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RACK MODULE ENHANCEMENTS OF COUNTING INVARIANTS

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Abstract

We introduce a modified rack algebra $\mathbb{Z}[X]$ for racks X with finite rack rank N. We use representations of $\mathbb{Z}[X]$ into rings, known as *rack modules*, to define enhancements of the rack counting invariant for classical and virtual knots and links. We provide computations and examples to show that the new invariants are strictly stronger than the unenhanced counting invariant and are not determined by the Jones or Alexander polynomials.

1. Introduction

First introduced in [1], a *quandle module* is a representation of an associative algebra $\mathbb{Z}[Q]$ known as the *quandle algebra*, defined from a quandle Q. In [3] quandle modules were used to define knot and link invariants including some enhancements of the quandle counting invariant. In [7] a generalized notion of rack modules defined in terms of trunks was introduced.

In this paper we introduce a modified form of the rack algebra for racks with finite rack rank and use the resulting representations into modules over finite rings to define new enhancements of the rack counting invariant $\Phi_X^{\mathbb{Z}}$ from [9]. These new enhancements specialize to $\Phi_X^{\mathbb{Z}}$ but are able to distinguish many knots and links with the same $\Phi_X^{\mathbb{Z}}$ values, and hence are strictly stronger invariants. We provide examples which demonstrate that the new enhanced invariants can distinguish knots with the same Jones and Alexander polynomials.

The paper is organized as follows. In Section 2 we review the basics of racks and the rack counting invariant. In Section 3 we define the rack algebra and resulting rack modules. In Section 4 we use finite rack modules to define new enhancements of the rack counting invariant. In Section 5 we give examples and describe our methods of computation of the new invariants. In Section 6 we collect a few open questions and suggest directions for future work.

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2. Rack basics

In this section, we review some rack basics needed for subsequent sections. We begin with a definition from [4].

DEFINITION 1. A *rack* is a set X with an operation $\triangleright : X \times X \to X$ satisfying the following axioms:
(i) for all $x, y \in X$, there exists a *unique* $z \in X$ such that $x = z \triangleright y$, and
(ii) for all $x, y, z \in X$, the equation

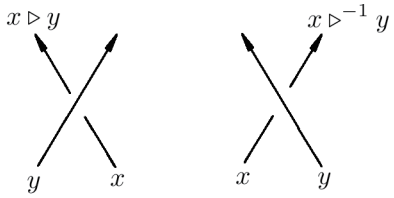
$$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$$

is satisfied.

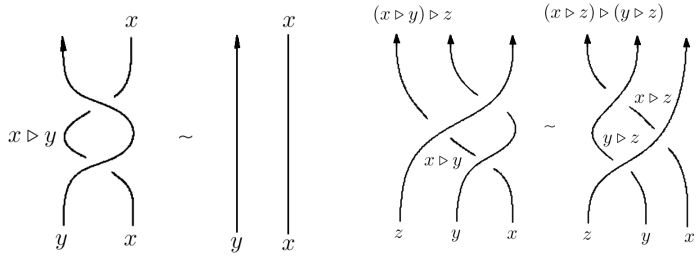
Note that Axiom (i) is equivalent to the following:
(i′) there exists a second operation $\triangleright^{-1} : X \times X \to X$ such that for all $x, y \in X$

$$(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y.$$

As shown below, the operation $\triangleright$ denotes a crossing from right to left given the positive direction of the over-crossing strand. Conversely, the operation $\triangleright^{-1}$ denotes a crossing from the opposite direction.

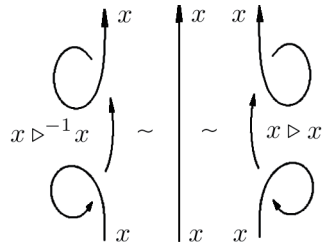


If we regard $x, y \in X$ as two labels on arcs in a knot diagram, as shown below, we see that axioms (i) and (ii) correspond to Reidemeister moves II and III.



The equivalence relation on knot diagrams generated by the Reidemeister II and III moves is known as *regular isotopy*, and labelings of links by racks are preserved

by regular isotopy moves. Rack colorings are also preserved by the *blackboard-framed type* I *moves*, which preserve the writhe or *blackboard-framing*:



EXAMPLE 1 (Constant action racks). Given a set X, we can define a binary operation $x \triangleright y = \sigma(x)$, where $x, y \in X$ and the bijection $\sigma\colon X \to X$ is any permutation in the symmetric group S_X on X. Then under the operation $\triangleright$, X is a rack, since $\triangleright^{-1} = \sigma^{-1}(x)$ and

$$(x \triangleright y) \triangleright z = \sigma(x \triangleright y) = \sigma^2(x) = \sigma(x \triangleright z) = (x \triangleright z) \triangleright (y \triangleright z).$$

EXAMPLE 2 ((t, s)-racks). A (t, s)-rack is a module X over $\ddot{\Lambda} = \mathbb{Z}[t^{\pm 1}, s]/(s^2 - (1-t)s)$ under the operations

$$x \triangleright y = tx + sy \quad \text{and} \quad x \triangleright^{-1} y = t^{-1}x - t^{-1}sy.$$

Convenient examples of (t, s)-rack structures include:

- $X = \mathbb{Z}_n$ with a choice of $s \in \mathbb{Z}_n$ and $t \in \mathbb{Z}_n^*$ satisfying $s^2 = (1-t)s$, e.g. $\mathbb{Z}_4$ with $t = 1$ and $s = 2$,
- $X = \ddot{\Lambda}/I$ for an ideal I, e.g. $\ddot{\Lambda}/(t^2 + 1)$,
- X any abelian group with an automorphism $t\colon X \to X$ and an endomorphism $s\colon X \to X$ satisfying $s^2 = (Id - t)s = s(Id - t)$.

