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Osaka J. Math.
48 (2011), 1085–1094

SOME GENERALIZATIONS OF HALPHEN'S EQUATIONS

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(Received May 24, 2010)

Abstract

Halphen's equations are given by a remarkable polynomial vector field in $\mathbf{C}^3$ having only single-valued solutions, defined in domains bounded by a circle or by a line. By generalizing the Lie-theoretic principle behind Halphen's equations and borrowing some facts from the theory of deformations of Fuchsian groups, we exhibit a family of polynomial vector fields in $\mathbf{C}^3$ having only single-valued solutions. The solutions of vector fields within this family are defined in domains which had not been previously observed as domains of definition of solutions of polynomial vector fields in $\mathbf{C}^3$. For example, we obtain polynomial vector fields having solutions defined in domains that are bounded by a fractal curve.

1. Introduction

In the complex domain, the solutions of an ordinary differential equation may be multivalued, as it is the case for the differential equation $t\,dy/dt = 1$, whose solution is given by the logarithm. Even at the level of complex differential equations given by polynomial vector fields in $\mathbf{C}^n$, there is no full understanding of the obstructions that a vector field must overcome in order to have a single valued solution. Even if a vector field does have a single-valued solution, it is difficult to know, a priori, its properties. If a solution is single-valued, we may extend the domain of $\mathbf{C}$ where it is defined in order to obtain a maximal domain (open subset of $\mathbf{C}$) where the solution is defined (this can be taken as the definition of *single-valued solution*). What can be said about the nature of this maximal domain? If we restrict to the class of polynomial vector fields in $\mathbf{C}^n$ where *every* solution is single-valued (this is, the vector fields that are *semicomplete* in the sense of Rebelo [20]) then, at least in $\mathbf{C}^2$, we have the following result: *In a semicomplete polynomial vector field in* $\mathbf{C}^2$, *every solution is defined in the complement of a countable set of points* [10, Corollary C].

We do not have a complete picture of semicomplete polynomial vector fields in $\mathbf{C}^3$. We know, however, that the domains where the solutions of such vector fields are defined are not as simple as the domains appearing in lower dimensions. A remarkable

2000 Mathematics Subject Classification. Primary 34M05, 34M15, 32G15; Secondary 17B66, 57S30.

vector field in $\mathbf{C}^3$ was introduced by Darboux [7] and studied by Halphen [12] in 1881:

$$(1)\quad X = (z_1z_2 - z_2z_3 + z_1z_3)\frac{\partial}{\partial z_1} + (z_2z_3 - z_1z_3 + z_1z_2)\frac{\partial}{\partial z_2} + (z_2z_3 - z_1z_2 + z_1z_3)\frac{\partial}{\partial z_3}.$$

Halphen showed that X is semicomplete and that its solutions are defined in domains having a circle (or a line) as a natural boundary. Equations (1) appear in many settings, such as reductions of Yang–Mills equations [5] and dynamics of magnetic monopoles [2] and have connections with modular forms and Ramanujan's functions [18], [23], to cite only a few. There are many generalizations of these equations, the first ones considered by Halphen himself [11], providing many more semicomplete vector fields in $\mathbf{C}^3$. We will not try to list all these generalizations. Let us just mention that some were given in [8] and generalizations where the ambient space is no longer $\mathbf{C}^3$ but a three-dimensional singular affine variety have appeared in [6] and [19]. However, in all the semicomplete generalizations we know of, the solutions are still meromorphic functions defined in domains having, again, a circle or a line as natural boundary. These natural boundaries arise as boundaries of components of the discontinuity domain of a Fuchsian group.

Do there exist semicomplete polynomial vector fields in $\mathbf{C}^3$ having solutions defined in other kinds of domains? We will exhibit a family of vector fields whose solutions are defined in components of the discontinuity domains of Kleinian groups which are no longer Fuchsian:

Theorem 1. *For every $\rho \in \mathbf{C} \setminus \{0, 1\}$, there exists a nonempty open bounded subset $\mathcal{B}_\rho \subset \mathbf{C}$ such that the vector field*

$$(2)\quad \begin{aligned} X_\rho(\alpha) = {} & [z_1^2 + z_2z_3(z_2 - z_3)(\{\alpha - \rho\}z_2 + \rho\{1 - \alpha\}z_3)]\frac{\partial}{\partial z_1} \\ & + z_2(z_1 + \rho z_2z_3 - \rho z_3^2)\frac{\partial}{\partial z_2} + z_3(z_1 + z_2^2 - z_2z_3)\frac{\partial}{\partial z_3} \end{aligned}$$

is semicomplete (*has only single-valued solutions*) *for every $\alpha \in \bar{\mathcal{B}}_\rho$. A solution of any such $X_\rho(\alpha)$ is defined in the complement of ∞ in $\mathbf{CP}^1$ of an invariant component of the discontinuity domain of either a quasifuchsian, a totally degenerate or a cusp Kleinian group* (*and all these situations appear*).

