Nishihara, K.
Osaka J. Math.
49 (2012), 331–348

ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR A SYSTEM OF SEMILINEAR HEAT EQUATIONS AND THE CORRESPONDING DAMPED WAVE SYSTEM

KENJI NISHIHARA

(Received October 25, 2010)

Abstract

Consider the Cauchy problem for a system of weakly coupled heat equations, whose typical one is

$$\begin{cases} u_t - \Delta u = |v|^{p-1}v, \\ v_t - \Delta v = |u|^{q-1}u, \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N, \end{cases}$$

with $p, q \geq 1$, $pq > 1$. When p, q satisfy $\max((p+1)/(pq-1), (q+1)/(pq-1)) < N/2$, the exponents p, q are supercritical. In this paper we assort the supercritical exponent case to two cases. In one case both p and q are bigger than the Fujita exponent $\rho_F(N) = 1+2/N$, while in the other case $\rho_F(N)$ is between p and q. In both cases we obtain the time-global and unique existence of solutions for small data and their asymptotic behaviors. These observation will be applied to the corresponding system of the damped wave equations in low dimensional space.

1. Introduction

We consider the Cauchy problem for the weakly coupled system of heat equations

$$\text{(1.1)} \qquad \begin{cases} u_t - \Delta u = g(v), \\ v_t - \Delta v = f(u), & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, v)(0, x) = (u_0, v_0)(x), & x \in \mathbf{R}^N, \end{cases}$$

and the Cauchy problem for the corresponding system of damped wave equations

$$\text{(1.2)} \qquad \begin{cases} u_{tt} - \Delta u + u_t = g(v), \\ v_{tt} - \Delta v + v_t = f(u), & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, v, u_t, v_t)(0, x) = (u_0, v_0, u_1, v_1)(x), & x \in \mathbf{R}^N. \end{cases}$$

Here the typical examples of (f, g) are

$$\text{(1.3)} \qquad (f(u), g(v)) = (|u|^{q-1}u, |v|^{p-1}v), (|u|^q, |v|^p) \quad \text{etc.}$$

2010 Mathematics Subject Classification. Primary 35K45; Secondary 35B40.

for $p, q \geq 1,\ pq > 1$. For (1.1) and (1.2) there are many literatures [1, 3, 4, 16, 17] and [12, 14, 15, 18, 19] etc. respectively. See also references therein. Our final aim is to consider (1.2), but, by the "diffusion phenomenon" it is essential to investigate (1.1).

In [3] Escobedo and Herrero showed that for nonnegative, continuous and bounded data (u_0, v_0) and $(f(u), g(v)) = (u^q, v^p)$ with $p, q > 0,\ pq > 1$, the exponents p, q satisfying

$$\alpha := \max\left(\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right) = \frac{N}{2} \tag{1.4}$$

are critical and that

- if $\alpha \geq N/2$, then any nontrivial, nonnegative solutions to (1.1) blow up within a finite time,
- if $\alpha < N/2$, then both global solutions and blow-up solutions coexist, when the data are not restricted to be small.

In [18] Sun and Wang showed that, for $(f(u), g(v)) = (|u|^q, |v|^p)$ and $N = 1, 3$, the solutions to (1.2) with suitably small data globally exist when $\alpha < N/2$, while the global solutions do not exist for suitable data when $\alpha \geq N/2$.

In this paper we discuss the precise behaviors of solutions to both (1.1) and (1.2), including the optimal decay rates and the asymptotic profile, for small data when $\alpha < N/2$. Moreover, we observe the relation between the critical exponents for the system and the critical exponent

$$\rho_F = \rho_F(N) = 1 + \frac{2}{N} \tag{1.5}$$

for the scalar semilinear heat equation

$$w_t - \Delta w = |w|^{\rho-1}w, \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N. \tag{1.6}$$

As well-known, any nontrivial and nonnegative solution w blows up within a finite time when $\rho \leq \rho_F(N)$, while the solution globally exists for small data when $\rho > \rho_F(N)$ ([5, 6, 20]). The critical exponent is called the Fujita exponent, named after Fujita's pioneering work [5].

Here and after we assume without loss of generality

$$1 \leq p \leq q, \quad pq > 1, \tag{1.7}$$

and consider the small data global existence of solutions to (1.1) and (1.2). By (1.4) and (1.7) the supercritical exponents p, q are given by

$$\frac{q+1}{pq-1} < \frac{N}{2}. \tag{1.8}$$

Hence

$$q\left(p-\frac{2}{N}\right) > 1+\frac{2}{N} \quad \text{or} \quad q(p+1-\rho_F(N)) > \rho_F(N). \tag{1.9}$$

By (1.9)

$$p > \frac{2}{N} \quad \text{or} \quad p+1 > \rho_F(N). \tag{1.10}$$

If both p and q are less than or equal to $\rho_F(N)$, then

$$q\left(p-\frac{2}{N}\right) \le \rho_F(N)\cdot\left(\rho_F(N)-\frac{2}{N}\right) = \rho_F(N),$$

which contradicts (1.9). Hence either p or q is greater than $\rho_F(N)$ and, by (1.7), $q > \rho_F(N)$. Thus, the supercritical exponents p, q in (1.8) are decomposed to two cases:
CASE I $\quad q \ge p > \rho_F(N)$,
CASE II $\quad q > \rho_F(N) \ge p > \rho_F(N) - 1$ and $q > (p+1-\rho_F(N))^{-1}\rho_F(N)$.
The case $p \le \rho_F(N)$ happens in the supercritical case, which is a different point from the scalar heat equation. We also believe that the Cases I and II are more understandable than (1.8), related to the Fujita exponent.

The solution (u, v) to (1.1) is obtained by those of the integral equations

$$\begin{aligned} u(t,x) &= \int G(t,x-y)u_0(y)\,dy + \int_0^t\!\!\int G(t-\tau,x-y)g(v(\tau,y))\,dy\,d\tau, \\ v(t,x) &= \int G(t,x-y)v_0(y)\,dy + \int_0^t\!\!\int G(t-\tau,x-y)f(u(\tau,y))\,dy\,d\tau, \end{aligned} \tag{1.11}$$

where the domain $\mathbf{R}^N$ of integration is often abbreviated and the Gauss kernel is given by

$$G(t,x) = (4\pi t)^{-N/2}e^{-|x|^2/(4t)}, \quad |x|^2 = x_1^2+\cdots+x_N^2. \tag{1.12}$$

By (1.11) we can obtain a unique time-global solution and its asymptotic profile for small data in Case I. In [15] the Case I is treated for much more general system. But, our system is simple, which helps us to understand more interesting Case II. So, we first state the result in Case I.

Denoting $L^r \times L^r = L^r$ etc. simply without confusions, our first theorem is the following.

