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ON THE CONSTRUCTIONS OF FREE AND LOCALLY STANDARD $\mathbb{Z}_2$-TORUS ACTIONS ON MANIFOLDS

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Abstract

We introduce an elementary way of constructing principal $(\mathbb{Z}_2)^m$-bundles over compact smooth manifolds. In addition, we will define a general notion of locally standard $(\mathbb{Z}_2)^m$-actions on closed manifolds for all $m \geq 1$, and then give a general way to construct all such $(\mathbb{Z}_2)^m$-actions from the orbit space. Some related topology problems are also studied.

1. Introduction

If the group $(\mathbb{Z}_2)^m$ acts freely and smoothly on a closed manifold M^n, the orbit space Q^n is also a closed manifold. We can think of M^n either as a principal $(\mathbb{Z}_2)^m$-bundle over Q^n or as a regular covering over Q^n with deck transformation group $(\mathbb{Z}_2)^m$. In algebraic topology, we have a standard way to recover M^n from Q^n using the universal covering space of Q^n and the monodromy map of the covering (see [1]). However, it is not very easy to visualize the total space of the covering using that construction. Considering the speciality of $(\mathbb{Z}_2)^m$, it is desirable to have a new way of constructing such regular coverings from the orbit spaces which can really help us to visualize the total space more easily. In this paper, such a construction will be given with the name *glue-back construction*.

Another source of nice $\mathbb{Z}_2$-torus actions on manifolds are locally standard actions (see [2]). Suppose M^n is a closed manifold with a smooth locally standard $(\mathbb{Z}_2)^n$-action, let $X^n = M^n/(\mathbb{Z}_2)^n$ be the orbit space and $\pi\colon M^n \to X^n$ be the orbit map. It is well known that X^n is a nice n-manifold with corners, and if the action is not free, X^n will have boundary. The $(\mathbb{Z}_2)^n$-action determines a *characteristic function* ν_π (taking values in $(\mathbb{Z}_2)^n$) on the facets of X^n, which encodes the information of isotropy subgroups of the non-free orbits. In particular, when X^n is a convex simple polytope, there is a standard construction to recover M^n (up to equivariant homeomorphism) from the characteristic function ν_π on X^n (see [2]). But in general, if $H^1(X^n, \mathbb{Z}_2)$ is not trivial, we need an additional piece of data to recover M^n—a principal $(\mathbb{Z}_2)^n$-bundle ξ_π over

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X^n which encodes the information of the free orbits of the action (see [3]). However, the bundle information ξ_π has a quite different flavor from the characteristic function ν_π and is not so easy to be visualized in the orbit space X^n. In this paper, we will combine the characteristic function ν_π and the $(\mathbb{Z}_2)^n$-bundle ξ_π on X^n into a composite $(\mathbb{Z}_2)^n$-valued colorings (λ_π, μ_π) on a new manifold U^n (called a $\mathbb{Z}_2$-*core* of X^n), which is a nice manifold with corners obtained from X^n (but not uniquely). And up to equivariant homeomorphisms, we can recover M^n from the composite $(\mathbb{Z}_2)^n$-coloring (λ_π, μ_π) on U^n from a generalized glue-back construction.

Moreover, we can define a general notion of locally standard $(\mathbb{Z}_2)^m$-action on n-dimensional manifolds for all $m \geq 1$, which includes all free $(\mathbb{Z}_2)^m$-actions on n-dimensional manifolds. The glue-back construction can be applied in this general setting as well. So actually we do not assume $m = n$ at all in this paper.

The paper is organized as following. In Section 2, we will explain how to get a $\mathbb{Z}_2$-core V^n from a closed manifold Q^n and introduce an important structure on V^n called *involutive panel structure*. We will introduce several definitions concerning this structure to make our subsequent discussions precise and convenient. Some explicit examples will be analyzed to illustrate these definitions. In Section 3, we will introduce the glue-back construction from a $\mathbb{Z}_2$-core V^n of Q^n with a $(\mathbb{Z}_2)^m$-colorings. And we will show that any principal $(\mathbb{Z}_2)^m$-bundles over Q^n can be obtained in this way. Also the glue-back construction makes sense for any nice manifold with corners equipped with an involutive panel structure. Some properties of this construction will be studied along with some explicit examples. In Section 4, we will generalize the notion of $\mathbb{Z}_2$-core and glue-back construction to compact manifolds with boundary as well. Then in Section 5 we define a general notion of locally standard $(\mathbb{Z}_2)^m$-actions on closed n-manifolds for any $m \geq 1$ and apply the glue-back construction to this general setting. Especially, the notion of involutive panel structure is used to unify all our constructions. In addition, we will state some classification theorems of locally standard $(\mathbb{Z}_2)^m$-actions on closed n-manifolds up to (weak) equivariant homeomorphisms. In Section 6, we will discuss how to get some topological information (e.g. the number of connected components and orientability) of the glue-back construction of locally standard $(\mathbb{Z}_2)^m$-actions from the $(\mathbb{Z}_2)^m$-colorings. In the end, we will propose some problem for the further study.

The main idea of the paper is inspired by the description of locally standard $\mathbb{Z}_2$-torus manifolds in [3]. An aim of this paper is to establish a framework for studying general locally standard $(\mathbb{Z}_2)^m$-actions on n-manifolds in the future. In particular, the author will use the glue-back construction to study the Halperin–Carlsson conjecture for free $(\mathbb{Z}_2)^m$-actions on compact manifolds in a sequel paper. Also, the involutive panel structure defined in this paper might have some independent value.

In this paper, we denote the quotient group $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}_2$ and always think of $(\mathbb{Z}_2)^m$ as an additive group. In addition, we will use the following conventions:

(1) any manifold and submanifold in this paper is smooth;

(2) we always identify an embedded submanifold with its image;
(3) any $(\mathbb{Z}_2)^m$-actions on manifolds in this paper are smooth and effective.

2. $\mathbb{Z}_2$-core of a closed manifold and involutive panel structure

Suppose M^n is an n-dimensional closed manifold with a free $(\mathbb{Z}_2)^m$-action (m is an arbitrary positive integer), let $Q^n = M^n/(\mathbb{Z}_2)^m$ be the orbit space and $\pi\colon M^n \to Q^n$ be the orbit map. Then Q^n is also a closed manifold. In addition, we always assume Q^n is connected in this paper.

We can consider the orbit map $\pi\colon M^n \to Q^n$ either as a regular $(\mathbb{Z}_2)^m$ covering or as a principal $(\mathbb{Z}_2)^m$-bundle map. Note that M^n may not be connected in general.

It is well-known that up to bundle isomorphism, principal $(\mathbb{Z}_2)^m$-bundles over Q^n are one-to-one correspondent with elements of $H^1(Q^n, (\mathbb{Z}_2)^m)$. Then $\pi : M^n \to Q^n$ determines an element

$$\Lambda_\pi \in H^1(Q^n, (\mathbb{Z}_2)^m) \cong \mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m).$$

From another viewpoint, as a regular covering space, $\pi : M^n \to Q^n$ is determined by its monodromy map $\mathcal{H}_\pi : \pi_1(Q^n) \to (\mathbb{Z}_2)^m$. Since $(\mathbb{Z}_2)^m$ is an abelian group, we get an induce group homomorphism $\mathcal{H}_\pi^{ab}\colon H_1(Q^n, \mathbb{Z}_2) \to (\mathbb{Z}_2)^m$ which is exactly the Λ_π above. Moreover, by the Poincaré duality, we have $H_{n-1}(Q^n, \mathbb{Z}_2) \cong H_1(Q^n, \mathbb{Z}_2)$. So we obtain a group homomorphism $\Lambda_\pi^*\colon H_{n-1}(Q^n, \mathbb{Z}_2) \to (\mathbb{Z}_2)^m$.

The above analysis suggests us to construct a new geometric object from Q^n which can carry all the information of Λ_π (or Λ_π^*). First, we recall a well-known theorem in algebraic topology.

Theorem 2.1 (Hopf). *Let $f : M^m \to N^n$ be a smooth map between closed oriented manifolds and $L^{n-p} \subset N^n$ a closed, oriented submanifold of codimension p such that f is transverse to L. Write $u \in H^p(N)$ for the Poincaré dual of $[L]_N$, that is, $u \cap [N] = [L]_N$. Then $[f^{-1}(L)]_M = f^*(u) \cap [M]$. In other words: If u is Poincaré dual to $[L]_N$, then $f^*(u) \in H^p(M)$ is Poincaré dual to $[f^{-1}(L)]_M$. If using $\mathbb{Z}_2$ coefficient, we do not need to assume that M^m and N^n are orientable.*

Proof. Use the naturality of the Thom class of the tangent bundle. □

If $H^1(Q^n, \mathbb{Z}_2) = 0$, then any principal $(\mathbb{Z}_2)^m$-bundle over Q^n is trivial. So in the rest of this paper, we always assume $H^1(Q^n, \mathbb{Z}_2) \neq 0$. let $\{\varphi_1, \dots, \varphi_k\}$ be a basis of $H^1(Q^n,\mathbb{Z}_2)$, and let $\{\alpha_1,\dots,\alpha_k\}$ be the basis of $H_{n-1}(Q^n,\mathbb{Z}_2)$ that is dual to $\{\varphi_1,\dots,\varphi_k\}$ under the Poincaré duality.

Lemma 2.2. *$\alpha_1, \dots, \alpha_k$ can be represented by codimension one connected embedded submanifolds of Q^n.*

Proof. Since $H^1(Q^n, \mathbb{Z}_2) \cong [Q^n, K(\mathbb{Z}_2, 1)] = [Q^n, \mathbb{R}P^\infty] = [Q^n, \mathbb{R}P^{n+1}]$, an element $\varphi \in H^1(Q^n, \mathbb{Z}_2)$ corresponds to a homotopy class of maps $[f]\colon Q^n \to \mathbb{R}P^{n+1}$ such that $\varphi = f^*(\Phi)$ where Φ is a generator of $H^1(\mathbb{R}P^{n+1}, \mathbb{Z}_2)$. Then the $\mathbb{Z}_2$-homology class represented by a canonically embedded $\mathbb{R}P^n \subset \mathbb{R}P^{n+1}$ is the Poincaré dual of Φ. We can always assume that f is smooth and transverse to $\mathbb{R}P^n$. Then by above theorem, $\Sigma = f^{-1}(\mathbb{R}P^n)$ is a codimension 1 embedded submanifold in Q^n which is Poincaré dual to φ. So we can find codimension one embedded submanifolds $\Sigma_1, \dots, \Sigma_k$ which are Poincaré dual to $\varphi_1, \dots, \varphi_k$ respectively. In addition, we can always choose $\Sigma_1, \dots, \Sigma_k$ to be connected. Indeed, for $n = 1, 2$, this is obviously true. And when $n \geq 3$, if some Σ_i is not connected, we can connect all its components via thin tubes in Q^n, which will not change the homology class of Σ_i in $H_{n-1}(Q^n, \mathbb{Z}_2)$. $\square$

A collection of codimension-one embedded closed submanifolds $\{\Sigma_1, \dots, \Sigma_k\}$ is called a *$\mathbb{Z}_2$-cut system* of Q^n if they satisfy the following conditions:

(1) the homology classes $[\Sigma_1], \dots, [\Sigma_k]$ form a $\mathbb{Z}_2$-linear basis of $H_{n-1}(Q^n, \mathbb{Z}_2)$.

(2) $\Sigma_1, \dots, \Sigma_k$ are in general position in Q^n which means that:

(a) $\Sigma_1, \dots, \Sigma_k$ intersect transversely with each other and,

(b) if $\Sigma_{i_1} \cap \cdots \cap \Sigma_{i_s}$ is not empty, then it is an embedded submanifold of Q^n with codimension s.

Now we choose a small tubular neighborhood $N(\Sigma_i)$ of each Σ_i in Q^n, and then remove the interior of each $N(\Sigma_i)$ from Q^n. The manifold that we get is:

$$V^n = Q^n - \bigcup_{i=1}^{k} int(N(\Sigma_i))$$

which is called a *$\mathbb{Z}_2$-core* of Q^n from cutting Q^n open along $\Sigma_1, \dots, \Sigma_k$. The boundary of V^n is $\partial\big(\bigcup_i N(\Sigma_i)\big)$. We call $\partial N(\Sigma_i)$ the *cut section* of Σ_i in Q^n.

