and the map $p_{\mathcal{B},t}^1$ is injective. If we take a metric on the manifold X, we have the adjoint operator d_k^* and the Green operator G_t of the complex $\#_{\mathcal{B},t}$. Since $\dim H^1(\#_{\mathcal{B},t})$ is a constant, the Green operator G_t depends smoothly on the parameter t. We define a section B_t of $\mathbf{E}_t^0(X)$ by

$$B_t = -d_0^* G_t d\dot{A}_t. \tag{3.2.8}$$

Then from the Hodge theory of the elliptic complex, we have

$$dB_t = -d\dot{A}_t. \tag{3.2.9}$$

Since B_t is written as $E_t \cdot \Phi_t$ for $E_t \in T \oplus T^*$, we obtain $v_t + \tilde{\gamma}_t = E_t$ such that a smooth family $\{v_t + \tilde{\gamma}_t\}$ satisfies the equation (3.2.7). By solving the equation $(f_t^{-1})^* \dot{f}_t^* \Phi_t = \mathcal{L}_{v_t}\Phi$, we have the smooth family $\{f_t\}$ with $f_0 = \mathrm{id}$. We also obtain γ_t solving the equation $(f_t^{-1})^* \dot{\gamma}_t = \tilde{\gamma}_t$. Hence we have $\{f_t\}$ and $\{d\gamma_t\}$ which satisfy the equation (3.2.2). □

3.3. Construction of deformations. This subsection is devoted to proof of Theorem 3.2.7.

Proof of Theorem 3.2.7. Let X be an n-dimensional, compact manifold with a $\mathcal{B}(V)$-structure Φ. We take a Riemannian metric on X. (Note that this metric is independent of the structure Φ.) The bundle $\mathrm{G_{cl}}(X) = \mathrm{G_{cl}}(T \oplus T^*)$ acts on the fibre bundle $\mathcal{B}(X)$ transitively. Hence every global section $\mathcal{E}_{\mathcal{B}}(X)$ is written as $g \cdot \Phi$ for a section g of $\mathrm{G_{cl}}(T \oplus T^*)$. The subset $\mathrm{G_{cl0}}(T \oplus T^*)$ with the identity of $\mathrm{G_{cl}}(T \oplus T^*)$ is given by

$$\mathrm{G_{cl0}}(T \oplus T^*) = \{e^a \mid a \in \mathrm{CL}^2(T \oplus T^*)\}. \tag{3.3.1}$$

Hence every deformation of Φ in $\mathcal{E}_{\mathcal{B}}(X)$ is given by $e^a \cdot \Phi$ for a section a of $\mathrm{CL}^2(T \oplus T^*)$. In order to obtain a deformation of Φ in $\tilde{\mathfrak{M}}_{\mathcal{B}}(X)$, we introduce a formal power series in t:

$$a(t) = a_1 t + \frac{1}{2!} a_2 t^2 + \frac{1}{3!} a_3 t^2 + \cdots, \tag{3.3.2}$$

each a_i is a section of $\mathrm{CL}^2(T \oplus T^*)$. We define a formal power series $g(t)$ by

$$g(t) = \exp(a(t)) \in \mathrm{G_{cl0}}(T \oplus T^*)[[t]]. \tag{3.3.3}$$

The group $\mathrm{G_{cl0}}(T \oplus T^*)$ acts on differential forms and we have

$$\begin{aligned} e^{a(t)} \cdot \Phi &= \Phi + a(t) \cdot \Phi + \frac{1}{2!} a(t) \cdot a(t) \cdot \Phi + \cdots \\ &= \Phi + (a_1 \cdot \Phi)t + \frac{1}{2!}((a_2 + a_1 \cdot a_1) \cdot \Phi)t^2 + \cdots . \end{aligned} \tag{3.3.4}$$

The equation that we want to solve is,

$$de^{a(t)} \cdot \Phi = 0. \tag{eq$_*$}$$

At first we take a_1 such that $da_1 \cdot \Phi = 0$ as an initial condition. It follows from Lemma 2.1.7 that we have

$$e^{-a(t)} \cdot d \cdot e^{a(t)} = \exp(\mathrm{Ad}_{a(t)})d, \tag{3.3.5}$$

where $\exp(\mathrm{Ad}_{a(t)})d$ is the operator acting on differential forms which is defined by the power series in t:

$$\begin{aligned}\exp(\mathrm{Ad}_{a(t)})d &= d + \frac{1}{k!}\sum_{k=1}^{\infty} \mathrm{Ad}^k_{a(t)}\, d \\ &= d + [d, a(t)] + \frac{1}{2!}[[d, a(t)], a(t)] + \cdots \\ &= d + [d, a_1]t + \frac{1}{2!}([d, a_2] + [[d, a_1], a_1])t^2 + \cdots,\end{aligned}$$

where $\mathrm{Ad}^k_{a(t)}\, d = [\mathrm{Ad}^{k-1}_{a(t)}\, d, a(t)]$. Hence the (eq$_*$) is equivalent to the equation

$$(\exp(\mathrm{Ad}_{a(t)})d)\Phi = 0. \tag{$\widetilde{\mathrm{eq}}_*$}$$

Then it follows from Proposition 2.1.5 that $\exp(\mathrm{Ad}_{a(t)})d$ is a Clifford–Lie operator of order 3 and we have

$$(\exp(\mathrm{Ad}_{a(t)})d)\Phi \in \mathbf{E}^2(X). \tag{3.3.6}$$

From (3.3.5), we have

$$de^{a(t)} \cdot \Phi = e^{a(t)} \cdot (\exp(\mathrm{Ad}_{a(t)})d)\Phi. \tag{3.3.7}$$

