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PATHWISE UNIQUENESS FOR SINGULAR SDEs DRIVEN BY STABLE PROCESSES

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Abstract

We prove pathwise uniqueness for stochastic differential equations driven by non-degenerate symmetric α-stable Lévy processes with values in $\mathbb{R}^d$ having a bounded and β-Hölder continuous drift term. We assume $\beta > 1 - \alpha/2$ and $\alpha \in [1, 2)$. The proof requires analytic regularity results for the associated integro-differential operators of Kolmogorov type. We also study differentiability of solutions with respect to initial conditions and the homeomorphism property.

1. Introduction

In this paper we prove a pathwise uniqueness result for the following SDE

$$X_t = x + \int_0^t b(X_s)\,ds + L_t, \quad x \in \mathbb{R}^d,\ t \geq 0, \tag{1.1}$$

where $b\colon \mathbb{R}^d \to \mathbb{R}^d$ is bounded and β-Hölder continuous and $L = (L_t)$ is a non-degenerate d-dimensional symmetric α-stable Lévy process ($L_0 = 0$, P-a.s.) and $d \geq 1$.

Currently, there is a great interest in understanding pathwise uniqueness for SDEs when b is not Lipschitz continuous or, more generally, when b is singular enough so that the corresponding deterministic equation (1.1) with $L = 0$ is not well-posed. A remarkable result in this direction was proved by Veretennikov in [25] (see also [28] for $d = 1$). He was able to prove uniqueness when b: $\mathbb{R}^d \to \mathbb{R}^d$ is only Borel and bounded and L is a standard d-dimensional Wiener process. This result has been generalized in various directions in [9], [13], [27], [6], [7], [5], [8].

The situation changes when L is not a Wiener process but is a symmetric α-stable process, $\alpha \in (0, 2)$. Indeed, when $d = 1$ and $\alpha < 1$, Tanaka, Tsuchiya and Watanabe prove in [24, Theorem 3.2] that even a bounded and β-Hölder continuous b is not enough to ensure pathwise uniqueness if $\alpha + \beta < 1$ (they consider drifts like $b(x) = \operatorname{sign}(x)(|x|^\beta \wedge 1)$ and initial condition $x = 0$). On the other hand, when $d = 1$ and $\alpha \geq 1$, they show pathwise uniqueness for any continuous and bounded b.

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In this paper we prove pathwise uniqueness in any dimension $d \geq 1$, assuming that $\alpha \geq 1$ and b is bounded and β-Hölder continuous with $\beta > 1-\alpha/2$. Our proof is different from the one in [24] and is inspired by [7]. The assumptions on the α-stable Lévy process L which we consider are collected in Section 2 (see in particular Hypothesis 1). Here we only mention two significant examples which satisfy our hypotheses. The first is when $L = (L_t)$ is a standard α-stable process (symmetric and rotationally invariant), i.e., the characteristic function of the random variable L_t is

$$E[e^{i\langle L_t, u\rangle}] = e^{-tc_\alpha |u|^\alpha}, \quad u \in \mathbb{R}^d,\ t \geq 0, \tag{1.2}$$

where c_α is a positive constant. The second example is $L = (L_t^1, \dots, L_t^d)$, where $L^1, \dots, L^d$ are independent one-dimensional symmetric stable processes of index α. In this case

$$E[e^{i\langle L_t, u\rangle}] = e^{-tk_\alpha(|u_1|^\alpha + \cdots + |u_d|^\alpha)}, \quad u \in \mathbb{R}^d,\ t \geq 0, \tag{1.3}$$

where k_α is a positive constant. Martingale problems for SDEs driven by $(L_t^1, \dots, L_t^d)$ have been recently studied (see [3] and references therein).

We prove the following result.

Theorem 1.1. *Let L be a symmetric α-stable process with $\alpha \in [1, 2)$, satisfying* Hypothesis 1 (*see* Section 2). *Assume that $b \in C_b^\beta(\mathbb{R}^d; \mathbb{R}^d)$ for some $\beta \in (0, 1)$ such that*

$$\beta > 1 - \frac{\alpha}{2}.$$

Then pathwise uniqueness holds for equation (1.1). *Moreover, if $X^x = (X_t^x)$ denotes the solution starting at $x \in \mathbb{R}^d$, we have*:

(i) *for any $t \geq 0$, $p \geq 1$, there exists a constant $C(t, p) > 0$ (depending also on α, β and $L = (L_t)$) such that*

$$E\Big[\sup_{0 \le s \le t} |X_s^x - X_s^y|^p\Big] \leq C(t, p)|x - y|^p, \quad x, y \in \mathbb{R}^d; \tag{1.4}$$

(ii) *for any $t \geq 0$, the mapping: $x \mapsto X_t^x$ is a homeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$, P-a.s.*;

(iii) *for any $t \geq 0$, the mapping: $x \mapsto X_t^x$ is a C^1-function on $\mathbb{R}^d$, P-a.s.*

All these assertions require that L is non-degenerate. Estimate (1.4) replaces the standard Lipschitz-estimate which holds without expectation E when b is Lipschitz continuous. Assertion (ii) is the so-called homeomorphism property of solutions (we refer to [1], [19] and [14]; see also [20] for the case of Log-Lipschitz coefficients).

Note that existence of strong solutions for (1.1) follows easily by a compactness argument (see the comment before Lemma 4.1). On the other hand, existence of weak solutions when b is only measurable and bounded is proved in [15]. Since $C_b^{\beta'}(\mathbb{R}^d, \mathbb{R}^d) \subset C_b^{\beta}(\mathbb{R}^d, \mathbb{R}^d)$ when $0 < \beta \leq \beta'$, our uniqueness result holds true for any $\alpha \geq 1$ when $\beta \in (1/2, 1)$. Theorem 1.1 implies the existence of a stochastic flow (see Remark 4.4).

The proof of the main result is given in Section 4. As in [7] our method is based on an Itô–Tanaka trick which requires suitable analytic regularity results. Such results are proved in Section 3. They provide global Schauder estimates for the following resolvent equation on $\mathbb{R}^d$

$$\lambda u - \mathcal{L}u - b \cdot Du = g, \tag{1.5}$$

where $\lambda > 0$ and $g \in C_b^{\beta}(\mathbb{R}^d)$ are given and we assume $\alpha \geq 1$ and $\alpha + \beta > 1$. Here $\mathcal{L}$ is the generator of the Lévy process L (see (2.5), [1] and [22]). If L satisfies (1.2) then $\mathcal{L}$ coincides with the fractional Laplacian $-(-\triangle)^{\alpha/2}$ on infinitely differentiable functions f with compact support (see [22, Example 32.7]), i.e., for any $x \in \mathbb{R}^d$,

$$-(-\triangle)^{\alpha/2} f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x) - 1_{\{|y| \leq 1\}} y \cdot Df(x)) \frac{\tilde{c}_\alpha}{|y|^{d+\alpha}} \, dy. \tag{1.6}$$

It is simpler to prove Schauder estimates for (1.5) when $\alpha > 1$. In such a case, assuming in addition that $\mathcal{L} = -(-\triangle)^{\alpha/2}$, i.e., L is a standard α-stable process, these estimates can be deduced from the theory of fractional powers of sectorial operators (see [16]). We also mention [2, Section 7.3] where Schauder estimates are proved when $\alpha > 1$ and $\mathcal{L}$ has the form (1.6) but with variable coefficients, i.e., $\tilde{c}_\alpha = \tilde{c}_\alpha(x, y)$. The limit case $\alpha = 1$ in (1.5) requires a special attention even for the fractional Laplacian $\mathcal{L} = -(-\triangle)^{1/2}$. Indeed in this case $\mathcal{L}$ is of the "same order" of $b \cdot D$. To treat $\alpha = 1$, we use a localization procedure which is based on Theorem 3.3 where Schauder estimates are proved in the case of $b(x) = k$, for any $x \in \mathbb{R}^d$, showing that the Schauder constant is independent of k (the case $\alpha < 1$ is discussed in Remark 3.5).

In order to prove Theorem 1.1, in Section 4 we apply Itô's formula to $u(X_t)$, where $u \in C_b^{\alpha+\beta}$ comes from Schauder estimates for (1.5) when $g = b$ (in such case (1.5) must be understood componentwise). This is needed to perform the Itô–Tanaka trick and find a new equation for X_t in which the singular term $\int_0^t b(X_s)\, ds$ of (1.1) is replaced by more regular terms. Then uniqueness and (1.4) follow by L^p-estimates for stochastic integrals. Such estimates require Lemma 4.1 and the condition $\alpha/2 + \beta > 1$. In addition, properties (ii) and (iii) are obtained transforming (1.1) into a form suitable for applying the results in [14].

We will use the letter c or C with subscripts for finite positive constants whose precise value is unimportant; the constants may change from proposition to proposition.

2. Preliminaries and notation

General references for this section are [1], [21, Chapter 2], [22] and [26].

Let $\langle u, v\rangle$ (or $u \cdot v$) be the euclidean inner product between u and $v \in \mathbb{R}^d$, for any $d \geq 1$; moreover $|u| = \langle u, u\rangle^{1/2}$. If $D \subset \mathbb{R}^d$ we denote by 1_D the indicator function of D. The Borel σ-algebra of $\mathbb{R}^d$ will be indicated by $\mathcal{B}(\mathbb{R}^d)$. All the measures considered in the sequel will be positive and Borel. A measure γ on $\mathbb{R}^d$ is called symmetric if $\gamma(D) = \gamma(-D)$, $D \in \mathcal{B}(\mathbb{R}^d)$.

Let us fix $\alpha \in (0, 2)$. In (1.1) we consider a d-dimensional *symmetric α-stable process* $L = (L_t)$, $d \geq 1$, defined on a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and $\mathcal{F}_t$-adapted; the stochastic basis satisfies the usual assumptions (see [1, p. 72]). Recall that L is a Lévy process (i.e., it is continuous in probability, it has stationary increments, càdlàg trajectories, $L_t - L_s$ is independent of $\mathcal{F}_s$, $0 \leq s \leq t$, and $L_0 = 0$) with the additional property that the characteristic function of L_t verifies

$$E[e^{i\langle L_t,u\rangle}] = e^{-t\psi(u)}, \quad \psi(u) = -\int_{\mathbb{R}^d} (e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle 1_{\{|y|\leq 1\}}(y))\nu(dy), \tag{2.1}$$

$u \in \mathbb{R}^d$, $t \geq 0$, where ν is a measure such that

$$\nu(D) = \int_{\mathbb{S}} \mu(d\xi) \int_0^\infty 1_D(r\xi)\frac{dr}{r^{1+\alpha}}, \quad D \in \mathcal{B}(\mathbb{R}^d), \tag{2.2}$$

for some symmetric, non-zero finite measure μ concentrated on the unitary sphere $\mathbb{S} = \{y \in \mathbb{R}^d : |y| = 1\}$ (see [22, Theorem 14.3]).

