Murphy, T.
Osaka J. Math.
49 (2012), 125–132

TOTALLY GEODESIC SUBMANIFOLDS OF REGULAR SASAKIAN MANIFOLDS

THOMAS MURPHY

(Received August 16, 2010)

Abstract

Let Q^{m+1} denote the family of regular Sasakian manifolds whose base manifold M^{2m} is a compact symmetric space. We provide a classification of the totally geodesic submanifolds of Q^{m+1} which are invariant, anti-invariant of maximal dimension or contact CR with respect to the Sasakian structure. Such submanifolds are closely related to complex and totally real totally geodesic submanifolds of the Hermitian symmetric space M^{2m}.

1. Introduction

Totally geodesic submanifolds of a Riemannian manifold are a fundamental object of study in submanifold geometry, generalizing the geodesics of the manifold to higher dimensional submanifolds. Their classification for a given manifold is a central problem and examples of totally geodesic submanifolds are explicitly known in some very special cases. For space forms all totally geodesic submanifolds were classified by Cartan (see [1] for an overview). The classification is also tractable in the case of symmetric spaces. Here the classification is equivalent to the classification of Lie triple systems, which is an algebraic problem. All the totally geodesic submanifolds of non-spherical rank one symmetric spaces were classified by Wolf [13] and [14]; we note that they are all distinguished with respect to the naturally defined geometric structures. For example, in quaternionic projective space $\mathbb{H}P^n$ the maximal totally geodesic submanifolds are $\mathbb{H}P^{n-1}$ and $\mathbb{C}P^n$. The first is quaternionic, i.e. invariant under a local section J_1, J_2, J_3 of the quaternionic-Kähler structure $\mathcal{J}$, and the second is totally complex with respect to one of the local sections, i.e. $J_1(T_pS) \subset T_pS$ and $J_i(T_pS) \subset \nu_p(S)$, the normal bundle of S, for $i = 2, 3$ and all points $p \in S$. In recent years the classification has been completed in the rank two case by Klein (see [6], [7], [8] and [9]), who found an error in the previous classification of totally geodesic submanifolds in compact rank two symmetric spaces due to Chen and Nagano. In a series of papers he resolved the errors and the classification is now watertight. Aside from two exceptional examples and one exceptional family (none of which are maximal unless the ambient symmetric space has dimension four) all of these in complex two-plane

2000 Mathematics Subject Classification. Primary 53C40; Secondary 53C25.

Grassmannians are again distinguished with respect to its natural geometric structures (the complex structure J and the quaternionic-Kähler structure $\mathcal{J}$). Hence, for symmetric spaces with additional geometric structures it is important to study the totally geodesic submanifolds which respect these structures.

The other fruitful field of study has been Hermitian symmetric spaces (M, J). Most known examples here are either totally real (i.e. for $X \in \Gamma(TS)$, $JX \in \Gamma(\nu(S))$) or else complex (i.e. $JX \in \Gamma(TS)$ for $X \in \Gamma(TS)$). Complex totally geodesic submanifolds are completely classified in Hermitian symmetric spaces due to a theorem of Ihara [10]. Leung's work [11] provides a classification of all real forms of Hermitian symmetric spaces (i.e. totally real, totally geodesic submanifolds S of M with $\dim_{\mathbb{R}}(S) = \dim_{\mathbb{C}}(M)$, but in general totally real totally geodesic submanifolds are more difficult to understand.