In particular, there exist semicomplete polynomial vector fields in $\mathbf{C}^3$ whose solutions are defined in simply connected domains bounded by a fractal Jordan curve, or in simply connected domains whose complement is uncountable (for example with non-empty interior) but whose boundary is not a Jordan curve. To our best knowledge, the family (2) gives the only examples of semicomplete polynomial vector fields in $\mathbf{C}^3$ where such phenomena appear.

A key point in Halphen's work is an invariance property that can be infinitesimally expressed as follows: X and the vector fields $L = \sum_i z_i \partial/\partial z_i$ and $Z = \sum_i \partial/\partial z_i$ satisfy the relations

$$[L, X] = X, \quad [L, Z] = -Z, \quad [Z, X] = 2L. \tag{3}$$

Our vector fields enjoy an analogue property and are, in this sense, generalizations of Halphen's equations. For the proof of our theorem we will use the approach developed in [9]. It is geometric and is expressed in terms of (G, X)-structures on manifolds in the sense of Thurston [22]. We will use some facts about Kleinian groups and, more specifically, the deformations of Fuchsian groups, that may be found in [16, 3, 15].

2. Preliminaries

Let G be a Lie group acting faithfully and transitively on a complex manifold X via a fixed action $\Phi\colon G \times X \to X$. A (G,X)-structure on a manifold M is an atlas for its complex structure taking values on X and having changes of coordinates in G. In other words, there exists a covering $\{U_i\}$ of M and charts $\phi_i\colon U_i \to X$ (biholomorphisms unto their image) such that $\phi_j \circ \phi_i^{-1}\colon \phi_i(U_i \cap U_j) \to X$ agrees, in each connected component of its domain, with $\Phi(g, \cdot\,)$ for a unique $g \in G$. A chart for a (G, X)-structure can be globalized in the universal covering $\tilde{M}$ of M: we have a *developing map* $\mathcal{D}\colon \tilde{M} \to X$ and a *monodromy* morphism $\mu\colon \pi_1(M) \to G$ that satisfy the relation

$$\mathcal{D}(\alpha \cdot p) = \mu(\alpha)\mathcal{D}(p). \tag{4}$$

Reciprocally, a (G, X)-structure may be recovered from the couple $(\mathcal{D}, \mu)$.

A *projective structure* on a (complex) curve C is a $(\mathrm{PSL}_2(\mathbf{C}), \mathbf{CP}^1)$-structure (with the action of $\mathrm{PSL}_2(\mathbf{C})$ on $\mathbf{CP}^1$ by fractional linear transformations). The projective structures on a curve C form an affine space directed by the vector space of holomorphic quadratic differentials on C: Let $\{(U_i, \phi_i)\}$ and $\{(V_i, \psi_i)\}$ be charts for two projective structures on a curve C. Their difference is a quadratic differential, given in $U_i \cap V_j$ by $\{f(s), s\}ds^2$, where

$$\{f, s\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the *Schwarzian derivative* and $f = \psi_j \circ \phi_i^{-1}$ (the quadratic differential is globally well-defined). Reciprocally, given a projective structure on C and a quadratic differential Q, we may build a new projective structure such that the difference with the original one is exactly Q: if, in a chart of the projective structure, Q is given by $q(z)\,dz^2$, a chart for the new projective structure will be given by any solution to the differential equation $\{\phi(z), z\} = q(z)$.