The measure ν is called the Lévy (intensity) measure of L and (2.1) is the Lévy–Khintchine formula. The measure ν is a σ-finite measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |y|^2)\nu(dy) < \infty$, with $1 \wedge |\cdot| = \min(1, |\cdot|)$. Formula (2.2) implies that (2.1) can be rewritten as

$$\begin{aligned}\psi(u) &= -\int_{\mathbb{R}^d} (\cos(\langle u, y\rangle) - 1)\nu(dy)\\ &= -\int_{\mathbb{S}} \mu(d\xi) \int_0^\infty \frac{\cos(\langle u, r\xi\rangle) - 1}{r^{1+\alpha}}\, dr = c_\alpha \int_{\mathbb{S}} |\langle u, \xi\rangle|^\alpha \mu(d\xi), \quad u \in \mathbb{R}^d\end{aligned} \tag{2.3}$$

(see also [22, Theorem 14.13]). The measure μ is called the spectral measure of the stable process L. In this paper we make the following non-degeneracy assumption (cf. [23] and [22, Definition 24.16]).

HYPOTHESIS 1. The support of the spectral measure μ is not contained in a proper linear subspace of $\mathbb{R}^d$.

It is not difficult to show that Hypothesis 1 is equivalent to the following assertion: there exists a positive constant C_α such that, for any $u \in \mathbb{R}^d$,

$$\psi(u) \geq C_\alpha |u|^\alpha. \tag{2.4}$$

Condition (2.4) is also assumed in [11, Proposition 2.1]. To see that (2.4) implies Hypothesis 1, we argue by contradiction: if $\mathrm{Supp}(\mu) \subset (M \cap \mathbb{S})$ where M is the hyperplane containing all vectors orthogonal to some $u_0 \neq 0$, then $\psi(u_0) = 0$. To show the converse, note that Hypothesis 1 implies that for any $v \in \mathbb{R}^d$ with $|v| = 1$, we have $\psi(v) > 0$ (indeed, otherwise, we would have $\mu(\{\xi \in \mathbb{S} \colon |\langle v, \xi\rangle| > 0\}) = 0$ and so $\mathrm{Supp}(\mu) \subset \{\xi \in \mathbb{S} \colon \langle v, \xi\rangle = 0\}$ which contradicts the hypothesis). By using a compactness argument, we deduce that (2.4) holds for any $u \in \mathbb{R}^d$ with $|u| = 1$. Then, writing, for any $u \in \mathbb{R}^d$, $u \neq 0$, $\int_{\mathbb{S}} |\langle u, \xi\rangle|^\alpha \mu(d\xi) = |u|^\alpha \int_{\mathbb{S}} |\langle u/|u|, \xi\rangle|^\alpha \mu(d\xi)$, we obtain easily (2.4).

The infinitesimal generator $\mathcal{L}$ of the process L is given by

$$\mathcal{L} f(x) = \int_{\mathbb{R}^d} \big(f(x+y) - f(x) - 1_{\{|y|\le 1\}} \langle y, Df(x)\rangle\big)\nu(dy), \quad f \in C_c^\infty(\mathbb{R}^d), \tag{2.5}$$

where $C_c^\infty(\mathbb{R}^d)$ is the space of all infinitely differentiable functions with compact support (see [1, Section 6.7] and [22, Section 31]). Let us consider the two examples of α-stable processes mentioned in Introduction which satisfy Hypothesis 1. The first is when L is a standard α-stable process, i.e., $\psi(u) = c_\alpha |u|^\alpha$. In this case ν has density $C_\alpha/|x|^{d+\alpha}$ with respect to the Lebesgue measure in $\mathbb{R}^d$. Moreover the spectral measure μ is the normalized surface measure on $\mathbb{S}$ (i.e., μ gives a uniform distribution on $\mathbb{S}$; see [21, Section 2.5] and [22, Theorem 14.14]).

The second example is $L = (L_t^1, \dots, L_t^d)$, see (1.3). In this case $\psi(u) = k_\alpha(|u_1|^\alpha + \dots + |u_d|^\alpha)$ and the Lévy measure ν is more singular since it is concentrated on the union of the coordinates axes, i.e., ν has density

$$c_\alpha \left(1_{\{x_2=0,\dots,x_d=0\}} \frac{1}{|x_1|^{1+\alpha}} + \dots + 1_{\{x_1=0,\dots,x_{d-1}=0\}} \frac{1}{|x_d|^{1+\alpha}} \right)$$

with respect to the Lebesgue measure. The spectral measure μ is a linear combination of Dirac measures, i.e. $\mu = \sum_{k=1}^d (\delta_{e_k} + \delta_{-e_k})$, where (e_k) is the canonical basis in $\mathbb{R}^d$. The generator is

$$\mathcal{L} f(x) = \sum_{k=1}^d \int_{\mathbb{R}} [f(x+se_k) - f(x) - 1_{\{|s|\le 1\}} s\, \partial_{x_k} f(x)] \frac{c_\alpha}{|s|^{1+\alpha}}\, ds, \quad f \in C_c^\infty(\mathbb{R}^d).$$

Let us fix some notation on function spaces. We define $C_b(\mathbb{R}^d; \mathbb{R}^k)$, for integers $k,d \ge 1$, as the set of all functions $f \colon \mathbb{R}^d \to \mathbb{R}^k$ which are bounded and continuous. It is a Banach space endowed with the supremum norm $\|f\|_0 = \sup_{x\in\mathbb{R}^d} |f(x)|$, $f \in C_b(\mathbb{R}^d; \mathbb{R}^k)$. Moreover, $C_b^\beta(\mathbb{R}^d; \mathbb{R}^k)$, $\beta \in (0,1)$, is the subspace of all β-Hölder continuous functions f, i.e., f verifies

$$[f]_\beta := \sup_{x,y\in\mathbb{R}^d x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty. \tag{2.6}$$

$C_b^\beta(\mathbb{R}^d; \mathbb{R}^k)$ is a Banach space with the norm $\|\cdot\|_\beta = \|\cdot\|_0 + [\,\cdot\,]_\beta$. If $k = 1$, we set $C_b^\beta(\mathbb{R}^d; \mathbb{R}^k) = C_b^\beta(\mathbb{R}^d)$. Let $C_b^0(\mathbb{R}^d, \mathbb{R}^k) = C_b(\mathbb{R}^d, \mathbb{R}^k)$ and $[\,\cdot\,]_0 = \|\cdot\|_0$. For any $n \geq 1$, $\alpha \in [0, 1)$, we say that $f \in C_b^{n+\alpha}(\mathbb{R}^d)$ if $f \in C^n(\mathbb{R}^d) \cap C_b^\alpha(\mathbb{R}^d)$ and, for all $j = 1, \dots, n$, the (Fréchet) derivatives $D^j f \in C_b^\alpha(\mathbb{R}^d; (\mathbb{R}^d)^{\otimes(j)})$. The space $C_b^{n+\alpha}(\mathbb{R}^d)$ is a Banach space endowed with the norm $\|f\|_{n+\alpha} = \|f\|_0 + \sum_{k=1}^n \|D^k f\|_0 + [D^n f]_\alpha$, $f \in C_b^{n+\alpha}(\mathbb{R}^d)$. Finally, we will also consider the Banach space $C_0(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ of all continuous functions vanishing at infinity endowed with the norm $\|\cdot\|_0$.

REMARK 2.1. Hypothesis 1 (or condition (2.4)) is equivalent to the following Picard's type condition (see [17]): there exists $\alpha \in (0, 2)$ and $C_\alpha > 0$, such that the following estimate holds, for any $\rho > 0$, $u \in \mathbb{R}^d$ with $|u| = 1$,

$$\int_{\{|\langle u,y\rangle| \leq \rho\}} |\langle u, y\rangle|^2 \nu(dy) \geq C_\alpha \rho^{2-\alpha}.$$

The equivalence follows from the computation

$$\begin{aligned}\int_{\{|\langle u,y\rangle| \leq \rho\}} |\langle u, y\rangle|^2 \nu(dy) &= \int_{\mathbb{S}} |\langle u, \xi\rangle|^2 \mu(d\xi) \int_0^\infty 1_{\{|\langle u,\xi\rangle| \leq \rho/r\}} r^{1-\alpha}\, dr \\ &= \rho^{2-\alpha} \int_{\mathbb{S}} |\langle u, \xi\rangle|^2 \mu(d\xi) \int_{|\langle u,\xi\rangle|}^\infty \frac{ds}{s^{3-\alpha}} = \frac{\rho^{2-\alpha}}{2-\alpha} \int_{\mathbb{S}} |\langle u, \xi\rangle|^\alpha \mu(d\xi).\end{aligned}$$

The Picard's condition is usually imposed on the Lévy measure ν of a non-necessarily stable Lévy process L in order to ensure that the law of L_t, for any $t > 0$, has a C^∞-density with respect to the Lebesgue measure.

3. Some analytic regularity results

In this section we prove existence of regular solutions to (1.5). This will be achieved through Schauder estimates and will be important in Section 4 to prove uniqueness for (1.1).

We will use the following three properties of the α-stable process L (in the sequel μ_t denotes the law of L_t, $t \geq 0$).

(a) $\mu_t(A) = \mu_1(t^{-1/\alpha} A)$, for any $A \in \mathcal{B}(\mathbb{R}^d)$, $t > 0$ (this scaling property follows from (2.1) and (2.3));

(b) μ_t has a density p_t with respect to the Lebesgue measure, $t > 0$; moreover $p_t \in C^1(\mathbb{R}^d)$ and its spatial derivative $Dp_t \in L^1(\mathbb{R}^d, \mathbb{R}^d)$ (this is a consequence of Hypothesis 1);

(c) for any $\sigma > \alpha$, we have by (2.2)

$$\int_{\{|x| \leq 1\}} |x|^\sigma \nu(dx) < \infty. \tag{3.1}$$

The fact that (b) holds can be deduced by an argument of [23, Section 3]. Actually, Hypothesis 1 implies the following stronger result.

Lemma 3.1. *For any* $\alpha \in (0,2)$, $t > 0$, *the density* $p_t \in C^\infty(\mathbb{R}^d)$ *and all derivatives* $D^k p_t$ *are integrable on* $\mathbb{R}^d$, $k \ge 1$.