If $s = 1 - t$ then our (t, s)-rack is a quandle, known as an *Alexander quandle*.

DEFINITION 2. Analogous to a Cayley table for a group, a *rack matrix* describes an operation $\triangleright$ on a finite rack $X = \{x_1, x_2, \ldots, x_n\}$. The entry k of the rack matrix in the i-th row and j-th column is defined to be the subscript of x_k, where $x_k = x_i \triangleright x_j$.

EXAMPLE 3. The following are the rack matrices of the constant action rack $X = \{x_1, x_2, x_3, x_4\}$ with $\sigma = (1234)$ and the (t, s)-rack $Y = \mathbb{Z}_4$ with $t = 1$ and $s = 2$:

$$M_X = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad M_Y = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 4 & 2 & 4 & 2 \\ 1 & 3 & 1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}.$$

Every tame blackboard-framed oriented knot or link L has a *fundamental rack* $FR(L)$ consisting of equivalence classes of rack words in a set of generators corresponding one-to-one with the set of arcs in a diagram of L modulo the equivalence relation generated by the rack axiom relations and the crossing relations in L. If we wish to specify explicitly the writhe vector or blackboard-framing vector of L, we will write $FR(L, \mathbf{w})$ where $\mathbf{w} = (w_1, \ldots, w_n)$ and w_k is the writhe of the k-th component of L.

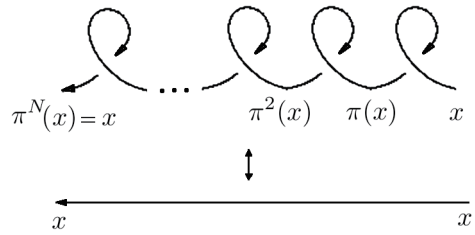
For a given framed link L and rack X, the number of rack labelings of the arcs in L by rack elements is an invariant of framed isotopy known as the *basic counting invariant*, denoted $|\mathrm{Hom}(FR(L), X)|$ since each labeling determines a unique rack homomorphism from $FR(L)$ to X.

DEFINITION 3. Let X be a rack. For every element $x \in X$, the *rack rank* of x, $N(x)$, is the minimal integer N such that $\pi^N(x) = x$ where the *kink map* $\pi : X \to X$ is defined by $\pi(x) = x \triangleright x$. The *rack rank* of X, $N(X)$, is the least common multiple (lcm) of the rank ranks of $x \in X$:

$$N(X) = \mathrm{lcm}\{N(x) \mid x \in X\}.$$

If X is finite, then the rack rank is the exponent of the kink map π considered as an element of the symmetric group S_X.

Rack labelings of knot and link diagrams by a rack X are also preserved by the *N-phone cord move* where N is the rack rank of X:



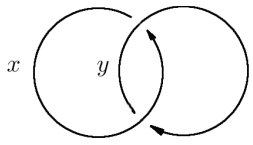
In [9] it is noted that if a rack X has rack rank N, then the basic counting invariants are periodic in N on each component, and thus the sum of these basic counting invariants over a complete set of writhe vectors mod N is an invariant of L up to ambient isotopy.

DEFINITION 4. The *integral rack counting invariant* of L with respect to X is defined as:

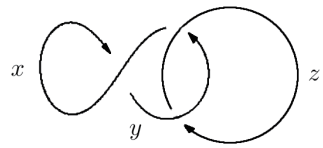
$$\Phi_X^{\mathbb{Z}}(L) = \sum_{\mathbf{w} \in W} |\mathrm{Hom}(FR(L, \mathbf{w}), X)|$$

where X is a finite rack, L is a link with c components, $N = N(X)$ and $W = (\mathbb{Z}_N)^c$.

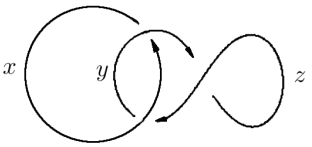
EXAMPLE 4. Let us find the set of rack labelings of the Hopf link H by the constant action rack $X_{(12)} = \{1, 2\}$ with $\sigma = (12)$. The labeling rule here says that when crossing under a strand with any label, a 1 switches to a 2 and vice-versa. In particular, in this rack we always have $x \neq x \triangleright y$ for all x, y. The rack rank is 2, so we need to find rack labelings over diagrams of L with all writhe vectors in $(\mathbb{Z}_2)^2 = \{(0,0),(0,1),(1,0),(1,1)\}$. The diagrams of the Hopf link with writhe vectors $(0,0)$, $(0,1)$ and $(1, 0)$ have no valid rack labelings by X, since each of these diagrams requires a relation of the form $x = x \triangleright y$:



$$x = x \triangleright y$$
$$y = y \triangleright x$$

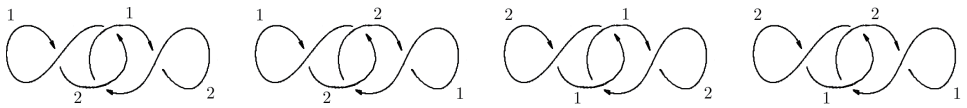


$$z = z \triangleright y$$
$$x = y \triangleright z$$
$$y = x \triangleright x$$



$$x = x \triangleright y$$
$$y = z \triangleright x$$
$$z = y \triangleright z$$

while the (1, 1) diagram has four valid X-labelings:



Hence, the integral rack counting invariant of the Hopf link with respect to X is $\Phi_X^{\mathbb{Z}}(H) = 0 + 0 + 0 + 4 = 4$.