Theorem 1.1 (Case I). *Suppose that* $q \ge p > \rho_F(N)$ *and*

$$\begin{aligned} |f(u_1)-f(u_2)| &\le C(|u_1|^{q-1}+|u_2|^{q-1})|u_1-u_2| \quad (u_1,u_2 \in N(0)), \\ |g(v_1)-g(v_2)| &\le C(|v_1|^{p-1}+|v_2|^{p-1})|v_1-v_2| \quad (v_1,v_2 \in N(0)), \end{aligned} \tag{1.13}$$

for small neighborhood $N(0)$ *of* $0 \in \mathbf{R}$. *If* $(u_0, v_0) \in L^1 \cap L^\infty$ *and* $\|u_0, v_0\|_{L^1\cap L^\infty} = \|u_0, v_0\|_{L^1} + \|u_0, v_0\|_{L^\infty}$ *is suitably small, then there exists a unique time-global solution* $(u, v) \in [C([0, \infty); L^1 \cap L^\infty)]^2$ *to* (1.1), *which satisfies for* $1 \le r \le \infty$

$$\|(u, v)(t, \cdot)\|_{L^r} \le C(t+1)^{-(N/2)(1-1/r)}, \quad t \ge 0. \tag{1.14}$$

Moreover, as $t \to \infty$,

$$\|(u - \theta_1 G, v - \theta_2 G)(t, \cdot)\|_{L^r} = o(t^{-(N/2)(1-1/r)}), \tag{1.15}$$

where

$$\begin{aligned}(\theta_1, \theta_2) \\ = \left(\int u_0(x)\,dx + \int_0^\infty \int g(v(t,x))\,dx\,dt, \int v_0(x)\,dx + \int_0^\infty \int f(u(t,x))\,dx\,dt\right).\end{aligned} \tag{1.16}$$

For the proofs below the following inequality plays an important role.

Lemma 1.1 (Hausdorff and Young). *Let* $1 \le p, q, r \le \infty$ *with* $1/r = 1/p + 1/q - 1$. *If* $f \in L^p$ *and* $g \in L^q$, *then* $(f * g)(x) = \int_{\mathbf{R}^N} f(x-y)g(y)\,dy \in L^r$ *and*

$$\|f * g\|_{L^r} \le C\|f\|_{L^p}\|g\|_{L^q}. \tag{1.17}$$

In the Case II we heuristically observe the decay rates of solutions. Since $q > \rho_F(N)$, we once assume

$$v_0 \in L^1 \cap L^\infty \quad \text{and} \quad \int_0^\infty \|f(u(t,\cdot))\|_{L^1\cap L^\infty}\,dt \le C\int_0^\infty \|u(t,\cdot)\|^q_{L^q\cap L^\infty}\,dt < \infty. \tag{1.18}$$

Then, by (1.17), $(1.11)_2$ (which implies the second equation of (1.11)) yields

$$\|v(t)\|_{L^r} \le C(t+1)^{-(N/2)(1-1/r)}, \quad 1 \le r \le \infty. \tag{1.19}$$

Here and after, by C denote a generic constant independent of time t, whose value is changed from a line to the next line. Therefore, denoting $\rho_F(N)$ simply by ρ_F, we have

$$\begin{aligned}&\left\|\int_0^t \int G(t-\tau, x-y)g(v(\tau,y))\,dy\,d\tau\right\|_{L^1_x} \\ &\le C\int_0^t \|v(\tau)\|^p_{L^p}\,d\tau \le C\int_0^t (\tau+1)^{-(N/2)(p-1)}\,d\tau \\ &\le \begin{cases} C(t+1)^{(N/2)(\rho_F - p)}, & \rho_F - 1 < p < \rho_F, \\ C\log(t+2), & p = \rho_F, \end{cases}\end{aligned} \tag{1.20_1}$$

and

$$(1.20)_2 \qquad \begin{aligned} &\left\| \int_0^t \int G(t-\tau, x-y) g(v(\tau,y))\, dy\, d\tau \right\|_{L^\infty_x} \\ &\le C \int_0^{t/2} (1+t-\tau)^{-N/2} \|v(\tau)\|^p_{L^p \cap L^\infty}\, d\tau + \int_{t/2}^t 1 \cdot \|v(\tau)\|^p_{L^\infty}\, d\tau \\ &\le \begin{cases} C(t+1)^{-(N/2)(p+1-\rho_F)}, & \rho_F - 1 < p < \rho_F, \\ C(t+1)^{-N/2} \log(t+2), & p = \rho_F. \end{cases} \end{aligned}$$

That is, L^1-norm of u may grow up and L^∞-norm decays. Hence, when $\rho_F - 1 < p < \rho_F(N)$, $\left\| \int_0^t \int G(t-\tau, x-y) g(v(\tau,y))\, dy\, d\tau \right\|_{L^r_x} \le C$ if r satisfies

$$-\frac{N}{2}(p+1-\rho_F)\cdot(r-1) + \frac{N}{2}(\rho_F - p) = 0$$

by interpolation. Then

$$r = \left(p - \frac{2}{N}\right)^{-1} = (p+1-\rho_F)^{-1}.$$

Therefore, define r_0 by

$$(1.21) \qquad r_0 = \begin{cases} (p+1-\rho_F)^{-1}, & \rho_F - 1 < p < \rho_F, \\ 1+\delta, & p = \rho_F, \end{cases}$$

and assume

$$(1.22) \qquad u_0 \in L^{r_0},$$

where $\delta > 0$ is so small as $q > \rho_F(N)\cdot(1+\delta)$. Note that $(N/2r_0)(q - r_0) > 1$ not only for $p = \rho_F$, $r_0 = 1+\delta$ but also for $p < \rho_F$ since $q > r_0\rho_F$ in Case II. Thus, by $(1.11)_1$ and (1.20),

$$(1.23) \qquad \|u(t)\|_{L^r} \le C(t+1)^{-(N/2)(1/r_0 - 1/r)}, \quad r_0 \le r \le \infty.$$

Then, by applying (1.23) to $(1.11)_2$, we get

$$(1.24)_1 \qquad \begin{aligned} \|v(t)\|_{L^1} &\le \|v_0\|_{L^1} + C \int_0^t \|u(\tau)\|^q_{L^q}\, d\tau \\ &\le \|v_0\|_{L^1} + C \int_0^t (\tau+1)^{-N/2r_0(q-r_0)}\, d\tau \\ &\le C \quad \text{by } (N/2r_0)(q-r_0) > 1, \end{aligned}$$

and

$$
\begin{aligned}
\|v(t)\|_{L^\infty} &\le C(t+1)^{-N/2}\|v_0\|_{L^1\cap L^\infty} \\
&\quad + C\int_0^{t/2}(1+t-\tau)^{-N/2}\|u(\tau)\|^q_{L^q\cap L^\infty}\,d\tau + C\int_{t/2}^t \|u(\tau)\|^q_{L^\infty}\,d\tau \\
&\le C(t+1)^{-N/2} + C\int_0^{t/2}(1+t-\tau)^{-N/2}(\tau+1)^{-(N/2r_0)(q-r_0)}\,d\tau \qquad (1.24)_\infty \\
&\quad + \int_{t/2}^t (\tau+1)^{-Nq/2r_0}\,d\tau \\
&\le C(t+1)^{-N/2} \quad \text{by } (N/2r_0)(q-r_0) > 1.
\end{aligned}
$$

Hence we again obtain (1.18) which was once assumed. This fact implies the global existence of solutions to (1.11) in a suitable space. In fact, we have the following theorem.