Proof. We only show that $p_t \in C^\infty(\mathbb{R}^d)$ and $Dp_t \in L^1(\mathbb{R}^d, \mathbb{R}^d)$, following [23]; arguing in a similar way one can obtain the full assertion. By (2.4), we know that $e^{-t\psi(u)} \le e^{-C_\alpha t|u|^\alpha}$, $u \in \mathbb{R}^d$, and so by the inversion formula of Fourier transform (see [22, Proposition 2.5]) μ_t has a density $p_t \in L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$,

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x,z\rangle} e^{-t\psi(z)}\, dz, \quad x \in \mathbb{R}^d,\ t > 0. \tag{3.2}$$

Note that (a) implies that $p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha}x)$. Thanks to (2.4) one can differentiate infinitely many times under the integral sign and verifies that $p_t \in C^\infty(\mathbb{R}^d)$. Let us fix $j = 1,\dots,d$ and check that the partial derivative $\partial_{x_j} p_t \in L^1(\mathbb{R}^d)$. By the scaling property (a) it is enough to consider $t = 1$. By writing $\psi = \psi_1 + \psi_2$,

$$\psi_1(u) = -\int_{\{|y|\le 1\}} (\cos(\langle u, y\rangle) - 1)\nu(dy), \quad \psi_2 = \psi - \psi_1,$$
$$\partial_{x_j} p_1(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x,z\rangle}((-iz_j)e^{-\psi_1(z)})e^{-\psi_2(z)} dz, \quad x \in \mathbb{R}^d.$$

We find easily that $\psi_1 \in C^\infty(\mathbb{R}^d)$ and so, using also (2.4) we deduce that $-iz_j e^{-\psi_1(z)}$ is in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. In particular, there exists $f_1 \in L^1(\mathbb{R}^d)$ such that the Fourier transform $\hat{f}_1(z) = (-iz_j)e^{-\psi_1(z)}$. On the other hand (see [22, Section 8]), there exists an infinitely divisible probability measure γ on $\mathbb{R}^d$ such that the Fourier transform $\hat{\gamma}(z) = e^{-\psi_2(z)}$. By [22, Proposition 2.5] we infer that $\widehat{f_1 * \gamma} = \hat{f}_1 \cdot \hat{\gamma}$. By the inversion formula we deduce that $\partial_{x_j} p_1(x) = (f_1 * \gamma)(x)$ and this proves that $\partial_{x_j} p_1 \in L^1(\mathbb{R}^d)$. □

Remark that (c) implies that the expression of $\mathcal{L}f$ in (2.5) is meaningful for any $f \in C_b^{1+\gamma}(\mathbb{R}^d)$ if $1 + \gamma > \alpha$. Indeed $\mathcal{L}f(x)$ can be decomposed into the sum of two integrals, over $\{|y| > 1\}$ and over $\{|y| \le 1\}$ respectively. The first integral is finite since f is bounded. To treat the second one, we can use the estimate

$$\begin{aligned} &|f(y+x) - f(x) - y \cdot Df(x)| \\ &\quad \le \int_0^1 |Df(x+ry) - Df(x)|\,|y|\, dr \le [Df]_\gamma |y|^{1+\gamma}, \quad |y| \le 1. \end{aligned} \tag{3.3}$$

Note that $\mathcal{L}f \in C_b(\mathbb{R}^d)$ if $f \in C_b^{1+\gamma}(\mathbb{R}^d)$ and $1 + \gamma > \alpha$.

The next result is a maximum principle. A related result is in [10, Section 4.5]. This will be used to prove uniqueness of solutions to (1.5) as well as to study existence.

Proposition 3.2. *Let* $\alpha \in (0,2)$*. If* $u \in C_b^{1+\gamma}(\mathbb{R}^d)$, $1+\gamma > \alpha$*, is a solution to* $\lambda u - \mathcal{L}u - b \cdot Du = g$*, with* $\lambda > 0$ *and* $g \in C_b(\mathbb{R}^d)$*, then*

$$\|u\|_0 \le \frac{1}{\lambda}\|g\|_0, \quad \lambda > 0. \tag{3.4}$$

Proof. Since $-u$ solves the same equation of u with g replaced by $-g$, it is enough to prove that $u(x) \le \|g\|_0/\lambda$, $x \in \mathbb{R}^d$. Moreover, possibly replacing u by $u - \inf_{x\in\mathbb{R}^d} u(x)$, we may assume that $u \ge 0$.

Now we show that there exists $c_1 > 0$ such that, for any $\epsilon > 0$ we can find $u_\epsilon \in C_b^{1+\gamma}(\mathbb{R}^d)$ with $\|u_\epsilon\|_0 = \max_{x\in\mathbb{R}^d}|u_\epsilon(x)|$ and also

$$\|u - u_\epsilon\|_{1+\gamma} < \epsilon c_1.$$

To this purpose let $x_\epsilon \in \mathbb{R}^d$ be such that $u(x_\epsilon) > \|u\|_0 - \epsilon$ and take a test function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi(x_\epsilon) = 1$, $0 \le \phi \le 1$, and $\phi(x) = 0$ if $|x - x_\epsilon| \ge 1$. One checks that $u_\epsilon(x) = u(x) + 2\epsilon\phi(x)$ verifies the assumptions. Let us define the operator $\mathcal{L}_1 = \mathcal{L} + b \cdot D$ and write

$$\lambda u_\epsilon(x) - \mathcal{L}_1 u_\epsilon(x) = g(x) + \lambda(u_\epsilon(x) - u(x)) - \mathcal{L}_1(u_\epsilon - u)(x).$$

Let y_ϵ be one point in which u_ϵ attains its global maximum. Since clearly $\mathcal{L}_1 u_\epsilon(y_\epsilon) \le 0$, we have (using also (3.3))

$$\lambda\|u_\epsilon\|_0 = \lambda u_\epsilon(y_\epsilon) \le \|g\|_0 + C\|u - u_\epsilon\|_{1+\gamma} \le \|g\|_0 + Cc_1\epsilon.$$

Letting $\epsilon \to 0^+$, we get (3.4). □

Next we prove Schauder estimates for (1.5) when b *is constant*. The case of $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$ will be treated in Theorem 3.4. We stress that the constant c in (3.6) is independent of $b = k$.

The condition $\alpha + \beta > 1$ which we impose is needed to have a regular C^1-solution u. On the other hand, the next result holds more generally without the hypothesis $\alpha + \beta < 2$. This is assumed just to simplify the proof and it is not restrictive in the study of pathwise uniqueness for (1.1). Indeed since $C_b^{\beta'}(\mathbb{R}^d, \mathbb{R}^d) \subset C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$ when $0 < \beta \le \beta'$, it is enough to study uniqueness when β satisfies $\beta < 2 - \alpha$.

Theorem 3.3. *Assume* Hypothesis 1. *Let* $\alpha \in (0,2)$ *and* $\beta \in (0,1)$ *be such that* $1 < \alpha + \beta < 2$*. Then, for any* $\lambda > 0$, $k \in \mathbb{R}^d$, $g \in C_b^\beta(\mathbb{R}^d)$*, there exists a unique solution*

$u = u_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ *to the equation*

$$\lambda u - \mathcal{L}u - k \cdot Du = g \tag{3.5}$$

on $\mathbb{R}^d$ ($\mathcal{L}$ *is defined in* (2.5)). *In addition there exists a constant* c *independent of* g, u, k *and* $\lambda > 0$ *such that*

$$\lambda \|u\|_0 + \lambda^{(\alpha+\beta-1)/\alpha} \|Du\|_0 + [Du]_{\alpha+\beta-1} \le c\|g\|_\beta. \tag{3.6}$$

Proof. Equation (3.5) is meaningful for $u \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ with $\alpha + \beta > 1$ thanks to (3.3). Moreover, uniqueness follows from Proposition 3.2.

To prove the result, we use the semigroup approach as in [4]. To this purpose, we introduce the α-stable Markov semigroup (P_t) acting on $C_b(\mathbb{R}^d)$ and associated to $\mathcal{L} + k \cdot Du$, i.e.,

$$P_t f(x) = \int_{\mathbb{R}^d} f(z + tk) p_t(z - x)\, dz, \quad t > 0,\ f \in C_b(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

where p_t is defined in (3.2), and $P_0 = I$. Then we consider the bounded function $u = u_\lambda$,

$$u(x) = \int_0^\infty e^{-\lambda t} P_t g(x)\, dt, \quad x \in \mathbb{R}^d. \tag{3.7}$$

We are going to show that u belongs to $C_b^{\alpha+\beta}(\mathbb{R}^d)$, verifies (3.6) and solves (3.5).

PART I. We prove that $u \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ and that (3.6) holds. First note that $\lambda\|u\|_0 \le \|g\|_0$ since (P_t) is a contraction semigroup. Then, using the scaling property $p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x)$, we arrive at

$$|DP_t f(x)| \le \frac{t^{-1/\alpha}}{t^{d/\alpha}} \int_{\mathbb{R}^d} |f(z + tk)|\, |Dp_1(t^{-1/\alpha} z - t^{-1/\alpha} x)|\, dz \le \frac{c_0 \|f\|_0}{t^{1/\alpha}}, \tag{3.8}$$

$t > 0$, $f \in C_b(\mathbb{R}^d)$, where $c_0 = \|Dp_1\|_{L^1(\mathbb{R}^d)}$, and so we find the estimate

$$\|DP_t f\|_0 \le \frac{c_0}{t^{1/\alpha}} \|f\|_0, \quad f \in C_b(\mathbb{R}^d),\ t > 0. \tag{3.9}$$

By interpolation theory we know that $(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d))_{\beta,\infty} = C_b^\beta(\mathbb{R}^d)$, $\beta \in (0, 1)$, see for instance [16, Chapter 1]; interpolating the previous estimate with the estimate $\|DP_t f\|_0 \le \|Df\|_0$, $t \ge 0$, $f \in C_b^1(\mathbb{R}^d)$, we obtain

$$\|DP_t f\|_0 \le \frac{c_1}{t^{(1-\beta)/\alpha}} \|f\|_\beta, \quad t > 0,\ f \in C_b^\beta(\mathbb{R}^d), \tag{3.10}$$

with $c_1 = c_1(c_0, \beta)$. In a similar way, we also find

$$\|D^2 P_t f\|_0 \leq \frac{c_2}{t^{(2-\beta)/\alpha}} \|f\|_\beta, \quad t > 0, \ f \in C_b^\beta(\mathbb{R}^d). \tag{3.11}$$

Using (3.10) and the fact that $(1-\beta)/\alpha < 1$, we can differentiate under the integral sign in (3.7) and prove that there exists $Du(x) = Du_\lambda(x)$, $x \in \mathbb{R}^d$. Moreover Du_λ is bounded on $\mathbb{R}^d$ and we have, for any $\lambda > 0$ with $\tilde{c}$ independent of λ, u, k and g,

$$\lambda^{(\alpha+\beta-1)/\alpha} \|Du\|_0 \leq \tilde{c} \|g\|_\beta$$

(we have used that $\int_0^\infty e^{-\lambda t} t^{-\sigma}\, dt = c/\lambda^{1-\sigma}$, for $\sigma < 1$ and $\lambda > 0$).

It remains to prove that $Du \in C_b^\theta(\mathbb{R}^d, \mathbb{R}^d)$, where $\theta = \alpha - 1 + \beta \in (0,1)$. We proceed as in the proof of [2, Proposition 4.2] and [18, Theorem 4.2].

Using (3.10), (3.11) and the fact that $2-\beta > \alpha$, we find, for any $x, x' \in \mathbb{R}^d$, $x \neq x'$,

$$\begin{aligned} |Du(x) - Du(x')| &\leq C\|g\|_\beta \left(\int_0^{|x-x'|^\alpha} \frac{1}{t^{(1-\beta)/\alpha}}\, dt + \int_{|x-x'|^\alpha}^\infty \frac{|x-x'|}{t^{(2-\beta)/\alpha}}\, dt \right) \\ &\leq c_3 \|g\|_\beta |x - x'|^\theta, \end{aligned}$$

and so $[Du]_{\alpha-1+\beta} \leq c_3\|g\|_\beta$, where c_3 is independent of g, u, k and λ.

PART II. We prove that u solves (3.5), for any $\lambda > 0$. We use the fact that the semigroup (P_t) is strongly continuous on the Banach space $C_0(\mathbb{R}^d)$; see [1, Section 6.7] and [22, Section 31].