We denote by $(P(t))_{[i]}$ the i-th homogeneous part of a power series $P(t)$ in t. Then from (3.3.7), we have

$$(de^{a(t)} \cdot \Phi)_{[k]} = \sum_{\substack{i,j\geq 0\\ k=i+j}} (e^{a(t)})_{[i]}(\exp(\mathrm{Ad}_{a(t)})d)_{[j]}\Phi. \tag{3.3.8}$$

Since $da_1 \cdot \Phi = 0$, we have

$$(\exp(\mathrm{Ad}_{a(t)})d)_{[0]} \cdot \Phi = (\exp(\mathrm{Ad}_{a(t)})d)_{[1]} \cdot \Phi = 0. \tag{3.3.9}$$

Thus it suffices to determine a_k satisfying ($\widetilde{\mathrm{eq}}_*$) by induction on k. We assume that $a_1, \ldots, a_{k-1}$ have been determined so that

$$(\exp(\mathrm{Ad}_{a(t)})d)_{[l]}\Phi = 0, \quad (l = 0, 1, \ldots, k-1). \tag{3.3.10}$$

Then it follows from (3.3.8) that

$$(de^{a(t)} \cdot \Phi)_{[k]} = (\exp(\mathrm{Ad}_{a(t)})d)_{[k]}\Phi. \tag{3.3.11}$$

Then form (3.3.6), we see that

$$(de^{a(t)} \cdot \Phi)_{[k]} \in \mathbf{E}^2(X). \tag{3.3.12}$$

The k-th part $(de^{a(t)} \cdot \Phi)_{[k]}$ is written as

$$(de^{a(t)} \cdot \Phi)_{[k]} = \frac{1}{k!} da_k \cdot \Phi + \mathrm{Ob}_k, \tag{3.3.13}$$

where Ob_k $(= \mathrm{Ob}_k(a_1, \dots, a_{k-1}))$ is the non-linear term depending only on $a_1, \dots, a_{k-1}$. Since $da_k \cdot \Phi \in d\mathbf{E}^1(X) \subset \mathbf{E}^2(X)$, it follows from (3.3.13) that

$$\mathrm{Ob}_k \in \mathbf{E}^2(X). \tag{3.3.14}$$

Since Ob_k is d-exact, we have the cohomology class $[\mathrm{Ob}_k] \in H^2(\#_{\mathcal{B}})$. Then we have

Lemma 3.3.1. *There exists a section a_k satisfying $(de^{a(t)} \cdot \Phi)_{[k]} = 0$ if and only if the class $[\mathrm{Ob}_k] \in H^2(\#_{\mathcal{B}})$ vanishes.*

Proof. The equation $(de^{a(t)} \cdot \Phi)_{[k]} = 0$ is written as

$$\frac{1}{k!} da_k \cdot \Phi = -\mathrm{Ob}_k, \tag{3.3.15}$$

where Ob_k only depends on $a_1, \dots, a_{k-1}$. The L.H.S. of (3.3.15) is an element of the image $d\mathbf{E}^1(X)$ in the complex $\#_{\mathcal{B}}$:

$$\cdots \xrightarrow{d_{-1}} \mathbf{E}^0 \xrightarrow{d_0} \mathbf{E}^1 \xrightarrow{d_1} \mathbf{E}^2 \xrightarrow{d_2} \cdots.$$

The R.H.S. of (3.3.15) is also a d_2-closed element of $\mathbf{E}^2$ which yields the class $[\mathrm{Ob}_k] \in H^2(\#_{\mathcal{B}})$. If we have a_k satisfying the equation (3.3.15), then the class $[\mathrm{Ob}_k]$ vanishes. The complex $\#_{\mathcal{B}}$ is an elliptic complex and we have the Green operator $G_{\#_{\mathcal{B}}}$ of the complex $\#_{\mathcal{B}}$. If the class $[\mathrm{Ob}_k]$ vanishes, we can obtain a_k by using the Green operator:

$$\frac{1}{k!} a_k \cdot \Phi = -d^* G_{\#_{\mathcal{B}}}(\mathrm{Ob}_k) \in \mathbf{E}^1.$$

Then a_k satisfies the equation (3.3.15). □

We call $[\mathrm{Ob}_k]$ the k-th obstruction class. (Note that $[\mathrm{Ob}_k]$ can be defined if the lower obstruction classes vanish.) Since Ob_k is d-exact, we have that the class $[\mathrm{Ob}_k] \in$

$H^2(\#_{\mathcal{B}})$ is contained in the kernel of the map $p^2_{\mathcal{B}}$. Hence if the map $p^2_{\mathcal{B}} \colon H^2(\#_{\mathcal{B}}) \to H^*_{\mathrm{dR}}(X)$ is injective then $[\mathrm{Ob}_k]$ vanishes. Hence from (3.3.11), we have a_k satisfying $(\exp(\mathrm{Ad}_{a(t)})d)_{[k]}\Phi = 0$. By induction, we have a formal power series $a(t)$ which is a solution of the equation ($\widetilde{\mathrm{eq}}_*$). The rest is to show the convergence of the power series $a(t)$. The convergence can be shown essentially by the same method as in [6]. We also have the smoothness of solutions by the standard elliptic regularity method. Hence the result follows. $\square$