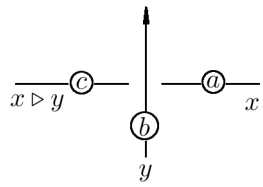
3. The rack algebra and rack modules

In this section we recall and slightly reformulate the notions of the rack algebra and rack modules from [1, 3, 7].

Let X be a finite rack with rack rank N. We would like to define an associative algebra $\mathbb{Z}[X]$ determined by X. One way to do this is to modify the (t,s)-rack structure described in Section 2, giving each pair $(x, y) \in X \times X$ its own invertible $t_{x,y}$ and generic $s_{x,y}$. Let $\Omega[X]$ be the free $\mathbb{Z}$-algebra generated by these $s_{x,y}$ and $t_{x,y}^{\pm 1}$. The *rack algebra* of X, $\mathbb{Z}[X]$, will be a quotient of $\Omega[X]$ by a certain ideal I.

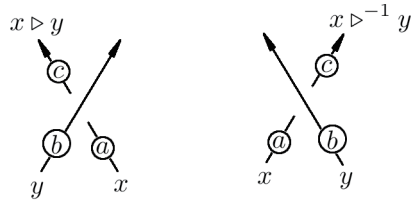
To see the multiplicative structure of $\mathbb{Z}[X] = \Omega[X]/I$, it is helpful to see a geometric interpretation. Let L be a fixed knot or link diagram with a fixed rack labeling by X. We would like to consider secondary labelings of L by elements of the rack algebra $\mathbb{Z}[X]$. In [3] these secondary labels are pictured as "beads" on the strands of

the knot or link. The basic bead-labeling rule is the usual (t, s)-rack rule, with the exception that the beads at a crossing use the specific $t_{x,y}$ and $s_{x,y}$ associated to the rack labels on the right-hand underarc and overarc as illustrated.



$$c = t_{x,y}a + s_{x,y}b$$

REMARK 5. Note that this rule does not make use of the orientation of the understrand. With rack label x on the underarc on the right-hand side when the overarc is oriented upward and rack label y on the overstrand, the rack algebra labels are a on the right-hand underarc, b on the overarc and $c = t_{x,y}a + s_{x,y}b$ on the left-hand overarc.



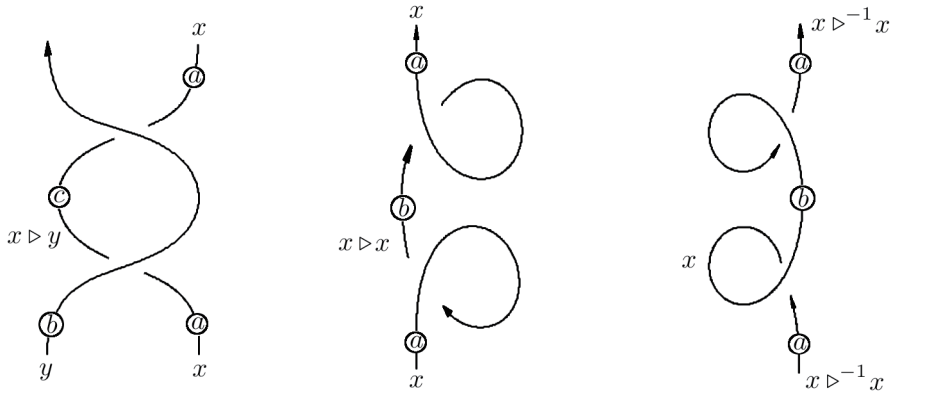
$$c = t_{x,y}a + s_{x,y}b \qquad c = \overline{t_{x,y}}a + \overline{s_{x,y}}b$$

We can also (as in [3]) formulate the rack algebra labeling rule in terms of the inbound arcs at a negative crossing; first we define elements $\bar{t}_{x,y} = t^{-1}_{x\triangleright^{-1}y,y}$ and $\bar{s}_{x,y} = -t^{-1}_{x\triangleright^{-1}y,y}s_{x\triangleright^{-1}y,y}$. Then we can express the label on the outbound underarc at a negative crossing with inbound rack colors x, y as

$$c = \bar{t}_{x,y}a + \bar{s}_{x,y}b = t^{-1}_{x\triangleright^{-1}y,y}a - t^{-1}_{x\triangleright^{-1}y,y}s_{x\triangleright^{-1}y,y}b.$$