Let $\mathcal{A}$: $D(\mathcal{A}) \subset C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ be its generator. By [22, Theorem 31.5]) $C_0^2(\mathbb{R}^d) \subset D(\mathcal{A})$ and moreover $\mathcal{A} f = \mathcal{L} f + k \cdot Df$ if $f \in C_0^2(\mathbb{R}^d)$ (we say that f belongs to $C_0^2(\mathbb{R}^d)$ if $f \in C_b^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ and all its first and second partial derivatives belong to $C_0(\mathbb{R}^d)$).

We first show the assertion assuming in addition that $g \in C_0^2(\mathbb{R}^d)$. It is easy to check that u belongs to $C_0^2(\mathbb{R}^d)$ as well. To this purpose, one can use the estimates $\|D^k P_t g\|_0 \leq \|D^k g\|_0$, $t \geq 0$, $k = 1, 2$, and the dominated convergence theorem. On the other hand, by the Hille–Yosida theorem we know that $u \in D(\mathcal{A})$ and $\lambda u - \mathcal{A} u = g$. Thus we have found that u solves (3.5).

Let us prove the assertion when $g \in C_b^2(\mathbb{R}^d)$. Note that also $u \in C_b^2(\mathbb{R}^d)$. We consider a function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi(0) = 1$ and introduce $g_n(x) = \psi(x/n) g(x)$, $x \in \mathbb{R}^d$, $n \geq 1$. It is clear that $g_n, u_n \in C_0^2(\mathbb{R}^d)$ (u_n is given in (3.7) when g is replaced by g_n). We know that

$$\lambda u_n(x) - \mathcal{L} u_n(x) - k \cdot D u_n(x) = g_n(x), \quad x \in \mathbb{R}^d. \tag{3.12}$$

It is easy to see that there exists $C > 0$ such that $\|g_n\|_2 \leq C$, $n \geq 1$, and moreover g_n and Dg_n converge pointwise to g and Dg respectively. It follows that also $\|u_n\|_2$ is uniformly bounded and moreover u_n and Du_n converge pointwise to u and Du respectively. Using also (3.3), we can apply the dominated convergence theorem and deduce that

$$\lim_{n\to\infty} \mathcal{L}u_n(x) = \mathcal{L}u(x), \quad x \in \mathbb{R}^d.$$

Passing to the limit in (3.12), we obtain that u is a solution to (3.5).

Let now $g \in C_b^\beta(\mathbb{R}^d)$. Take any $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $0 \le \phi \le 1$ and $\int_{\mathbb{R}^d} \phi(x)\,dx = 1$. Define $\phi_n(x) = n^d \phi(xn)$ and $g_n = g * \phi_n$. Note that $(g_n) \subset C_b^\infty(\mathbb{R}^d) = \bigcap_{k\ge1} C_b^k(\mathbb{R}^d)$ and $\|g_n\|_\beta \le \|g\|_\beta$, $n \ge 1$. Moreover, possibly passing to a subsequence still denoted by (g_n), we may assume that

$$g_n \to g \quad \text{in} \quad C^{\beta'}(K). \tag{3.13}$$

for any compact set $K \subset \mathbb{R}^d$ and $0 < \beta' < \beta$ (see p. 37 in [12]). Let u_n be given in (3.7) when g is replaced by g_n. By the first part of the proof, we know that

$$\|u_n\|_{\alpha+\beta} \le C\|g_n\|_\beta \le C\|g\|_\beta,$$

where C is independent of n. It follows that, possibly passing to a subsequence still denoted with (u_n), we have that $u_n \to u$ in $C^{\alpha+\beta'}(K)$, for any compact set $K \subset \mathbb{R}^d$ and $\beta' > 0$ such that $1 < \alpha + \beta' < \alpha + \beta$. Arguing as before, we can pass to the limit in $\lambda u_n(x) - \mathcal{L}u_n(x) - k \cdot Du_n(x) = g_n(x)$ and obtain that u solves (3.5). The proof is complete. □

Now we extend Theorem 3.3 to the case in which b is Hölder continuous. We can only do this when $\alpha \ge 1$ (see also Remark 3.5). To prove the result when $\alpha = 1$ we adapt the localization procedure which is well known for second order uniformly elliptic operators with Hölder continuous coefficients (see [12]). This technique works in our situation since in estimate (3.6) the constant is independent of $k \in \mathbb{R}^d$.

We also need the following interpolatory inequalities (see [12, p. 40, (3.3.7)]); for any $t \in [0,1)$, $0 \le s \le r < 1$, there exists $N = N(d,k,r,t)$ such that if $f \in C_b^{r+t}(\mathbb{R}^d, \mathbb{R}^k)$, then

$$[f]_{s+t} \le N[f]_{r+t}^{s/r}[f]_t^{1-s/r}, \tag{3.14}$$

where $[f]_{s+t}$ is defined as in (2.6) if $0 < s+t < 1$, $[f]_0 = \|f\|_0$, $[f]_1 = \|Df\|_0$, and $[f]_{s+t} = [Df]_{s+t-1}$ if $1 < s+t < 2$. By (3.14) we deduce, for any $\epsilon > 0$,

$$[f]_{s+t} \le \tilde{N}\epsilon^{r-s}[f]_{r+t} + \tilde{N}\epsilon^{-s}[f]_t, \quad f \in C_b^{r+t}(\mathbb{R}^d, \mathbb{R}^k). \tag{3.15}$$

Theorem 3.4. *Assume* Hypothesis 1. *Let* $\alpha \ge 1$ *and* $\beta \in (0,1)$ *be such that* $1 < \alpha + \beta < 2$. *Then, for any* $\lambda > 0$, $g \in C_b^\beta(\mathbb{R}^d)$, *there exists a unique solution* $u = u_\lambda \in$

$C_b^{\alpha+\beta}(\mathbb{R}^d)$ *to the equation*

$$\lambda u - \mathcal{L}u - b \cdot Du = g \tag{3.16}$$

on $\mathbb{R}^d$. *Moreover, for any* $\omega > 0$, *there exists* $c = c(\omega)$, *independent of* g *and* u, *such that*

$$\lambda \|u\|_0 + [Du]_{\alpha+\beta-1} \le c\|g\|_\beta, \quad \lambda \ge \omega. \tag{3.17}$$

Finally, we have $\lim_{\lambda\to\infty} \|Du_\lambda\|_0 = 0$.

Proof. Uniqueness and estimate $\lambda\|u\|_0 \le \|g\|_0$, $\lambda > 0$, follow from the maximum principle (see Proposition 3.2). Moreover, the last assertion follows from (3.17) using (3.14). Indeed, with $t = 0$, $s = 1$, $r = \alpha + \beta$, we obtain, for $\lambda \ge \omega$,

$$[Du_\lambda]_0 = [u_\lambda]_1 \le N[Du_\lambda]_{\alpha+\beta-1}^{1/(\alpha+\beta)}[u_\lambda]_0^{1-1/(\alpha+\beta)} \le N\tilde{c}\lambda^{-(\alpha+\beta-1)/(\alpha+\beta)}\|g\|_\beta,$$

where $\tilde{c} = \tilde{c}(\omega)$. Letting $\lambda \to \infty$, we get the assertion.

Let us prove existence and estimate $[Du]_{\alpha+\beta-1} \le c\|g\|_\beta$, for $\lambda \ge \omega$, with $\omega > 0$ fixed. We treat $\alpha > 1$ and $\alpha = 1$ separately.

PART I (the case $\alpha > 1$). In the sequel we will use the estimate

$$\|lf\|_\theta \le \|l\|_0\|f\|_\theta + \|f\|_0[l]_\theta, \quad l,\ f \in C_b^\theta(\mathbb{R}^d),\ \theta \in (0, 1). \tag{3.18}$$

Writing $\lambda u(x) - \mathcal{L}u(x) = g(x) + b(x) \cdot Du(x)$, and using (3.6) and (3.18), we obtain the following a priori estimate (assuming that $u \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ is a solution to (3.16))

$$\begin{aligned}[Du]_{\alpha+\beta-1} &\le C\|g\|_\beta + C\|b \cdot Du\|_\beta \\ &\le C\|g\|_\beta + C\|b\|_\beta\|Du\|_0 + C\|b\|_0[Du]_\beta,\end{aligned} \tag{3.19}$$

where C is independent of $\lambda > 0$. Combining the interpolatory estimates (see (3.15) with $t = 0$, $s = 1 + \beta$, $r = \alpha + \beta$)

$$[Du]_\beta \le \tilde{N}\epsilon^{\alpha-1}[Du]_{\alpha+\beta-1} + \tilde{N}\epsilon^{-(1+\beta)}\|u\|_0, \quad \epsilon > 0,$$

and $\|Du\|_0 \le \tilde{N}\epsilon^{\alpha+\beta-1}[Du]_{\alpha+\beta-1} + \tilde{N}\epsilon^{-1}\|u\|_0$ (recall that $\alpha + \beta > 1 + \beta$) with the maximum principle, we get for ϵ small enough the a priori estimate

$$\begin{aligned}[Du]_{\alpha+\beta-1} &\le c_1(\|g\|_\beta + C(\epsilon)\|u\|_0) \\ &\le c_1\left(\|g\|_\beta + \frac{C(\epsilon)}{\lambda}\|g\|_0\right) \le c_1\left(\|g\|_\beta + \frac{C(\epsilon)}{\omega}\|g\|_0\right) \le C_1\|g\|_\beta,\end{aligned} \tag{3.20}$$

for any $\lambda \geq \omega$. Now to prove the existence of a $C_b^{\alpha+\beta}$-solution, we use the continuity method (see, for instance, [12, Section 4.3]). Let us introduce

$$\lambda u(x) - \mathcal{L}u(x) - \delta b(x) \cdot Du(x) = g(x), \tag{3.21}$$

$x \in \mathbb{R}^d$, where $\delta \in [0,1]$ is a parameter. Let us define $\Gamma = \{\delta \in [0,1]$: there is a unique solution $u = u_\delta \in C_b^{\alpha+\beta}(\mathbb{R}^d)$, for any $g \in C_b^\beta(\mathbb{R}^d)\}$.

Clearly Γ is not empty since $0 \in \Gamma$. Fix $\delta_0 \in \Gamma$ and rewrite (3.21) as

$$\lambda u(x) - \mathcal{L}u(x) - \delta_0 b(x) \cdot Du(x) = g(x) + (\delta - \delta_0) b(x) \cdot Du(x).$$

Introduce the operator $S: C_b^{\alpha+\beta}(\mathbb{R}^d) \to C_b^{\alpha+\beta}(\mathbb{R}^d)$. For any $v \in C_b^{\alpha+\beta}(\mathbb{R}^d)$, $u = Sv$ is the unique $C_b^{\alpha+\beta}$-solution to $\lambda u(x) - \mathcal{L}u(x) - \delta_0 b(x) \cdot Du(x) = g(x) + (\delta - \delta_0) b(x) \cdot Dv(x)$.

By using (3.20), we get $\|Sv_1 - Sv_2\|_{\alpha+\beta} \leq 2|\delta - \delta_0| \cdot \tilde{c}_1 \|b\|_\beta \|v_1 - v_2\|_{\alpha+\beta}$. By choosing $|\delta - \delta_0|$ small enough, S becomes a contraction and it has a unique fixed point which is the solution to (3.21). A compactness argument shows that $\Gamma = [0, 1]$. The assertion is proved.

PART II (the case $\alpha = 1$). As before, we establish the existence of a $C_b^{1+\beta}(\mathbb{R}^d)$-solution, by using the continuity method. This requires the a priori estimate (3.20) for $\alpha = 1$.