4. Generalized Calabi–Yau and generalized SU(n)-structures

4.1. Generalized Calabi–Yau structures. Let V be a real vector space of dimension $2n$ and $\mathcal{J}(V)$ the set of complex structures on V. We denote by $\bigwedge_J^{n,0} V^*_{\mathbb{C}}$ the space of complex forms of type $(n,0)$ with respect to $J \in \mathcal{J}(V)$. Let $\mathfrak{P}(V)$ be the set of pairs consisting of complex structures J and a non-zero complex form of type $(n,0)$:

$$\mathfrak{P}(V) := \left\{ (J, \Omega_J) \;\middle|\; J \in \mathcal{J}(V),\ 0 \neq \Omega_J \in \bigwedge_J^{n,0} V^*_{\mathbb{C}} \right\}.$$

Then we have the projection π_2 from $\mathfrak{P}(V)$ to complex n-forms $\bigwedge^n V^*_{\mathbb{C}}$:

$$\pi_2 \colon \mathfrak{P}(V) \to \bigwedge^n V^*_{\mathbb{C}}.$$

DEFINITION 4.1.1. A complex n-form Ω_V on V is an $\mathrm{SL}_n(\mathbb{C})$-*structure* if Ω_V belongs to the image $\pi_2(\mathfrak{P}(V))$. The set of $\mathrm{SL}_n(\mathbb{C})$-structures on V is denoted by $\mathcal{A}_{\mathrm{SL}}(V)$.

Hence each $\mathrm{SL}_n(\mathbb{C})$-structure Ω_V is a complex form of type $(n,0)$ with respect to a complex structure $J \in \mathcal{J}(V)$. Conversely for each $\mathrm{SL}_n(\mathbb{C})$-structure Ω_V we define a complex subspace $\ker \Omega_V$ by

$$\ker \Omega_V := \{ v \in V_{\mathbb{C}} \mid i_v \Omega_V = 0 \}.$$

Then the complexified vector space $V_{\mathbb{C}}$ is decomposed into $\ker \Omega_V$ and the conjugate space $\overline{\ker \Omega_V}$:

$$V_{\mathbb{C}} = \ker \Omega_V \oplus \overline{\ker \Omega_V}. \tag{4.1.1}$$

The decomposition (4.1.1) gives the complex structure J on V such that Ω_V is the complex form of type $(n,0)$ with respect to J. Then we have the map from the set of $\mathrm{SL}_n(\mathbb{C})$-structures to the set of complex structures:

$$\mathcal{A}_{\mathrm{SL}}(V) \to \mathcal{J}(V).$$

An $\mathrm{SL}_n(\mathbb{C})$-structure Ω_V is written as $\Omega_V = \theta^1 \wedge \cdots \wedge \theta^n$, where $\{\theta^1, \dots, \theta^n\}$ is a basis of the space of complex forms of type (1, 0) with respect to J. Then it follows that the real linear group $\mathrm{GL}(V)$ acts on $\mathcal{A}_{\mathrm{SL}}(V)$ transitively with isotropy group $\mathrm{SL}_n(\mathbb{C})$ and $\mathcal{A}_{\mathrm{SL}}(V)$ is the orbit which is described as the homogeneous space:

$$\mathcal{A}_{\mathrm{SL}}(V) = \mathrm{GL}(V)/\mathrm{SL}_n(\mathbb{C}).$$

The real Clifford group $\mathrm{G}_{\mathrm{cl}}(V \oplus V^*)$ acts on $\bigwedge^* V^* \otimes \mathbb{C}$ by the spin representation as in Section 1. When we consider complex forms as pairs of real forms, we can apply the viewpoint in Section 3 and then the Calabi–Yau structures naturally arise as $\mathcal{B}_{\mathrm{SL}}(V)$-structures.

DEFINITION 4.1.2. Let $\mathcal{B}_{\mathrm{SL}}(V)$ be the orbit of G_{cl} including $\mathrm{SL}_n(\mathbb{C})$-structures $\mathcal{A}_{\mathrm{SL}}(V)$,

$$\mathcal{A}_{\mathrm{SL}}(V) \subset \mathcal{B}_{\mathrm{SL}}(V).$$

We call an element ϕ_V of $\mathcal{B}_{\mathrm{SL}}(V)$ is a *generalized Calabi–Yau structure* on V (i.e., non-degenerate, pure spinor) and $\mathcal{B}_{\mathrm{SL}}(V)$ *the orbit of generalized Calabi–Yau structures*,

Let X be a compact manifold of dimension $2n$. Then by applying the construction as in Section 3, we define $\mathcal{B}_{\mathrm{SL}}(V)$-structures on X to be generalized geometric structures corresponding to the orbit $\mathcal{B}_{\mathrm{SL}}(V)$. The $\mathcal{B}_{\mathrm{SL}}(V)$-structures coincide with the generalized Calabi–Yau structures introduced by Hitchin [14] since the set of non-degenerate, pure spinors of $V \oplus V^*$ forms the orbit $\mathcal{B}_{\mathrm{SL}}(V)$. We shall apply our deformation theory to generalized Calabi–Yau structures. A $\mathcal{B}_{\mathrm{SL}}(V)$-structure ϕ on X gives the complex $\#_{\mathcal{B}_{\mathrm{SL}}}$ of vector bundles $\{\mathbf{E}^k_{\mathrm{SL}}\}$ on X:

$$0 \to \mathbf{E}^{-1}_{\mathrm{SL}} \to \mathbf{E}^0_{\mathrm{SL}} \to \mathbf{E}^1_{\mathrm{SL}} \to \mathbf{E}^2_{\mathrm{SL}} \to \cdots.$$