Sporadic new examples have been recently found in other symmetric spaces, and there has been progress on the classification of such submanifolds (see [7] for a survey). There has been no progress at all to date on the classification of totally geodesic submanifolds for non-symmetric spaces, as the existing techniques which reduce it to an algebraic problem do not generalize. This paper aims to address this deficit by investigating regular Sasakian manifolds Q^{m+1} (which are circle bundles over Hermitian manifolds) whose base M^{2m} is also a symmetric space. Hence their base space is a Hermitian symmetric space, and all real forms and complex totally geodesic submanifolds in the base space are known. The totally geodesic submanifolds of Q^{m+1} which satisfy similar geometric conditions to those of totally real and complex submanifolds in Hermitian symmetric spaces are classified in this paper. Q^{m+1} does not have a complex structure, but it does have a Sasakian structure φ. The analogous class of submanifolds $S \subset Q^{m+1}$ which respect this structure are those which are invariant (i.e. for $X \in \Gamma(TS)$, $\varphi(X) \in \Gamma(TS)$) and those which are anti-invariant (i.e. for $X \in \Gamma(TS)$, $\varphi(X) \in \Gamma(\nu(S))$, the normal bundle of S). The third distinguished family of submanifolds that have been investigated are contact CR submanifolds. We define a submanifold $S \subset Q^{m+1}$ to be contact CR if there exists a distribution $\mathcal{D}$ of non-zero dimension such that there is a decomposition

$$TS = \{U\} \oplus \mathcal{D} \oplus \mathcal{D}^{\perp}$$

with $\varphi(\mathcal{D}) = \mathcal{D}$ and $\varphi(\mathcal{D}^{\perp}) \in \nu(S)$, the normal bundle of S. Here $\{U\}$ denotes the distribution spanned by the Killing vector field U tangential to the fibres of the submersion. Our objective is to establish:

Theorem. *Let $\pi : (Q^{m+1}, \varphi) \to (M^{2m}, J)$ denote the canonical submersion from a regular Sasakian manifold to a Hermitian symmetric space.*

- *S is an invariant totally geodesic submanifold of Q^{m+1} if and only if it is locally isometric to $\pi^{-1}(P)$, where P is a complex totally geodesic submanifold of M^{2m}.*

- *S is an anti-invariant totally geodesic submanifold of Q^{m+1} of maximal dimension m if and only if it is locally isometric to $\bar{H}$, where $\bar{H}$ is a horizontal lift of a real form of M^{2m}.*
- *There are no contact CR submanifolds of Q^{m+1} which are totally geodesic.*

If we assume further that S is a closed submanifold it is easy to extend this local classification to deduce that if S is invariant it is isometric to $\pi^{-1}(P)$ for P a closed totally geodesic complex submanifold of M^{2m}. If S is anti-invariant the horizontal lift $\bar{H}$ might not patch together to give a closed submanifold in Q^{m+1}, so one can only conclude that it is locally isometric to a horizontal lift of the same totally real, totally geodesic submanifold of M^{2m}. Applying the work of Ihara mentioned above together with this result yields an explicit classification for invariant submanifolds. Leung's work implies that the anti-invariant totally geodesic submanifolds of Q^{m+1} of maximal dimension m are also classified as a corollary of this theorem.

For symmetric spaces of low rank, specifically of rank one or two, there are complete explicit classifications of totally geodesic submanifolds [13], [7], and one may easily read off from this a list of real forms and complex totally geodesic submanifolds. In these cases, the theorem yields an explicit list of the invariant and m-dimensional anti-invariant totally geodesic submanifolds of the corresponding regular Sasakian manifolds. As an illustration of this, consider the case $\pi: SU(m+2)/SU(m)SU(2) \to G_2(\mathbb{C}^{m+2})$, $m > 3$, where the base space is the complex two-plane Grassmannian, a rank-two symmetric space. From the list in [7] there are three families of submanifolds in the first class, namely $\pi^{-1}(P)$ where P is isometric to a neighbourhood of $\mathbb{C}P_1^l$, $G_2(\mathbb{C}^{l+2})$, where in both cases $1 \le l \le n$, and finally $\mathbb{C}P_1^{l_1} \times \mathbb{C}P_1^{l_2}$, where $l_1 + l_2 = m$. There is also an exceptional example, where P is isometric to a neighbourhood of an $G_2^+(\mathbb{R}^5)$. For the second class, S is a horizontal lift of a real form of the base space. These are locally isometric to one of; $\mathbb{R}P_1^m$, $\mathbb{R}P_{1/\sqrt{5}}^m$, $\mathbb{R}P_{1/2}^m$, $\mathbb{C}P_{1/2}^m$, $\mathbb{H}P_{1/2}^m$, $G_2(\mathbb{R}^{m+2})$, or $\mathbb{R}P_1^{l_1} \times \mathbb{R}P_1^{l_2}$ where $l_1 + l_2 = m$. The subscripts here refer to nonstandard scalings of the metric; we refer the reader to [8] for explanations of this notation and further details.