Let $u \in C_b^{1+\beta}(\mathbb{R}^d)$ be a solution. Let $r > 0$. Consider a function $\xi \in C_c^\infty(\mathbb{R}^d)$ such that $\xi(x) = 1$ if $|x| \leq r$ and $\xi(x) = 0$ if $|x| > 2r$.

Let now $x_0 \in \mathbb{R}^d$ and define $\rho(x) = \xi(x - x_0)$, $x \in \mathbb{R}^d$, and $v = u\rho$. One can easily check that

$$\begin{aligned} \mathcal{L}v(x) &= \rho(x)\mathcal{L}u(x) + u(x)\mathcal{L}\rho(x) \\ &\quad + \int_{\mathbb{R}^d} (\rho(x+y) - \rho(x))(u(x+y) - u(x))\nu(dy), \quad x \in \mathbb{R}^d. \end{aligned} \tag{3.22}$$

We have

$$\lambda v(x) - \mathcal{L}v(x) - b(x_0) \cdot Dv(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x), \quad x \in \mathbb{R}^d,$$

where

$$\begin{aligned} &f_1(x) = \rho(x) g(x), \quad f_2(x) = (b(x) - b(x_0)) \cdot Dv(x), \\ &f_3(x) = -u(x)[\mathcal{L}\rho(x) + b(x) \cdot D\rho(x)], \\ &f_4(x) = -\int_{\mathbb{R}^d} (\rho(x+y) - \rho(x))(u(x+y) - u(x))\nu(dy), \quad x \in \mathbb{R}^d. \end{aligned}$$

By Theorem 3.3 we know that

$$[Dv]_\beta \leq C_1(\|f_1\|_\beta + \|f_2\|_\beta + \|f_3\|_\beta + \|f_4\|_\beta), \tag{3.23}$$

where the constant C_1 is independent of x_0 and λ. Let us consider the crucial term f_2. By (3.18) we find

$$\| f_2\|_\beta \le \left(\sup_{x\in B(x_0,2r)} |b(x) - b(x_0)| \right) [Dv]_\beta + \|Dv\|_0 \|b\|_\beta.$$

Let us fix r small enough such that $C_1 \sup_{x\in B(x_0,2r)} |b(x) - b(x_0)| < 1/2$. We get

$$[Dv]_\beta \le 2C_1(\| f_1\|_\beta + \|Dv\|_0\|b\|_\beta + \| f_3\|_\beta + \| f_4\|_\beta). \tag{3.24}$$

Note that $\| f_1\|_\beta \le C(r)\|g\|_\beta$. By the interpolatory estimates (3.15) and the maximum principle, arguing as in (3.20), we arrive at

$$[Dv]_\beta \le C_2(\|g\|_\beta + \| f_3\|_\beta + \| f_4\|_\beta),$$

for any $\lambda \ge \omega$. Let us estimate f_4. To this purpose we introduce the following non-local linear operator T

$$T f(x) = \int_{\mathbb{R}^d} (\rho(x + y) - \rho(x))(f(x + y) - f(x))\nu(dy), \quad f \in C^1_b(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

One can easily check that T is continuous from $C^1_b(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$ and from $C^{1+\beta}_b(\mathbb{R}^d)$ into $C^1_b(\mathbb{R}^d)$. To this purpose we only remark that, for any $x \in \mathbb{R}^d$,

$$\begin{aligned} |DT f(x)| &\le 5\|\rho\|_2\| f\|_1\left(\int_{\{|y|\le 1\}} |y|^2\nu(dy) + \int_{\{|y|>1\}} \nu(dy)\right) \\ &\quad + 5\|\rho\|_1\| f\|_{1+\beta}\left(\int_{\{|y|\le 1\}} |y|^{1+\beta}\nu(dy) + \int_{\{|y|>1\}} \nu(dy)\right), \quad f \in C^{1+\beta}_b(\mathbb{R}^d). \end{aligned}$$

By interpolation theory we know that

$$(C^1_b(\mathbb{R}^d), C^{1+\beta}_b(\mathbb{R}^d))_{\beta,\infty} = C^{1+\beta^2}_b(\mathbb{R}^d),$$

see [16, Chapter 1], and so we get that T is continuous from $C^{1+\beta^2}_b(\mathbb{R}^d)$ into $C^\beta_b(\mathbb{R}^d)$ (see [16, Theorem 1.1.6]). Since $f_4 = -Tu$, we obtain the estimate

$$\| f_4\|_\beta \le C_3\|u\|_{1+\beta^2}.$$

We have $\| f_4\|_\beta + \| f_3\|_\beta \le c_3(r)\|u\|_{1+\beta^2}$ and so

$$[Dv]_\beta \le C_4(\|g\|_\beta + \|u\|_{1+\beta^2}),$$

where C_4 is independent of $\lambda \ge \omega$. It follows that $[Du]_{C^\beta(B(x_0,r))} \le C_4(\|g\|_\beta + \|u\|_{1+\beta^2})$, where $B(x_0, r)$ is the ball of center x_0 and radius $r > 0$. Since C_4 is independent of x_0, we obtain

$$[Du]_\beta \le C_4(\|g\|_\beta + \|u\|_{1+\beta^2}),$$

for any $\lambda \geq \omega$. Using again (3.15) and the maximum principle, we get the a priori estimate (3.20) for $\alpha = 1$. The proof is complete. □

REMARK 3.5. In contrast with Theorem 3.3, in Theorem 3.4 we can not show existence of $C_b^{\alpha+\beta}$-solutions to (3.16) when $\alpha < 1$. The difficulty is evident from the a priori estimate (3.19). Indeed, starting from

$$[Du]_{\alpha+\beta-1} \leq C\|g\|_\beta + C\|b\|_\beta \|Du\|_0 + C\|b\|_0 [Du]_\beta,$$

we cannot continue, since $\alpha < 1$ gives $Du \in C_b^\theta$ with $\theta = \alpha + \beta - 1 < \beta$. Roughly speaking, when $\alpha < 1$, the perturbation term $b \cdot Du$ is of order larger than $\mathcal{L}$ and so we are not able to prove the desired a priori estimates.

4. The main result

We briefly recall basic facts about Poisson random measures which we use in the sequel (see also [1], [14], [19], [26]). The Poisson random measure N associated with the α-stable process $L = (L_t)$ in (1.1) is defined by

$$N((0,t] \times U) = \sum_{0<s\leq t} 1_U(\triangle L_s) = \#\{0 < s \leq t : \triangle L_s \in U\},$$

for any Borel set U in $\mathbb{R}^d \setminus \{0\}$, i.e., $U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $t > 0$. Here $\triangle L_s = L_s - L_{s-}$ denotes the jump size of L at time $s > 0$. The compensated Poisson random measure $\tilde{N}$ is defined by $\tilde{N}((0,t] \times U) = N((0,t] \times U) - t\nu(U)$, where ν is given in (2.2) and $0 \notin \bar{U}$. Recall the Lévy–Itô decomposition of the process L (see [1, Theorem 2.4.16] or [14, Theorem 2.7]). This says that

$$L_t = \hat{b}t + \int_0^t \int_{\{|x|\leq 1\}} x\tilde{N}(ds,dx) + \int_0^t \int_{\{|x|>1\}} xN(ds,dx), \quad t \geq 0, \tag{4.1}$$

where $\hat{b} = E\big[L_1 - \int_0^1 \int_{\{|x|>1\}} xN(ds,dx)\big]$. Note that in our case, since ν is symmetric, we have $\hat{b} = 0$.

The stochastic integral $\int_0^t \int_{\{|x|\leq 1\}} x\tilde{N}(ds,dx)$ is the compensated sum of small jumps and is an L^2-martingale. The process $\int_0^t \int_{\{|x|>1\}} xN(ds,dx) = \int_{(0,t]} \int_{\{|x|>1\}} xN(ds,dx) = \sum_{0<s\leq t,\, |\triangle L_s|>1} \triangle L_s$ is a compound Poisson process.

Let $T > 0$. The predictable σ-field $\mathcal{P}$ on $\Omega \times [0,T]$ is generated by all left-continuous adapted processes (defined on the same stochastic basis fixed in Section 2). Let $U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. In the sequel, we will always consider a $\mathcal{P} \times \mathcal{B}(U)$-measurable mapping $F : [0,T] \times U \times \Omega \to \mathbb{R}^d$.

If $0 \notin \bar{U}$, then $\int_0^T \int_U F(s,x)N(ds,dx) = \sum_{0<s\leq T} F(s,\triangle L_s) 1_U(\triangle L_s)$ is a random finite sum.

If $E\int_0^T ds\int_U |F(s,x)|^2\nu(dx)<\infty$, then one can define the stochastic integral

$$Z_t=\int_0^t\int_U F(s,x)\tilde{N}(ds,dx),\quad t\in[0,T]$$

(here we do not assume $0\notin\bar{U}$). The process $Z=(Z_t)$ is an L^2-martingale with a càdlàg modification. Moreover, $E|Z_t|^2=E\int_0^t ds\int_U |F(s,x)|^2\nu(dx)$ (see [14, Lemma 2.4]). We will use the following L^p-estimates (see [14, Theorem 2.11] or the proof of Proposition 6.6.2 in [1]); for any $p\ge 2$, there exists $c(p)>0$ such that

$$\begin{aligned}E\Big[\sup_{0<s\le t}|Z_s|^p\Big]&\le c(p)E\bigg[\bigg(\int_0^t ds\int_U |F(s,x)|^2\nu(dx)\bigg)^{p/2}\bigg]\\&\quad+c(p)E\bigg[\int_0^t ds\int_U |F(s,x)|^p\nu(dx)\bigg],\quad t\in[0,T]\end{aligned}\tag{4.2}$$

(the inequality is obvious if the right-hand side is infinite).

Let us recall the concept of (*strong*) *solution* which we consider. A solution to the SDE (1.1) is a càdlàg $\mathcal{F}_t$-adapted process $X^x=(X_t^x)$ (defined on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge 0},P)$ fixed in Section 2) which solves (1.1) P-a.s., for $t\ge 0$.

It is easy to show the existence of a solution to (1.1) using the fact that b is bounded and continuous. We may argue at ω fixed. Let us first consider $t\in[0,1]$. By introducing $v(t)=X_t-L_t$, we get the equation

$$v(t)=x+\int_0^t b(v(s)+L_s)\,ds.$$

Approximating b with smooth drifts b_n we find solutions $v_n\in C([0,1];\mathbb{R}^d)$. By the Ascoli–Arzela theorem, we obtain a solution to (1.1) on $[0,1]$. The same argument works also on the time interval $[1,2]$ with a random initial condition. Iterating this procedure we can construct a solution for all $t\ge 0$.

The proof of Theorem 1.1 requires some lemmas. We begin with a deterministic result.