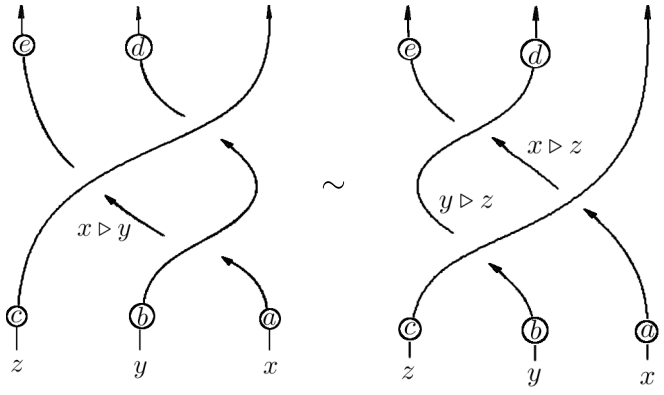
We would like to find the conditions required to make labelings of a rack-labeled diagram by rack algebra elements invariant under rack-labeled blackboard-framed isotopy. We will then define the rack algebra $\mathbb{Z}[X]$ by taking a quotient by the ideal I generated by the elements we must kill to obtain invariance under X-colored blackboard-framed Reidemeister moves. In [1, 3] the special case where X is a quandle, i.e. a rack of rack rank $N(X) = 1$, is considered. To extend this definition to the case of racks with finite rank $N \geq 1$, the Reidemeister moves we need are the standard type II and III moves together with the blackboard-framed type I moves and the N-phone cord move.

The Reidemeister II and blackboard-framed type I moves do not impose any conditions due to our choice of crossing rule:



$$c = t_{x,y}a + s_{x,y}b \qquad b = (t_{x,x} + s_{x,x})a \qquad b = t_{x\triangleright^{-1}x,x}b + s_{x\triangleright^{-1}x,x}a$$

The Reidemeister III move gives us three conditions which we recognize as indexed versions of the (t, s)-rack conditions:



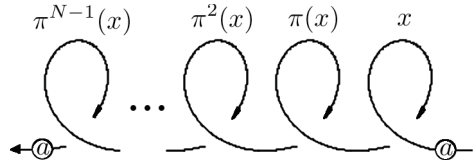
Comparing coefficients on e, we must have

$$t_{x\triangleright y,z}t_{x,y} = t_{x\triangleright z,y\triangleright z}t_{x,z}, \quad t_{x\triangleright y,z}s_{x,y} = s_{x\triangleright z,y\triangleright z}t_{y,z}$$

and

$$s_{x\triangleright y,z} = s_{x\triangleright z,y\triangleright z}s_{y,z} + t_{x\triangleright z,y\triangleright z}s_{x,z}.$$

Finally, for defining counting invariants we will need N-phone cord move compatibility.



Going though the kth kink multiplies our rack algebra element by a factor of $t_{\pi^k(x),\pi^k(x)} + s_{\pi^k(x),\pi^k(x)}$, so N-phone cord invariance requires that the product of these factors must equal 1:

$$\prod_{k=0}^{N-1}(t_{\pi^k(x),\pi^k(x)} + s_{\pi^k(x),\pi^k(x)}) = 1.$$

Thus, we have:

DEFINITION 5. Let X be a rack with rack rank N and let $\Omega[X]$ be the free $\mathbb{Z}$-algebra generated by elements $t_{x,y}^{\pm 1}$ and $s_{x,y}$ where $x, y \in X$. Then the *rack algebra* of X, denoted $\mathbb{Z}[X]$, is $\mathbb{Z}[X] = \Omega[X]/I$ where I is the ideal generated by elements of the form

- $t_{x\triangleright y,z}t_{x,y} - t_{x\triangleright z,y\triangleright z}t_{x,z}$,
- $t_{x\triangleright y,z}s_{x,y} - s_{x\triangleright z,y\triangleright z}t_{y,z}$,
- $s_{x\triangleright y,z} - s_{x\triangleright z,y\triangleright z}s_{y,z} - t_{x\triangleright z,y\triangleright z}s_{x,z}$ and
- $1 - \prod_{k=0}^{N-1}(t_{\pi^k(x),\pi^k(x)} + s_{\pi^k(x),\pi^k(x)})$

A *rack module over* X or a $\mathbb{Z}[X]$-*module* is a representation of $\mathbb{Z}[X]$, i.e., an abelian group G with isomorphisms $t_{x,y}\colon G \to G$ and endomorphisms $s_{x,y}\colon G \to G$ such that each of the above maps is zero.

EXAMPLE 6. Let X be a rack and let Y be a (t, s)-rack. Then Y is a $\mathbb{Z}[X]$-module via $t_{x,y} = t$ and $s_{x,y} = s$ for all x, y if and only if $N(X) = N(Y)$.

REMARK 7. Let X be a rack of rack rank N and let G be an abelian group with isomorphisms $t_{x,y}\colon G \to G$ and endomorphisms $s_{x,y}\colon G \to G$. The rack module condition is equivalent to the condition that the cartesian product $X \times G$ is a rack of rack rank N under the operation

$$(x, g) \triangleright (y, h) = (x \triangleright y, t_{x,y}(g) + s_{x,y}(h)).$$

Such a rack is usually known as an *extension rack* of X.

For the purpose of enhancing the rack counting invariant, we want to find examples of finite rack modules. Probably the simplest way to achieve this is the following: let X be a finite rack and let R be a finite ring with identity, e.g. $\mathbb{Z}_n$ or, for a

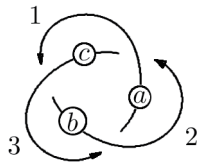
non-commutative example, the ring $M_m(\mathbb{Z}_n)$ of $m \times m$ matrices over $\mathbb{Z}_n$. We can give any R-module V the structure of a $\mathbb{Z}[X]$-module by choosing elements $t_{x,y} \in R^*$ and $s_{x,y} \in R$ satisfying the rack module relations for all $x, y \in X$.