We denote by $H^*(\#_{\mathcal{B}_{\mathrm{SL}}})$ the cohomology group of the complex $\#_{\mathcal{B}_{\mathrm{SL}}}$. Let L_ϕ be the vector bundle on X which is defined by

$$L_\phi = \{E \in (T \oplus T^*) \otimes \mathbb{C} \mid E \cdot \phi = 0\}.$$

Then we have a decomposition:

$$(T \oplus T^*) \otimes \mathbb{C} = L_\phi \oplus \overline{L_\phi}, \tag{4.1.2}$$

where $\overline{L_\phi}$ is the conjugate bundle of L_ϕ. The decomposition (4.1.2) gives rise to the generalized complex structure $\mathcal{J}_\phi$ which is defined by

$$\mathcal{J}_\phi(E) = \begin{cases} +\sqrt{-1}E, & (E \in L_\phi), \\ -\sqrt{-1}E, & (E \in \overline{L_\phi}). \end{cases}$$

We call $\mathcal{J}_\phi$ the induced generalized complex structure. We denote by $\bigwedge^i \overline{L_\phi}$ the i-th wedge product of $\overline{L_\phi}$ which acts on ϕ by the Clifford multiplication. Then we define a vector bundle U^i_ϕ by

$$U_\phi^{-n+i} := \bigwedge^i \overline{L_\phi} \cdot \phi,$$

for $i = 0, \dots, 2n$. The bundle U_ϕ^{-n} is the line bundle generated by ϕ. The vector bundle $\mathbf{E}^k_{\mathrm{SL}}$ is described in terms of U^i_ϕ.

Lemma 4.1.3. *We have the following identification as real vector bundles*:

$$\mathbf{E}^0_{\mathrm{SL}} \cong U_\phi^{-n+1}, \tag{4.1.3}$$

$$\mathbf{E}^1_{\mathrm{SL}} \cong U_\phi^{-n} \oplus U_\phi^{-n+2}, \tag{4.1.4}$$

$$\mathbf{E}^2_{\mathrm{SL}} \cong U_\phi^{-n+1} \oplus U_\phi^{-n+3}. \tag{4.1.5}$$

In general we have

$$\mathbf{E}^{2k-1}_{\mathrm{SL}} \cong \bigoplus_{i=0}^{k} U_\phi^{-n+2i}, \tag{4.1.6}$$

$$\mathbf{E}^{2k}_{\mathrm{SL}} \cong \bigoplus_{i=0}^{k} U_\phi^{-n+2i+1}. \tag{4.1.7}$$

Proof. We consider the complex form $\phi = \phi^{\mathrm{Re}} + \sqrt{-1}\phi^{\mathrm{Im}}$ as the pair of real forms $(\phi^{\mathrm{Re}}, \phi^{\mathrm{Im}})$. Then applying the construction in Section 3, we have the vector bundles $\mathbf{E}^k_{\mathrm{SL}}$ generated by

$$\mathbf{E}^k_{\mathrm{SL}} = \{(a \cdot \phi^{\mathrm{Re}},\ a \cdot \phi^{\mathrm{Im}}) \mid a \in \mathrm{CL}^k\}.$$

Then we have the complex form $a \cdot \phi^{\mathrm{Re}} + \sqrt{-1} a \cdot \phi^{\mathrm{Im}} = a \cdot \phi$. From the decomposition (4.1.2), we have the identification:

$$\mathrm{CL}^{2k} \otimes \mathbb{C} \cong \mathrm{CL}^{2k}(L_\phi \oplus \overline{L_\phi}) \cong \bigoplus_{i=0}^{k} \bigwedge^{2l} (L_\phi \oplus \overline{L_\phi}).$$

Since $L_\phi \cdot \phi = \{0\}$, We have an identification:

$$\mathbf{E}^{2k-1}_{\mathrm{SL}} = \mathrm{CL}^{2k} \cdot \phi \cong \bigoplus_{l=0}^{k} \bigwedge^{2l} \overline{L_\phi} \cdot \phi \tag{4.1.8}$$

$$= \bigoplus_{l=0}^{k} U_\phi^{-n+2l}. \tag{4.1.9}$$

Similarly we have $\mathbf{E}^{2k}_{\mathrm{SL}} \cong \bigoplus_{i=0}^{k} U_{\phi}^{-n+2i+1}$. $\square$

Proposition 4.1.4. *The complex $\#_{\mathcal{B}_{\mathrm{SL}}}$ is elliptic, that is, the orbit $\mathcal{B}_{\mathrm{SL}}$ is an elliptic orbit.*