Lemma 4.1. *Let $\gamma\in[0,1]$ and $f\in C_b^{1+\gamma}(\mathbb{R}^d)$. Then for any $u,v\in\mathbb{R}^d$, $x\in\mathbb{R}^d$, with $|x|\le 1$, we have*

$$|f(u+x)-f(u)-f(v+x)+f(v)|\le c_\gamma\|f\|_{1+\gamma}|u-v||x|^\gamma,\quad \textit{with}\quad c_\gamma=3^{1-\gamma}.$$

Proof. For any $x\in\mathbb{R}^d$, $|x|\le 1$, define the linear operator $T_x\colon C_b^1(\mathbb{R}^d)\to C_b^1(\mathbb{R}^d)$,

$$T_x f(u)=f(u+x)-f(u),\quad f\in C_b^1(\mathbb{R}^d),\ u\in\mathbb{R}^d.$$

Since $\|T_x f\|_0 \le \|Df\|_0|x|$ and $\|D(T_x f)\|_0 \le 2\|Df\|_0$, it follows that T_x is continuous and $\|T_x f\|_1 \le (2+|x|)\|f\|_1$, $f \in C_b^1(\mathbb{R}^d)$. Similarly, T_x is continuous from $C_b^2(\mathbb{R}^d)$ into $C_b^1(\mathbb{R}^d)$ and

$$\|T_x f\|_1 \le |x|\|f\|_2, \quad f \in C_b^2(\mathbb{R}^d).$$

By interpolation theory $(C_b^1(\mathbb{R}^d), C_b^2(\mathbb{R}^d))_{\gamma,\infty} = C_b^{1+\gamma}(\mathbb{R}^d)$, see for instance [16, Chapter 1]; we deduce that, for any $\gamma \in [0,1]$, T_x is continuous from $C_b^{1+\gamma}(\mathbb{R}^d)$ into $C_b^1(\mathbb{R}^d)$ (cf. [16, Theorem 1.1.6]) with operator norm less than or equal to $(2+|x|)^{1-\gamma}|x|^\gamma$.

Since $|x| \le 1$, we obtain that $\|T_x f\|_1 \le c_\gamma |x|^\gamma \|f\|_{1+\gamma}$, $f \in C_b^{1+\gamma}(\mathbb{R}^d)$. Now the assertion follows noting that, for any $u, v \in \mathbb{R}^d$,

$$|f(u+x) - f(u) - f(v+x) + f(v)| = |T_x f(u) - T_x f(v)| \le \|DT_x f\|_0 |u-v|.$$

The proof is complete. □

In the sequel we will consider the following resolvent equation on $\mathbb{R}^d$

$$\lambda u - \mathcal{L}u - Du \cdot b = b, \tag{4.3}$$

where $b \colon \mathbb{R}^d \to \mathbb{R}^d$ is given in (1.1), $\mathcal{L}$ in (2.5) and $\lambda > 0$ (the equation must be understood componentwise, i.e., $\lambda u_i - \mathcal{L}u_i - b \cdot Du_i = b_i$, $i = 1, \dots, d$). The next two results hold for SDEs of type (1.1) when b is only continuous and bounded.

Lemma 4.2. *Let $\alpha \in (0,2)$ and $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ in (1.1). Assume that, for some $\lambda > 0$, there exists a solution $u \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ to (4.3) with $\gamma \in [0,1]$, and moreover*

$$1 + \gamma > \alpha.$$

Let $X = (X_t)$ be a solution of (1.1) starting at $x \in \mathbb{R}^d$. We have, P-a.s., $t \ge 0$,

$$\begin{aligned} &u(X_t) - u(x) \\ &\quad = x - X_t + L_t + \lambda \int_0^t u(X_s)\,ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-})]\tilde{N}(ds, dx). \end{aligned} \tag{4.4}$$

Proof. First note that the stochastic integral in (4.4) is meaningful thanks to the estimate

$$\begin{aligned} &E \int_0^t ds \int_{\mathbb{R}^d} |u(X_{s-} + x) - u(X_{s-})|^2 \nu(dx) \\ &\quad \le 4t\|u\|_0^2 \int_{\{|x|>1\}} \nu(dx) + t\|u\|_1^2 \int_{\{|x|\le 1\}} |x|^2 \nu(dx) < \infty. \end{aligned} \tag{4.5}$$

The assertion is obtained applying Itô's formula to $u(X_t)$ (for more details on Itô's formula see [1, Theorem 4.4.7] and [14, Section 2.3]).

Let us fix $i = 1, \dots, d$ and set $u_i = f$. A difficulty is that Itô's formula is usually stated assuming that $f \in C^2(\mathbb{R}^d)$. However, in the present situation in which L is α-stable, using (3.1), one can show that Itô's formula holds for $f(X_t)$ when $f \in C_b^{1+\gamma}(\mathbb{R}^d)$. We give a proof of this fact.

We assume that $\gamma > 0$ (the proof with $\gamma = 0$ is similar). By convolution with mollifiers, as in (3.13) we obtain a sequence $(f_n) \subset C_b^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $C^{1+\gamma'}(K)$, for any compact set $K \subset \mathbb{R}^d$ and $0 < \gamma' < \gamma$. Moreover, $\| f_n \|_{1+\gamma} \le \| f \|_{1+\gamma}$, $n \ge 1$. Let us fix $t > 0$. By Itô's formula for $f_n(X_t)$ we find, P-a.s.,

$$
\begin{aligned}
& f_n(X_t) - f_n(x) \\
& = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [f_n(X_{s-} + x) - f_n(X_{s-})] \tilde{N}(ds, dx) \\
& \quad + \int_0^t ds \int_{\mathbb{R}^d} [f_n(X_{s-} + x) - f_n(X_{s-}) - 1_{\{|x| \le 1\}} x \cdot D f_n(X_{s-})] \nu(dx) \\
& \quad + \int_0^t b(X_s) \cdot D f_n(X_s) ds.
\end{aligned}
\tag{4.6}
$$

It is not difficult to pass to the limit as $n \to \infty$; we show two arguments which are needed. To deal with the integral involving ν, one can apply the dominated convergence theorem, thanks to the following estimate similar to (3.3),

$$
|f_n(X_{s-} + x) - f_n(X_{s-}) - x \cdot D f_n(X_{s-})| \le [Df]_\gamma |x|^{1+\gamma}, \quad |x| \le 1
$$

(recall that $\int_{\{|x| \le 1\}} |x|^{1+\gamma} \nu(dx) < \infty$ since $1 + \gamma > \alpha$). To pass to the limit in the stochastic integral with respect to $\tilde{N}$, one uses the isometry formula

$$
\begin{aligned}
& E \left| \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [f_n(X_{s-} + x) - f_n(X_{s-}) - f(X_{s-} + x) + f(X_{s-})] \tilde{N}(ds, dx) \right|^2 \\
& = \int_0^t ds \int_{\{|x| \le 1\}} E |f_n(X_{s-} + x) - f(X_{s-} + x) - f_n(X_{s-}) + f(X_{s-})|^2 \nu(dx) \\
& \quad + \int_0^t ds \int_{\{|x| > 1\}} E |f_n(X_{s-} + x) - f(X_{s-} + x) - f_n(X_{s-}) + f(X_{s-})|^2 \nu(dx).
\end{aligned}
\tag{4.7}
$$

Arguing as in (4.5), since $\| f_n \|_{1+\gamma} \le \| f \|_{1+\gamma}$, $n \ge 1$, we can apply the dominated convergence theorem in (4.7). Letting $n \to \infty$ in (4.7) we obtain 0. Finally, we pass to the limit in probability in (4.6) and obtain Itô's formula when $f \in C_b^{1+\gamma}(\mathbb{R}^d)$.

Noting that, for any $i = 1, \dots, d$,

$$
\mathcal{L} u_i(y) = \int_{\mathbb{R}^d} [u_i(y + x) - u_i(y) - 1_{\{|x| \le 1\}} x \cdot D u_i(y)] \nu(dx), \quad y \in \mathbb{R}^d,
$$

and using that u solves (4.3), i.e., $\mathcal{L}u + b \cdot Du = \lambda u - b$, we can replace in the Itô formula for $u(X_t)$ the term

$$\int_0^t \mathcal{L}u(X_s)\,ds + \int_0^t Du(X_s)b(X_s)\,ds$$
$$= \sum_{i=1}^d \left(\int_0^t \mathcal{L}u_i(X_s)\,ds + \int_0^t Du_i(X_s) \cdot b(X_s) ds \right) e_i$$

with $-\int_0^t b(X_s)\,ds + \lambda \int_0^t u(X_s)\,ds = x - X_t + L_t + \lambda \int_0^t u(X_s)\,ds$ and obtain the assertion. □

The proof of Theorem 1.1 will be a consequence of the following result.

Theorem 4.3. *Let* $\alpha \in (0, 2)$ *and* $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ *in* (1.1). *Assume that, for some* $\lambda > 0$, *there exists a solution* $u = u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ *to the equation* (4.3) *with* $\gamma \in [0, 1]$, *such that* $c_\lambda = \|Du_\lambda\|_0 < 1/3$. *Moreover, assume that*

$$2\gamma > \alpha.$$

Then the SDE (1.1), *for every* $x \in \mathbb{R}^d$, *has a unique solution* (X_t^x).

Moreover, assertions (i), (ii) *and* (iii) *of* Theorem 1.1 *hold.*

Proof. Note that $2\gamma > \alpha$ implies the condition $1 + \gamma > \alpha$ of Lemma 4.2.

We provide a direct proof of pathwise uniqueness and assertion (i). This uses Lemmas 4.2 and 4.1 together with L^p-estimates for stochastic integrals (see (4.2)). Statements (ii) and (iii) will be obtained by transforming (1.1) in a form suitable for applying the results in [14, Chapter 3].

Let us fix $t > 0$, $p \geq 2$ and consider two solutions X and Y of (1.1) starting at x and $y \in \mathbb{R}^d$ respectively. Note that X_t is not in L^p if $p \geq \alpha$ (compare with [14, Theorem 3.2]) but the difference $X_t - Y_t$ is a bounded process. Pathwise uniqueness and (1.4) (for any $p \geq 1$) follow if we prove

$$E\left[\sup_{0 \leq s \leq t} |X_s - Y_s|^p\right] \leq C(t)|x - y|^p, \quad x, y \in \mathbb{R}^d, \tag{4.8}$$

with a positive constant $C(t)$ independent of x and y. Indeed in the special case of $x = y$ estimate (4.8) gives uniqueness of solutions.