Theorem 1.2 (Case II). *Let* $q > \rho_F(N) \ge p > \rho_F(N) - 1$ *and* $q > (p + 1 - \rho_F(N))^{-1}\rho_F(N)$. *Suppose that, for* r_0 *defined by* (1.21),

$$
(1.25)\quad (u_0, v_0) \in (L^{r_0} \cap L^\infty) \times (L^1 \cap L^\infty) \quad \textit{and} \quad \|u_0\|_{L^{r_0}\cap L^\infty} + \|v_0\|_{L^1\cap L^\infty} \quad \textit{is small.}
$$

Then there exists a unique solution $(u, v) \in C([0,\infty);\, L^{r_0} \cap L^\infty) \times C([0,\infty);\, L^1 \cap L^\infty)$, *which satisfies*

$$
(1.26)\qquad
\begin{aligned}
\|u(t)\|_{L^r} &\le C(t+1)^{-(N/2)(1/r_0-1/r)}, \quad r_0 \le r \le \infty, \\
\|v(t)\|_{L^r} &\le C(t+1)^{-(N/2)(1-1/r)}, \quad 1 \le r \le \infty.
\end{aligned}
$$

More precisely, for θ_2 *defined in* (1.16) *and* $1 \le r \le \infty$

$$
(1.27)\qquad \|(v - \theta_2 G)(t,\,\cdot)\|_{L^r} = o(t^{-(N/2)(1-1/r)}) \quad \textit{as} \quad t \to \infty,
$$

and, though $u(t,\,\cdot)$ *is not necessarily in* L^1, $(u - G * u_0)(t,\,\cdot)$ *is in* L^1 *and*

$$
(1.28)\qquad \|(u - G * u_0)(t,\,\cdot)\|_{L^r} \le
\begin{cases}
Ct^{-(N/2)(1/r_0-1/r)}, & p < \rho_F(N), \\
Ct^{-(N/2)(1-1/r)}\log t, & p = \rho_F(N),
\end{cases}
\quad \textit{as} \quad t \to \infty
$$

for $1 \le r \le \infty$. *Moreover, if* $g(v) \ge C^{-1}|v|^p$ *in* $v \in N(0)$, *the estimate from below*

$$
(1.28)'\qquad \|(u - G * u_0)(t,\,\cdot)\|_{L^r} \ge
\begin{cases}
C^{-1}t^{-(N/2)(1/r_0-1/r)}, & p < \rho_F(N), \\
C^{-1}t^{-(N/2)(1-1/r)}\log t, & p = \rho_F(N),
\end{cases}
$$

for $t \ge t_1 \gg 1$ *and* $1 \le r \le \infty$ *holds, too, provided that* $\theta_1 > 0$.

Similar consideration to the system for heat equations can be applied to the Cauchy problem (1.2) for the system of damped wave equations in low dimensional space. By

$w = [S_N(t)h](x)$, denote the solution to

$$\begin{cases} w_{tt} - \Delta w + w_t = 0, & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (w, w_t)(0, x) = (0, h)(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.29}$$

then the solution (u, v) to (1.2) is given by the integral equations

$$\begin{aligned} u(t, \cdot) &= S_N(t)(u_0 + u_1) + \partial_t(S_N(t)u_0) + \int_0^t S_N(t-\tau)g(v(\tau, \cdot))\,d\tau, \\ v(t, \cdot) &= S_N(t)(v_0 + v_1) + \partial_t(S_N(t)v_0) + \int_0^t S_N(t-\tau)f(u(\tau, \cdot))\,d\tau. \end{aligned} \tag{1.30}$$

DEFINITION 1.1. If $(u, v)(t, x)$ is the solution to the integral equations (1.30), then we call (u, v) the weak solution to the system (1.2).

The solution for the damped wave equation has the diffusion phenomenon, that is, the solution behaves like that for the corresponding diffusion equation as time tends to infinity. In fact, this is observed by the explicit formula of $S_N(t)h$ in each space dimension $N = 1, 2, 3$:

$(1.31)_1$

$$\begin{aligned} &[S_1(t)h](x) \\ &= \frac{e^{-t/2}}{2} \int_{|z|\le t} I_0\left(\frac{1}{2}\sqrt{t^2 - |z|^2}\right) h(x+z)\,dz \\ &= e^{-t/2} \cdot \frac{1}{2} \int_{|z|\le t} h(x+z)\,dz + \frac{e^{-t/2}}{2} \int_{|z|\le t} \left(I_0\left(\frac{1}{2}\sqrt{t^2 - |z|^2}\right) - 1\right) h(x+z)\,dz, \end{aligned}$$

$(1.31)_2$

$$\begin{aligned} &[S_2(t)h](x) \\ &= \frac{e^{-t/2}}{2\pi} \int_{|z|\le t} \frac{\cosh(\sqrt{t^2 - |z|^2}/2)}{\sqrt{t^2 - |z|^2}} h(x+z)\,dz \\ &= e^{-t/2} \cdot \frac{1}{2\pi} \int_{|z|\le t} \frac{h(x+z)}{\sqrt{t^2 - |z|^2}}\,dz + \frac{e^{-t/2}}{2\pi} \int_{|z|\le t} \frac{\cosh(\sqrt{t^2 - |z|^2}/2) - 1}{\sqrt{t^2 - |z|^2}} h(x+z)\,dz, \end{aligned}$$

$(1.31)_3$

$$\begin{aligned} &[S_3(t)h](x) \\ &= \frac{e^{-t/2}}{4\pi t} \partial_t \int_{|z|\le t} I_0\left(\frac{1}{2}\sqrt{t^2 - |z|^2}\right) h(x+z)\,dz \\ &= e^{-t/2} \cdot \frac{t}{4\pi} \int_{|\omega|=1} h(x + t\omega)\,d\omega + \frac{e^{-t/2}}{8\pi} \int_{|z|\le t} I_1\left(\frac{1}{2}\sqrt{t^2 - |z|^2}\right) \frac{h(x+z)}{\sqrt{t^2 - |z|^2}}\,dz. \end{aligned}$$

Here I_ν ($\nu = 0, 1, 2, \ldots$) is the modified Bessel function of order ν, which is given by

$$I_\nu(y) = \sum_{m=0}^{\infty} \frac{1}{m!\,(m+\nu)!}\Big(\frac{y}{2}\Big)^{2m+\nu}.$$