Proof. Since there is the inclusion $\mathrm{CL}^{k-2} \subset \mathrm{CL}^{k}$, we have the inclusion $\mathbf{E}^{k-2}_{\mathrm{SL}} \subset \mathbf{E}^{k}_{\mathrm{SL}}$ and then the quotient is given by

$$\mathbf{E}^{k}_{\mathrm{SL}}/\mathbf{E}^{k-2}_{\mathrm{SL}} \cong U_{\phi}^{-n+k+1},$$

for $k \geq 1$. We define a complex $\tilde{\#}_{\mathcal{B}_{\mathrm{SL}}} = \{\tilde{\mathbf{E}}^{\bullet}_{\mathrm{SL}}\}$ by replacing $\mathbf{E}^{-1}$ by $\mathbf{E}^{-1} \otimes \mathbb{C}$, that is,

$$\tilde{\mathbf{E}}^{k} = \begin{cases} \mathbf{E}^{-1} \otimes \mathbb{C}, & (k = -1), \\ \mathbf{E}^{k}, & (k \neq -1). \end{cases}$$

Then there is a map [2] by shifting its degree from $*$ to $* + 2$:

$$\tilde{\mathbf{E}}^{k}_{\mathrm{SL}} \overset{[2]}{\mapsto} \tilde{\mathbf{E}}^{k+2}_{\mathrm{SL}}.$$

Thus we have the short exact sequence:

$$0 \to \tilde{\mathbf{E}}^{\bullet}_{\mathrm{SL}} \xrightarrow{[2]} \tilde{\mathbf{E}}^{\bullet+2} \to U_{\phi}^{\bullet} \to 0, \tag{4.1.10}$$

which yields the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & \mathbf{E}^{-1}_{\mathrm{SL}} \otimes \mathbb{C} \arrow[r] \arrow[d] & \mathbf{E}^{0}_{\mathrm{SL}} \arrow[r] \arrow[d] & \mathbf{E}^{1}_{\mathrm{SL}} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathbf{E}^{-1}_{\mathrm{SL}} \otimes \mathbb{C} \arrow[r] \arrow[d, "\cong"] & \mathbf{E}^{0}_{\mathrm{SL}} \arrow[r] \arrow[d, "\cong"] & \mathbf{E}^{1}_{\mathrm{SL}} \arrow[r] \arrow[d] & \mathbf{E}^{2}_{\mathrm{SL}} \arrow[r] \arrow[d] & \mathbf{E}^{3}_{\mathrm{SL}} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & U_{\phi}^{-n} \arrow[r, "\overline{\partial}"'] & U_{\phi}^{-n+1} \arrow[r, "\overline{\partial}"'] & U_{\phi}^{-n+2} \arrow[r, "\overline{\partial}"'] \arrow[d] & U_{\phi}^{-n+3} \arrow[r, "\overline{\partial}"'] \arrow[d] & U_{\phi}^{-n+4} \arrow[r] \arrow[d] & \cdots \\
 & & & 0 & 0 & 0 &
\end{tikzcd}$$

It follows from $U_{\phi}^{-n+i} = \bigwedge^{i} \overline{L_{\phi}} \cdot \phi$ that the quotient complex $(U_{\phi}^{-n+\bullet}, \overline{\partial})$ is an elliptic complex. Hence from the commutative diagram, we see that the complex $\#_{\mathcal{B}_{\mathrm{SL}}}$ is elliptic by induction on degree k. $\square$

The complex $(U^p_\phi, \overline{\partial})$ is the deformation complex of generalized complex structures which is introduced in [11]. Then the exterior derivative d acting on U^p_ϕ is decomposed into two projections ∂ and $\overline{\partial}$, that is,

$$d = \partial + \overline{\partial},$$

$$U^{p-1}_\phi \xleftarrow{\partial} U^p_\phi \xrightarrow{\overline{\partial}} U^{p+1}_\phi.$$

We define an operator $d^{\mathcal{J}}$ by

$$d^{\mathcal{J}} := \sqrt{-1}(\overline{\partial} - \partial).$$

The $dd^{\mathcal{J}}$-property is introduced and discussed in [14], [12], [2]:

DEFINITION 4.1.5. A generalized complex manifold $(X, \mathcal{J})$ satisfies the $dd^{\mathcal{J}}$-*property* if and only if the following three conditions are equivalent:

- $a \in \bigwedge^* T^*$ is d-closed and $d^{\mathcal{J}}$-exact,
- $a \in \bigwedge^* T^*$ is d-exact and $d^{\mathcal{J}}$-closed,
- $a = dd^{\mathcal{J}} b$ for some $b \in \bigwedge^* T^*$.

Theorem 4.1.6. *Let ϕ be a generalized Calabi–Yau structure on a compact manifold X with the induced generalized complex structure $\mathcal{J}_\phi$. If the generalized complex structure $\mathcal{J}_\phi$ satisfies the $dd^{\mathcal{J}}$-property, we have unobstructed deformations of ϕ as generalized Calabi–Yau structures which are parametrized by an open set of the cohomology group $H^1(\#_{\mathcal{B}_{\mathrm{SL}}})$. Further the period map P from the space of deformations of ϕ to the de Rham cohomology group is locally injective, i.e., the local Torelli type theorem holds.*

Proof. Since U^p_ϕ is the eigenspace of the action of $\mathcal{J}_\phi$ with eigenvalue $\sqrt{-1}p$, we have the decomposition $\bigwedge^* T^* = \bigoplus_{p=-n}^n U^p_\phi$. If an exact form $da^{(m)}$ is an element of U^{m-1}_ϕ for $a^{(m)} \in U^m_\phi$, we have $\partial d a^{(m)} = \partial\overline{\partial} a^{(m)} = 0$. Hence applying the $dd^{\mathcal{J}}$-property we have

$$da^{(m)} = dd^{\mathcal{J}} b = 2\sqrt{-1}\overline{\partial}\partial b = 2\sqrt{-1} d\partial b, \tag{4.1.11}$$

for $b \in U^{m-1}_\phi$. Then we have $da^{(m)} = d\gamma$ for $\gamma = 2\sqrt{-1}\partial b \in U^{m-2}_\phi$. From our decomposition, a form a is written as

$$a = \sum_{p=-n}^{m} a^{(p)},$$

where $a^{(p)} \in U^p_\phi$ for some m. If da is an element of $\sum_{p=-n}^k U^p_\phi$, then applying the $dd^{\mathcal{J}}$-property successively, we have $da = db$ for $b \in \sum_{p=-n}^{k-1} U^p_\phi$. Similarly if $da \in$