We have from Lemma 4.2, P-a.s.,

$$
\begin{aligned}
X_t - Y_t &= [x-y] + [u(x)-u(y)] + [u(Y_t)-u(X_t)] \\
&\quad + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} [u(X_{s-}+x) - u(X_{s-}) - u(Y_{s-}+x) + u(Y_{s-})] \tilde{N}(ds, dx) \\
&\quad + \lambda \int_0^t [u(X_s) - u(Y_s)]\, ds.
\end{aligned} \tag{4.9}
$$

Since $\|Du\|_0 \le 1/3$, we have $|u(X_t) - u(Y_t)| \le (1/3)|X_t - Y_t|$. It follows the estimate $|X_t - Y_t| \le (3/2)\Lambda_1(t) + (3/2)\Lambda_2(t) + (3/2)\Lambda_3(t) + (3/2)\Lambda_4$, where

$$
\begin{aligned}
\Lambda_1(t) &= \left| \int_0^t \int_{\{|x|>1\}} [u(X_{s-}+x) - u(X_{s-}) - u(Y_{s-}+x) + u(Y_{s-})] \tilde{N}(ds, dx) \right|, \\
\Lambda_2(t) &= \lambda \int_0^t |u(X_s) - u(Y_s)|\, ds, \\
\Lambda_3(t) &= \left| \int_0^t \int_{\{|x|\le 1\}} [u(X_{s-}+x) - u(X_{s-}) - u(Y_{s-}+x) + u(Y_{s-})] \tilde{N}(ds, dx) \right|, \\
\Lambda_4 &= |x-y| + |u(x) - u(y)| \le \frac{4}{3}|x-y|.
\end{aligned}
$$

Note that, P-a.s.,

$$
\sup_{0\le s\le t} |X_s - Y_s|^p \le C_p |x-y|^p + C_p \sum_{k=1}^{3} \sup_{0\le s\le t} \Lambda_k(s)^p.
$$

The main difficulty is to estimate $\Lambda_3(t)$. Let us first consider the other terms. By the Hölder inequality

$$
\sup_{0\le s\le t} \Lambda_2(s)^p \le c_1(p) t^{p-1} \int_0^t \sup_{0\le s\le r} |X_s - Y_s|^p\, dr.
$$

By (4.2) with $U = \{x \in \mathbb{R}^d : |x| > 1\}$ we find

$$
\begin{aligned}
&E\left[\sup_{0\le s\le t} \Lambda_1(s)^p\right] \\
&\le c(p) E\left[\left(\int_0^t ds \int_{\{|x|>1\}} |u(X_{s-}+x) - u(Y_{s-}+x) + u(Y_{s-}) - u(X_{s-})|^2 \nu(dx)\right)^{p/2}\right] \\
&\quad + c(p) E \int_0^t ds \int_{\{|x|>1\}} |u(X_{s-}+x) - u(Y_{s-}+x) + u(Y_{s-}) - u(X_{s-})|^p \nu(dx).
\end{aligned}
$$

Using $|u(X_{s-}+x) - u(Y_{s-}+x) + u(Y_{s-}) - u(X_{s-})| \le (2/3)|X_{s-} - Y_{s-}|$ and the Hölder inequality, we get

$$E\left[\sup_{0\le s\le t}\Lambda_1(s)^p\right]\le C_1(p)(1+t^{p/2-1})$$
$$\cdot\left(\int_{\{|x|>1\}}\nu(dx)+\left(\int_{\{|x|>1\}}\nu(dx)\right)^{p/2}\right)\int_0^t E\left[\sup_{0\le s\le r}|X_s-Y_s|^p\right]dr.$$

Let us treat $\Lambda_3(t)$. This requires the condition $2\gamma>\alpha$. By using (4.2) with $U=\{x\in\mathbb{R}^d : |x|\le 1,\ x\ne 0\}$ and also Lemma 4.1, we get

$$E\left[\sup_{0\le s\le t}\Lambda_3(s)^p\right]\le c(p)\|u\|_{1+\gamma}^p E\left[\left(\int_0^t ds\int_{\{|x|\le 1\}}|X_s-Y_s|^2|x|^{2\gamma}\nu(dx)\right)^{p/2}\right]$$
$$+c(p)\|u\|_{1+\gamma}^p E\int_0^t ds\int_{\{|x|\le 1\}}|X_s-Y_s|^p|x|^{\gamma p}\nu(dx).$$

We obtain

$$E\left[\sup_{0\le s\le t}\Lambda_3(s)^p\right]$$
$$\le C_2(p)(1+t^{p/2-1})\|u\|_{1+\gamma}^p$$
$$\cdot\left(\left(\int_{\{|x|\le 1\}}|x|^{2\gamma}\nu(dx)\right)^{p/2}+\int_{\{|x|\le 1\}}|x|^{\gamma p}\nu(dx)\right)\int_0^t E\left[\sup_{0\le s\le r}|X_s-Y_s|^p\right]dr,$$

where $\int_{\{|x|\le 1\}}|x|^{p\gamma}\nu(dx)<+\infty$, since $p\ge 2$ and $2\gamma>\alpha$. Collecting the previous estimates, we arrive at

$$E\left[\sup_{0\le s\le t}|X_s-Y_s|^p\right]\le C_p|x-y|^p+C_4(p)(1+t^{p-1})\int_0^t E\left[\sup_{0\le s\le r}|X_s-Y_s|^p\right]dr.$$

Applying the Gronwall lemma we obtain (4.8) with $C(t)=C_p\exp(C_4(p)(1+t^{p-1}))$. The assertion is proved.

Now we establish the homeomorphism property (ii) (cf. [14, Chapter 3], [1, Chapter 6] and [19, Section V.10]).

First note that, since $\|Du\|_0<1/3$, the classical Hadamard theorem (see [19, p. 330]) implies that the mapping $\psi:\mathbb{R}^d\to\mathbb{R}^d$, $\psi(x)=x+u(x)$, $x\in\mathbb{R}^d$, is a C^1-diffeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Moreover, $D\psi^{-1}$ is bounded on $\mathbb{R}^d$ and $\|D\psi^{-1}\|_0\le 1/(1-c_\lambda)<3/2$ thanks to

$$D\psi^{-1}(y)=[I+Du(\psi^{-1}(y))]^{-1}=\sum_{k\ge 0}(-Du(\psi^{-1}(y)))^k,\quad y\in\mathbb{R}^d. \tag{4.10}$$

Let $r \in (0, 1)$ and introduce the SDE

$$
(4.11)\qquad \begin{aligned} Y_t &= y + \int_0^t \tilde{b}(Y_s)\,ds \\ &\int_0^t \int_{\{|z|\le r\}} g(Y_{s-}, z)\tilde{N}(ds, dz) + \int_0^t \int_{\{|z|>r\}} g(Y_{s-}, z) N(ds, dz), \quad t \ge 0, \end{aligned}
$$

where $\tilde{b}(y) = \lambda u(\psi^{-1}(y)) - \int_{\{|z|>r\}} [u(\psi^{-1}(y) + z) - u(\psi^{-1}(y))]\nu(dz)$ and

$$
g(y, z) = u(\psi^{-1}(y) + z) + z - u(\psi^{-1}(y)), \quad y \in \mathbb{R}^d,\ z \in \mathbb{R}^d.
$$

Note that (4.11) is a SDE of the type considered in [14, Section 3.5]. Due to the Lipschitz condition, there exists a unique solution $Y^y = (Y_t^y)$ to (4.11). Moreover, using (4.4) and the formula

$$
L_t = \int_0^t \int_{\{|x|\le r\}} x\tilde{N}(ds, dx) + \int_0^t \int_{\{|x|>r\}} x N(ds, dx), \quad t \ge 0
$$

(due to the fact that ν is symmetric) it is not difficult to show that

$$
(4.12)\qquad \psi(X_t^x) = Y_t^{\psi(x)}, \quad x \in \mathbb{R}^d,\ t \ge 0.
$$

Thanks to (4.12) to prove our assertion, it is enough to show the homeomorphism property for Y_t^y. To this purpose, we will apply [14, Theorem 3.10] to equation (4.11). Let us check its assumptions.

Clearly, $\tilde{b}$ is Lipschitz continuous and bounded. Let us consider [14, condition (3.22)]. For any $y \in \mathbb{R}^d$, $z \in \mathbb{R}^d$, $|g(y, z)| \le |z|(1 + \|Du\|_0) \le K(z)$, with $K(z) = (4/3)|z|$ (recall that $\int_{|z|\le 1} |z|^2 \nu(dz) < \infty$); further by Lemma 4.1 and (4.10) we have, for any $y, y' \in \mathbb{R}^d$, $z \in \mathbb{R}^d$ with $|z| \le 1$,

$$
|g(y, z) - g(y', z)| \le L(z)|y - y'| \quad \text{where} \quad L(z) = C_1 \|u\|_{1+\gamma} |z|^\gamma,
$$

with $\int_{|z|\le 1} L(z)^2 \nu(dz) < \infty$, since $2\gamma > \alpha$. Note that we may fix $r > 0$ small enough in (4.11) in order that $K(r) + L(r) < 1$ (according to [14, Section 3.5], this condition is needed to study the homeomorphism property for equation (4.11) without $\int_0^t \int_{\{|z|>r\}} g(Y_{s-}, z) N(ds, dz)$; see also [14, Remark 1, Section 3.4]).

By [14, Theorem 3.10] in order to get the homeomorphism property, it remains to check that, for any $z \in \mathbb{R}^d$, the mapping:

$$
(4.13)\qquad y \mapsto y + g(y, z) \quad \text{is a homeomorphism from } \mathbb{R}^d \text{ onto } \mathbb{R}^d.
$$

Let us fix z. To verify the assertion, we will again apply the Hadamard theorem. We have

$$
D_y g(y, z) = [Du(\psi^{-1}(y) + z) - Du(\psi^{-1}(y))][D\psi^{-1}(y)]
$$

and so by (4.10) (since $\|Du\|_0 < 1/3$) we get $\|D_y g(\cdot,z)\|_0 \le 2c_\lambda/(1-c_\lambda) < 1$. We have obtained (4.13). By [14, Theorem 3.10] the homeomorphism property for Y_t^y follows and this gives the assertion.

Now we show that, for any $t \ge 0$, *the mapping*: $x \mapsto X_t^x$ *is of class* C^1 *on* $\mathbb{R}^d$, P-a.s. (see (iii)).

We fix $t > 0$ and a unitary vector e_k of the canonical basis in $\mathbb{R}^d$. We will show that there exists, P-a.s., the partial derivative $\lim_{s\to 0}(X_t^{x+se_k} - X_t^x)/s = D_{e_k}X_t^x$ and, moreover, that the mapping $x \mapsto D_{e_k}X_t^x$ is continuous on $\mathbb{R}^d$, P-a.s.

Let us consider the process $Y^y = (Y_t^y)$ which solves the SDE (4.11). If we prove that the mapping $y \mapsto Y_t^y$ is of class C^1 on $\mathbb{R}^d$, P-a.s., then we have proved the assertion. Indeed, P-a.s.,

$$D_{e_k}X_t^x = [D\psi^{-1}(Y_t^{\psi(x)})][DY_t^{\psi(x)}]D_{e_k}\psi(x), \quad x \in \mathbb{R}^d.$$

We rewrite (4.11) as

$$Y_t = y + \lambda \int_0^t u(\psi^{-1}(Y_r))\,dr + \int_0^t\int_{\mathbb{R}^d\setminus\{0\}} h(Y_{r-}, z)\tilde{N}(dr, dz) + L_t, \tag{4.14}$$

$t \ge 0$, $y \in \mathbb{R}^d$, where

$$h(y, z) = u(\psi^{-1}(y) + z) - u(\psi^{-1}(y)) = g(y, z) - z,$$

and note that the statement of [14, Theorem 3.4] about the differentiability property holds for SDEs of the form (4.14), provided that the coefficients $\lambda u \circ \psi^{-1}$ and h satisfy [14, conditions (3.1), (3.2), (3.8) and (3.9)]. Indeed the presence of L_t in the equation does not give rise to any difficulty. To check this fact, remark that, for any $t \ge 0$, $y \in \mathbb{R}^d$, $s \ne 0$, we have the equality

$$\begin{aligned}\frac{Y_t^{y+se_k} - Y_t^y}{s} = e_k + \Bigg(\lambda \int_0^t \frac{u(\psi^{-1}(Y_r^{y+se_k})) - u(\psi^{-1}(Y_r^y))}{s}\,dr \\ + \int_0^t\int_{\mathbb{R}^d\setminus\{0\}} \frac{h(Y_{r-}^{y+se_k}, z) - h(Y_{r-}^y, z)}{s}\tilde{N}(dr, dz)\Bigg),\end{aligned}$$

where L_t is disappeared. Thus we can apply the same argument which is used to prove [14, Theorem 3.4] (see also the proof of [14, Theorem 3.3]), i.e., we can provide estimates for

$$E\left[\sup_{0\le t\le T}\left|\frac{Y_t^{y+se_k} - Y_t^y}{s}\right|^p\right] \quad \text{and} \quad E\left[\sup_{0\le t\le T}\left|\frac{Y_t^{y+se_k} - Y_t^y}{s} - \frac{Y_t^{y'+s'e_k} - Y_t^{y'}}{s'}\right|^p\right],$$

$p \ge 2$, $s, s' \ne 0$, $y, y' \in \mathbb{R}^d$, by using (4.2) and the Gronwall lemma (remark that in [14] the term $s^{-1}(Y_t^{y+se_k} - Y_t^y)$ is denoted by $N_t(y,s)$), and then apply the Kolmogorov criterion in order to prove that $y \mapsto Y_t^y$ is of class C^1 on $\mathbb{R}^d$, P-a.s.