From the D'Alembert, Poisson and Kirchhoff formulas for the wave equations without dissipation, denoted by $[W_N(t)h](x)$, (1.31) has the form

$$S_N(t)h = e^{-t/2}\cdot W_N(t)h + J_{0N}(t)h, \quad N = 1, 2, 3, \tag{1.32}$$

together with its derivative

$$(1.33)_{1,2} \quad \begin{aligned} \partial_t(S_N(t)h) &= e^{-t/2}\Big\{-\frac{1}{2}W_N(t)h + \partial_t(W_N(t)h)\Big\} + \partial_t(J_{0N}(t)h) \\ &=: e^{-t/2}\cdot \tilde{W}_N(t)h + J_{1N}(t)h, \quad N = 1, 2, \end{aligned}$$

$$(1.33)_3 \quad \begin{aligned} \partial_t(S_3(t)h) &= e^{-t/2}\Big\{\Big(-\frac{1}{2}+\frac{t}{8}\Big)W_3(t)h + \partial_t(W_3(t)h)\Big\} \\ &\quad + \int_{|z|\le t}\partial_t\Bigg[\frac{e^{-t/2}I_1((1/2)\sqrt{t^2-|z|^2})}{8\pi\sqrt{t^2-|z|^2}}\Bigg]h(x+z)\,dz \\ &=: e^{-t/2}\cdot\tilde{W}_N(t)h + J_{1N}(t)h, \quad N = 3. \end{aligned}$$

For the solution formulas of damped wave equations, refer [2]. For the decomposition and the following estimates on the operators $J_{iN}(t)$ ($i = 1, 2$) and $W_N(t)$, $\tilde{W}_N(t)$, see [10] for $N = 1$, [8] for $N = 2$, and [13] for $N = 3$, where the properties of modified Bessel functions play an important role. See also [7, 19, 14] for $N = 2$ and [11] for general dimension N, where the method of Fourier transformation is applied.

Lemma 1.2. *Let J_{iN} ($i = 1, 2$) and W_N, $\tilde{W}_N$ with $N = 1, 2, 3$ be defined in* (1.30)–(1.33), *respectively. Then it holds that, for $1 \le q \le p \le \infty$*

$$\|J_{0N}(t)h\|_{L^p} \le C(t+1)^{-(N/2)(1/q-1/p)}\|h\|_{L^q}, \quad t \ge 0, \tag{1.34}$$

$$\|(J_{0N}(t) - e^{t\Delta})h\|_{L^p} \le Ct^{-(N/2)(1/q-1/p)-1}\|h\|_{L^q}, \quad t > 0, \tag{1.35}$$

$$\|J_{1N}(t)h\|_{L^p} \le C(t+1)^{-(N/2)(1/q-1/p)-1}\|h\|_{L^q}, \quad t \ge 0, \tag{1.36}$$

and, for $1 \le r \le \infty$

$$\|W_N(t)h\|_{L^r} \le C(t+1)\|h\|_{L^1\cap L^\infty}, \quad t \ge 0, \tag{1.37}$$

$$\|\tilde{W}_N(t)h\|_{L^r} \le C(t+1)\|h\|_{W^{[N/2],1}\cap W^{[N/2],\infty}}, \quad t \ge 0, \tag{1.38}$$

where $e^{t\Delta}h = G(t, \cdot\,) * h$ and $W^{m,r} = \{h;\, \partial_x^\alpha h \in L^r\ (|\alpha| \le m)\}$ with $\|h\|^2_{W^{m,r}} = \sum_{|\alpha|=0}^{m}\|\partial_x^\alpha h\|^2_{L^r}$.

By (1.35) we call $J_{0N}(t)h$ the "diffusion part", and $e^{-t/2}W_N(t)h$ the "wave part". Thus, the solution $S_N(t)h$ is decomposed to the wave part decaying rapidly and the diffusion part, and so we expect that the solution for the damped wave equation behaves like that for the corresponding diffusion equation. Thus, we will have the global existence of solutions to (1.30) when $N = 1, 2, 3$, though suitable regularity assumptions on the data are necessary.

However, when $N \geq 4$, $W_N(t)h$ includes ∇h etc. and (1.37) is not available. For example, when $N = 4$, we have

$$\|W_4(t)h\|_{L^r} \leq C(t+1)^2\|h, \nabla h\|_{L^1\cap L^\infty}.$$

Hence, it is difficult to obtain the solution to (1.30) by the iteration method, because the regularity problem happens.

Theorem 1.3 (Case I). *Suppose* (1.13) *with* $q \geq p > \rho_F(N)$ *with* $N = 1, 2, 3$ *and*

$$(1.39)\qquad \begin{aligned} &(u_0, u_1), (v_0, v_1) \in (W^{[N/2],1} \cap W^{[N/2],\infty}) \times (L^1 \cap L^\infty) =: Y_1 \\ &\text{with their norms}\quad \|u_0, u_1\|_{Y_1} + \|v_0, v_1\|_{Y_1} \ll 1, \end{aligned}$$

where $\|u_0, u_1\|_{Y_1} = \|u_0\|_{W^{[N/2],1}\cap W^{[N/2],\infty}} + \|u_1\|_{L^1\cap L^\infty}$. *Then there exists a unique global weak solution* $(u, v) \in [C([0, \infty); L^1 \cap L^\infty)]^2$ *to* (1.2), *which satisfies* (1.14) *and* (1.15) *with*

(1.16)′

$$(\theta_1, \theta_2) = \left(\int (u_0+u_1)(x)\,dx + \int_0^\infty\!\!\int g(v(t,x))\,dx\,dt, \int (v_0+v_1)(x)\,dx + \int_0^\infty\!\!\int f(u(t,x))\,dx\,dt\right).$$

Theorem 1.4 (Case II). *Let* $q > \rho_F(N) \geq p > \rho_F(N) - 1$ *and* $q > (p + 1 - \rho_F(N))^{-1}\rho_F(N)$ *with* $N = 1, 2, 3$. *Suppose that, for* r_0 *defined by* (1.21),

$$\begin{aligned} &(u_0, u_1) \in (W^{[N/2],r_0} \cap W^{[N/2],\infty}) \times (L^{r_0} \cap L^\infty) =: Y_2, \quad (v_0, v_1) \in Y_1 \\ &\text{with}\quad \|u_0, u_1\|_{Y_2} + \|v_0, v_1\|_{Y_1} \ll 1, \end{aligned}$$

where $\|u_0, u_1\|_{Y_2} = \|u_0\|_{W^{[N/2],r_0}\cap W^{[N/2],\infty}} + \|u_1\|_{L^{r_0}\cap L^\infty}$. *Then there exists a unique weak solution* $(u, v) \in C([0, \infty); L^{r_0} \cap L^\infty) \times C([0, \infty); L^1 \cap L^\infty)$, *which satisfies* (1.26), (1.27) *and* (1.28) *with* θ_2 *in* (1.16)′. *Moreover, if* $g(v) \geq C^{-1}|v|^p$ *in* $N(0)$, *then the estimate* (1.28)′ *from below holds for* $t \geq t_1 \gg 1$ *provided that* $\theta_1 > 0$ *in* (1.16)′.

For more general systems the Case I is treated in [15], and both Theorem 1.1 and Theorem 1.3 are essentially included in their results.