$\bigwedge^{\text{even}} T^*$ (resp. $da \in \bigwedge^{\text{odd}} T^*$) then applying the $dd^{\mathcal{J}}$-property we see that $da = db$ for $b \in \bigwedge^{\text{odd}} T^*$ (resp. $b \in \bigwedge^{\text{even}} T^*$). Hence it follows from Lemma 4.1.3 that if $da \in \mathbf{E}^k_{\text{SL}}$ then $da = db$ for $b \in \mathbf{E}^{k-1}_{\text{SL}}$ $(k \geq 1)$. It implies that the map $p^k_{\mathcal{B}}\colon H^k(\#_{\mathcal{B}_{\text{SL}}}) \to H^*_{\text{dR}}(X)$ is injective for $k \geq 1$. □

Gualtieri also shows that the $dd^{\mathcal{J}}$-property holds for generalized Kähler structures [12]. By applying his theorem, we obtain

Theorem 4.1.7. *Let ϕ be a generalized Calabi–Yau structure on X with the induced generalized complex structure $\mathcal{J}_\phi$. If there exists another generalized complex structure $\mathcal{I}$ such that the pair $(\mathcal{I}, \mathcal{J}_\phi)$ gives a generalized Kähler structure on X, then ϕ is a topological structure.*

Proof. The result follows from the proof of Theorem 4.1.6. □

We denote by $H^\bullet(\tilde{\#}_{\mathcal{B}_{\text{SL}}})$ the cohomology group of the complex $\tilde{\#}_{\mathcal{B}_{\text{SL}}} = \{\tilde{\mathbf{E}}^\bullet_{\text{SL}}\}$. The short exact sequence (4.1.10) in the proof of Proposition 4.1.4 gives the long exact sequence:

$$\cdots \to H^{-1}(\tilde{\#}_{\mathcal{B}_{\text{SL}}}) \to H^1(\tilde{\#}_{\mathcal{B}_{\text{SL}}}) \to H^2_{\overline{\partial}}(U^*_\phi) \to H^0(\tilde{\#}_{\mathcal{B}_{\text{SL}}}) \to H^2(\#_{\mathcal{B}_{\text{SL}}}),$$

where $H^k_{\overline{\partial}}(U^\bullet_\phi)$ denotes the cohomology group of the complex $U^\bullet_\phi$:

$$H^k_{\overline{\partial}}(U^\bullet_\phi) = (\ker \overline{\partial}\colon U^{-n+k}_\phi \to U^{-n+k+1}_\phi)/\overline{\partial}(U^{-n+k-1}_\phi).$$

In particular, $H^2_{\overline{\partial}}(U^\bullet_\phi)$ is the infinitesimal tangent space of deformations of generalized complex structures (cf. [6], [11]). It follows from the the $dd^{\mathcal{J}}$-property that the map $H^k(\tilde{\#}_{\mathcal{B}_{\text{SL}}})$ to $H^{k+1}_{\overline{\partial}}(U^\bullet_\phi)$ is surjective. Thus we have the short exact sequence:

$$0 \to H^{-1}(\tilde{\#}_{\mathcal{B}_{\text{SL}}}) \to H^1(\tilde{\#}_{\mathcal{B}_{\text{SL}}}) \to H^2_{\overline{\partial}}(U^*_\phi) \to 0. \tag{4.1.12}$$

Then we have

Proposition 4.1.8. *Let ϕ be a generalized Calabi–Yau structure on X and $\mathcal{J}_\phi$ the induced generalized complex structure. If $\mathcal{J}_\phi$ satisfies the $dd^{\mathcal{J}}$-property, then deformations of $\mathcal{J}_\phi$ as generalized complex structures are unobstructed and small deformations are induced from deformations of generalized Calabi–Yau structures.*

Proof. The exact sequence (4.1.12) implies that the map $\#_{\mathcal{B}_{\text{SL}}} H^1(\tilde{\#}_{\mathcal{B}_{\text{SL}}}) \to H^2_{\overline{\partial}}(U^\bullet_\phi)$ is surjective. The cohomology group $H^1(\tilde{\#}_{\mathcal{B}_{\text{SL}}})$ is the infinitesimal tangent space of deformations of generalized Calabi–Yau structures and $H^2_{\overline{\partial}}(U^\bullet_\phi)$ is the one of generalized complex structures. Thus it follows from Theorem 4.1.6 that deformations of $\mathcal{J}_\phi$ as generalized complex structures are unobstructed and small deformations are induced from deformations of generalized Calabi–Yau structures. □

REMARK 4.1.9. Li [22] showed the following result: Let (X, ϕ) be a generalized Calabi–Yau manifold. If there is another generalized complex structure $\mathcal{I}$ such that the pair $(\mathcal{I}, \mathcal{J}_\phi)$ is a generalized Kähler structure, then small deformations of $\mathcal{J}_\phi$ as generalized complex structures are unobstructed and parametrized by $H^2_{\bar\partial}(U^*_\phi)$. Li used the deformation theory developed by [11] and solved the generalized Maurer–Cartan equation to obtain unobstructed deformations of generalized complex structures. By using Theorem 4.1.7 and Proposition 4.1.8, we can give a different proof of Li's result (see [7], [9] for more detail about the relation between our deformation theory of generalized Calabi–Yau structures and the deformation theory of generalized complex structures).