We can express a $\mathbb{Z}[X]$-module structure on R conveniently by giving a block matrix $M_R = [M_t | M_s]$ with block matrices M_t and M_s indexed by elements of $X = \{x_1, \ldots, x_n\}$, i.e. the elements in row i column j of M_t and M_s respectively are t_{x_i,y_j} and s_{x_i,y_j}.

EXAMPLE 8. Let X be the constant action rack on the set $\{1,2\}$ with permutation $\sigma = (12)$, let $R = \mathbb{Z}_3$ and set $s_{11} = s_{22} = 1$, $s_{12} = s_{21} = 2$ and $t_{11} = t_{12} = t_{21} = t_{22} = 1$. It is straightforward to verify that the rack algebra relations are satisfied, and thus we have a finite $\mathbb{Z}[X]$-module structure on $\mathbb{Z}_3$ with rack module matrix given by

$$M_R = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{array}\right].$$

EXAMPLE 9. Let L be a fixed oriented link diagram with arcs $a_1, \ldots, a_n$ and let f be a rack labeling of L by X. We can regard $f \in \mathrm{Hom}(FR(L), X)$ as a rack homomorphism from the fundamental rack of L into X or as a link diagram with a fixed rack labeling. Such a labeling determines a rack module $\mathbb{Z}[f] = \mathbb{Z}[X][a_1, \ldots, a_n]/C$ where C is the submodule determined by the crossing relations at the crossings of L. For instance, the trefoil knot diagram below with the pictured rack coloring defines a rack module with the listed presentation matrix $M_{\mathbb{Z}[f]}$.



$$M_{\mathbb{Z}[f]} = \left[\begin{array}{ccc} t_{13} & -1 & s_{13} \\ -1 & s_{32} & t_{32} \\ s_{21} & t_{21} & -1 \end{array}\right].$$

REMARK 10. We note that in the case where X is the trivial quandle with one element, we have $t_{1,1} = t$, $s_{1,1} = s$ and $(t + s) = 1$ implies $s = 1 - t$, and $M_{\mathbb{Z}[f]}$ is a presentation matrix for the Alexander quandle of L.

4. Rack module enhancements of the counting invariant

In this section we use finite rack modules to enhance the rack counting invariant.

Let X be a finite rack with rack rank N, L an oriented link of c ordered components, and $W = (\mathbb{Z}_N)^c$ the set of writhe vectors of L mod N. Recall from Section 2 that the *integral rack counting invariant* of L with respect to X is the sum

$$\Phi_X^{\mathbb{Z}}(L) = \sum_{\mathbf{w} \in W} |\mathrm{Hom}(FR(L, \mathbf{w}), X)|$$

counting the total number of rack labelings of L by X over a complete period of writhes mod N.

Now let R be a $\mathbb{Z}[X]$-module. For each rack labeling $f \in \mathrm{Hom}(FR(L, \mathbf{w}), X)$ of L by X, we can ask how many associated R-labelings of the diagram there are. Such a labeling is a homomorphism of $\mathbb{Z}[X]$-modules $\phi\colon \mathbb{Z}[f] \to R$. We can find the set of all such labelings from the presentation matrix of $\mathbb{Z}[f]$ by substituting in the values of $t_{x,y}$ and $s_{x,y}$ in R and finding the solution space of the resulting homogeneous system of linear equations in R. Denote this set by $\mathrm{Hom}(\mathbb{Z}[f], R)$.

By construction, the set $\mathrm{Hom}(\mathbb{Z}[f], R)$ of R-labelings of f is invariant under X-colored blackboard-framed Reidemeister moves and N-phone cord moves, and thus gives us an invariant signature of the X-labeling. The multiset of such signatures over the set of all X-labelings $\{\mathrm{Hom}(FR(L, \mathbf{w}), X)\colon \mathbf{w} \in W\}$ gives us a natural enhancement of the rack counting invariant.

DEFINITION 6. Let X be a finite rack with rack rank N, L an oriented link of c ordered components, $W = (\mathbb{Z}_N)^c$ and R a finite $\mathbb{Z}[X]$-module. The *rack module enhanced multiset* of L with respect to X and R is the multiset

$$\Phi^M_{X,R}(L) = \{\mathrm{Hom}(\mathbb{Z}[f], R)\colon\ f \in \mathrm{Hom}(FR(L, \mathbf{w}), X),\ \mathbf{w} \in W\}.$$

For ease of comparison, we can take the cardinality of each of the sets of rack module homomorphisms to get a multiset of integers, the *rack module enhanced counting multiset*

$$\Phi^{M,\mathbb{Z}}_{X,R}(L) = \{|\mathrm{Hom}(\mathbb{Z}[f], R)|\colon\ f \in \mathrm{Hom}(FR(L, \mathbf{w}), X),\ \mathbf{w} \in W\}.$$

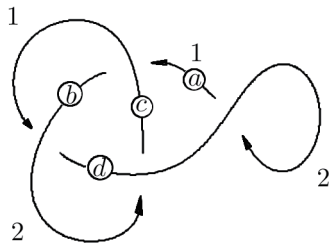
The generating function of $\Phi^{M,\mathbb{Z}}_{X,R}(L)$ is a polynomial link invariant, the *rack module enhanced counting invariant*

$$\Phi_{X,R}(L) = \sum_{\mathbf{w}\in W}\left(\sum_{f\in \mathrm{Hom}(FR(L,\mathbf{w}),X)} u^{|\mathrm{Hom}(\mathbb{Z}[f],R)|}\right).$$