Let G be a complex Lie group, $\mathfrak{g}$ the Lie algebra of its left-invariant vector fields. Let M be a complex manifold having the same dimension as G and let $\mathfrak{X}(M)$ be its Lie

algebra of holomorphic vector fields. For the natural action of G to the left upon itself by left translations, a (G,G)-structure (or left G *translation structure*) is equivalent to a representation $f\colon \mathfrak{g} \to \mathfrak{X}(M)$ such that $f(V)|_p \neq 0$ for every $V \in \mathfrak{g}$ and $p \in M$. In fact, given a left G translation structure $\{U_i, \phi_i\}$ on M, for $V \in \mathfrak{g}$ define $f(V)$ on U_i as the pull-back $\phi_i^*(V)$. In the opposite direction, if $f\colon \mathfrak{g} \to \mathfrak{X}(M)$ is such that $f(V)|_p \neq 0$ for every $V \in \mathfrak{g}$, Lie's third theorem [21] guarantees that there exists a neighborhood $U \subset M$ of p and a diffeomorphism $\phi\colon U \to G$ such that $D\phi(V) = f(V)$, whose germ at p is unique up to a left translation in G.

3. The geometry of the vector fields

We will set, in order to simplify notation, $X = X_\rho(\alpha)$. Together with X, the vector fields $L = z_1\,\partial/\partial z_1 + (1/2)\,\partial/\partial z_2 + (1/2)\,\partial/\partial z_3$ and $Z = \partial/\partial z_1$ satisfy the relations (3). In this way, if $\mathfrak{W}$ denotes the Lie algebra of vector fields in $\mathbf{C}^3$ generated by X, L and Z and $\mathfrak{sl}_2(\mathbf{C})$ denotes the Lie algebra of *left* invariant vector fields in $\mathrm{SL}_2(\mathbf{C})$ (identifying each vector field to its value at the identity), the unique linear mapping $\psi : \mathfrak{W} \to \mathfrak{sl}_2(\mathbf{C})$ for which

$$\psi(X) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi(L) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \psi(Z) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{5}$$

is a Lie algebra isomorphism. In this way, X comes with a representation of $\mathfrak{sl}_2(\mathbf{C})$ into the Lie algebra of polynomial vector fields in $\mathbf{C}^3$. Let Ω be the locus of linear independence of L, Z and X, which is invariant by X. Its complement is the zero locus of $z_2 z_3 (z_2 - z_3)(z_2 - \rho z_3)$.

The solutions of X with initial condition in the complement of Ω are rational. In the coordinates $(x_1, x_2, 0)$ for $\{z_3 = 0\}$; $(x_1, 0, x_2)$ for $\{z_2 = 0\}$; (x_1, x_2, x_2) for $\{z_2 = z_3\}$; $(x_1 + \rho(\rho-1)x_2^2, \rho x_2, x_2)$ for $\{z_2 = \rho z_3\}$, the restriction of X is $x_1^2\,\partial/\partial x_1 + x_1 x_2\,\partial/\partial x_2$. For the initial condition (x_1^0, x_2^0), the solution is $(x_1^0/(1 - t x_1^0), x_2^0/(1 - t x_1^0))$ and is thus single valued (in restriction to the complement of Ω, the vector field is semicomplete). In view of this, we will focus exclusively on the solutions with initial condition in Ω.

For any discrete group $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$, the vector field $\psi(X)^\Gamma$ induced by $\psi(X)$ on $\Gamma\backslash\mathrm{SL}_2(\mathbf{C})$ is complete. This implies that for any open subset $U \subset \Gamma\backslash\mathrm{SL}_2(\mathbf{C})$, the restriction of $\psi(X)^\Gamma$ to U is semicomplete. To prove Theorem 1 we will prove that, for some values of (α, ρ), there is a discrete group $\Gamma_{\alpha,\rho} \subset \mathrm{SL}_2(\mathbf{C})$ and an embedding $i_{\alpha,\rho}\colon \Omega \to \Gamma\backslash\mathrm{SL}_2(\mathbf{C})$ that maps $X|_\Omega$ to the restriction to the image of $\psi(X)^\Gamma$. In this situation, the solutions of X with initial condition in Ω are single-valued and hence, for every $p \in \mathbf{C}^3$, the solution of X with initial condition p will be single-valued.