4.2. Generalized SU(n)-structures. Let ω_V be a real 2-form on a real vector space V of dimension $2n$. As in Section 4.1, an $\mathrm{SL}_n(\mathbb{C})$-structure Ω_V gives rise to the complex structure J on V and then the associated bilinear form g_V is given by

$$g_V(u, v) = \omega_V(Ju, v), \quad (u, v \in V).$$

DEFINITION 4.2.1. A pair (Ω_V, ω_V) is *an* SU(n)*-structure* on V if the following three conditions hold:

(1) $\Omega_V \wedge \omega_V = 0$,

(2) $\Omega_V \wedge \overline{\Omega_V} = c_n \omega_V^n$, where c_n is a constant which depends only on n and $\overline{\Omega_V}$ denotes the complex conjugate of Ω_V,

(3) The associated bilinear form g_V is positive-definite.

The condition (1) implies that ω_V is a form of type (1, 1) with respect to J and then it follows from (3) that ω_V is a Hermitian form. The equation (2) is called the Monge–Ampère condition. Let $\mathcal{A}_{\mathrm{SU}}(V)$ be the set of SU(n)-structures on V. Then the real linear group GL(V) acts on $\mathcal{A}_{\mathrm{SU}}(V)$ transitively with the isotropy group SU(n). Hence $\mathcal{A}_{\mathrm{SU}}(V)$ is the orbit of GL(V) which is described as the homogeneous space:

$$\mathcal{A}_{\mathrm{SU}}(V) = \mathrm{GL}(V)/\mathrm{SU}(n).$$

We consider the pair $(\Omega_V, e^{\sqrt{-1}\omega_V})$ for an SU(n)-structure (Ω_V, ω_V) which consists of two non-degenerate, pure spinor Ω_V and $e^{\sqrt{-1}\omega_V}$, where

$$e^{\sqrt{-1}\omega_V} = 1 + \sqrt{-1}\omega_V + \frac{1}{2!}(\sqrt{-1}\omega_V)^2 + \frac{1}{3!}(\sqrt{-1}\omega_V)^3 + \cdots.$$

DEFINITION 4.2.2. The orbit $\mathcal{B}_{\mathrm{SU}}(V)$ of the Clifford group $\mathrm{G_{cl}}$ through the pair $(\Omega_V, e^{\sqrt{-1}\omega_V})$ is called *the generalized* SU(n) *orbit*. An element $(\phi_{V,0}, \phi_{V,1})$ of the orbit

$\mathcal{B}_{\mathrm{SU}}(V)$ is *a generalized* SU(n)*-structure on* V. Note that the orbit $\mathcal{B}_{\mathrm{SU}}(V)$ is embedded into the space of pairs of complex forms $\bigwedge^* V^*_{\mathbb{C}} \oplus \bigwedge^* V^*_{\mathbb{C}}$. Let X be a compact manifold of dimension $2n$. Then as in Section 3, we define *generalized* SU(n)*-structures on* X to be $\mathcal{B}_{\mathrm{SU}}(V)$-structures on X.

Let (ϕ_0, ϕ_1) be a generalized SU(n)-structure on a compact manifold X of dimension $2n$. Since it consists of two generalized Calabi–Yau structures ϕ_0 and ϕ_1, we obtain the pair $(\mathcal{J}_0, \mathcal{J}_1)$ of the induced generalized complex structures on X. Since the set of generalized Kähler structures also forms an orbit of G_{cl}, it turns out that the pair $(\mathcal{J}_0, \mathcal{J}_1)$ is a generalized Kähler structure. By applying the $dd^{\mathcal{J}}$-property, we obtain the following theorem on deformations of generalized SU(n)-structures:

Theorem 4.2.3. *The orbit $\mathcal{B}_{SU}(V)$ of generalized* SU(n)*-structures is an elliptic and topological orbit on X.*

Theorem 4.2.3 implies the following:

Theorem 4.2.4. *Let $\Phi = (\phi_0, \phi_1)$ be a generalized* SU(n)*-structure on a compact manifold X of dimension $2n$. Then we obtain unobstructed deformations of Φ as generalized* SU(n)*-structures which are parametrized by an open set of the cohomology group $H^1(\#_{\mathcal{B}_{\mathrm{SU}}})$. Further the period map of the moduli space $\mathfrak{M}_{\mathrm{SU}}(X)$ is locally injective, i.e., the local Torelli type theorem holds.*