Our plan of this paper is simple. In the next section we consider the system of heat equations and prove Theorem 1.1 and Theorem 1.2. In Section 3 we treat the system of damped wave equations on the same line as the system of heat equations, using the decomposition of solution formula and Lemma 1.2.

2. System of heat equations

In this section we show the global existence of solution (u, v) and its behavior for small data.

Proof of Theorem 1.1. As stated above, the proof is essentially included in [15]. But, we sketch the proof for our simpler system, which helps us to prove Theorem 1.2 in the Case II.

Define the solution space

$$X := [C([0,\infty); L^1 \cap L^\infty)]^2 \quad \text{with} \quad \|u, v\|_X = \|u\|_{X_1} + \|v\|_{X_2}, \tag{2.1}$$

where $X_1 = X_2$ and for $i = 1, 2$

$$\|w\|_{X_i} = \sup_{[0,\infty)} \{\|w(t)\|_{L^1} + (t+1)^{N/2}\|w(t)\|_{L^\infty}\}. \tag{2.2}$$

The approximate sequence $\{(u^{(n)}, v^{(n)})\}$ $(n = 0, 1, 2, \ldots)$ in X is defined by

$$(u^{(0)}, v^{(0)})(t,x) = \left(\int G(t, x-y)u_0(y)\,dy, \int G(t, x-y)v_0(y)\,dy\right) \tag{2.3}$$

$$\begin{aligned} u^{(n+1)}(t,x) &= u^{(0)}(t,x) + \int_0^t \int G(t-\tau, x-y) g(v^{(n)}(\tau, y))\,dy\,d\tau, \\ v^{(n+1)}(t,x) &= v^{(0)}(t,x) + \int_0^t \int G(t-\tau, x-y) f(u^{(n)}(\tau, y))\,dy\,d\tau. \end{aligned} \tag{2.4}$$

We seek for the solution in

$$X_{2\varepsilon} = \{(u, v) \in X;\ \|u\|_{X_1} \le 2\varepsilon,\ \|v\|_{X_2} \le 2\varepsilon\} \tag{2.5}$$

for sufficiently small $\varepsilon > 0$. It suffices to show the following three assertions:
(i) if $\|u_0, v_0\|_{L^1\cap L^\infty} \ll 1$, then $(u^{(0)}, v^{(0)}) \in X_\varepsilon$,
(ii) if $(u^{(n)}, v^{(n)}) \in X_{2\varepsilon}$, then $(u^{(n+1)}, v^{(n+1)}) \in X_{2\varepsilon}$,
(iii) $\|u^{(n+1)} - u^{(n)}, v^{(n+1)} - v^{(n)}\|_X \le (1/2)\|u^{(n)} - u^{(n-1)}, v^{(n)} - v^{(n-1)}\|_X$ $(n = 1, 2, \ldots)$.
By (i), (ii), $(u^{(n)}, v^{(n)}) \in X_{2\varepsilon}$ for all n, so that (iii) holds and $\{(u^{(n)}, v^{(n)})\}$ is the Cauchy sequence in X. Thus, we have the desired global solution together with decay rates.

It is easy to show

$$(t+1)^{(N/2)(1-1/r)}\|(G * u_0)(t)\|_{L^r} \le c_0\|u_0\|_{L^1\cap L^\infty}, \quad t \ge 0$$

etc. for some constant $c_0 > 0$ and $1 \le r \le \infty$, which imply (i). For (ii),

$$\begin{aligned}\|u^{(n+1)}(t)\|_{L^1} &\le \|u^{(0)}(t)\|_{L^1} + C\int_0^t \|v^{(n)}(\tau)\|_{L^p}^p \, d\tau \\ &\le \|u^{(0)}(t)\|_{L^1} + C\int_0^t (\tau+1)^{-(N/2)(p-1)} \, d\tau \cdot \|v^{(n)}\|_{X_2}^p \\ &\le \|u^{(0)}(t)\|_{L^1} + C\|v^{(n)}\|_{X_2}^p,\end{aligned}$$

and, when $N = 1, 2$ and $N \ge 3$ with $p \ge 2$,

$$\begin{aligned}\|u^{(n+1)}(t)\|_{L^\infty} &\le \|u^{(0)}(t)\|_{L^\infty} + C\int_0^t (1+t-\tau)^{-N/2}\|v^{(n)}(\tau)\|_{L^1\cap L^\infty}^p \, d\tau \\ &\le \|u^{(0)}(t)\|_{L^\infty} + C\int_0^t (1+t-\tau)^{-N/2}(1+\tau)^{-(N/2)(p-1)} \, d\tau \|v^{(n)}\|_{X_2}^p \\ &\le \|u^{(0)}(t)\|_{L^\infty} + C(t+1)^{-N/2}\|v^{(n)}\|_{X_2}^p,\end{aligned}$$

since $(N/2)(p-1) > 1$ and $(N/2)(p-1) \ge N/2$ if $p \ge 2$. When $N \ge 3$, $\rho_F(N) < q < 2$ and $p' := p/(p-1) > 1$,

$$\begin{aligned}&\|u^{(n+1)}(t)\|_{L^\infty} \\ &\le \|u^{(0)}(t)\|_{L^\infty} + C\int_0^{t/2} (1+t-\tau)^{-N/2}\|v^{(n)}(\tau)\|_{L^1\cap L^\infty}^p \, d\tau \\ &\quad + C\int_{t/2}^t (1+t-\tau)^{-(N/2)(1/p')}\||v^{(n)}(\tau)|^p\|_{L^{p'}\cap L^\infty} \, d\tau \\ &\le \|u^{(0)}(t)\|_{L^\infty} + C\biggl(\int_0^{t/2} (1+t-\tau)^{-N/2}(1+\tau)^{-(N/2)(p-1)} \, d\tau \\ &\qquad + \int_{t/2}^t (1+t-\tau)^{-(N/2)(p-1)}(1+\tau)^{-N/2} \, d\tau\biggr)\|v^{(n)}(\tau)\|_{X_2}^p, \\ &\le \|u^{(0)}(t)\|_{L^\infty} + C(t+1)^{-N/2}\|v^{(n)}\|_{X_2}^p.\end{aligned}$$

Hence we have

$$\begin{aligned}\|u^{(n+1)}\|_{X_1} &\le \|u^{(0)}\|_{X_1} + C\|v^{(n)}\|_{X_2}^p \\ &\le (1 + C_1(2\varepsilon)^{p-1}2)\varepsilon,\end{aligned}$$

and, similarly,

$$\begin{aligned}\|v^{(n+1)}\|_{X_2} &\le \|v^{(0)}\|_{X_2} + C\|u^{(n)}\|_{X_1}^q \\ &\le (1 + C_2(2\varepsilon)^{q-1}2)\varepsilon.\end{aligned}$$

Taking $\varepsilon > 0$ so small as

$$\max\{C_1(2\varepsilon)^{p-1}2, C_2(2\varepsilon)^{q-1}2\} \le 1,$$

we have (ii). By the assumption (1.13), (iii) holds similar to (ii). Thus we obtain the solution (u, v) as a limit in X.