This work was undertaken as part of a Ph.D. under the supervision of Prof Jürgen Berndt at University College Cork, Ireland. The author thanks him and Sebastian Klein for their helpful comments. The author was supported in the course of this research by a postgraduate fellowship of the Irish Research Council for Science, Engineering and Technology.

2. Proof of main result

To begin, well-known facts about the geometry of regular Sasakian manifolds are briefly summarized. A good general reference is [4]. Let M^{2m} denote a compact Hermitian symmetric space, where G is the identity component of the full isometry group. This is a connected semisimple Lie group. The stabilizer K has one-dimensional center U_1, and so K is diffeomorphic to HU_1. The homogeneous space $Q^{m+1} = G/H$ is a regular

Sasakian manifold of dimension $2m+1$ and the canonical projection $\pi\colon Q \to M$ is the corresponding Riemannian submersion with U_1 fibres. This paper is concerned with the geometry of the submanifolds of Q^{m+1}.

The metric $\langle\,\cdot\,,\,\cdot\,\rangle$ is invariant along the fibre of the circle bundle. Denote the corresponding Killing vector field on Q^{m+1} by U, whose value at p is denoted $U(p)$. There is a foliation $\mathcal{F}$ on Q^{m+1} induced by the flow of the unit Killing vector field U and the maximal integral curves of U are closed geodesics in Q^{m+1}. $\mathcal{F}$ is a totally geodesic Riemannian foliation on Q^{m+1}, with $Q^{m+1}/\mathcal{F}$ isometric to (M^{2m}, J). Setting $\varphi := -\nabla U$, where ∇ denotes the Levi-Civita connection of $(Q^{m+1}, \langle\cdot,\cdot\rangle)$, we also have that $(\varphi, U, \langle\,\cdot\,, U\rangle)$ is a Sasakian structure on Q^{m+1}, and it is related to the complex structure on the base space by the fundamental formula

$$J\pi_* = \pi_*\varphi. \tag{2.1}$$

Define the horizontal distribution $\mathcal{H}(p) := \{X \in T_pQ^{m+1}\colon \langle X, U(p)\rangle = 0\}$. Then we note without proof that $\varphi(\mathcal{H}) \subset \mathcal{H}$ and $\varphi^2(X) = -X + \langle X, U\rangle U$ (see [4]). Moreover,

$$\langle\varphi(X), \varphi(Y)\rangle = \langle X, Y\rangle - \langle X, U\rangle\langle Y, U\rangle.$$

It follows from this that that $\langle\varphi(X), Y\rangle = -\langle X, \varphi(Y)\rangle$ if one of $X, Y \in \Gamma(\mathcal{H})$. We are now ready to prove the theorem.

Proof. Let S be a totally geodesic submanifold in (Q^{m+1}, g) of dimension n. We will work in a neighbourhood W of a point $q \in S$ in what follows. Suppose further that S is an invariant or anti-invariant submanifold of Q^{m+1} that is not in one of the following two families:

(1) $U(p) \in T_pS$ for all $p \in W$, or else

(2) $U(p) \perp T_pS$ for all $p \in W$,

i.e. $U(p)$ is neither perpendicular to nor contained in T_pS for all $p \in W$. Choose a local framing of Q^{m+1} constructed in the following manner. At each point $p \in W$, $U|_{TS}(p) := U(p)|_{T_pS}$ denotes the vector field obtained by projecting U at each point $p \in S$ to the vector subspace T_pS. Multiply $U|_{TS}$ by the function $f \in C^\infty(W)$ so that $\|f(p)U|_{TS}(p)\| = 1$ and set $X(p) = f(p)U|_{TS}(p)$. Then choose an orthonormal framing of $TS|_W$: $\{X, Y_1, \ldots, Y_{n-1}\}$ where by construction $Y_i \in \Gamma(\mathcal{H})$. The notation $\Gamma(TS)|_W$ will denote the restriction of the vector field to W, so that on all points $q \in W$, $X(q) \in T_qS$, etc. Since $U \notin \Gamma(TS)|_W$ there is also a nonzero orthogonal projection at each point p to the normal bundle $\nu_p(S)$ which is again normalized to give a unit normal vector field called ξ_1. Then complete our framing by choosing an orthonormal frame field of $\nu(S)|_W$: $\{\xi_1, \ldots, \xi_{m-n}\}$. By our choice of framing on W we also have that

$$U(p) = f_1(p)X(p) + f_2(p)\xi_1(p) \tag{2.2}$$

for nonzero functions $f_1, f_2 \in C^\infty(W)$. By re-choosing a smaller neighbourhood W of p if necessary, both functions may be chosen to be nonzero at all points $p \in W$.

Observe that $\langle\varphi(Y), U\rangle = -\langle\nabla_Y U, U\rangle = 0$ so, from Equation (2.2)

$$\langle\varphi(Y_i), X\rangle = \frac{-f_2(p)}{f_1(p)}\langle\varphi(Y_i), \xi_1\rangle. \tag{2.3}$$

Assume now that S is anti-invariant of dimension $n = m$. From the fact $\varphi(X) = -(f_2(p)/f_1(p))\varphi(\xi_1)$ it follows also that $\varphi(X) \perp \xi_1$ and so

$$\varphi(X) \in Span\{\xi_2, \ldots, \xi_m\},$$

where the framing is chosen by setting $\xi_j = \varphi(Y_{j-1})$, $j \geq 2$. But $\langle\varphi(X), \xi_j\rangle = \langle\varphi(X), \varphi(Y_{j-1})\rangle = \langle-\varphi^2(X), Y_{j-1}\rangle$, because $Y_{j-1} \in \mathcal{H}$. This is $\langle X - f_1 U, Y_{j-1}\rangle$ which is zero by construction. Hence $\varphi(X) \in \Gamma(TS)$, contrary to assumption. It follows from Equation (2.3) alone that S cannot be invariant for all n. If it were, then $\langle\varphi(X), Y_j\rangle \neq 0$ for at least one $j \geq 1$ and then applying Equation (2.3) leads to an immediate contradiction.

The upshot of this is that there are exactly two possibilities:

(1) $U \in \Gamma(TS)|_W$, or else

(2) $U \in \Gamma(\nu S)|_W$.

In Case (1), $U(q) \in T_q S$ and so $U(p) \in T_p S$ for all $p \in W$. Otherwise, at some point q_0 we would have $U \notin T_{q_0} S$. But then, repeating the same argument as above at the point q_0, since we are not in Case (1), and S is totally geodesic, Case (2) holds at q_0: so $U \in \nu_{q_0}(S)$. Connecting q and q_0 by a path ϕ, let U' be the orthogonal projection of U onto $TS|_W$. Case (1) means $\|U'\| = 1$, Case (2) means $\|U'\| = 0$. For continuity reasons, the continuous function $\|U'\|\colon S \to \mathbb{R}$ cannot jump from being 1 to being 0 as one travels along ϕ, so S cannot jump from being in Case (1) to being in Case (2).

For Case (1), the fact that $U \in \Gamma(TS)|_W$ implies that $\pi(S)|_W$ is a submanifold of M. This follows from the equation $\pi_*[\bar{X}, \bar{Y}] = [X, Y] \circ \pi$ for basic lifts of two vector fields $X, Y \in \Gamma(T\pi(S))$, the Frobenius theorem and the fact $S \subset Q$. Choosing η to be a unit normal vector field to S, $\eta \in \Gamma(\mathcal{H})$ and so induces a unit normal vector field $\pi_*\eta$ on $\pi(S)|_W$, and every unit normal vector field arises that way. The next step is to establish that $\pi(S)$ is totally geodesic. If it were not, then there would exists a vector field $B \in \Gamma(T\pi(S))$ such that $\nabla^M_B \xi = \lambda B$ with $\lambda \neq 0$, where $\xi \in \nu(T\pi(S))$ is a unit normal vector field. Here ∇^M denotes the Levi-Civita connection for M^{2m} and similarly ∇^Q denotes the Levi-Civita connection of Q^{m+1}. But then by the fundamental equations for a Riemannian submersion