Notice that the projection $\eta_i\colon \partial N(\Sigma_i) \to \Sigma_i$ is a double cover, either trivial or nontrivial. Let $\bar{\tau}_i$ be the generator of the deck transformation of η_i. Then $\bar{\tau}_i$ is a free involution on $\partial N(\Sigma_i)$, i.e. $\bar{\tau}_i$ is a homeomorphism with no fixed point and $\bar{\tau}_i^2 = id$.

The boundary of V^n is tessellated by $(n-1)$-dimensional compact connected manifolds (with boundary) called *facets* of V^n. Any connected component of the intersection of some facets is called a (*closed*) *face* of V^n. Since $\Sigma_1, \dots, \Sigma_k$ are in general position in Q^n, so V^n is a *nice manifold with corners*, which means that each codimension l face of V^n is in the intersection of exactly l facets. For a comprehensive introduction of manifolds with corners and related concepts, see [4] and [5].

REMARK 2.3. V^n might not have *vertices* (0-dimensional strata) on the boundary. for example, if $Q^n = S^{n-1} \times S^1$ $(n \geq 3)$, cutting Q^n along $S^{n-1} \times \{1\}$ gives a $\mathbb{Z}_2$-core $V^n = S^{n-1} \times [0, 1]$ of Q^n whose boundary consists of two disjoint S^{n-1}.

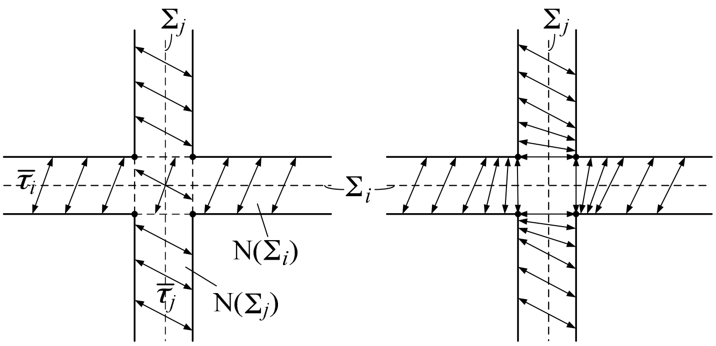

Fig. 1. Local deformation of $\bar{\tau}_i$'s.

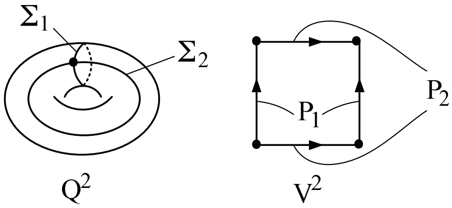

Fig. 2. A $\mathbb{Z}_2$-core of torus.

In addition, we call the union of facets of V^n that belong to $\partial N(\Sigma_i)$ a *panel*, denoted by P_i (see Fig. 2). So $\{P_1, \dots, P_k\}$ forms a panel structure on V^n. Recall that a *panel structure* on a topological space Y is a locally finite family of closed subspaces $\{Y_\alpha\}_{\alpha\in\mathcal{A}}$ indexed by some set $\mathcal{A}$. Each Y_α is called a *panel* of Y (see [5]).

Notice that the involution $\bar{\tau}_i$ may not map $P_i \subset \partial N(\Sigma_i)$ into P_i. This is because that there might be some $N(\Sigma_j)$ so that $\bar{\tau}_i$ and $\bar{\tau}_j$ do not commute at the intersections $\partial N(\Sigma_i) \cap \partial N(\Sigma_j)$ (see the left picture in Fig. 1). But the following lemma shows that we can always deform $\bar{\tau}_i$ and $\bar{\tau}_j$ locally by isotopies to make them commute at $\partial N(\Sigma_i) \cap \partial N(\Sigma_j)$.

Lemma 2.4. *We can deform $\bar{\tau}_i$'s around the intersections of $\partial N(\Sigma_i)$'s so that after the deformations, we have*:
(i) *each $\bar{\tau}_i$ is still a free involution $\partial N(\Sigma_i) \to \partial N(\Sigma_i)$ and the quotient $\partial N(\Sigma_i)/\langle x \sim \bar{\tau}_i(x)\rangle \cong \Sigma_i$;*
(ii) *for any $1 \le i, j \le k$, $\partial N(\Sigma_i) \cap \partial N(\Sigma_j)$ becomes an invariant set of both $\bar{\tau}_i$ and $\bar{\tau}_j$;*
(iii) *for any point $x \in \partial N(\Sigma_i) \cap \partial N(\Sigma_j)$, $\bar{\tau}_i(\bar{\tau}_j(x)) = \bar{\tau}_j(\bar{\tau}_i(x))$.*

Proof. For $\forall p \in \Sigma_i$, let $T_p\Sigma_i$ be the tangent plane of Σ_i at p in Q^n. Suppose $\Sigma_{i_1} \cap \cdots \cap \Sigma_{i_s}$ is nonempty. For any $p \in \Sigma_{i_1} \cap \cdots \cap \Sigma_{i_s}$, there exists an open neighborhood U of p and a homeomorphism $\phi\colon U \to \mathbb{R}^n$ such that $\phi(p) = 0$ and for any $i \in \{i_1, \dots, i_s\}$, we have

- $\phi(\Sigma_i \cap U)$ is the coordinate hyperplane $H_i = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = 0\}$. So $\phi(\Sigma_{i_1} \cap \cdots \cap \Sigma_{i_s} \cap U) = H_{i_1} \cap \cdots \cap H_{i_s}$;
- $\phi(N(\Sigma_i) \cap U) = H_i \times [-1, 1] \subset \mathbb{R}^n$;
- in the chart (U, ϕ), $\bar{\tau}_i$ defines a homeomorphism $f_i\colon H_i \times \{1\} \to H_i \times \{-1\}$.

Then we can deform $\bar{\tau}_i$'s via some isotopies in U such that they satisfy our requirements (i)–(iii) locally in U. Indeed, let r_i be the reflection of $\mathbb{R}^n$ about the hyperplane H_i. Then we can isotope f_i such that for any $j \in \{i_1, \dots, i_s\}$ with $j \neq i$, we have

$$f_i(x) = r_i(x), \quad \text{for any} \quad x \in H_j \times \{\pm 1\} \cap H_i \times \{1\}. \tag{1}$$

Then these $\bar{\tau}_i$'s obviously meet our requirements. Moreover, since the (i)–(iii) are coordinate-independent properties, we can carry out the deformations of these $\bar{\tau}_i$'s chart by chart around $\Sigma_{i_1} \cap \cdots \cap \Sigma_{i_s}$ until the (i)–(iii) are satisfied at all places. In addition, we should do the deformation of $\bar{\tau}_i$'s in the charts around the higher degree intersection points first, then extend to the charts around lower degree intersection points. In the end, we will get $\bar{\tau}_i$'s which satisfy all the requirements (i)–(iii). We remark that doing the isotopy of $\bar{\tau}_i$'s in a chart might slightly alter what we have previous done in another chart, but since the (i)–(iii) are coordinate-independent properties, the altering will not cause any inconsistency in our construction. □

After the local deformations of $\bar{\tau}_i$'s described in the preceding lemma, the restriction of each $\bar{\tau}_i$ on $P_i \subset \partial N(\Sigma_i)$ defines a free involution on P_i, denoted by τ_i. Because of the existence of these τ_i's, we call the set of panels of V^n an *involutive panel structure*. We will always assume that V^n has this involutive panel structure in the rest of the paper. Note that $\{\tau_i\colon P_i \to P_i\}_{1 \le i \le k}$ satisfy:

- τ_i maps a face f of P_i to a face f' of P_i (it is possible that $f' = f$ though);
- $\tau_i(P_i \cap P_j) \subset P_i \cap P_j$ for all $1 \le i, j \le k$;
- $\tau_i \circ \tau_j = \tau_j \circ \tau_i\colon P_i \cap P_j \to P_i \cap P_j$ for all $1 \le i, j \le k$.

For any $I = \{i_1, \dots, i_s\} \subset \{1, \dots, k\}$ with $|I| = s \ge 1$, we define:

$$P_I := P_{i_1} \cap \cdots \cap P_{i_s} \subset V^n, \quad \Sigma_I := \Sigma_{i_1} \cap \cdots \cap \Sigma_{i_s} \subset Q^n. \tag{2}$$

When $I = \varnothing$, we define $P_\varnothing = V^n$ and $\Sigma_\varnothing = Q^n$. If P_I with $|I| \ge 2$ is nonempty, it is called a *subpanel* of V^n. Notice that the P_I is empty whenever $|I| > n$. Although Q^n is assumed to be connected, the Σ_I may not be connected.

For any point $x \in P_i$, let $x^*_{P_i} = \tau_i(x) \in P_i$. We call $x^*_{P_i}$ the *twin point* of x in P_i. Obviously, $x^*_{P_i} \neq x$ since τ_i here is free.

Generally, for a face $f \subset P_i$, if the face $f^*_{P_i} = \tau_i(f)$ is disjoint from f, it is called the *twin face* of f in P_i. Otherwise, f is called *self-involutive* in P_i in the sense that $\tau_i(f) = f$. In particular, if a facet F of V^n is not self-involutive, it has a unique twin facet F^* which belongs to the same panel as F. We call $\hat{F} := F \cup F^*$ a *facet pair*.

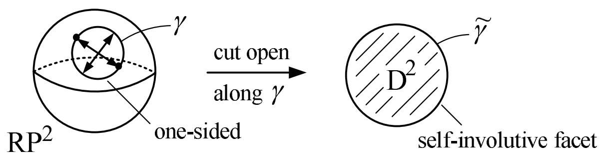

Fig. 3. A $\mathbb{Z}_2$-core of real projective plane.

As an embedded submanifold of Q^n, Σ_i could be two-sided or one-sided, which is determined by the orientability of the normal bundle of Σ_i in Q^n. If Σ_i is two-sided, any facet F in P_i has a twin facet F^* (see Fig. 2). But if Σ_i is one-sided, some facet in P_i might be self-involutive (see Fig. 3).

REMARK 2.5. The facets in the same panel of V^n are pairwise disjoint since each Σ_i in the $\mathbb{Z}_2$-cut system has no self-intersections. But a panel of V^n may consist of more than one facet pair (see the Example 1 below).

If we identify any points of V^n with all their twin points in V^n, we will get a manifold denoted by $\hat{Q}^n$. Let $\varrho\colon V^n \to \hat{Q}^n$ be the quotient map.

Lemma 2.6. *There exists a homeomorphism $h\colon \hat{Q}^n \to Q^n$ with $h(\varrho(P_I)) = \Sigma_I$ for any $I \subset \{1, \dots, k\}$.*

Proof. By our construction of τ_i, it is easy to see that $\varrho(P_i) \cong \Sigma_i$ for $1 \le \forall i \le k$. In addition, there exists a neighborhood $N(\partial V^n)$ of ∂V^n in V^n with $N(\partial V^n) \cong \partial V^n \times [0, \varepsilon]$ so that $\varrho(N(\partial V^n)) \subset \hat{Q}^n$ is homeomorphic to $\bigcup_{i=1}^k N(\Sigma_i) \subset Q^n$. Let $U^n = Q^n - int\big(\bigcup_{i=1}^k N(\Sigma_i)\big)$. Then we can think of $\hat{Q}^n$ (or Q^n) as the gluing of $N(\partial V^n)$ (or $\bigcup_{i=1}^k N(\Sigma_i)$) with U^n along their boundary, that is:

$$\hat{Q}^n = \varrho(N(\partial V^n)) \bigcup_{\varphi_1} U^n, \quad Q^n = \left(\bigcup_{i=1}^k N(\Sigma_i)\right) \bigcup_{\varphi_2} U^n,$$

where $\varphi_1\colon \partial(\varrho(N(\partial V^n))) \to \partial U^n$ and $\varphi_2\colon \partial\big(\bigcup_{i=1}^k N(\Sigma_i)\big) \to \partial U^n$ are homeomorphisms. If we identify $\partial(\varrho(N(\partial V^n)))$ with $\partial\big(\bigcup_{i=1}^k N(\Sigma_i)\big)$, φ_1 and φ_2 are actually isotopic because the local deformations we make on τ_i's are all isotopies of homeomorphisms. So we can construct a homeomorphism $h\colon \hat{Q}^n \to Q^n$ from an isotopy between φ_1 and φ_2, which satisfies our requirement. $\square$

We call $\rho = h \circ \varrho\colon V^n \to Q^n$ the *restoring map* of V^n. Then $P_I = \rho^{-1}(\Sigma_I)$ for any $I \subset \{1, \dots, k\}$. Obviously, for any point x in the relative interior of $P_{i_1} \cap \cdots \cap P_{i_s}$, we have:

$$\rho^{-1}(\rho(x)) = \{\tau_{i_s}^{\varepsilon_s} \circ \cdots \circ \tau_{i_1}^{\varepsilon_1}(x);\ \varepsilon_j \in \{0, 1\}, 1 \le j \le s\},$$

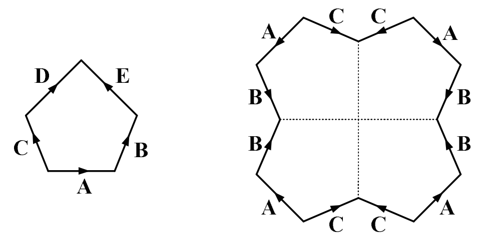

Fig. 4. A $\mathbb{Z}_2$-core of $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

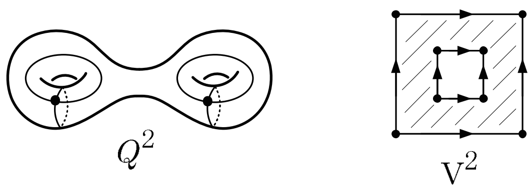

Fig. 5. A $\mathbb{Z}_2$-core of $T^2 \# T^2$.

where $\tau^0_{i_j} := id$. It is easy to see that $\rho^{-1}(\rho(x))$ consists of exactly 2^s different points in V^n. Any point $x' \in \rho^{-1}(\rho(x))$ (including x itself) is called a *duplicate point* of x in V^n.