We now show the asymptotic behavior following [9]. First,

$$
\begin{aligned}
&(4\pi t)^{N/2}\left|\int G(t, x-y)u_0(y)\,dy - \left(\int u_0(y)\,dy\right)G(t, x)\right| \\
&\le (4\pi t)^{N/2}\int |G(t, x-y) - G(t, x)|\,|u_0(y)|\,dy \\
&\le \int_{|y|\le t^{1/4}} |e^{-|x-y|^2/(4t)} - e^{-|x|^2/(4t)}|\,|u_0(y)|\,dy + \int_{|y|>t^{1/4}} 2|u_0(y)|\,dy \\
&\to 0 \quad \text{as} \quad t\to\infty.
\end{aligned} \tag{2.6}
$$

Secondly,

$$
\begin{aligned}
&t^{N/2}\left|\int_0^t\int G(t-\tau, x-y)g(v(\tau, y))\,dy\,d\tau - \left(\int_0^\infty\int g(v(\tau, y))\,dy\,d\tau\right)G(t, x)\right| \\
&\le Ct^{N/2}\int_0^{t/2}\int |G(t-\tau, x-y) - G(t, x)|\,|v(\tau, y)|^p\,dy\,d\tau \\
&\quad + Ct^{N/2}\int_{t/2}^t \|v(\tau)\|_{L^\infty}^p\,d\tau + C\int_{t/2}^\infty \|v(\tau)\|_{L^\infty}^{p-1}\|v(\tau)\|_{L^1}\,d\tau \\
&\le Ct^{N/2}\left(\int_{\Omega_1} + \int_{\Omega_2}\right)|G(t-\tau, x-y) - G(t, x)|\,|v(\tau, y)|^p\,dy\,d\tau \\
&\quad + Ct^{N/2}\left(\frac{t}{2}+1\right)^{-(N/2)p} + C\int_{t/2}^\infty (\tau+1)^{-(N/2)(p-1)}\,d\tau.
\end{aligned} \tag{2.7}
$$

Here, the last two terms tend to zero as $t\to\infty$, and, for $0<\delta<1/2$ the domain of integration Ω_i $(i=1, 2)$ are defined as

$$
\Omega_1 = [0, \delta t]\times\{y\in\mathbf{R}^N;\ |y|\le \delta t^{1/2}\}, \quad \Omega_2 = ([0, t/2]\times\mathbf{R}^N)\setminus\Omega_1.
$$

Then, because of $\int_0^\infty\int |v(\tau, y)|^p\,dy\,d\tau<\infty$,

$$
\begin{aligned}
t^{N/2}\int_{\Omega_2} &\le Ct^{N/2}\int_{\Omega_2}(t-\tau)^{-N/2}|v(\tau, y)|^p dy\,d\tau + C\int_{\Omega_2}|v(\tau, y)|^p\,dy\,d\tau \\
&\le C\int_{\Omega_2}|v(\tau, y)|^p\,dy\,d\tau \to 0 \quad \text{as} \quad t\to\infty.
\end{aligned} \tag{2.8}
$$

In Ω_1, by setting $\tau = ts$ and $y=\sqrt{t}z$,

$$
\begin{aligned}
&t^{N/2}\cdot\sup_{(\tau, y)\in\Omega_1}|G(t-\tau, x-y) - G(t, x)| \\
&= \sup_{0\le s\le\delta,\ 0\le|z|\le\delta}\left|(4\pi(1-s))^{-N/2}e^{-|x/\sqrt{t}-z|^2/(4(1-s))} - (4\pi)^{-N/2}e^{-|x/\sqrt{t}|^2/4}\right|.
\end{aligned}
$$

Hence for any small $\eta > 0$, there exists $\delta_0 = \delta_0(\eta)$ independent of $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^N$ such that, if $0 < \delta < \delta_0$, then

$$t^{N/2} \cdot \sup_{(\tau,y)\in\Omega_1} |G(t-\tau, x-y) - G(t,x)| < \eta. \tag{2.9}$$

Thus, when $r = \infty$, by (2.6)–(2.9) we have (1.15) for u, and (1.15) for v samely. When $r = 1$, it is easier to show (1.15), which is omitted. □

Proof of Theorem 1.2. Define the solution space by

$$\begin{aligned} &X' = C([0,\infty); L^{r_0} \cap L^\infty) \times C([0,\infty); L^1 \cap L^\infty) \\ &\text{with} \quad \|u, v\|_{X'} = \|u\|_{X_1'} + \|v\|_{X_2}, \end{aligned} \tag{2.10}$$

where $\|\cdot\|_{X_2}$ is the same as in (2.2) and

$$\|u\|_{X_1'} = \sup_{[0,\infty)} \{\|u(t)\|_{L^{r_0}} + (t+1)^{N/2r_0}\|u(t)\|_{L^\infty}\}. \tag{2.11}$$

The approximate sequence $\{(u^{(n)}, v^{(n)})\}$ is defined by (2.3)–(2.4). The solution (u, v) to (1.11) is sought in

$$X'_{2\varepsilon} = \{(u,v) \in X';\ \|u\|_{X_1'} \le 2\varepsilon,\ \|v\|_{X_2} \le 2\varepsilon\} \tag{2.12}$$

for suitably small $\varepsilon > 0$. It is almost the same as in Theorem 1.1 to show that $\{(u^{(n)}, v^{(n)})\}$ is the Cauchy sequence in X'. So, we omit the details. Once we obtain (1.26), then $\int_0^\infty \int f(u(t,y))\,dy\,d\tau < \infty$, and (1.27) follows from the same line as in (2.6)–(2.9). We now estimate

$$I_u(t,x) := (u - G * u_0)(t,x) = \int_0^t \int G(t-\tau, x-y) g(v(\tau,y))\,dy\,d\tau. \tag{2.13}$$

For L^1-estimate we have

$$\begin{aligned} \|I_u(t)\|_{L^1} &\le C \int_0^t \|v(\tau)\|_{L^\infty}^{p-1} \|v(\tau)\|_{L^1}\,d\tau \le C \int_0^t (\tau+1)^{-(N/2)(p-1)}\,d\tau \\ &\le \begin{cases} C(t+1)^{-(N/2)(p-1)+1}, & p < \rho_F(N), \\ C\log(t+2), & p = \rho_F(N), \end{cases} \end{aligned} \tag{2.14$_1$}$$

and, for L^∞-estimate,

$(2.14)_\infty$

$$\begin{aligned}\|I_u(t)\|_{L^\infty} &\le C\int_{t/2}^t \|v(\tau)\|_{L^\infty}^p\,d\tau + C\int_0^{t/2}(1+t-\tau)^{-N/2}\|v(\tau)\|_{L^\infty}^{p-1}\|v(\tau)\|_{L^1}\,d\tau\\ &\le C(t+1)^{-(N/2)p+1} + C(t+1)^{-(N/2)}\cdot\begin{cases}(t+1)^{-(N/2)(p-1)+1}, & p<\rho_F(N),\\ \log(t+2), & p=\rho_F(N),\end{cases}\\ &\le \begin{cases}C(t+1)^{-(N/2)(p+1-\rho_F)}, & p<\rho_F(N),\\ C(t+1)^{-(N/2)}\log(t+2), & p=\rho_F(N).\end{cases}\end{aligned}$$

Thus we have (1.28). Note that L^∞-norm of $I_u(t)$ decays since $p>\rho_F(N)-1$.