Since Ω has three linearly independent vector fields satisfying the relations (3), it is endowed with a left $\mathrm{SL}_2(\mathbf{C})$ translation structure. Fix a point $p \in \Omega$. We have the universal covering $\Pi\colon (\tilde{\Omega}, \tilde{p}) \to (\Omega, p)$, a unique developing map $\mathcal{D}\colon (\tilde{\Omega}, \tilde{p}) \to$

$(\mathrm{SL}_2(\mathbf{C}), e)$ and the corresponding monodromy morphism $\mu\colon \pi_1(\Omega) \to \mathrm{SL}_2(\mathbf{C})$—that we will denote by $\mu_\rho[\alpha]$ whenever we need to stress the dependence upon the parameters. Let $P = \left\{\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}\right\} \subset \mathrm{SL}_2(\mathbf{C})$. It is a closed subgroup of $\mathrm{SL}_2(\mathbf{C})$ generated by $\psi(L)$ and $\psi(Z)$. The Lie subalgebra of $\mathfrak{W}$ generated by L and Z integrates into a free holomorphic action to the right of P on Ω given by

$$(z_1, z_2, z_3)\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = (a^2 z_1 - ac, a z_2, a z_3).$$

The Lie subalgebra of $\mathfrak{sl}_2(\mathbf{C})$ generated by $\psi(L)$ and $\psi(Z)$ integrates into the action by right translations of P.

Let A be the orbit of this action that contains p and let $\Sigma_\rho = \mathbf{C} \setminus \{0, 1, \rho\}$. The orbits of the action are fibers of the fibration $A \to \Omega \to \Sigma_\rho$, given by the restriction of $\omega(z_1, z_2, z_3) = z_2/z_3$. Since $\pi_2(\Sigma_\rho) = 0$, from the homotopy long exact sequence associated to the fibration, we obtain the short exact sequence

$$0 \to \pi_1(A) \to \pi_1(\Omega) \to \pi_1(\Sigma_\rho) \to 0. \tag{6}$$

We claim that $\pi_1(A) \subset \ker \mu$. This is equivalent to the fact that $\mathcal{D}$ is well defined in a neighborhood of A. The developing map $\mathcal{D}$ is (locally) equivariant with respect to the above action of P on Ω and with respect to the action by right translations of P on $\mathrm{SL}_2(\mathbf{C})$. Let $\tau : U \to (\Omega, p)$ a germ of solution of X with initial condition p (defined in some suitable neighborhood U of 0 in $\mathbf{C}$). The mapping τ parametrizes locally the fibers close to A. For each point q close to A there exists a unique $(t, B) \in U \times P$ such that $\tau(t)$ and q are in the same fiber and such that, under the above action of P, $qB \in \tau(U)$. We have that $\mathcal{D}(q) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} B^{-1}$ is well-defined in a neighborhood of A and thus $\pi_1(A) \subset \ker \mu$.

Let $\bar{\Pi}\colon \bar{\Omega} \to \Omega$ be the Galois covering associated to $\pi_1(A)$, with the group of deck transformations $\pi_1(\Omega)/\pi_1(A) = \pi_1(\Sigma_\rho)$. We have a well-defined mapping $\bar{\mu}\colon \pi_1(\Sigma_\rho) \to \mathrm{SL}_2(\mathbf{C})$ induced by μ and a well defined developing map $\bar{\mathcal{D}}\colon \bar{\Omega} \to \mathrm{SL}_2(\mathbf{C})$ induced by $\mathcal{D}$. The couple $(\bar{\mathcal{D}}, \bar{\mu})$ satisfies relation (4). If $\Gamma = \mu(\pi_1(\Omega))$, we have

$$\begin{array}{ccc} \bar{\Omega} & \xrightarrow{\bar{\Pi}} & \Omega \\ \downarrow{\scriptstyle \bar{\mathcal{D}}} & & \downarrow{\scriptstyle i} \\ \mathrm{SL}_2(\mathbf{C}) & \xrightarrow{\Gamma\backslash} & \Gamma\backslash\mathrm{SL}_2(\mathbf{C}). \end{array}$$

The quotient $\Gamma\backslash\mathrm{SL}_2(\mathbf{C})$ is a non-Hausdorff manifold (Hausdorff if Γ is discrete) and i, defined as the class in $\Gamma\backslash\mathrm{SL}_2(\mathbf{C})$ of $\bar{\mathcal{D}} \circ \bar{\Pi}^{-1}$, is an immersion that is well-defined in view of relation (4).

Suppose that $\bar{\mathcal{D}}$ is injective. By the monodromy formula (4), $\bar{\mu}$ is a faithful representation. Also, the action of Γ on $\mathrm{SL}_2(\mathbf{C})$ preserves $\bar{\mathcal{D}}(\bar{\Omega})$ and the restriction of this

action to $\bar{\mathcal{D}}(\bar{\Omega})$ is free and properly discontinuous (in particular, $\Gamma \subset \mathrm{SL}_2(\mathbf{C})$ is a discrete subgroup). The quotient is, as a manifold with a left $\mathrm{SL}_2(\mathbf{C})$ translation structure, isomorphic to Ω and thus, in this case, Ω identifies to a subset of $\Gamma\backslash\mathrm{SL}_2(\mathbf{C})$, identifying X to $\psi(X)$, and X is thus semicomplete. This proves that *X is semicomplete if $\bar{\mathcal{D}}$ is injective*.