Proof of Theorems 4.2.3 and 4.2.4. Let (ϕ_0, ϕ_1) be a generalized SU(n)-structure with the generalized Kähler structure $(\mathcal{J}_0, \mathcal{J}_1)$ on X. We denote by $\#_{\mathcal{B}_{\mathrm{SU}}} = \{\mathbf{E}^*_{\mathrm{SU}}, d\}$ the deformation complex of generalized SU(n)-structure (ϕ_0, ϕ_1). Then it suffices to show that each map

$$p^i_{\mathcal{B}_{\mathrm{SU}}} \colon H^i(\#_{\mathcal{B}_{\mathrm{SU}}}) \to \bigoplus^2 H^*_{\mathrm{dR}}(X, \mathbb{C})$$

is injective for $i = 1, 2$. We have the eigenspace decomposition of $\bigwedge^* T^* = \bigoplus_{p=-n}^n U^p_{\phi_j}$ for each $j = 0, 1$. Since $[\mathcal{J}_0, \mathcal{J}_1] = 0$, we then have the simultaneous decomposition into eigenspaces:

$$\bigwedge^* T^* = \bigoplus_{\substack{|p+q| \le n \\ p+q \equiv n \ (\mathrm{mod}\ 2)}} U^{p,q},$$

where $U^{p,q} = U^p_{\phi_0} \cap U^q_{\phi_1}$. Each $\mathbf{E}^i_{\mathrm{SU}}$ consists of pairs of differential forms and then the projection π_1 to the first component induces the map from $\#_{\mathcal{B}_{\mathrm{SU}}}$ to $\#_{\mathcal{B}_{\mathrm{SL}}}$. We denote by $K^\bullet = (K^\bullet, d)$ the complex defined by the kernel of π_1. Then we have a short exact sequence:

$$0 \to K^\bullet \to E^\bullet_{\mathrm{SU}} \xrightarrow{\pi_1} E^\bullet_{\mathrm{SL}} \to 0, \tag{4.2.1}$$

that is,

$$
\begin{tikzcd}
 & K^0 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & K^2 \arrow[r] \arrow[d] & \cdots \\
\mathbf{E}_{\mathrm{SU}}^{-1} \arrow[r] \arrow[d] & \mathbf{E}_{\mathrm{SU}}^{0} \arrow[r] \arrow[d] & \mathbf{E}_{\mathrm{SU}}^{1} \arrow[r] \arrow[d] & \mathbf{E}_{\mathrm{SU}}^{2} \arrow[r] \arrow[d] & \cdots \\
\mathbf{E}_{\mathrm{SL}}^{-1} \arrow[r] & \mathbf{E}_{\mathrm{SL}}^{0} \arrow[r] & \mathbf{E}_{\mathrm{SL}}^{1} \arrow[r] & \mathbf{E}_{\mathrm{SL}}^{2} \arrow[r] & \cdots .
\end{tikzcd}
$$

If $E \cdot \phi_1 = 0$ for real $E \in T \oplus T^*$, then we see that $E = 0$. It implies that $K^0 \cong \{0\}$. Similarly K^1 and K^2 are respectively given by

$$K^1 \cong U^{0,-n+2}, \tag{4.2.2}$$

$$K^2 \cong U^{1,-n+1} \oplus U^{-1,-n+1} \oplus U^{1,-n+3} \oplus U^{-1,-n+3}. \tag{4.2.3}$$

The complex (K^*, d) is a subcomplex of the full de Rham complex and we have the map $p_K^i : H^i(K^*) \to H^*_{\mathrm{dR}}(X)$. By applying the Hodge decomposition of generalized Kähler manifold in [12], it turns out that p_K^i is injective for each i (cf. Section 1.3 of [7]). We denote by $S^\bullet$ the full de Rham complex as in Section 3, where $\mathrm{S}^i = S = \bigoplus_{j=0}^{2n} \bigwedge^j T^*X$, for all i. Then there is the splitting short exact sequence:

$$0 \to S^\bullet \to S^\bullet \oplus S^\bullet \to S^\bullet \to 0,$$

where $S^\bullet \oplus S^\bullet$ is the direct sum of $S^\bullet$ and $S^\bullet$. The short exact sequence (4.2.1) is a subsequence of the splitting short exact sequence:

$$
\begin{tikzcd}
0 \arrow[r] & K^* \arrow[r] \arrow[d] & E_{\mathrm{SU}}^\bullet \arrow[r] \arrow[d] & E_{\mathrm{SL}}^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^\bullet \arrow[r] & S^\bullet \oplus S^\bullet \arrow[r] & S^\bullet \arrow[r] & 0
\end{tikzcd}
\tag{4.2.4}
$$

Hence we have the diagram of long exact sequences:

$$
\begin{tikzcd}
\cdots \arrow[r] & H^i(K^\bullet) \arrow[r] \arrow[d, "p_K^i"'] & H^i(\#_{\mathcal{B}_{\mathrm{SU}}}) \arrow[r] \arrow[d, "p^i_{\mathcal{B}_{\mathrm{SU}}}"'] & H^i(\#_{\mathcal{B}_{\mathrm{SL}}}) \arrow[r] \arrow[d, "p^i_{\mathcal{B}_{\mathrm{SL}}}"] & \cdots \\
0 \arrow[r] & H^*_{\mathrm{dR}}(X) \arrow[r] & H^*_{\mathrm{dR}}(X) \oplus H^*_{\mathrm{dR}}(X) \arrow[r] & H^*_{\mathrm{dR}}(X) \arrow[r] & 0,
\end{tikzcd}
$$

where the sequence at the top is the long exact sequence given by the short exact sequence (4.2.1). Since $p^i_{\mathcal{B}_{\mathrm{SL}}}$ and p^i_K are injective, it follows that $p^i_{\mathcal{B}_{\mathrm{SU}}}$ is injective for i. Hence the results follows. □

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