Moreover, when $g(v)\ge C^{-1}|v|^p$, we have the estimate (1.28)′ from below. In fact, by (1.27)

$$v(t,x)-\theta_2 G(t,x) =: h(t,x),\quad \|h(t)\|_{L^r} = o(t^{-(N/2)(1-1/r)}),\quad t\to\infty, \tag{2.15}$$

for $1\le r\le\infty$. Hence

$$\begin{aligned}\|I_u(t)\|_{L^1} &= \int_0^t\int g(v(\tau,y))\,dy\,d\tau \ge C^{-1}\int_0^t\int|v(\tau,y)|^p dy\,d\tau\\ &\ge C^{-1}\theta_2^p\int_0^t\int G(\tau,y)^p dy\,d\tau - C\int_0^t\int h(\tau,y)^p dy\,d\tau\\ &\ge \begin{cases}C^{-1}t^{-(N/2)(p-1)+1}, & p<\rho_F(N),\\ C^{-1}\log t, & p=\rho_F(N),\end{cases}\quad t\ge t_1\gg 1,\end{aligned}$$

provided that $\theta_2>0$. Note that $-(N/2)(p-1)+1=-(N/2)(p-\rho_F)=-(N/2)(1/r_0-1)>0$ and L^1-norm of $I_u(t)$ grows up. The L^∞-estimate from below is similar to the above. □

3. System of damped wave equations

Denote the solution $w=[S_N(t)h](x)$ to (1.29), then the weak solution (u,v) to (1.2) is the solution to the integral equation (1.30) by Definition 1.1. As we observe in Introduction, when $N=1,2,3$, $S_N(t)h$ and its derivative are decomposed in the following forms:

$$S_N(t)h = e^{-t/2}\cdot W_N(t)h + J_{0N}(t)h, \tag{1.32}$$

$$\partial_t(S_N(t)h) = e^{-t/2}\cdot \tilde W_N(t)h + J_{1N}(t)h. \tag{1.33}$$

The L^p-L^q estimates on $W_N(t)h$, $\tilde{W}_N(t)h$, $J_{0N}(t)h$, $J_{1N}(t)h$ are given in Lemma 1.2. Therefore we rewrite (1.30) to

(3.1)$_1$

$$\begin{aligned} u(t,\ \cdot\ ) &= [e^{-t/2}(W_N(t)(u_0+u_1)+\tilde{W}(t)u_0)+J_{0N}(t)(u_0+u_1)+J_{1N}(t)u_0] \\ &\quad + \int_0^t e^{-(t-\tau)/2}\cdot W_N(t-\tau)g(v(\tau,\ \cdot\ ))\,d\tau + \int_0^t J_{0N}(t-\tau)g(v(\tau,\ \cdot\ ))\,d\tau \\ &=: D_0(u_0,u_1)(t,\ \cdot\ )+G_W(v)(t,\ \cdot\ )+G_J(v)(t,\ \cdot\ ), \end{aligned}$$

(3.1)$_2$

$$\begin{aligned} v(t,\ \cdot\ ) &= [e^{-t/2}(W_N(t)(v_0+v_1)+\tilde{W}(t)v_0)+J_{0N}(t)(v_0+v_1)+J_{1N}(t)v_0] \\ &\quad + \int_0^t e^{-(t-\tau)/2}\cdot W_N(t-\tau)f(u(\tau,\ \cdot\ ))\,d\tau + \int_0^t J_{0N}(t-\tau)f(u(\tau,\ \cdot\ ))\,d\tau \\ &=: D_0(v_0,v_1)(t,\ \cdot\ )+F_W(u)(t,\ \cdot\ )+F_J(u)(t,\ \cdot\ ). \end{aligned}$$

The approximate solution $\{(u^{(n)}, v^{(n)})\}$ $(n = 0, 1, 2, \ldots)$ is defined by

$$(u^{(0)}, v^{(0)})(t,x) = \big(D_0(u_0,u_1), D_0(v_0,v_1)\big)(t,x), \tag{3.2$_0$}$$

$$\begin{aligned} u^{(n+1)}(t,x) &= u^{(0)}(t,x)+G_W(v^{(n)})(t,x)+G_J(v^{(n)})(t,x), \\ v^{(n+1)}(t,x) &= v^{(0)}(t,x)+F_W(u^{(n)})(t,x)+F_J(u^{(n)})(t,x). \end{aligned} \tag{3.2$_n$}$$

Sketch of the proof of Theorem 1.3. As same as the proof of Theorem 1.1, we define the solution space X by (2.1) and (2.2), and seek for the solution to (3.1) in $X_{2\varepsilon}$ defined in (2.5) for small $\varepsilon > 0$. To do so, it suffices to show the following three assertions:
(i)$'$ If $\|u_0,u_1\|_{Y_1}+\|v_0,v_1\|_{Y_1} \ll 1$, then $(u^{(0)}, v^{(0)}) \in X_\varepsilon$.
and (ii), (iii). By (1.34), (1.36)–(1.38) in Lemma 1.2, for $r = 1, \infty$

$$\begin{aligned} \|u^{(0)}(t)\|_{L^r} &\le C_0(t+1)^{-(N/2)(1-1/r)}\|u_0,u_1\|_{Y_1}, \quad t \ge 0, \\ \|v^{(0)}(t)\|_{L^r} &\le C_0(t+1)^{-(N/2)(1-1/r)}\|v_0,v_1\|_{Y_1}, \quad t \ge 0, \end{aligned}$$

for some positive constant C_0. Hence

$$\|u^{(0)}, v^{(0)}\|_X \le C_0(\|u_0,u_1\|_{Y_1}+\|v_0,v_1\|_{Y_1}), \tag{3.3}$$

and so (i)$'$ holds. Since

$$\begin{aligned} \|G_W(v^{(n+1)})(t)\|_{L^1} &\le C\int_0^t e^{-(t-\tau)/2}(t-\tau+1)\|v^{(n)}(\tau)\|_{L^\infty}^{p-1}\|v^{(n)}(\tau)\|_{L^1}\,d\tau \\ &\le C\int_0^t (\tau+1)^{-(N/2)(p-1)}\,d\tau\cdot\|v^{(n)}\|_X^p \le C(2\varepsilon)^p, \end{aligned}$$

$$\|G_J(v^{(n+1)})(t)\|_{L^1} \le C\int_0^t (\tau+1)^{-(N/2)(p-1)}\,d\tau\cdot\|v^{(n)}\|_X^p \le C(2\varepsilon)^p,$$