Theorem 1. *For any finite rack X and rack module R, $\Phi^M_{X,R}(L)$, $\Phi_{X,R}(L)$ and $\Phi_{X,R}(L)$ are invariants of classical and virtual knots and links.*

EXAMPLE 11. Let X and R be the rack and rack module in Example 8. Every knot has exactly two labelings by X in even writhe and zero labelings in odd writhe, so the integral rack counting invariant for any knot is $\Phi^{\mathbb{Z}}_X(K) = 2$. However, this smallest possible non-quandle rack with a choice of $\mathbb{Z}[X]$-module structure on $\mathbb{Z}_3$ nevertheless detects the difference between the two smallest non-trivial knots, the trefoil 3_1 and the figure-eight knot 4_1.

The X-colored trefoil knot below has the listed $\mathbb{Z}[f]$ presentation matrix.

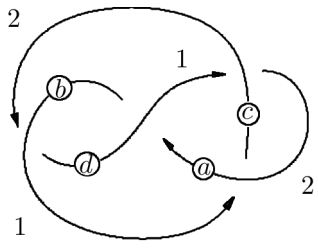


$$M_{\mathbb{Z}[f]} = \begin{bmatrix} -1 & 0 & 0 & t_{2,2}+s_{2,2} \\ t_{1,1} & -1 & s_{1,1} & 0 \\ 0 & s_{1,2} & t_{1,2} & -1 \\ 0 & t_{2,2} & -1 & s_{2,2} \end{bmatrix}.$$

Substituting in the values of $t_{x,y}$ and $s_{x,y}$ in R and thinking of the matrix as a coefficient matrix for a homogeneous system, we find there are 9 total rack algebra labelings; note that the symmetry in the indices in M_R implies that the other X-labeling yields the same number of R-labelings, for an invariant value of $\phi_{X,R}(3_1) = 2u^9$

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Comparing this with the X-colored figure eight knot 4_1, we find that the space of $\mathbb{Z}[X]$-colorings has only three elements:



$$M_{\mathbb{Z}[f]} = \begin{bmatrix} t_{2,2} & 0 & s_{2,2} & -1 \\ t_{2,1} & -1 & 0 & s_{2,1} \\ 0 & s_{2,1} & t_{2,1} & -1 \\ s_{2,2} & -1 & t_{2,2} & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ 0 & 2 & 1 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, we have $\Phi_{X,R}(4_1) = 2u^3 \neq 2u^9 = \Phi_{X,R}(3_1)$, and the rack module enhanced invariant is strictly stronger than the rack counting invariant.

REMARK 12. In the special case when R is a commutative ring and $t_{x,y}$ and $s_{x,y}$ are elements of R, we do not need R to be finite in order to have a well-defined computable enhanced invariant. Replacing the the number of R-labelings with the dimension of the space of R-labelings yields a computable invariant we might call the

dimension-enhanced rack module counting invariant:

$$\Phi^{\dim}_{X,R}(L) = \sum_{\mathbf{w}\in W}\left(\sum_{f\in \mathrm{Hom}(FR(L,\mathbf{w}),X)} u^{\dim(\mathrm{Hom}(\mathbb{Z}[f],R))}\right).$$

Indeed, if R is a PID, $\Phi^{\dim}_{X,R}$ carries the same information as $\Phi_{X,R}$, though if R has torsion then $\Phi_{X,R}$ will generally contain more information.

REMARK 13. Let X be a rack and R a finite $\mathbb{Z}[X]$-module. As noted in Remark 7, the cartesian product $X \times R$ is a rack under

$$(x, a) \triangleright (y, b) = (x \triangleright y, t_{x,y}a + s_{x,y}b).$$

The rack module enhanced counting invariant $\Phi_{X,R}$ is then related to the integral rack counting invariant with respect to the extension rack $X \times R$ by

$$\Phi^{\mathbb{Z}}_{X\times R}(L) = \sum_{H\in\Phi^{M}_{X,R}(L)} |H|,$$

i.e., $\Phi^{\mathbb{Z}}_{X\times R}(L)$ is the sum of the exponents in the terms of $\Phi_{X,R}$. In particular, we can understand $\Phi_{X,R}$ as an enhancement of $\Phi^{\mathbb{Z}}_{X\times R}$ defined by collecting together the labelings of L by the extension rack $X \times R$ which project to the same rack labeling of L by X.

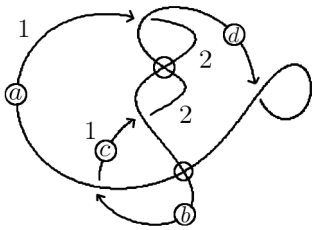
REMARK 14. Note that all of the preceding extends to virtual knots and links in the usual way, i.e., by ignoring any virtual crossings. See [6] for more.

5. Computations and Examples

In this section we collect a few examples of the rack module enhanced invariant.