EXAMPLE 1. Suppose Q^n is a small cover over some simple polytope. It is well known that the $\mathbb{Z}_2$-homology classes of Q^n can all be represented by some special embedded submanifolds of Q^n, called *facial submanifolds* (see [2] and [6]). And cutting Q^n open along a collection of facial submanifolds of Q^n will give us a connected $\mathbb{Z}_2$-core V^n of Q^n. Fig. 4 shows such an example in dimension 2 where $Q^2 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ is a small cover over a pentagon. A $\mathbb{Z}_2$-core of Q^2 is an octagon where the four edges marked by "**A**" belong to the same panel.

REMARK 2.7. A closed connected manifold Q^n may have a $\mathbb{Z}_2$-core V^n with $H^1(V^n, \mathbb{Z}_2) \neq 0$. For example, the $\mathbb{Z}_2$-core of $Q^2 = T^2 \# T^2$ shown in Fig. 5 is homeomorphic to an annulus.

Next, we define a general notion of involutive panel structure for any nice manifolds with corners. The involutive panel structure on a $\mathbb{Z}_2$-core V^n constructed above is just a special case of this general notion.

DEFINITION 2.8 (Involutive panel structure). Suppose W^n is a nice manifold with corners (may not be connected). Suppose the boundary of W^n is the union of several panels $P_1, \ldots, P_k$ which satisfy the following conditions:
(a) each panel P_i is a disjoint union of facets of W^n and each facet is contained in exactly one panel;

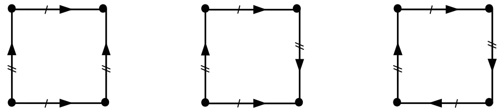

Fig. 6. Three different involutive panel structures on a square.

(b) there is an involution τ_i on each P_i (i.e. τ_i is a homeomorphism with $\tau_i^2 = id_{P_i}$) which sends a face $f \subset P_i$ to a face $f' \subset P_i$ (it is possible that $f' = f$);
(c) for all $i \neq j$, $\tau_i(P_i \cap P_j) \subset P_i \cap P_j$ and $\tau_i \circ \tau_j = \tau_j \circ \tau_i \colon P_i \cap P_j \to P_i \cap P_j$.

Then we say that W^n has an *involutive panel structure* defined by $\{P_i, \tau_i\}_{1 \le i \le k}$ on the boundary. Note here, we do not require that the involution τ_i on P_i is free.

Similar to the $\mathbb{Z}_2$-core V^n, for any $x \in P_i \subset W^n$, we call $\tau_i(x)$ the twin point x in P_i. Moreover, if x is in the relative interior of $P_{i_1} \cap \cdots \cap P_{i_s}$, any $\tau_{i_s}^{\varepsilon_s} \circ \cdots \circ \tau_{i_1}^{\varepsilon_1}(x)$ where $\varepsilon_i \in \{0, 1\}$ is called a *duplicate point* of x in W^n. But in this case, it is not necessarily that x has exactly 2^s duplicate points (even if each τ_i on P_i is free, see Fig. 10). Also we can define subpanels for W^n as in (2).

REMARK 2.9. A nice manifold with corners W^n may admit many different involutive panel structures on the boundary (for example see Fig. 6).

EXAMPLE 2. Suppose X^n is a nice manifold with corners, and let $F_1, \ldots, F_l$ be all the facets of X^n. We can think of X^n having a *trivial involutive panel structure* which is defined by: for any $1 \le i \le l$, $P_i = F_i$ and the involution $\tau_i = id_{F_i} \colon F_i \to F_i$. In this case, we call each P_i a *reflexive panel* of X^n. Obviously, in the trivial involutive panel structure, any point of X^n has only one duplicate point—itself.

In general, suppose $\{P_i, \tau_i\}_{1 \le i \le k}$ is an involutive panel structure on a nice manifold with corners W^n. If the $\tau_i \colon P_i \to P_i$ is the identity map, we call P_i a *reflexive panel* of W^n.

EXAMPLE 3. Suppose V^n is a $\mathbb{Z}_2$-core of Q^n. Use the notations above, for any panel P_i of V^n, P_i itself is a nice manifold with corners and has an involutive panel structure on its boundary induced from V^n, which is given by:

$$\{P_j \cap P_i;\ \tau_j|_{P_j \cap P_i} \colon P_j \cap P_i \to P_j \cap P_i \text{ for } 1 \le \forall j \le k,\ j \neq i\}.$$

More generally, for any $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, k\}$, the subpanel P_I is an $(n-s)$-dimensional nice manifold with corners (may not be connected), and P_I has an involutive panel structure on its boundary induced from V^n which is given by:

$$\{P_j \cap P_I \neq \varnothing;\ \tau_j|_{P_j \cap P_I} \colon P_j \cap P_I \to P_j \cap P_I \text{ for } 1 \le \forall j \le k,\ j \notin I\}.$$

Obviously, the $\mathbb{Z}_2$-core of a closed manifold Q^n is far from unique. The topological type of a $\mathbb{Z}_2$-core depends on the corresponding $\mathbb{Z}_2$-cut system in Q^n. For an arbitrary $\mathbb{Z}_2$-cut system of Q^n, it is fairly possible that the corresponding $\mathbb{Z}_2$-core of Q^n is not connected. But we can prove the following statement (which will not be used in any other place in this paper).

Theorem 2.10. *For any closed connected manifold Q^n, there always exists a connected $\mathbb{Z}_2$-core for Q^n.*

Proof. For $n = 1$ and 2, the statement is obviously true. So assume $n \geq 3$ in the rest of the proof.

First, let us choose a $\mathbb{Z}_2$-cut system $\Sigma_1, \ldots, \Sigma_k$ of Q^n with each Σ_i being connected. We claim that each Σ_i is non-separating in Q^n. let $\{[\Gamma_1],\ldots,[\Gamma_k]\} \subset H_1(Q^n,\mathbb{Z}_2)$ be the dual basis of $\{[\Sigma_1], \ldots, [\Sigma_k]\} \subset H_{n-1}(Q^n, \mathbb{Z}_2)$ under the $\mathbb{Z}_2$-intersection form of Q^n, i.e.

$$\#(\Gamma_i \cap \Sigma_j) = \delta_{ij} \mod 2.$$

So the curve Γ_i must intersect Σ_i odd number of times. Let $N(\Sigma_i)$ be a small tubular neighborhood of Σ_i in Q^n. Since Σ_i is connected, $Q^n - int(N(\Sigma_i))$ is either connected or has exactly two connected-component. but the later case contradicts $\#(\Gamma_i \cap \Sigma_i) = 1 \mod 2$. So Σ_i must be non-separating in Q^n.

Let Q^n_j be the manifold we get by cutting Q^n open along $\{\Sigma_1, \ldots, \Sigma_j\}$, i.e.

$$Q^n_j = Q^n - \bigcup_{i=1}^{j} int(N(\Sigma_i)).$$

In addition, for $j + 1 \leq \forall i \leq k$, let $\Sigma_i^{(j)} := \Sigma_i \cap Q^n_j$ and $\Gamma_i^{(j)} := \Gamma_i \cap Q^n_j$.

Assume Q^n_j is connected and we cut Q^n_j open along $\Sigma^{(j)}_{j+1}$. Since the relative intersection number of $\Sigma^{(j)}_{j+1}$ and $\Gamma^{(j)}_{j+1}$ in $H_*(Q^n_j, \partial Q^n_j, \mathbb{Z}_2)$ is 1 (mod 2), if $\Sigma^{(j)}_{j+1}$ is connected in Q^n_j, then $\Sigma^{(j)}_{j+1}$ must be non-separating in Q^n_j for the same reason as above. If $\Sigma^{(j)}_{j+1}$ is not connected in Q^n_j, we can connect all the components of $\Sigma^{(j)}_{j+1}$ via some thin tubes in Q^n_j which are transverse to other $\Sigma_i^{(j)}$'s. This operation will change the original Σ_{j+1} in Q^n simultaneously, but it will not change the homology class of Σ_{j+1} in $H_{n-1}(Q^n, \mathbb{Z}_2)$. Now, cutting Q^n_j open along the new $\Sigma^{(j)}_{j+1}$, we get a nice manifold with corners Q^n_{j+1} which remains connected.

By iterating the above argument from $j = 1$ to $j = k$, we will get a connected nice manifold with corners V^n. By definition, V^n is the $\mathbb{Z}_2$-core of Q^n from cutting Q^n open along a $\mathbb{Z}_2$-cut system $\{\Sigma'_1, \ldots, \Sigma'_k\}$, which is obtained from the original $\mathbb{Z}_2$-cut system by some homology preserving operations. □

3. Construction of free $(\mathbb{Z}_2)^m$-actions on closed manifolds

Suppose $\pi\colon M^n \to Q^n$ is a principal $(\mathbb{Z}_2)^m$-bundle over a closed connected manifold Q^n. Let V^n be a $\mathbb{Z}_2$-core of Q^n from cutting Q^n along a $\mathbb{Z}_2$-cut system $\{\Sigma_1, \dots, \Sigma_k\}$ in Q^n (we do not assume that V^n is connected). We have shown that the principal $(\mathbb{Z}_2)^m$-bundle π is classified by an element

(3) $$\Lambda_\pi \in H^1(Q^n, (\mathbb{Z}_2)^m) \cong \mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m).$$

By the Poincaré duality, there is an isomorphism $\psi\colon H_{n-1}(Q^n, \mathbb{Z}_2) \cong H_1(Q^n, \mathbb{Z}_2)$. So we get an element $\Lambda^*_\pi \in \mathrm{Hom}(H_{n-1}(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$ defined by:

$$\begin{aligned}\Lambda^*_\pi\colon \{[\Sigma_1], \dots, [\Sigma_k]\} &\to (\mathbb{Z}_2)^m,\\ [\Sigma_i] &\mapsto \Lambda_\pi(\psi([\Sigma_i])).\end{aligned}$$

Let $P_i \subset \partial V^n$ be the panel corresponding to Σ_i. So we have a map

$$\begin{aligned}\lambda_\pi\colon \{P_1, \dots, P_k\} &\to (\mathbb{Z}_2)^m,\\ P_i &\mapsto \Lambda^*_\pi([\Sigma_i]) = \Lambda_\pi(\psi([\Sigma_i]))\end{aligned}$$

λ_π is called the *associated* $(\mathbb{Z}_2)^m$*-coloring* of $\pi\colon M^n \to Q^n$ on V^n. In general, any map $\lambda\colon \{P_1, \dots, P_k\} \to (\mathbb{Z}_2)^m$ is called a $(\mathbb{Z}_2)^m$*-coloring on* V^n, and any element in $(\mathbb{Z}_2)^m$ is called a *color*.