Let us check that $\lambda u \circ \psi^{-1}$ and h satisfy the assumptions of [14, Theorem 3.4] (i.e., respectively, [14, conditions (3.1), (3.2), (3.8) and (3.9)]). Conditions (3.1) and (3.2) are easy to check. Indeed $\lambda u(\psi^{-1}(\cdot))$ is Lipschitz continuous on $\mathbb{R}^d$ and, moreover, thanks to Lemma 4.1 and to the boundeness of $D\psi^{-1}$,

$$|h(y,z) - h(y',z)| \le C\|u\|_{1+\gamma}(1_{\{|z|\le 1\}}|z|^\gamma + 1_{\{|z|>1\}})|y-y'|, \quad z \in \mathbb{R}^d,$$

$y, y' \in \mathbb{R}^d$, with $\int_{\mathbb{R}^d}(1_{\{|z|\le 1\}}|z|^\gamma + 1_{\{|z|>1\}})^p \nu(dz) < \infty$, for any $p \ge 2$. In addition, $|h(y,z)| \le L_0(z)$, $z \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, where, since $\|Du\|_0 < 1/3$,

$$L_0(z) = \frac{1}{3}1_{\{|z|\le 1\}}|z| + 2\|u\|_0 1_{\{|z|>1\}} \quad \text{with} \quad \int_{\mathbb{R}^d} L_0(z)^p \nu(dz) < \infty, \quad p \ge 2.$$

Assumptions [14, (3.8) and (3.9)] are more difficult to check. They require that there exists some $\delta > 0$ such that (setting $l(x) = \lambda u(\psi^{-1}(x))$)

$$\text{(4.15)} \qquad \begin{aligned} &(1) \quad \sup_{y\in\mathbb{R}^d}|Dl(y))| < \infty; \quad |Dl(y) - Dl(y')| \le C|y-y'|^\delta, \quad y, y' \in \mathbb{R}^d. \\ &(2) \quad |D_y h(y,z))| \le K_1(z); \quad |D_y h(y,z) - D_y h(y',z)| \le K_2(z)|y-y'|^\delta, \end{aligned}$$

for any $y, y' \in \mathbb{R}^d$, $z \in \mathbb{R}^d$, with $\int_{\mathbb{R}^d} K_i(z)^p \nu(dz) < \infty$, for any $p \ge 2$, $i = 1, 2$. Such estimates are used in [14] in combination with the Kolmogorov continuity theorem to show the differentiability property.

Let us check (1) with $\delta = \gamma$, i.e., $Dl \in C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d)$. Since, for any $y \in \mathbb{R}^d$, $Dl(y) = \lambda Du(\psi^{-1}(y))D\psi^{-1}(y)$, we find that Dl is bounded on $\mathbb{R}^d$. Moreover, thanks to the following estimate (cf. (3.18))

$$[Dl]_\gamma \le \lambda\|Du\|_0[D\psi^{-1}]_\gamma + \lambda[Du]_\gamma\|D\psi^{-1}\|_0^{1+\gamma},$$

in order to prove the assertion it is enough to show that $[D\psi^{-1}]_\gamma < \infty$. Recall that for $d \times d$ real matrices A and B, we have $(I+A)^{-1} - (I+B)^{-1} = (I+A)^{-1}(B-A)(I+B)^{-1}$ (if $(I+A)$ and $(I+B)$ are invertible). We obtain, using also that $D\psi^{-1}$ is bounded,

$$\begin{aligned} |D\psi^{-1}(y) - D\psi^{-1}(y')| &= |[I + Du(\psi^{-1}(y))]^{-1} - [I + Du(\psi^{-1}(y'))]^{-1}| \\ &\le c_1[Du]_\gamma|y-y'|^\gamma, \quad y, y' \in \mathbb{R}^d \end{aligned}$$

and the proof of (1) is complete with $\gamma = \delta$. Let us consider (2). Clearly,

$$D_y h(y,z) = [Du(\psi^{-1}(y) + z) - Du(\psi^{-1}(y))]D\psi^{-1}(y)$$

verifies the first part of (2) with $K_1(z) = c_2\|Du\|_\gamma(1_{\{|z|\le 1\}}|z|^\gamma + 1_{\{|z|>1\}})$.

Let us deal with the second part of (2). We choose $\gamma' \in (0, \gamma)$ such that $2\gamma' > \alpha$ and first show that, for any $f \in C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d)$, we have

$$\text{(4.16)} \qquad [T_x f]_{\gamma-\gamma'} \le C[f]_\gamma|x|^{\gamma'}, \quad x \in \mathbb{R}^d,$$

where (as in Lemma 4.1) for any $x \in \mathbb{R}^d$, we define the mapping $T_x f : \mathbb{R}^d \to \mathbb{R}^d$ as $T_x f(u) = f(x+u) - f(u)$, $u \in \mathbb{R}^d$. Using also (3.14), we get

$$[T_x f]_{\gamma-\gamma'} \leq N[T_x f]_\gamma^{(\gamma-\gamma')/\gamma}[T_x f]_0^{1-(\gamma-\gamma')/\gamma} \leq cN[f]_\gamma |x|^{\gamma(1-(\gamma-\gamma')/\gamma)} \leq cN|x|^{\gamma'}[f]_\gamma,$$

for any $x \in \mathbb{R}^d$. By (4.16) we will prove (2) with $\delta = \gamma - \gamma' > 0$.

First consider the case when $|z| \leq 1$. By (4.16) with $Du = f$, we get

$$\begin{aligned}
&|D_y h(y,z) - D_y h(y',z)| \\
&= |Du(\psi^{-1}(y)+z) - Du(\psi^{-1}(y)) - Du(\psi^{-1}(y')+z) + Du(\psi^{-1}(y'))| \|D\psi^{-1}\|_0 \\
&\leq C_1[Du]_\gamma |y-y'|^\delta |z|^{\gamma'},
\end{aligned}$$

for any $y, y' \in \mathbb{R}^d$. Let now $|z| > 1$; we find, for $y, y' \in \mathbb{R}^d$ with $|y-y'| \leq 1$,

$$|D_y h(y,z) - D_y h(y',z)| \leq C_2[Du]_\gamma |y-y'|^\gamma \leq C_2[Du]_\gamma |y-y'|^{\gamma-\gamma'}.$$

On the other hand, if $|y-y'| > 1$, $|z| > 1$, $|D_y h(y,z) - D_y h(y',z)| \leq 4\|Du\|_0 |y-y'|^{\gamma-\gamma'}$. In conclusion, the second part of (2) is verified with $\delta = \gamma - \gamma'$ and

$$K_2(z) = C_3\|Du\|_\gamma (1_{\{|z|\leq 1\}}|z|^{\gamma'} + 1_{\{|z|>1\}}).$$

(note that $\int_{\mathbb{R}^d} K_2(z)^p \nu(dz) < \infty$, for any $p \geq 2$, since $2\gamma' > \alpha$). Since $C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d) \subset C_b^{\gamma-\gamma'}(\mathbb{R}^d, \mathbb{R}^d)$, we deduce that both (1) and (2) hold with $\delta = \gamma - \gamma'$.

Arguing as in [14, Theorem 3.4], we get that $y \mapsto Y_t^y$ is C^1, P-a.s., and this proves our assertion. We finally note that [14, Theorem 3.4] also provides a formula for $H_t^y = DY_t^y$, i.e.,

$$\begin{aligned}
H_t^y &= I + \lambda \int_0^t Du(\psi^{-1}(Y_s^y)) D\psi^{-1}(Y_s^y) H_s^y \, ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} (D_y h(Y_{s-}^y, z) H_{s-}^y) \tilde{N}(ds, dz), \quad t \geq 0, \ y \in \mathbb{R}^d.
\end{aligned}$$

The stochastic integral is meaningful, thanks to (2) in (4.15) and to the estimate $\sup_{0\leq s\leq t} E[|H_s|^p] < \infty$, for any $t > 0$, $p \geq 2$ (see [14, assertion (3.10)]). The proof is complete. □

Proof of Theorem 1.1. We may assume that $1-\alpha/2 < \beta < 2-\alpha$. We will deduce the assertion from Theorem 4.3.

Since $\alpha \geq 1$, we can apply Theorem 3.4 and find a solution $u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ to the resolvent equation (4.3) with $\gamma = \alpha - 1 + \beta \in (0,1)$. By the last assertion of Theorem 3.4, we may choose λ sufficiently large in order that $\|Du\|_0 = \|Du_\lambda\|_0 < 1/3$. The crucial assumption about γ and α in Theorem 4.3 is satisfied. Indeed $2\gamma = 2\alpha - 2 + 2\beta > \alpha$ since $\beta > 1 - \alpha/2$. By Theorem 4.3 we obtain the result. □

REMARK 4.4. Thanks to Theorem 1.1 we may define a stochastic flow associated to (1.1). To this purpose, note that by (ii) we have $X_t^x = \xi_t(x)$, $t \geq 0$, $x \in \mathbb{R}^d$, P-a.s., where ξ_t is a homeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Let ξ_t^{-1} be the inverse map. As in [14, Section 3.4], we set $\xi_{s,t}(x) = \xi_t \circ \xi_s^{-1}(x)$, $0 \leq s \leq t$, $x \in \mathbb{R}^d$.

The family $(\xi_{s,t})$ is a stochastic flow since verifies the following properties (P-a.s.):
(i) for any $x \in \mathbb{R}^d$, $(\xi_{s,t}(x))$ is a càdlàg process with respect to t and a càdlàg process with respect s;
(ii) $\xi_{s,t} \colon \mathbb{R}^d \to \mathbb{R}^d$ is an onto homeomorphism, $s \leq t$;
(iii) $\xi_{s,t}(x)$ is the unique solution to (1.1) starting from x at time s;
(iv) we have $\xi_{s,t}(x) = \xi_{u,t}(\xi_{s,u}(x))$, for all $0 \leq s \leq u \leq t$, $x \in \mathbb{R}^d$, and $\xi_{s,s}(x) = x$.

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