$$\nabla^Q_{\bar{B}}\bar{\xi} = \lambda\bar{B} + A^{ON}_{\bar{B}}\bar{\xi}$$

where A^{ON} is the O'Neill tensor for a Riemannian submersion, $\bar{B}$ denotes the horizontal lift of B, and $\bar{\xi}$ is the horizontal lift of ξ. $\bar{\xi}$ is a unit normal vector field to S. By definition A^{ON} takes values in the tangent space to the fibres, so the vector fields on the right hand side are normal to each other. The fact $\lambda \neq 0$ would contradict the fact

S is totally geodesic and hence $\pi(S)$ is totally geodesic. Since $U \in T_pS$ for all $p \in W$, therefore the integral curves of U (which are the fibres of the submersion) are in T_qS and so $\pi^{-1}(\pi(S)) = S$ on W. Now it is well-known that for a Riemannian submersion $\pi\colon Q \to M$ and $P \subset M$ a totally geodesic submanifold of the base space M, then $\pi^{-1}(P)$ is totally geodesic if and only if

$$A^{QN}_{\bar{Z}}\bar{\xi} = 0,$$

for $\bar{Z}$ a horizontal lift of a vector field Z tangent to P and $\bar{\xi}$ a horizontal lift of a unit vector field ξ normal to P. See [5], Theorem 2.9 for a proof of this. Calculating the scalar component of this tensor gives

$$\begin{aligned}\frac{1}{2}\langle[\bar{Z},\bar{\xi}],U\rangle &= \frac{1}{2}\big(\langle\nabla^Q_{\bar{Z}}\bar{\xi} - \nabla^Q_{\bar{\xi}}\bar{Z},U\rangle\big) = \frac{1}{2}\big(\langle\nabla^Q_{\bar{Z}}\bar{\xi},U\rangle - \langle\nabla^Q_{\bar{\xi}}\bar{Z},U\rangle\big)\\ &= \frac{-1}{2}\big(\langle\bar{\xi},\nabla^Q_{\bar{Z}}U\rangle - \langle\bar{Z},\nabla^Q_{\bar{\xi}}U\rangle\big) = \langle\bar{\xi},\varphi(\bar{Z})\rangle,\end{aligned}$$

using the facts that φ is skew-symmetric and $\bar{Z},\bar{\xi} \in \mathcal{H}_q$ for all points $q \in W$. Hence it vanishes if and only if $\varphi(TS) \subset TS$, and from Equation (2.1) it follows that it vanishes if and only if $P \subset M$ is complex. This proves the first case of the theorem.

The classification of all examples which fall into Case (2) is an elementary application of a theorem of Reckziegel [12], who showed that if one has a Riemannian submersion $\pi\colon Q \to M$ from a Sasakian manifold Q to a Kähler manifold M then every horizontal submanifold $\tilde{H}$ corresponds (under the map π) to a totally real submanifold H in M, and vice versa. Moreover, the the second fundamental forms are related by the formula

$$\alpha_H = \pi_*\alpha_{\tilde{H}}.$$

It is immediate from this that $\tilde{H}$ is totally geodesic if and only if H is and hence $\tilde{H} \subset Q^{m+1}$ is totally geodesic if and only if it is a horizontal lift of a totally geodesic totally real submanifold.