In addition, if we consider $\pi\colon M^n \to Q^n$ as a regular covering, the map $\Lambda_\pi\colon H_1(Q^n,\mathbb{Z}_2) \to (\mathbb{Z}_2)^m$ is just the abelianization of the monodromy map $\mathcal{H}_\pi\colon \pi_1(Q^n,q_0) \to (\mathbb{Z}_2)^m$, where $q_0 \in Q^n$ is a base point and $(\mathbb{Z}_2)^m$ is identified with the deck transformation group of this covering π. Indeed, let $\{[\Gamma_1], \dots, [\Gamma_k]\} \subset H_1(Q^n, \mathbb{Z}_2)$ be the dual basis of $\{[\Sigma_1], \dots, [\Sigma_k]\}$ under the $\mathbb{Z}_2$-intersection form of Q^n where each Γ_i is a closed curve that intersects all Σ_j's transversely. If we fix a point $x_0 \in \pi^{-1}(q_0)$, and let $\tilde{\Gamma}_i\colon [0, 1] \to M^n$ be a lifting of Γ_i with $\tilde{\Gamma}_i(0) = x_0$, then

(4) $$\tilde{\Gamma}_i(1) = \mathcal{H}_\pi(\Gamma_i) \cdot x_0 = \Lambda_\pi([\Gamma_i]) \cdot x_0.$$

Conversely, given an arbitrary $(\mathbb{Z}_2)^m$-coloring λ on V^n, we can construct a principal $(\mathbb{Z}_2)^m$-bundle over Q^n by the following rule:

(5) $$M(V^n, \{P_i, \tau_i\}, \lambda) := V^n \times (\mathbb{Z}_2)^m/\sim$$

where $(x, g) \sim (x', g')$ whenever $x' = \tau_i(x)$ for some P_i and $g' = g + \lambda(P_i) \in (\mathbb{Z}_2)^m$. It is easy to see that if x is in the interior of $P_{i_1} \cap \cdots \cap P_{i_s}$, $(x, g) \sim (x', g')$ if and only if $(x', g') = (\tau_{i_s}^{\varepsilon_s} \circ \cdots \circ \tau_{i_1}^{\varepsilon_1}(x), g + \varepsilon_1\lambda(P_1) + \cdots + \varepsilon_s\lambda(P_s))$ where $\varepsilon_j \in \{0, 1\}$ for $1 \le \forall j \le s$.

We call $M(V^n, \{P_i, \tau_i\}, \lambda)$ the *glue-back construction* from (V^n, λ). Also, we use $M(V^n, \lambda)$ to denote $M(V^n, \{P_i, \tau_i\}, \lambda)$ if there is no ambivalence about the involutive panel structure on V^n in the context.

Let $[(x, g)] \in M(V^n, \lambda)$ denote the equivalence class of (x, g) defined in (5). Then we can define a natural $(\mathbb{Z}_2)^m$-action on $M(V^n, \lambda)$ by:

$$g \cdot [(x, g_0)] := [(x, g + g_0)], \quad \forall x \in V^n,\ \forall g, g_0 \in (\mathbb{Z}_2)^m. \tag{6}$$

It is easy to check that the $(\mathbb{Z}_2)^m$-action is well defined. And for any element $g \neq 0 \in (\mathbb{Z}_2)^m$, $g \cdot [(x, g_0)] = [(x, g + g_0)] \neq [(x, g_0)]$. This is because:
(i) when x is in the interior of V^n, $(x, g + g_0)$ and (x, g_0) are not equivalent under $\sim$ for any $g \neq 0$;
(ii) when x is in the relative interior of $P_{i_1} \cap \cdots \cap P_{i_s}$, $(x, g + g_0) \sim (x, g_0)$ would force $(x, g + g_0) = \big(\tau_{i_s}^{\varepsilon_s} \circ \cdots \circ \tau_{i_1}^{\varepsilon_1}(x), g_0 + \varepsilon_1\lambda(P_1) + \cdots + \varepsilon_s\lambda(P_s)\big)$. Notice that $g \neq 0$ implies that at least one of the $\varepsilon_1, \ldots, \varepsilon_s$ is not 0. But since x has exactly 2^s duplicate points in V^n, $\tau_{i_s}^{\varepsilon_s} \circ \cdots \circ \tau_{i_1}^{\varepsilon_1}(x) \neq x$ as long as some $\varepsilon_j \neq 0$.

So the action of $(\mathbb{Z}_2)^m$ on $M(V^n, \lambda)$ defined by (6) is always a free group action. In the rest of this paper, we will always assume that $M(V^n, \lambda)$ is equipped with this free $(\mathbb{Z}_2)^m$-action.

REMARK 3.1. Since Q^n is smooth, the $M(V^n, \lambda)$ is naturally a smooth manifold and the natural $(\mathbb{Z}_2)^m$-action on $M(V^n, \lambda)$ defined in (6) is smooth.

REMARK 3.2. A similar idea to the glue-back construction was used to construct cyclic and infinite cyclic covering spaces of the complement of knots in S^3 (see [7]).

Theorem 3.3. *$M(V^n, \lambda)$ is a closed n-manifold and the orbit space of the free $(\mathbb{Z}_2)^m$-action on $M(V^n, \lambda)$ defined in* (6) *is homeomorphic to Q^n.*

Proof. Observe that each orbit of this $(\mathbb{Z}_2)^m$-action has some representative in $V^n \times \{0\}$. And for any point x in the relative interior of $P_{i_1} \cap \cdots \cap P_{i_s}$, a point $(x', 0) \in V^n \times \{0\}$ is in the same orbit as $(x, 0)$ under the above $(\mathbb{Z}_2)^m$-action if and only if $x' = \tau_{i_s}^{\varepsilon_s} \circ \cdots \circ \tau_{i_1}^{\varepsilon_1}(x)$ for some $\varepsilon_1, \ldots, \varepsilon_s \in \{0, 1\}$ (in other words, x' is a duplicate point of x in V^n). So the orbit space is homeomorphic to the space of gluing all points of V^n with their duplicate points together, which is homeomorphic to Q^n by Lemma 2.6. And since Q^n is a closed manifold, so is $M(V^n, \lambda)$. $\square$

Following are some explicit examples of free $(\mathbb{Z}_2)^m$-actions on manifolds from the glue-back construction.

EXAMPLE 4. A meridian and a longitude of the torus T^2 forms a $\mathbb{Z}_2$-cut system of T^2. The corresponding $\mathbb{Z}_2$-core of T^2 is a square V^2. Given a coloring of the edges

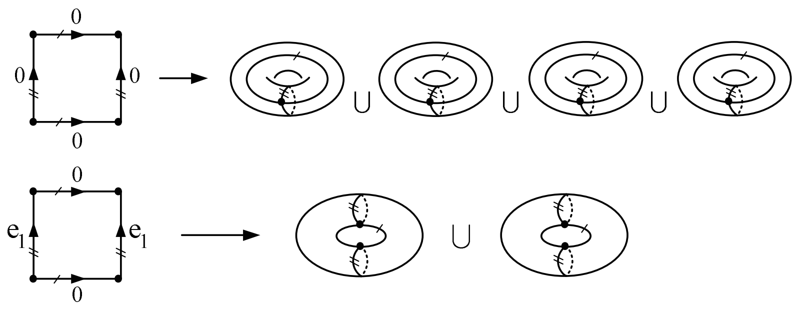

Fig. 7. Examples of the glue-back construction.

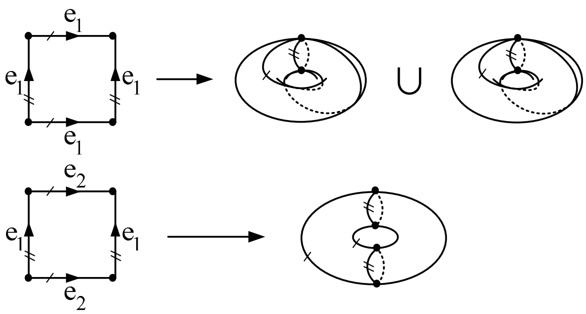

Fig. 8. Examples of the glue-back construction.

of V^2 by elements in $(\mathbb{Z}_2)^2 = \langle e_1\rangle \oplus \langle e_2\rangle$ such that opposite edges of V^2 are colored by the same element of $(\mathbb{Z}_2)^2$, we can construct all principal $(\mathbb{Z}_2)^2$-bundles over T^2 (see Figs. 7 and 8 for such examples).

EXAMPLE 5. Let M^2 be a disjoint union of two S^2. Fig. 9 shows a free $(\mathbb{Z}_2)^2$-action on M^2 whose orbit space is $\mathbb{R}P^2$. A $\mathbb{Z}_2$-core V^2 of $\mathbb{R}P^2$ is a disk with only one panel $P = \partial V^2$. Let $\{e_1, e_2\}$ be a basis of $(\mathbb{Z}_2)^2$. Then $M^2 \cong M(V^2, \lambda)$ where λ is a $(\mathbb{Z}_2)^2$-coloring on V^2 given by $\lambda(P) = e_1$ (or e_2).

More generally, for any nice manifold with corners W^n with an involutive panel structure $\{\tau_i\colon P_i \to P_i\}_{1\le i\le k}$, any map $\mu\colon \{P_1,\dots,P_k\} \to (\mathbb{Z}_2)^m$ is called a $(\mathbb{Z}_2)^m$-coloring on W^n. We can define the glue-back construction $M(W^n,\mu)$ by the same rule as in (5). Also we have a natural $(\mathbb{Z}_2)^m$-action on $M(W^n, \mu)$ defined by (6). But this $(\mathbb{Z}_2)^m$-action on $M(W^n, \mu)$ may not be free. Indeed, suppose $\theta_\mu\colon W^n \times (\mathbb{Z}_2)^m \to M(W^n, \mu)$ is the quotient map. For a point x in the relative interior of a codimension s face of W^n, it is possible that x has less than 2^s duplicate points in W^n. In that case, $\theta_\mu(x \times (\mathbb{Z}_2)^m)$ would consist of less than 2^m points, which implies that $\theta_\mu(x \times (\mathbb{Z}_2)^m)$ can not be a free orbit under the natural $(\mathbb{Z}_2)^m$-action on $M(W^n,\mu)$ defined in (6) (see the examples below).

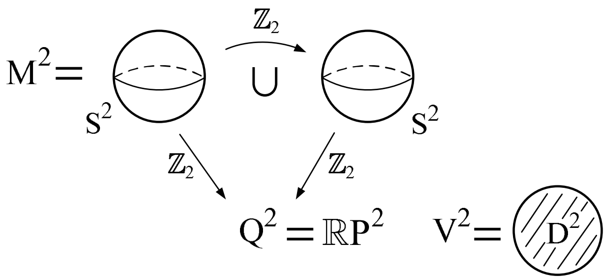

Fig. 9.

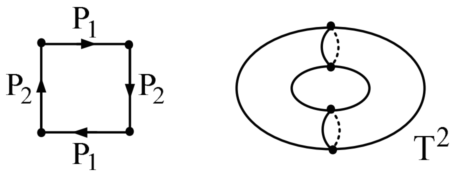

Fig. 10.

EXAMPLE 6. For a simple polytope V^n, consider each facet of V^n as a panel and V^n has the trivial involutive panel structure (see Example 2). Then a small cover over V^n can be thought of as the glue-back construction $M(V^n, \lambda)$ where λ is a *characteristic function* on V^n with value in $(\mathbb{Z}_2)^n$ (see [2]). But the natural action of $(\mathbb{Z}_2)^n$ defined by (6) on a small cover is exactly the locally standard action defined in [2], which is not free.

REMARK 3.4. From the Example 6, we can see that the significance of introducing the general notion of involutive panel structure in Definition 2.8 is that: it allows us to unify the constructions of free $(\mathbb{Z}_2)^m$-actions and non-free locally standard $(\mathbb{Z}_2)^m$-actions on manifolds from the orbit spaces (see Section 5 for details).

EXAMPLE 7. Suppose a square $[0,1]^2$ is equipped with an involutive panel structure as indicated by the arrows in Fig. 10. For a $(\mathbb{Z}_2)^2$-coloring λ defined by $\lambda(P_1) = e_1$, $\lambda(P_2) = e_2$ where $\{e_1, e_2\}$ is a basis of $(\mathbb{Z}_2)^2$, the glue-back construction $M([0,1]^2, \lambda)$ is homeomorphic to T^2. But the natural $(\mathbb{Z}_2)^2$-action on T^2 defined by (6) is not free.