EXAMPLE 15. Let X and R be the rack and rack module used in Examples 8 and 11. The virtual knots labeled 3_1 and 3_7 in the knot atlas [2] both have a Jones polynomial of 1. Since the rack X has a rack rank of 2 and each knot has writhe 1, we must also account for colorings of each knot with the addition of a positive kink. In fact, the only valid colorings of both virtual knots 3_1 and 3_7 by the rack are the colorings of the knots with even writhe. One possible X-coloring of the virtual knot

3_1 with its presentation matrix is:



$$M_{\mathbb{Z}[f]} = \begin{bmatrix} t_{2,2}^{-1} & -1 & 0 & -t_{2,2}^{-1}s_{2,2} \\ 0 & s_{1,2} & t_{1,2} & -1 \\ -t_{1,1}^{-1}s_{1,1} & t_{1,1}^{-1} & -1 & 0 \\ s_{2,1}-1 & 0 & 0 & t_{2,1} \end{bmatrix}.$$

Substituting the values from the rack module R yields a matrix that may be reduced as follows:

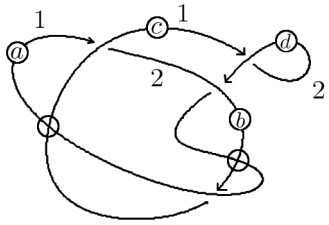
$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 2 & 1 & 2 \\ 2 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This row reduced matrix shows that there are three possible rack module colorings. In addition, the other possible coloring of this knot is symmetric to this coloring so the 3_1 virtual knot has a rack module enhanced counting invariant of $2u^3$.

In contrast, the 3_7 virtual knot has presentation matrix

$$M_{\mathbb{Z}[f]} = \begin{bmatrix} t_{2,1}^{-1} & -1 & -t_{2,1}^{-1}s_{2,1} & 0 \\ -1 & -t_{1,2}^{-1}s_{1,2} & 0 & t_{1,2}^{-1} \\ s_{2,1} & t_{2,1} & -1 & 0 \\ 0 & 0 & t_{1,2} & s_{1,2}-1 \end{bmatrix}$$

derived from the coloring:

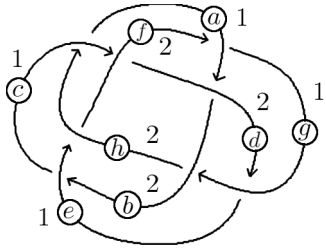


The module values yield a presentation matrix that may be reduced as follows:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix indicates that the 3_7 virtual knot with this coloring has 9 rack module colorings. The 3_7 knot also has one additional rack coloring, which is symmetric to the given coloring, so the rack module enhanced invariant for the 3_7 virtual knot is $2u^9$. Both 3_1 and 3_7 have a Jones polynomial of 1, yet their rack module enhanced counting invariants are different, $2u^3 \neq 2u^9$, which shows that the the rack module enhanced invariant is not determined by the Jones polynomial.

EXAMPLE 16. The rack module enhanced counting invariant is also not determined by the Alexander polynomial. Knots 8_{18} and 9_{24} both have the same Alexander polynomial, $-t^3+5t^2-10t+13-10t^{-1}+5t^{-2}-t^{-3}$. Using the two element constant action rack and module used in Examples 8, 11, and 15, the knot 8_{18} has two rack colorings, one of which is illustrated below.



The presentation matrix of 8_{18} is:

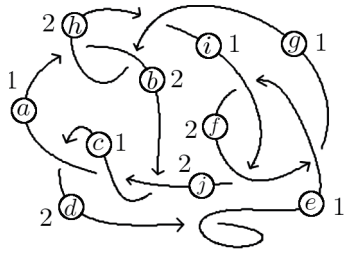
$$M_{\mathbb{Z}[f]} = \begin{bmatrix} s_{2,1} & 0 & 0 & 0 & 0 & t_{2,1} & -1 & 0 \\ t_{2,2}^{-1} & -1 & 0 & -t_{2,2}^{-1}s_{2,2} & 0 & 0 & 0 & 0 \\ 0 & -t_{2,2}^{-1}s_{2,2} & 0 & 0 & 0 & 0 & t_{2,2}^{-1} & -1 \\ 0 & t_{2,1} & -1 & 0 & s_{2,1} & 0 & 0 & 0 \\ -1 & 0 & s_{2,1} & 0 & 0 & 0 & 0 & t_{2,1} \\ 0 & 0 & t_{2,2}^{-1} & -1 & 0 & -t_{2,2}^{-1}s_{2,2} & 0 & 0 \\ 0 & 0 & 0 & t_{2,1} & -1 & 0 & s_{2,1} & 0 \\ 0 & 0 & 0 & 0 & t_{2,2}^{-1} & -1 & 0 & -t_{2,2}^{-1}s_{2,2} \end{bmatrix}$$

which can be reduced as shown after substituting in values from the module:

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix indicates that there are twenty-seven rack module colorings of the knot 8_{18}, yielding an invariant value of $2u^{27}$ since there is one additional rack coloring due to the symmetry of the rack and rack module used.