Finally, suppose $S \subset Q^{m+1}$ is a contact CR totally geodesic submanifold, and choose a neighbourhood W of a point $p \in S$. Recall that $TS = \{U\} \oplus \mathcal{D} \oplus \mathcal{D}^\perp$. Hence any CR contact submanifold would falls into class 1 by definition. It has already been shown that the only totally geodesic submanifolds in the first case are invariant, so it follows that there are no totally geodesic contact CR submanifolds of Q^{m+1}. □

In general it is more difficult to classify the totally real submanifolds of a Hermitian symmetric space than the complex ones, as Leung's work illustrates. The theorem above actually classifies all anti-invariant submanifolds of maximal dimension which fall into class 1 or 2, so it would be of interest to study those which are not in these classes.

REMARK. A theorem in [15] was brought to our attention after this work was completed. Here the following result is proven: Let $\pi\colon Q \to M$ be the usual fibering from a $(2m+1)$-dimensional Sasakian manifold Q to a $2m$-dimensional Kähler manifold M. Let S be an $(m+1)$-dimensional invariant submanifold of Q and N be an m-dimensional complex submanifold of M such that $\pi(S) = N$. Then S is totally geodesic in Q if, and only if, N is totally geodesic in M. Our result agrees with theirs in the invariant submanifold case, but is stronger as we have shown that the hypothesis that the projection of the invariant submanifold to the base space be a complex submanifold and the restriction on the dimensions of the submanifolds may be removed, as well as actually classifying the various possibilities for S.

References

[1] J. Berndt, S. Console and C. Olmos: Submanifolds and Holonomy, Chapman & Hall/CRC Research Notes in Mathematics **434**, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[2] J. Berndt: *Riemannian geometry of complex two-plane Grassmannians*, Rend. Sem. Mat. Univ. Politec. Torino **55** (1997), 19–83.

[3] A. Bejancu: Geometry of CR-Submanifolds, Mathematics and its Applications (East European Series) **23**, Reidel, Dordrecht, 1986.

[4] D.E. Blair: Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics **203**, Birkhäuser Boston, Boston, MA, 2002.

[5] M. Falcitelli, S. Ianus and A.M. Pastore: Riemannian Submersions and Related Topics, World Sci. Publishing, River Edge, NJ, 2004.

[6] S. Klein: *Totally geodesic submanifolds of the exceptional Riemannian symmetric spaces of rank* 2, Osaka J. Math. **47** (2010), 1077–1157.

[7] S. Klein: *Totally geodesic submanifolds in Riemannian symmetric spaces*; in Differential Geometry, World Sci. Publ., Hackensack, NJ, 136–145, 2009.

[8] S. Klein: *Totally geodesic submanifolds of the complex and the quaternionic* 2*-Grassmannians*, Trans. Amer. Math. Soc. **361** (2009), 4927–4967.

[9] S. Klein: *Totally geodesic submanifolds of the complex quadric*, Differential Geom. Appl. **26** (2008), 79–96.

[10] S. Ihara: *Holomorphic imbeddings of symmetric domains*, J. Math. Soc. Japan **19** (1967), 261–302.

[11] D.S.P. Leung: *Reflective submanifolds*, IV, *Classification of real forms of Hermitian symmetric spaces*, J. Differential Geom. **14** (1979), 179–185.

[12] H. Reckziegel: *A correspondence between horizontal submanifolds of Sasakian manifolds and totally real submanifolds of Kählerian manifolds*; in Topics in Differential Geometry, I, II (Debrecen, 1984), Colloq. Math. Soc. János Bolyai **46**, North-Holland, Amsterdam, 1063–1081, 1988.

[13] J.A. Wolf: *Geodesic spheres in Grassmann manifolds*, Illinois J. Math. **7** (1963), 425–446.

[14] J.A. Wolf: *Elliptic spaces in Grassmann manifolds*, Illinois J. Math. **7** (1963), 447–462.

[15] K. Yano and M. Kon: CR Submanifolds of Kaehlerian and Sasakian Manifolds, Progress in Mathematics **30**, Birkhäuser, Boston, MA, 1983.

School of Mathematical Sciences
University College Cork
Ireland

Current address:
Départment de Mathématique
Université Libre de Bruxelles
Boulevard du Triomphe
B-1050 Bruxelles
Belgique
e-mail: tmurphy@ulb.ac.be