For a nice manifold with corners W^n equipped with an involutive panel structure, if for any $s \geq 0$, any point in the relative interior of any codimension s face of W^n has exactly 2^s duplicate points in W^n, the involutive panel structure is called *perfect*. For example: the involutive panel structure on any $\mathbb{Z}_2$-core V^n of Q^n constructed from Lemma 2.4 above is perfect.

We can easily show that if the involutive panel structure on W^n is perfect, the natural $(\mathbb{Z}_2)^m$-action on $M(W^n, \lambda)$ defined by (6) is free for all $(\mathbb{Z}_2)^m$-coloring λ on

W^n. However, even if the involutive panel structure on W^n is not perfect, it is still possible that there exists some nontrivial $(\mathbb{Z}_2)^m$-coloring λ on W^n so that the natural $(\mathbb{Z}_2)^m$-action on $M(W^n, \lambda)$ is free. For example, although the involutive panel structure on the square in Fig. 10 is not perfect, if we color the two panels of the square both by $e_1 \in (\mathbb{Z}_2)^2$, we will obtain a disjoint union of two spheres $S^2 \cup S^2$ from the glue-back construction. Obviously, the natural $(\mathbb{Z}_2)^2$-action on this $S^2 \cup S^2$ is free.

Let V^n be a $\mathbb{Z}_2$-core of a closed manifold Q^n described as above. As in Example 3, we can think of a panel $P_i \subset V^n$ itself as a nice manifold with corners with an involutive panel structure defined by $\{P_j \cap P_i\,;\ 1 \le j \le k,\ j \neq i\}$. Then we have an induced $(\mathbb{Z}_2)^m$-coloring $\lambda^{in}_{P_i}$ of P_i defined by

$$\lambda^{in}_{P_i}(P_j \cap P_i) := \lambda(P_j), \quad \forall\, j \neq i,\ P_j \cap P_i \neq \varnothing.$$

Furthermore, for any $I = \{i_1, \dots, i_s\} \subset \{1, \dots, k\}$, the subpanel $P_I = P_{i_1} \cap \cdots \cap P_{i_s}$ has an involutive panel structure on its boundary defined by $\{P_j \cap P_I \neq \varnothing;\ 1 \le j \le k,\ j \notin I\}$. The *induced* $(\mathbb{Z}_2)^m$*-coloring* $\lambda^{in}_{P_I}$ of P_I is

$$\lambda^{in}_{P_I}(P_j \cap P_I) = \lambda(P_j) \in (\mathbb{Z}_2)^m, \quad 1 \le j \le k,\ j \notin I,\ P_j \cap P_I \neq \varnothing. \tag{7}$$

If we apply the glue-back construction (5) to $(P_I, \lambda^{in}_{P_I})$, we will get a closed manifold $M(P_I, \lambda^{in}_{P_I})$. Notice that when $|I| \ge 1$, by the definition of $M(P_I, \lambda^{in}_{P_I})$, the relative interior points of the 2^m copies of P_I are not glued together like they are in $M(V^n, \lambda)$. In fact, it is easy to see that $M(P_I, \lambda^{in}_{P_I})$ is homeomorphic to a disjoint union of 2^s copies of $\theta_\lambda^{-1}(\Sigma_I)$, where $\theta_\lambda \colon V^n \times (\mathbb{Z}_2)^m \to M(V^n, \lambda)$ is the quotient map defined in (5).

Theorem 3.5. *For any principal $(\mathbb{Z}_2)^m$-bundle $\pi \colon M^n \to Q^n$, let λ be the associated $(\mathbb{Z}_2)^m$-coloring of π on V^n. Then there is an equivariant homeomorphism from $M(V^n, \lambda)$ to M^n which covers the identity of Q^n.*

Proof. Let $\theta_\lambda \colon V^n \times (\mathbb{Z}_2)^m \to M(V^n, \lambda)$ be the quotient map and $\xi_\lambda \colon M(V^n, \lambda) \to Q^n$ be the orbit map of the natural $(\mathbb{Z}_2)^m$-action defined by (6). It suffice to show that ξ_λ and π defines the same monodromy map as regular coverings over Q^n. So let us first compute the monodromy $\mathcal{H}_{\xi_\lambda}([\Gamma])$ for any closed curve $\Gamma \colon [0, 1] \to Q^n$.

Suppose $\Gamma(t)$ meets $\Sigma_{i_1}, \dots, \Sigma_{i_r}$ consecutively in Q^n as the time t goes from 0 to 1. When cutting Q^n open along $\{\Sigma_1, \dots, \Sigma_k\}$, the cut-open image of Γ is $\gamma := \Gamma \cap V^n$. Note that the curve γ might be disconnected in V^n. When the time parameter increases, γ will meet the panels $P_{i_1}, \dots, P_{i_r}$ of V^n consecutively. Then when we glue the 2^m copies of V^n together in the glue-back construction, the curve γ in different copies of V^n are fit together which gives all the liftings of Γ in $M(V^n, \lambda)$. Indeed, if we choose the start point of a lifting of Γ in $\theta_\lambda(V^n \times g_0)$ where $g_0 \in (\mathbb{Z}_2)^m$, the end point of the lifting would be in $\theta_\lambda(V^n \times (g_0 + \lambda(P_{i_1}) + \cdots + \lambda(P_{i_r})))$. So the monodromy $\mathcal{H}_{\xi_\lambda}$ of ξ_λ is:

(8) $$\mathcal{H}_{\xi_\lambda}([\Gamma]) = \lambda(P_{i_1}) + \cdots + \lambda(P_{i_r}) \in (\mathbb{Z}_2)^m.$$

Now let $\{[\Gamma_1], \dots, [\Gamma_k]\} \subset H_1(Q^n, \mathbb{Z}_2)$ be the dual basis of $\{[\Sigma_1], \dots, [\Sigma_k]\} \subset H_{n-1}(Q^n, \mathbb{Z}_2)$ under the $\mathbb{Z}_2$-intersection form of Q^n. Then

(9) $$\#(\Gamma_i \cap \Sigma_j) = \delta_{ij} \mod 2.$$

We can assume that each $\Gamma_i : [0, 1] \to Q^n$ is a closed curve which intersects all Σ_j's transversely and starts at the same base point $q_0 \in Q^n$. Suppose $\gamma_i = \Gamma_i \cap V^n$ is the cut-open image of Γ_i in V^n. Then by (9), γ_i will meet P_i odd number of times and meet all other P_j $(j \neq i)$ even number of times. So by (8), we have:

(10) $$\mathcal{H}_{\xi_\lambda}([\Gamma_i]) = \lambda(P_i) = \mathcal{H}_\pi([\Gamma_i]), \quad 1 \leq i \leq k.$$

This implies that $\mathcal{H}_{\xi_\lambda} = \mathcal{H}_\pi$. So the theorem is proved. □

REMARK 3.6. For a $\mathbb{Z}_2$-core V^n of Q^n with $H^1(V^n, \mathbb{Z}_2) \neq 0$, a principal $(\mathbb{Z}_2)^m$-bundle over V^n is not necessarily trivial. If we apply the gluing rule (5) to an arbitrary principal $(\mathbb{Z}_2)^m$-bundle over V^n, we may get a principal $(\mathbb{Z}_2)^m$-bundle over Q^n too. The significance of Theorem 3.5 is that we can actually use the trivial $(\mathbb{Z}_2)^m$-bundle over V^n (i.e. $V^n \times (\mathbb{Z}_2)^m$) and the gluing rule (5) to construct all principal $(\mathbb{Z}_2)^m$-bundles over Q^n, which is enough for our purpose in this paper.

For a $(\mathbb{Z}_2)^m$-coloring λ on the panels $P_1, \dots, P_k$ of V^n, define:

(11) $$L_\lambda := \text{the subgroup of } (\mathbb{Z}_2)^m \text{ generated by } \{\lambda(P_1), \dots, \lambda(P_k)\},$$

(12) $$\operatorname{rank}(\lambda) := \dim_{\mathbb{Z}_2} L_\lambda.$$

Since λ encodes all the structural information of $M(V^n, \lambda)$, so any topological invariant of $M(V^n,\lambda)$ (e.g. homology groups) should be completely determined by (V^n,λ). But if we try to compute the $\mathbb{Z}_2$-homology groups of $M(V^n, \lambda)$ via the Serre-spectral sequence, the problem of twisted local coefficients could occur when the orbit space is not simply-connected. This problem is hard to get around in general. However, we can at least compute $H_0(M(V^n, \lambda), \mathbb{Z}_2)$, i.e. the number of connected components of $M(V^n, \lambda)$, from the $(\mathbb{Z}_2)^m$-coloring λ.

Theorem 3.7. *For any $(\mathbb{Z}_2)^m$-coloring λ of V^n, $M(V^n,\lambda)$ has $2^{m-\operatorname{rank}(\lambda)}$ connected components which are pairwise homeomorphic. Let $\theta_\lambda : V^n \times (\mathbb{Z}_2)^m \to M(V^n, \lambda)$ be the quotient map. Then each connected component of $M(V^n, \lambda)$ is homeomorphic to $\theta_\lambda(V^n \times L_\lambda)$. And there is a free action of $L_\lambda \cong (\mathbb{Z}_2)^{\operatorname{rank}(\lambda)}$ on each connected component of $M(V^n, \lambda)$ whose orbit space is Q^n.*

Proof. Let $\theta_\lambda\colon V^n \times (\mathbb{Z}_2)^m \to M(V^n, \lambda)$ be the quotient map defined by (5). Here we do not assume V^n is connected. Then by the definition of $M(V^n, \lambda)$, for two arbitrary connected components K, K' of V^n and $\forall g, g' \in (\mathbb{Z}_2)^m$, we have:
(a) $\theta_\lambda(K \times g)$ and $\theta_\lambda(K' \times g')$ are in the same connected component of $M(V^n, \lambda)$ if and only if there is a sequence

$$(K, g) = (K_0, g_0) \leftrightarrow (K_1, g_1) \leftrightarrow \cdots \leftrightarrow (K_{r-1}, g_{r-1}) \leftrightarrow (K_r, g_r) = (K', g')$$

where each K_i is a connected component of V^n, $g_i \in (\mathbb{Z}_2)^m$, and $\theta_\lambda(K_i \times g_i)$ and $\theta_\lambda(K_{i+1} \times g_{i+1})$ share an $(n-1)$-dimensional face in $M(V^n, \lambda)$.
(b) $\theta_\lambda(K \times g)$ and $\theta_\lambda(K' \times g')$ share an $(n-1)$-dimensional face in $M(V^n, \lambda)$ if and only if there is a facet F of K with its twin facet $F^* \subset K'$ and $g' - g = \lambda(F)$.
So if $\theta_\lambda(K \times g)$ and $\theta_\lambda(K' \times g')$ are in the same connected component of $M(V^n, \lambda)$, it is necessary that $g' \in g + L_\lambda$.

Conversely, we claim: for any $g' \in g + L_\lambda$, $\theta_\lambda(K \times g)$ and $\theta_\lambda(K' \times g')$ are always in the same connected component of $M(V^n, \lambda)$ for any connected components K and K' of V^n.

Indeed, since Q^n is connected, for any connected components K and K' of V^n, there always exists a sequence, $K = K_0, K_1, \ldots, K_{r-1}, K_r = K'$, such that some facet $F_{a_i} \subset K_i$ while $F^*_{a_i} \subset K_{i+1}$. So by the above argument, $\theta_\lambda(K \times g)$ lies in the same connected component as $\theta_\lambda(K' \times g^*)$ in $M(V^n, \lambda)$ for some $g^* \in g + L_\lambda$. Then it remains to show that $\theta_\lambda(K' \times g^*)$ and $\theta_\lambda(K' \times g')$ are always in the same connected component of $M(V^n, \lambda)$ whenever $g' - g^* \in L_\lambda$.

To see this, let $\Gamma\colon [0, 1] \to Q^n$ be an arbitrary closed curves based at a point $q_0 \in K' \subset V^n$. For any $g^* \in (\mathbb{Z}_2)^m$, there is a lifting of Γ in $M(V^n, \lambda)$ which goes from a point in $\theta_\lambda(K' \times g^*)$ to a point in $\theta_\lambda(K' \times (g^* + \mathcal{H}_{\xi_\lambda}([\Gamma])))$, where $\mathcal{H}_{\xi_\lambda}([\Gamma])$ is the monodromy of Γ with respect to the covering $\xi_\lambda\colon M(V^n, \lambda) \to Q^n$ (see (4)). So $\theta_\lambda(K' \times g^*)$ and $\theta_\lambda(K' \times (g^* + \mathcal{H}_{\xi_\lambda}([\Gamma])))$ are in the same connected component of $M(V^n, \lambda)$.