The knot 9_{24} requires a writhe adjustment by adding one positive kink in order to have a valid coloring by the rack, which yields the knot with presentation matrix shown below.



$$M_{\mathbb{Z}[f]} = \begin{bmatrix} t_{2,2}^{-1} & -1 & 0 & 0 & 0 & 0 & 0 & -t_{2,2}^{-1}s_{2,2} & 0 & 0 \\ 0 & -t_{2,2}^{-1}s_{2,2} & 0 & 0 & 0 & 0 & t_{2,2}^{-1} & -1 & 0 & 0 \\ 0 & t_{2,2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & s_{2,2} \\ -1 & 0 & s_{2,1} & 0 & 0 & 0 & 0 & 0 & 0 & s_{2,1} \\ s_{1,1} & 0 & t_{1,1} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{1,1}^{-1} & -t_{1,1}^{-1}s_{1,1}-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t_{1,1}^{-1}s_{1,1} & t_{1,1}^{-1} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{2,1}^{-1} & -1 & 0 & 0 & -t_{2,1}^{-1}s_{2,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -t_{2,2}^{-1}s_{2,2} & 0 & 0 & t_{2,2}^{-1} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_{2,1} & t_{2,1} & -1 & 0 \end{bmatrix}.$$

After substituting in the values from the rack module, the matrix can be reduced to the form:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The reduced matrix shows that there are nine colorings of this knot by the rack module. The other coloring of the knot by the rack yields an additional nine colorings so the

rack module enhanced invariant for the knot 9_{24} is $2u^9$. Since $2u^{27} \neq 2u^9$, the rack module enhanced counting invariant is not determined by the Alexander polynomial.

Our next examples give a sense of the effectiveness of some examples of $\Phi_{X,R}$ using racks X and rack modules $R = \mathbb{Z}_n$ chosen at random from the results of our computer searches.

EXAMPLE 17. Let X be the quandle with quandle matrix given by

$$M_X = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Via computations in python, we selected a $\mathbb{Z}[X]$-module structure on $R = \mathbb{Z}_5$ given by the rack module matrix

$$M_R = \left[\begin{array}{ccc|ccc} 4 & 2 & 3 & 2 & 1 & 1 \\ 1 & 4 & 2 & 2 & 2 & 3 \\ 1 & 3 & 4 & 3 & 2 & 2 \end{array}\right]$$

and computed the knot invariant for all prime knots with up to 8 crossings, all prime links with up to 7 crossings as listed in the knot atlas [2], the unknot U, the square knot SK and granny knot GK, and unlinks of two and three components U_2 and U_3. The results are collected in the table below.

$\Phi_{X,M}$	L
$3u^5$	$U, 5_2, 6_2, 6_3, 7_1, 7_2, 7_3, 7_5, 7_6, 8_1, 8_2, 8_3, 8_4, 8_6, 8_7, 8_{12}, 8_{13}, 8_{14}, 8_{17}$
	$L2a1, L4a1, L5a1, L6a4, L6n1, L7a3, L7a4, L7a6, L7n1, L7n2$
$3u^5 + 6u^{25}$	$3_1, 6_1, 7_7, 8_5, 8_{10}, 8_{11}, 8_{15}, 8_{19}, 8_{20}, L6a1, L6a3, L6a5, L7a1, L7a5$
$3u^5 + 12u^{25} + 12u^{125}$	SK, GK
$3u^{25}$	$4_1, 5_1, 8_8, 8_9, 8_{16}, L6a2, L7a2, L7a7$
$9u^{25}$	$7_4, U_2$
$27u^{25}$	$8_{18}, 8_{21}, 9_2, 9_{24}, U_3$

EXAMPLE 18. Let X be the rack with rack matrix given by

$$M_X = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}.$$

Via computations in python, we selected a $\mathbb{Z}[X]$-module structure on $R = \mathbb{Z}_5$ given by

$$M_R = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & 1 & 1 & 0 & 0 & 4 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

and computed the knot invariant for the same set of knots and link as in Example 17. For all of the knots in our list, the invariant value is $\Phi_{X,M} = 4u^5$. Our results for links are collected in the table below.

$\Phi_{X,M}$	L
$8u^5 + 8u^{25}$	$L2a1$, $L4a1$, $L6a1$, $L6a2$, $L6a3$, $L7a2$, $L7a5$, $L7a6$, $L7n1$
$16u^{25}$	U_2, $L5a1$, $L7a1$, $L7a3$, $L7a4$, $L7n2$
$48u^{25} + 16u^{125}$	$L6a4$, $L6a5$, $L6n1$, $L7a7$
$64u^{125}$	U_3

Our `python` and `C` code for computing rack modules and their invariants is available for download at `www.esotericka.org`.

6. Open questions

In this section we collect a few questions for future investigation.

- If X is a (t, s)-rack with rack rank $N = 1$, i.e. an Alexander quandle, then X is a rack module over the trivial quandle with one element X' with $t_{1,1} = t$ and $s_{1,1} = 1 - t$. In this scenario, the integral rack counting invariant $\Phi_X^{\mathbb{Z}}(L)$ is related to the rack module enhanced counting invariant $\Phi_{X',X}$ by

$$\Phi_{X',X}(L) = u^{\Phi_X^{\mathbb{Z}}(L)}.$$

How does this relationship generalize when X' is a larger (t, s)-rack?

- Other enhancements of the rack counting invariant using rack modules are possible; we have only scratched the surface with the most basic enhancement. What other enhancements of $\Phi_X^{\mathbb{Z}}$ using rack modules are possible?
- What enhancements of $\Phi_{X,M}$ are possible?
- What approach might we use to categorify these invariants?
- How is $\Phi_{X,M}$ related to other knot and link invariants?
- How do rack modules extend to the case of virtual racks, i.e. racks with a nontrivial action at virtual crossings?

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