$$\begin{aligned}\|G_W(v^{(n+1)})(t)\|_{L^\infty} &\le C\int_0^t e^{-(t-\tau)/2}(t-\tau+1)\|v^{(n)}(\tau)\|_{L^\infty}^p\,d\tau\\ &\le C\int_0^t e^{-(t-\tau)/2}(t-\tau+1)(\tau+1)^{-(N/2)p}\,d\tau\cdot\|v^{(n)}\|_X^p\\ &\le C(t+1)^{-N/2}(2\varepsilon)^p,\end{aligned}$$

and

$$\begin{aligned}&\|G_J(v^{(n+1)})(t)\|_{L^\infty}\\ &\le C\int_0^{t/2}(t-\tau+1)^{-N/2}\|v^{(n)}(\tau)\|_{L^\infty}^{p-1}\|v^{(n)}(\tau)\|_{L^1}\,d\tau + C\int_{t/2}^t \|v^{(n)}(\tau)\|_{L^\infty}^p\,d\tau\\ &\le C\left(\int_0^{t/2}(t-\tau+1)^{-N/2}(\tau+1)^{-(N/2)(p-1)}\,d\tau + \int_{t/2}^t(\tau+1)^{-(N/2)p}\,d\tau\right)\cdot\|v^{(n)}\|_X^p\\ &\le C(t+1)^{-N/2}(2\varepsilon)^p,\end{aligned}$$

we have the desired estimate

$$\|u^{(n+1)}\|_X \le \|u^{(0)}\|_X + C(2\varepsilon)^p \le (1+C_1(2\varepsilon)^{p-1}2)\varepsilon.$$

We also have the estimate on v, and the assertion (ii) by taking $\varepsilon>0$ as

$$\max\{C_1(2\varepsilon)^{p-1}2,\, C_2(2\varepsilon)^{q-1}2\}\le 1.$$

The assertion (iii) is almost similar. Thus we have the solution (u, v) in X satisfying (3.1), which decays with the rate (1.14).

For the asymptotic behavior the estimate (1.35) in Lemma 1.1 plays an important role. For examples,

$$\begin{aligned}&J_{0N}(t)(u_0+u_1) - \left(\int (u_0+u_1)(y)\,dy\right)G(t,\ \cdot)\\ &\quad= (J_{0N}(t)-e^{t\Delta})(u_0+u_1) + \int (G(t,\ \cdot - y) - G(t,\ \cdot))(u_0+u_1)(y)\,dy,\end{aligned}\tag{3.4}$$

and

(3.5)

$$\begin{aligned}&\int J_{0N}(t-\tau)g(v(\tau,\ \cdot))\,d\tau - \left(\int_0^\infty\int g(v(\tau,y))\,dy\,d\tau\right)G(t,\ \cdot)\\ &= \int_0^t (J_{0N}(t-\tau)-e^{(t-\tau)\Delta})g(v(\tau,\ \cdot))\,d\tau\\ &\quad+\left[\int_0^t\int G(t-\tau,\ \cdot - y)g(v(\tau,y))\,dy\,d\tau - \left(\int_0^\infty\int g(v(\tau,y))\,dy\,d\tau\right)G(t,\ \cdot)\right].\end{aligned}$$

Each first term in the right hand side of (3.4) and (3.5) decays fast by (1.35) and last terms in (3.4) and (3.5) decay fast, as shown in the proof of Theorem 1.1. The other terms in $(3.1)_1$ decays fast. Estimates on v are similar. Thus we have (1.15) with $(1.16)'$. □

The proof of Theorem 1.4 is a little bit complicated compared to that of Theorem 1.3, but it is almost similar to that of Theorem 1.2 by applying Lemma 1.2. So we omit the details.

ACKNOWLEDGEMENT. This work was supported in part by Grant-in-Aid for Scientific Research (C) 20540219 of Japan Society for the Promotion of Science.

References

[1] D. Andreucci, M.A. Herrero and J.J.L. Velázquez: *Liouville theorems and blow up behaviour in semilinear reaction diffusion systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 1–53.

[2] R. Courant and D. Hilbert: Methods of Mathematical Physics, II, Wiley, New York, 1989.

[3] M. Escobedo and M. A. Herrero: *Boundedness and blow up for a semilinear reaction-diffusion system*, J. Differential Equations **89** (1991), 176–202.

[4] M. Escobedo and H.A. Levine: *Critical blowup and global existence numbers for a weakly coupled system of reaction-diffusion equations*, Arch. Rational Mech. Anal. **129** (1995), 47–100.

[5] H. Fujita: *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I **13** (1966), 109–124.

[6] K. Hayakawa: *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan Acad. **49** (1973), 503–505.

[7] T. Hosono and T. Ogawa: *Large time behavior and* L^p-L^q *estimate of solutions of* 2*-dimensional nonlinear damped wave equations*, J. Differential Equations **203** (2004), 82–118.

[8] R. Ikehata, K. Nishihara and H. Zhao: *Global asymptotics of solutions to the Cauchy problem for the damped wave equation with absorption*, J. Differential Equations **226** (2006), 1–29.

[9] G. Karch: *Selfsimilar profiles in large time asymptotics of solutions to damped wave equations*, Studia Math. **143** (2000), 175–197.

[10] P. Marcati and K. Nishihara: *The* L^p-L^q *estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media*, J. Differential Equations **191** (2003), 445–469.

[11] T. Narazaki: L^p-L^q *estimates for damped wave equations and their applications to semi-linear problem*, J. Math. Soc. Japan **56** (2004), 585–626.

[12] T. Narazaki: *Global solutions to the Cauchy problem for the weakly coupled system of damped wave equations*, Discrete Contin. Dyn. Syst. Suppl. (2009), 592–601.

[13] K. Nishihara: L^p-L^q *estimates of solutions to the damped wave equation in* 3*-dimensional space and their application*, Math. Z. **244** (2003), 631–649.

[14] T. Ogawa and H. Takeda: *Global existence of solutions for a system of nonlinear damped wave equations*, Differential Integral Equations **23** (2010), 635–657, in press.

[15] T. Ogawa and H. Takeda: *Large time behavior of solutions for a system of nonlinear damped wave equations*, preprint.

[16] J. Rencławowicz: *Blow up, global existence and growth rate estimates in nonlinear parabolic systems*, Colloq. Math. **86** (2000), 43–66.

[17] S. Snoussi and S. Tayachi: *Global existence, asymptotic behavior and self-similar solutions for a class of semilinear parabolic systems*, Nonlinear Anal. **48** (2002), 13–35.

[18] F. Sun and M. Wang: *Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping*, Nonlinear Anal. **66** (2007), 2889–2910.

[19] H. Takeda: *Global existence and nonexistence of solutions for a system of nonlinear damped wave equations*, J. Math. Anal. Appl. **360** (2009), 631–650.

[20] F.B. Weissler: *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math. **38** (1981), 29–40.

Faculty of Political Science and Economics
Waseda University
Tokyo, 169-8050
Japan
e-mail: kenji@waseda.jp