Let $\Gamma_1, \ldots, \Gamma_k$ be some closed curves in Q^n based at q_0 so that $\{[\Gamma_1], \ldots, [\Gamma_k]\} \subset H_1(Q^n, \mathbb{Z}_2)$ is the dual basis of the $\{[\Sigma_1], \ldots, [\Sigma_k]\} \subset H_{n-1}(Q^n, \mathbb{Z}_2)$ under the $\mathbb{Z}_2$-intersection form of Q^n. Then by (10), we have

$$\begin{aligned} \operatorname{Im}(\mathcal{H}_{\xi_\lambda}) &= \{\mathcal{H}_{\xi_\lambda}([\Gamma]) \mid \Gamma\colon [0, 1] \to Q^n, \Gamma(0) = \Gamma(1) = q_0\} \\ &= \langle \mathcal{H}_{\xi_\lambda}([\Gamma_1]), \ldots, \mathcal{H}_{\xi_\lambda}([\Gamma_k]) \rangle \subset (\mathbb{Z}_2)^m \\ &= \langle \lambda(P_1), \ldots, \lambda(P_k) \rangle = L_\lambda. \end{aligned} \tag{13}$$

So any element of L_λ can be realized by $\mathcal{H}_{\xi_\lambda}([\Gamma])$ for some closed curve Γ. Then the above claim is proved.

So $\theta_\lambda(K \times g)$ and $\theta_\lambda(K' \times g')$ belong to the same connected component of $M(V^n, \lambda) \Leftrightarrow g' \in g + L_\lambda$. Since $\dim_{\mathbb{Z}_2} L_\lambda = \operatorname{rank}(\lambda)$, each connected component of $M(V^n, \lambda)$ is made up of $2^{\mathrm{rank}(\lambda)}$ copies of V^n from $V^n \times (\mathbb{Z}_2)^m$. Indeed, suppose $(\mathbb{Z}_2)^m = L_\lambda \oplus \langle \omega_1 \rangle \oplus \cdots \oplus \langle \omega_q \rangle$ where $q = m - \mathrm{rank}(\lambda)$. Then $M(V^n, \lambda)$ has $2^{m-\mathrm{rank}(\lambda)}$ connected components which are given by:

$$\theta_\lambda(V^n \times (L_\lambda + t_1\omega_1 + \cdots + t_q\omega_q)), t_i \in \{0, 1\}, \quad 1 \le i \le q,$$

each of which is equipped with a natural free action by $L_\lambda \cong (\mathbb{Z}_2)^{\mathrm{rank}(\lambda)}$ defined in (6) whose orbit space is Q^n. □

REMARK 3.8. In the above proof, let $\kappa : (\mathbb{Z}_2)^m \to L_\lambda \cong (\mathbb{Z}_2)^{\mathrm{rank}(\lambda)}$ be a quotient homomorphism. Then for any $(\mathbb{Z}_2)^m$-coloring λ on V^n, we can think of $\kappa \circ \lambda$ as a $(\mathbb{Z}_2)^{\mathrm{rank}(\lambda)}$-coloring on V^n. It is easy to see that each connected component K of $M(V^n, \lambda)$ is homeomorphic to $M(V^n, \kappa \circ \lambda)$.

In general, for $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, k\}$, the submanifold $\Sigma_I \subset Q^n$ may not be connected. Suppose S is a connected component of Σ_I. Then S is an $(n-s)$-dimensional embedded submanifold of Q^n. We also want to compute the number of components in $\xi_\lambda^{-1}(S)$, where $\xi_\lambda : M(V^n, \lambda) \to Q^n$ is the quotient map defined by (5).

REMARK 3.9. Two connected components S and S' of Σ_I may not be homeomorphic to each other, and the $\xi_\lambda^{-1}(S)$ and $\xi_\lambda^{-1}(S')$ may not have equal number of connected components either.

If we cut S open along the transversely intersected embedded submanifolds $\{\Sigma_j \cap S \neq \varnothing \mid j \notin I\}$, we will get a nice manifold with corners, denoted by V_S. Similar to the $\mathbb{Z}_2$-core of Q^n, we can construct a (perfect) involutive panel structure on the boundary of V_S from the cut sections of $\{\Sigma_j \cap S \neq \varnothing \mid j \notin I\}$. And any $(\mathbb{Z}_2)^m$-coloring λ on V^n induces a $(\mathbb{Z}_2)^m$-coloring λ_S^{in} on the panels of V_S by: if E is the panel of V_S corresponding to $\Sigma_j \cap S$,

$$\lambda_S^{in}(E) := \lambda(P_j).$$

It is easy to see that the glue-back construction $M(V_S, \lambda_S^{in})$ is homeomorphic to $\xi_\lambda^{-1}(S)$. But V_S may not be a $\mathbb{Z}_2$-core of S, since the homology classes of $\{\Sigma_j \cap S \neq \varnothing \mid j \notin I\}$ may not form a basis of $H_{n-s-1}(S, \mathbb{Z}_2)$. So we can not directly apply the formula in Theorem 3.7 to compute the number of components of $M(V_S, \lambda_S^{in})$. In fact, the number of components of $M(V_S, \lambda_S^{in})$ also depends on what homology classes are represented by $\{\Sigma_j \cap S \neq \varnothing \mid j \notin I\}$ in $H_{n-s-1}(S, \mathbb{Z}_2)$. So we need to modify the proof of Theorem 3.7 to deal with this case. In the following, we will treat this problem in a very general setting on Q^n.

Suppose $\{N_1, \ldots, N_r\}$ is an arbitrary collection of codimension one embedded submanifolds of a closed manifold Q^n which lie in general position. Cutting Q^n open along $N_1, \ldots, N_r$ gives us a nice manifold with corners W^n. As before, we can construct a (perfect) involutive panel structure $\hat{P}_1, \ldots, \hat{P}_r$ on the boundary of W^n from the cut sections of $N_1, \ldots, N_r$. In addition, suppose $\Gamma_1, \ldots, \Gamma_k$ are simple closed curves in Q^n whose homology classes form a basis of $H_1(Q^n, \mathbb{Z}_2)$, and each Γ_i intersects $\{N_1, \ldots, N_r\}$ transversely. Let $a_{ij} \in \mathbb{Z}_2$ be the mod 2 intersection number between Γ_i and N_j. Note that for any fixed j, $\{a_{ij}\}$ is completely determined by the homology class of N_j in $H_{n-1}(Q^n, \mathbb{Z}_2)$. Indeed, if $\{\Sigma_1, \ldots, \Sigma_k\}$ is a $\mathbb{Z}_2$-cut system of Q^n whose homology classes is a dual basis of $\{[\Gamma_1], \ldots, [\Gamma_k]\}$ under the $\mathbb{Z}_2$-intersection form, then for each $1 \leq j \leq r$, the homology class $[N_j] = \sum_i a_{ij}[\Sigma_i] \in H_{n-1}(Q^n)$.

For any $(\mathbb{Z}_2)^m$-coloring $\lambda \colon \{\hat{P}_1, \ldots, \hat{P}_r\} \to (\mathbb{Z}_2)^m$ on W^n, we can show that $M(W^n, \lambda)$ is a closed manifold with a natural free $(\mathbb{Z}_2)^m$-action defined by (6) whose orbit space is Q^n. The proof of this fact is exactly the same as Theorem 3.3, hence omitted. In addition, by a similar argument as in the proof Theorem 3.5, the monodromy of each Γ_i with respect to the covering $M(W^n, \lambda) \to Q^n$ is given by:

$$\hat{\mathcal{H}}_\lambda(\Gamma_i) := \sum_{j=1}^{r} a_{ij}\lambda(\hat{P}_j) \in (\mathbb{Z}_2)^m. \tag{14}$$

Now, we define a new $(\mathbb{Z}_2)^m$-coloring $\hat{\lambda}$ on the panels of W^n by:

$$\hat{\lambda}(\hat{P}_i) := \hat{\mathcal{H}}_\lambda(\Gamma_i), \quad 1 \leq i \leq r.$$

$$L_{\hat{\lambda}} := \text{the subgroup of } (\mathbb{Z}_2)^m \text{ generated by } \{\hat{\lambda}(\hat{P}_1), \ldots, \hat{\lambda}(\hat{P}_r)\}.$$

Theorem 3.10. *For any $(\mathbb{Z}_2)^m$-coloring λ on the panels of W^n, the number of connected components of $M(W^n, \lambda)$ equals 2^{m-l}, where $l = \dim_{\mathbb{Z}_2} L_{\hat{\lambda}}$. In addition, all the connected components of $M(W^n, \lambda)$ are homeomorphic to each other, and there is free $L_{\hat{\lambda}} \cong (Z_2)^l$-action on each component of $M(W^n, \lambda)$ whose orbit space is Q^n.*

Proof. The argument here is parallel to that in Theorem 3.7 except that in (13), the monodromy of Γ_i should be replaced by $\hat{\mathcal{H}}_\lambda(\Gamma_i)$ in (14). So the proof is left to the reader. □

4. Generalize to compact manifolds with boundary

We can generalize the notion of $\mathbb{Z}_2$-core and glue-back construction to any compact manifold with boundary. Suppose X^n is an n-dimensional compact connected nice manifold with corners and $H^1(X^n, \mathbb{Z}_2) \neq 0$. Let $\{F_1, \ldots, F_l\}$ be the set of facets of X^n. A $\mathbb{Z}_2$-*cut system* of X^n is a collection of $(n-1)$-dimensional embedded submanifolds $\Sigma_1, \ldots, \Sigma_k$ (possible with boundary) of X^n which satisfy:

(i) $\Sigma_1, \ldots, \Sigma_k$ are in general position in X^n; and

(ii) the (relative) homology classes $[\Sigma_1], \ldots, [\Sigma_k]$ form a $\mathbb{Z}_2$-linear basis of $H_{n-1}(X^n, \partial X^n, \mathbb{Z}_2) \cong H^1(X^n, \mathbb{Z}_2) \neq 0$.

Moreover, we can choose each Σ_i to be connected. If we cut X^n open along $\{\Sigma_1, \ldots, \Sigma_k\}$, we get a nice manifold with corners U^n, called a $\mathbb{Z}_2$-*core* of X^n. Similar to

Theorem 2.10, we can show that there always exists a connect $\mathbb{Z}_2$-core for X^n. Note that the boundary stratification of U^n is a mixture of the facets in the cut section of Σ_i and the facets from ∂X^n. So we define the panel structure on U^n by $\{P_1, \dots, P_k, P'_1, \dots, P'_l\}$, where P_i consists of the facets in the cut section of Σ_i and P'_j consists of the facets in the cut open image of F_j.

By a similar argument as in Lemma 2.4, we can construct a free involution τ_i on each P_i $(1 \le i \le k)$ which satisfies the conditions (a), (b) and (c) in Definition 2.8. If we do not define any involution on P'_j, we say that $\{\tau_i : P_i \to P_i\}_{1\le i\le k}$ along with $\{P'_1, \dots, P'_l\}$ is a *partial involutive panel structure* on the boundary of U^n.

Let $\mathcal{P}(U^n) = \{P_1, \dots, P_k\}$ be the set of all panels in U^n that are equipped with involutions. Any map from $\mathcal{P}(U^n)$ to $(\mathbb{Z}_2)^m$ is called a *$(\mathbb{Z}_2)^m$-coloring on U^n*. It is easy to see that the glue-back construction $M(U^n, \lambda)$ makes perfect sense for U^n with a $(\mathbb{Z}_2)^m$-coloring λ. Indeed, $M(U^n, \lambda)$ is got by glue 2^m copies of U^n only along those panels equipped with involutions according to the rule in (5).

By a parallel argument as Theorem 3.3, we can show that (6) defines a natural free $(\mathbb{Z}_2)^m$-action on $M(U^n, \lambda)$ whose orbit space is homeomorphic to X^n. And similarly, we can prove the following.

Theorem 4.1. *Suppose X^n is a compact connected nice manifold with corners and U^n is a $\mathbb{Z}_2$-core of X^n. Then we have*:
(1) *any principal $(\mathbb{Z}_2)^m$-bundle $\pi : M^n \to X^n$ determines a $(\mathbb{Z}_2)^m$-coloring λ_π on U^n*;
(2) *there is an equivariant homeomorphism from the $M(U^n, \lambda_\pi)$ to M^n which covers the identity of X^n.*

In addition, we can similarly define L_λ and $\operatorname{rank}(\lambda)$ for any $(\mathbb{Z}_2)^m$-coloring λ on U^n (see (11) and (12)) and extend the results in Theorem 3.7 to $M(U^n, \lambda)$ as well. Here we only give the statement below and leave the proof to the reader.

Theorem 4.2. *For any $(\mathbb{Z}_2)^m$-coloring λ on U^n, the $M(U^n, \lambda)$ has $2^{m-\operatorname{rank}(\lambda)}$ connected components which are pairwise homeomorphic. Let $\theta_\lambda\colon U^n \times (\mathbb{Z}_2)^m \to M(U^n, \lambda)$ be the quotient map. Then each connected component of $M(U^n, \lambda)$ is homeomorphic to $\theta_\lambda(U^n \times L_\lambda)$. And there is a free action of $L_\lambda \cong (\mathbb{Z}_2)^{\operatorname{rank}(\lambda)}$ on each connected component of $M(U^n, \lambda)$ whose orbit space is X^n.*

5. Locally standard $(\mathbb{Z}_2)^m$-action on closed n-manifolds

First, let us define the meaning of *standard action* of $(\mathbb{Z}_2)^m$ in dimension n for any $m \ge 1$. Suppose $g = (g_1, \dots, g_m)$ is an arbitrary element of $(\mathbb{Z}_2)^m$.
(1) If $m \le n$, the standard $(\mathbb{Z}_2)^m$-action on $\mathbb{R}^n$ is:

$$(x_1, \dots, x_n) \mapsto ((-1)^{g_1} x_1, \dots, (-1)^{g_m} x_m, x_{m+1}, \dots, x_n),$$

whose orbit space is $\mathbb{R}^{n,m}_+ := \{(x_1, \dots, x_n);\ x_i \ge 0 \text{ for } 1 \le \forall i \le m\}$.

(2) For $m > n$, the standard $(\mathbb{Z}_2)^m$-action on $\mathbb{R}^n \times (\mathbb{Z}_2)^{m-n}$ is:

$$((x_1, \dots, x_n), (h_1, \dots, h_{m-n}))$$
$$\mapsto (((-1)^{g_1} x_1, \dots, (-1)^{g_n} x_n), (g_{n+1} + h_1, \dots, g_m + h_{m-n})),$$

whose orbit space is $\mathbb{R}^{n,n}_+$.

Suppose $(\mathbb{Z}_2)^m$ acts effectively and smoothly on a closed n-manifold M^n. A *local isomorphism* of M^n with the standard action (defined above) consists of:
(1) a group automorphism $\sigma : (\mathbb{Z}_2)^m \to (\mathbb{Z}_2)^m$;
(2) a $(\mathbb{Z}_2)^m$-stable open set V in M^n and U in $\mathbb{R}^n$ (if $m \leq n$) or $\mathbb{R}^n \times (\mathbb{Z}_2)^{m-n}$ (if $m > n$);
(3) a σ-equivariant homeomorphism $f : V \to U$, i.e. $f(g \cdot v) = \sigma(g) \cdot f(v)$ for any $g \in (\mathbb{Z}_2)^m$ and $v \in V$.

A $(\mathbb{Z}_2)^m$-action on M^n is called *locally standard* if each point of M^n is in the domain of some local isomorphism. Then M^n is called a *locally standard* $(\mathbb{Z}_2)^m$*-manifold* over $M^n/(\mathbb{Z}_2)^m$. Note that here we generalize the notion of *locally standard* 2*-torus manifold* defined in [3] where m is required to be equal to n.

Now, suppose we have a locally standard $(\mathbb{Z}_2)^m$-action on a closed manifold M^n. Then the orbit space $X^n = M^n/(\mathbb{Z}_2)^m$ is a nice manifold with corners (in the rest of the paper, we always assume X^n is connected). Let $\pi : M^n \to X^n$ be the orbit map. Suppose the set of all facets of X^n is $\mathcal{F}(X^n) = \{F_1, \dots, F_l\}$. The *characteristic function* $\nu_\pi : \mathcal{F}(X^n) \to (\mathbb{Z}_2)^m$ of the action is defined by:

$$\nu_\pi(F_j) = \text{the element of } (\mathbb{Z}_2)^m \text{ that fixes } \pi^{-1}(F_j) \text{ pointwise.}$$

Observe that whenever $F_{j_1} \cap \dots \cap F_{j_s} \neq \varnothing$, $\{\nu_\pi(F_{j_1}), \dots, \nu_\pi(F_{j_s})\}$ should be linearly independent vectors in $(\mathbb{Z}_2)^m$ over $\mathbb{Z}_2$. And the isotropy group of the set $\pi^{-1}(F_{j_1} \cap \dots \cap F_{j_s})$ is the subgroup of $(\mathbb{Z}_2)^m$ generated by $\{\nu_\pi(F_{j_1}), \dots, \nu_\pi(F_{j_s})\}$.

In addition, M^n determines a principal $(\mathbb{Z}_2)^m$-bundle over X^n, denoted by ξ_π. If $H^1(X^n, \mathbb{Z}_2) = 0$, ξ_π is always trivial. So we assume $H^1(X^n, \mathbb{Z}_2) \neq 0$ in the rest of this section.

REMARK 5.1. If $m < n$, the dimension of any face of X^n is at least $n - m$.

We will see that the characteristic function ν_π and the principal $(\mathbb{Z}_2)^m$-bundle ξ_π encode all the structural information of the $(\mathbb{Z}_2)^m$-action, and we can classify locally standard $(\mathbb{Z}_2)^m$-manifolds $\pi : M^n \to X^n$ by ν_π and ξ_π up to some natural equivalence relations. The following discussions are parallel to those in [3].

First of all, we say that two locally standard $(\mathbb{Z}_2)^m$-manifolds M^n and N^n over X^n are *equivalent* if there is a homeomorphism $f : M^n \to N^n$ together with an element $\sigma \in \mathrm{GL}(m, \mathbb{Z}_2)$ such that
(1) $f(g \cdot x) = \sigma(g) \cdot f(x)$ for all $g \in (\mathbb{Z}_2)^m$ and $x \in M^n$, and

(2) f induces the identity map on the orbit space.

In addition, we call two locally standard $(\mathbb{Z}_2)^m$-manifolds M^n and N^n over X^n *equivariantly homeomorphic* if there is a homeomorphism $f: M^n \to N^n$ such that $f(g \cdot x) = g \cdot f(x)$ for all $g \in (\mathbb{Z}_2)^m$ and $x \in M^n$. Such a homeomorphism f is called an *equivariant homeomorphism* between M^n and N^n. Notice that f will induce a homeomorphism $h_f: X^n \to X^n$ which preserves the manifold with corners structure of X^n. But h_f may not be the identity map of X^n.

Let U^n be a $\mathbb{Z}_2$-core of X^n from cutting X^n open along a $\mathbb{Z}_2$-cut system $\{\Sigma_1, \ldots, \Sigma_k\}$ in X^n as described in Section 4. The panel structure of U^n is $\{P_1, \ldots, P_k, P'_1, \ldots, P'_l\}$ where P_i corresponds to the cut section of Σ_i and P'_j is the cut open image of the facet F_j, and there is a partial involutive panel structure $\tau_i: P_i \to P_i$ on U^n. Moreover, if we define $\tau'_j = id$: $P'_j \to P'_j$ for any $1 \le j \le l$, then $\{P_i, \tau_i\}_{1 \le i \le k}$ and $\{P'_j, \tau'_j\}_{1 \le j \le l}$ together define a complete involutive panel structure on U^n. We call $\{P_1, \ldots, P_k\}$ the *principal panels* of U^n and call $\{P'_1, \ldots, P'_l\}$ the *reflexive panels* of U^n. And we assume U^n having this involutive panel structure in the rest of this paper.

By Theorem 4.1, the principal bundle ξ_π determines a $(\mathbb{Z}_2)^m$-coloring λ_π on the set of principal panels $\{P_1, \ldots, P_k\}$, and the characteristic function ν_π induces a $(\mathbb{Z}_2)^m$-coloring μ_π on the reflexive panels $\{P'_1, \ldots, P'_l\}$ by $\mu_\pi(P'_j) = \nu_\pi(F_j)$, $1 \le j \le l$. So whenever $P'_{j_1} \cap \cdots \cap P'_{j_s} \neq \varnothing$, we should have:

$$\mu_\pi(P'_{j_1}), \ldots, \mu_\pi(P'_{j_s}) \text{ is linearly independent vectors in } (\mathbb{Z}_2)^m \text{ over } \mathbb{Z}_2. \tag{15}$$

The glue-back construction of U^n with respect to the composite $(\mathbb{Z}_2)^m$-coloring (λ_π, μ_π) on the panels of U^n gives us a closed manifold, denoted by $M(U^n, \lambda_\pi, \mu_\pi)$. The natural $(\mathbb{Z}_2)^m$-action on $M(U^n, \lambda_\pi, \mu_\pi)$ is also defined by:

$$g \cdot [(x, g_0)] := [(x, g + g_0)], \quad \forall x \in U^n,\ \forall g, g_0 \in (\mathbb{Z}_2)^m. \tag{16}$$

The following theorem is parallel to that in [3].

Theorem 5.2. *The action $(\mathbb{Z}_2)^m \curvearrowright M(U^n, \lambda_\pi, \mu_\pi)$ defined in* (16) *is locally standard and there is an equivariant homeomorphism from $M(U^n, \lambda_\pi, \mu_\pi)$ to M^n which covers the identity of X^n.*

Proof. It is easy to check the action is locally standard. And by a parallel argument as the proof of Theorem 3.3, the orbit space of $(\mathbb{Z}_2)^m \curvearrowright M(U^n, \lambda_\pi, \mu_\pi)$ is X^n. Moreover, $(\mathbb{Z}_2)^m \curvearrowright M(U^n, \lambda_\pi, \mu_\pi)$ defines the same principal $(\mathbb{Z}_2)^m$-bundle over X^n and the same characteristic function on X^n as the locally standard $(\mathbb{Z}_2)^m$-action on M^n. Then it is easy to construct an equivariant homeomorphism from $M(U^n, \lambda_\pi, \mu_\pi)$ to M^n which covers the identity of X^n. $\square$

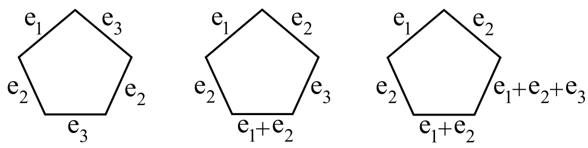

Fig. 11.

Denote by $\Xi(U^n, (\mathbb{Z}_2)^m)$ the set of all eligible composite $(\mathbb{Z}_2)^m$-colorings on U^n,

$$\Xi(U^n, (\mathbb{Z}_2)^m) := \{(\lambda, \mu) \mid \lambda \text{ is a } (\mathbb{Z}_2)^m\text{-coloring on the principal panels of } U^n; \\ \mu \text{ is a } (\mathbb{Z}_2)^m\text{-coloring on the reflexive panels of } U^n \\ \text{which satisfies the condition (15)}\}.$$

Then by Theorem 5.2, any locally standard $(\mathbb{Z}_2)^m$-action on a closed n-manifold with X^n as the orbit space can be obtained from U^n and some composite $(\mathbb{Z}_2)^m$-coloring $(\lambda, \mu) \in \Xi(U^n, (\mathbb{Z}_2)^m)$.

By the definition of U^n, it is easy to see that we can identify $\Xi(U^n, (\mathbb{Z}_2)^m)$ with $H^1(X^n,(\mathbb{Z}_2)^m)\times\mathcal{V}(X^n,(\mathbb{Z}_2)^m)$ as a set, where $\mathcal{V}(X^n,(\mathbb{Z}_2)^m)$ is the set of all characteristic functions on X^n, i.e.

$$\mathcal{V}(X^n, (\mathbb{Z}_2)^m) := \{\nu\colon \mathcal{F}(X^n) \to (\mathbb{Z}_2)^m;\ \nu(F_{j_1}), \dots, \nu(F_{j_s}) \text{ are linearly} \\ \text{independent vectors in } (\mathbb{Z}_2)^m \text{ whenever} \\ F_{j_1} \cap \cdots \cap F_{j_s} \neq \varnothing\}.$$

EXAMPLE 8. Suppose P^n is a convex simple polytope with m facets $\{F_1,\dots,F_m\}$. Let $\{e_1, \dots, e_m\}$ be a basis of $(\mathbb{Z}_2)^m$. If we color F_i by e_i, the glue-back construction for P^n with the trivial involutive panel structure (see Example 2) gives us a manifold $\mathbb{R}\mathcal{Z}_{P^n}$, called the *real moment-angle manifold* over P^n (see [2] and [6]). The $\mathbb{Z}_2$-coefficient equivariant cohomology ring of $\mathbb{R}\mathcal{Z}_{P^n}$ with respect to the natural $(\mathbb{Z}_2)^m$ action is isomorphic to the face ring of P^n (see [2]). The ordinary $\mathbb{Z}_2$-cohomology groups of $\mathbb{R}\mathcal{Z}_{P^n}$ were calculated by XiangYu Cao and Zhi Lü in [8].

EXAMPLE 9. In Fig. 11, we have three different $(\mathbb{Z}_2)^3$-colorings of a pentagon which is equipped with the trivial involutive panel structure (see Example 2). The glue-back construction for the left picture gives $T^2 \# T^2$ (connected sum of two tori). For the other two pictures, the glue-back constructions both give the connected sum of two Klein bottles (these examples are taken from [9]).

EXAMPLE 10. Let $X^2 = T^2 - P^2$ where P^2 is a zigzag polygon on T^2. In Fig. 12, we have three different $\mathbb{Z}_2$-cores of X^2, each of whose $\mathbb{Z}_2$-cut system consists of a (broken) longitude and a (broken) meridian of X^2. Then any locally standard $(\mathbb{Z}_2)^m$-action on a closed manifold with X^2 as the orbit space can be obtained by the

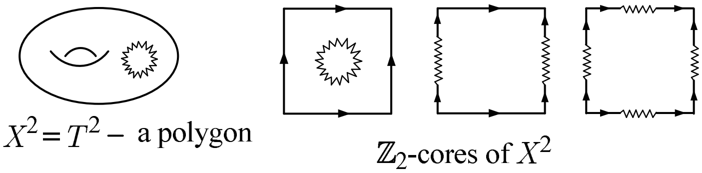

Fig. 12.

glue-back construction from any one of the three $\mathbb{Z}_2$-cores with some suitable composite $(\mathbb{Z}_2)^m$-coloring. Note that the zigzag boundary in each of the three $\mathbb{Z}_2$-cores denotes their reflexive panel which corresponds to the non-free orbits of a locally standard $(\mathbb{Z}_2)^m$-action.

Next, let us classify the locally standard $(\mathbb{Z}_2)^m$-manifolds over X^n up to the equivalence from the viewpoint of glue-back construction. By a similar argument as in [3], we can prove:

Theorem 5.3. *The set of equivalence classes in locally standard $(\mathbb{Z}_2)^m$-manifolds over X^n bijectively corresponds to the coset $\Xi(U^n, (\mathbb{Z}_2)^m)/\mathrm{GL}(m, \mathbb{Z}_2)$, where the* $\mathrm{GL}(m, \mathbb{Z}_2)$ *acts on $\Xi(U^n, (\mathbb{Z}_2)^m)$ via automorphisms of the coefficient group $(\mathbb{Z}_2)^m$.*

Moreover, similar to [3], we can classify locally standard $(\mathbb{Z}_2)^m$-manifolds over X^n up to equivariant homeomorphisms as following. Let $\mathrm{Aux}(X^n)$ be the group of self-homeomorphisms of X^n which preserve the manifold with corners structure of X^n. An element $h \in \mathrm{Aux}(X^n)$ will induce a permutation on $\mathcal{F}(X^n)$ denoted by $\Phi(h)\colon \mathcal{F}(X^n) \to \mathcal{F}(X^n)$. So h naturally acts on $\mathcal{V}(X^n, (\mathbb{Z}_2)^m)$ by sending any $\nu \in \mathcal{V}(X^n, (\mathbb{Z}_2)^m)$ to $\nu \circ \Phi(h)$.

Theorem 5.4. *Suppose $\pi\colon M^n \to X^n$ and $\pi'\colon N^n \to X^n$ are two locally standard $(\mathbb{Z}_2)^m$-manifolds over X^n. M^n and N^n are equivariantly homeomorphic if and only if there is an $h \in \mathrm{Aux}(X^n)$ such that $\nu_{\pi'} = \nu_\pi \circ \Phi(h)$ and $h^*(\xi_{\pi'}) = \xi_\pi$ where $h^*(\xi_{\pi'})$ is the pull-back bundle by h.*

Theorem 5.5. *The set of equivariant homeomorphism classes of all n-dimensional locally standard $(\mathbb{Z}_2)^m$-manifolds over X^n bijectively corresponds to the coset*

$$(H^1(X^n, (\mathbb{Z}_2)^m) \times \mathcal{V}(X^n, (\mathbb{Z}_2)^m))/\mathrm{Aux}(X^n)$$

where $\mathrm{Aux}(X^n)$ *acts diagonally on the two factors.*

In addition, we say that two locally standard $(\mathbb{Z}_2)^m$-manifolds M^n and N^n over X^n are *weakly equivariantly homeomorphic* if there is a homeomorphism $f\colon M^n \to N^n$ and an element $\sigma \in \mathrm{GL}(m,\mathbb{Z}_2)$ such that $f(g\cdot x) = \sigma(g)\cdot f(x)$ for all $g \in (\mathbb{Z}_2)^m$ and $x \in M^n$.

Theorem 5.6. *The set of weakly equivariant homeomorphism classes of all n-dimensional locally standard $(\mathbb{Z}_2)^m$-manifolds over X^n bijectively corresponds to the double coset*

$$\mathrm{GL}(m, \mathbb{Z}_2)\backslash(H^1(X^n, (\mathbb{Z}_2)^m) \times \mathcal{V}(X^n, (\mathbb{Z}_2)^m))/\ \mathrm{Aux}(X^n)$$

where both $\mathrm{GL}(m, \mathbb{Z}_2)$ *and* $\mathrm{Aux}(X^n)$ *act diagonally on the two factors.*

6. Some topological information of locally standard $(\mathbb{Z}_2)^m$-manifolds from the $(\mathbb{Z}_2)^m$-colorings

Suppose U^n is a $\mathbb{Z}_2$-core of a connected nice manifold with corners X^n, and the principal panels and reflexive panels of U^n are $\{P_1, \dots, P_k\}$ and $\{P_1', \dots, P_l'\}$ as described in the preceding section. Similar to Theorem 3.7, we can compute the number of connected components in any $M(U^n, \lambda, \mu)$ from the composite $(\mathbb{Z}_2)^m$-coloring (λ, μ) on U^n as following.

Theorem 6.1. *For any* $(\lambda, \mu) \in \Xi(U^n, (\mathbb{Z}_2)^m)$*, the number of connected components in* $M(U^n, \lambda, \mu)$ *is* $2^{m-\mathrm{rank}(\lambda,\mu)}$ *where*

$$\mathrm{rank}(\lambda, \mu) = \dim_{\mathbb{Z}_2}\langle \lambda(P_1), \dots, \lambda(P_k), \mu(P_1'), \dots, \mu(P_l')\rangle.$$

The connected components of $M(U^n, \lambda, \mu)$ *are pairwise homeomorphic, and there is an induced locally standard* $(\mathbb{Z}_2)^{\mathrm{rank}(\lambda,\mu)}$*-action on each component of* $M(U^n, \lambda, \mu)$ *whose orbit space is* X^n*.*

Proof. Suppose we glue the 2^m copies of U^n only along the principal panels first according to the coloring λ, we will get a manifold with boundary denoted by $M(U^n, \lambda)$. By the same argument as in the proof of Theorem 3.7, $M(U^n, \lambda)$ has $2^{m-\mathrm{rank}(\lambda)}$ connected components which are pairwise homeomorphic. Let $L_\lambda := \langle \lambda(P_1), \dots, \lambda(P_k)\rangle \subset (\mathbb{Z}_2)^m$ and let $\theta_\lambda\colon U^n \times (\mathbb{Z}_2)^m \to M(U^n, \lambda)$ be the quotient map. Then an arbitrary connected component of $M(U^n, \lambda)$ is of the following form

$$N_g = \bigcup_{g' \in g + L_\lambda} \theta_\lambda(U^n \times g') \quad \text{for some fixed} \quad g \in (\mathbb{Z}_2)^m.$$

So $M(U^n, \lambda) = N_{g_1} \cup \cdots \cup N_{g_r}$ where $r = 2^{m-\mathrm{rank}(\lambda)}$ and $g_{i'} - g_i \notin L_\lambda$ for any $1 \leq i, i' \leq r$. The boundary of $M(U^n, \lambda)$ consists of those facets from the reflexive panels of $U^n \times g's$.

Next, we glue the facets in the boundary of $M(U^n, \lambda)$ together according to (5) and the coloring μ on the reflexive panels, which will give us the $M(U^n, \lambda, \mu)$. Let $\theta_\mu\colon M(U^n, \lambda) \to M(U^n, \lambda, \mu)$ denote the corresponding quotient map. In addition, let $L_\mu = \langle \mu(P_1'), \dots, \mu(P_l')\rangle \subset (\mathbb{Z}_2)^m$. It is easy to see that $\theta_\mu(N_{g_i})$ and $\theta_\mu(N_{g_{i'}})$ are in the same connected component of $M(U^n, \lambda, \mu)$ if and only if $g_{i'} - g_i \in L_\mu$. So for any $g, g' \in (\mathbb{Z}_2)^m$, the two blocks $\theta_\mu(\theta_\lambda(U^n \times g))$ and $\theta_\mu(\theta_\lambda(U^n \times g'))$ are in the same connected component of $M(U^n,\lambda,\mu)$ if and only if $g'-g \in L_\lambda + L_\mu$. Since the $\mathbb{Z}_2$-dimension of $L_\lambda + L_\mu$ is rank(λ, μ), so each connected component of $M(U^n, \lambda, \mu)$ is the gluing of $2^{\mathrm{rank}(\lambda,\mu)}$ copies of U^n from the glue-back construction. Then $M(U^n, \lambda, \mu)$ has exactly $2^{m-\mathrm{rank}(\lambda,\mu)}$ connected components which are pairwise homeomorphic. Obviously, the restricted action of $(\mathbb{Z}_2)^m$ to $L_\lambda + L_\mu$ on each component of $M(U^n, \lambda, \mu)$ is locally standard. So our theorem is proved. $\square$

In addition, if X^n is orientable, using the same argument as the Theorem 1.7 in [10], we can prove the following.

Theorem 6.2. *For a basis $\{e_1, \dots, e_m\}$ of $(\mathbb{Z}_2)^m$, there is a group homomorphism $\epsilon\colon (\mathbb{Z}_2)^m \to \mathbb{Z}_2$ defined by $\epsilon(e_i) = 1$ for all i. Suppose X^n is orientable. Then $M(U^n,\lambda,\mu)$ is orientable if and only if there exists a basis $\{e_1,\dots,e_m\}$ of $(\mathbb{Z}_2)^m$ such that $\epsilon(\mu(P_1')) = \cdots = \epsilon(\mu(P_l')) = 1$. So in this case, the orientability of $M(U^n, \lambda, \mu)$ is determined only by the coloring μ on the reflexive panels.*

It was shown in [2] that the $\mathbb{Z}_2$-coefficient cohomology ring of any small cover can be computed from the combinatorial structure of the orbit space and the associated characteristic function. But for a general locally standard $(\mathbb{Z}_2)^m$-manifold M^n over X^n, it is not clear how to compute the $\mathbb{Z}_2$-homology group or cohomology ring of M^n from X^n. From our preceding discussions, the simplest case in this problem would be when X^n has a $\mathbb{Z}_2$-core U^n whose faces are all contractible. So we propose the following problem.

PROBLEM. for a locally standard $(\mathbb{Z}_2)^m$-manifold $M(U^n, \lambda, \mu)$ where $(\lambda, \mu) \in \Xi(U^n, (\mathbb{Z}_2)^m)$, if each face of U^n is contractible (or a $\mathbb{Z}_2$-homology ball), find some way to compute the $\mathbb{Z}_2$-homology group and cohomology ring of $M(U^n,\lambda,\mu)$ from the data (U^n, λ, μ).

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