The above diagram can be better understood when we "factor out" the action of P. Let $\bar{\omega}$ the foliation in $\bar{\Omega}$ obtained by the pullback of the fibers of ω by $\bar{\Pi}$. The quotient of $\bar{\Omega}$ under $\bar{\omega}$ is, by relation (6), the universal covering $\tilde{\Sigma}_\rho$ of Σ_ρ. The quotient of $\mathrm{SL}_2(\mathbf{C})$ under the multiplicative action to the right of P is $\mathbf{CP}^1$ and is given by

$$(7) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to [b : d], \quad \text{since} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} * & b\alpha^{-1} \\ * & d\alpha^{-1} \end{pmatrix}.$$

Moreover, the standard action to the left of $\mathrm{SL}_2(\mathbf{C})$ on $\mathbf{CP}^1$ is exactly the multiplicative action to the left of $\mathrm{SL}_2(\mathbf{C})$ on $\mathrm{SL}_2(\mathbf{C})/P$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} * & z_0 \\ * & z_1 \end{pmatrix} = \begin{pmatrix} * & az_0 + bz_1 \\ * & cz_0 + dz_1 \end{pmatrix}.$$

Since a leaf of $\bar{\omega}$ is mapped by $\bar{\mathcal{D}}$ to an orbit of the multiplicative action to the right of P upon $\mathrm{SL}_2(\mathbf{C})$, $\bar{\mathcal{D}}$ induces a mapping $\mathcal{D}^\flat\colon \tilde{\Sigma}_\rho \to \mathbf{CP}^1$ (a local biholomorphism) and $\bar{\mu}$ induces a representation $\mu^\flat\colon \pi_1(\Sigma_\rho) \to \mathrm{PSL}_2(\mathbf{C})$. The image of $\mu^\flat$ is the image $\Gamma^\flat$ of Γ under the projection from $\mathrm{SL}_2(\mathbf{C})$ to $\mathrm{PSL}_2(\mathbf{C})$. The previous commutative diagram becomes:

$$\begin{array}{ccc} \tilde{\Sigma}_\rho & \xrightarrow{\Pi} & \Sigma_\rho \\ \downarrow{\scriptstyle \mathcal{D}^\flat} & & \downarrow{\scriptstyle i^\flat} \\ \mathbf{CP}^1 & \xrightarrow{\Gamma^\flat\backslash} & \Gamma^\flat\backslash\mathbf{CP}^1. \end{array}$$

If $\bar{\mathcal{D}}$ is not injective then there are two different points p_1 and p_2 in $\bar{\Omega}$ such that $\bar{\mathcal{D}}(p_1) = \bar{\mathcal{D}}(p_2)$. The points belong to different fibers of ω (we have already established that $\bar{\mathcal{D}}$ is injective in restriction to each fiber). Hence, the fiber containing p_1 and the fiber containing p_2 have the same image and hence $\mathcal{D}^\flat$ is not injective. We have thus proved that *X is semicomplete if $\mathcal{D}^\flat$ is injective*. In this situation, the solution of X with initial condition p is, by formula (7), defined in the domain $\{T \in \mathbf{C};\, [T : 1] \in \mathcal{D}^\flat(\tilde{\Sigma}_\rho)\}$.

Since $\mathcal{D}^\flat$ and $\mu^\flat$ satisfy the relation (4), we have a projective structure on Σ_ρ that depends on α and that will be denoted by $\Sigma_\rho[\alpha]$. By equation (7), if $z(t) = (z_1(t), z_2(t), z_3(t))$ is a local solution to X, an inverse of a chart of the projective structure on Σ_ρ is given by $\omega \circ z(t)$. On the other hand, being a subset of $\mathbf{CP}^1$, there

is a natural projective structure on Σ_ρ. By comparing these two projective structures, we have

$$\begin{aligned}\{\omega^{-1}, s\}\, ds^2 &= \frac{1}{(\omega')^2}\{\omega(t), t\} \circ \omega^{-1}(s)\, ds^2 \\ &= \frac{s^4 - 4\alpha s^3 + 2(2\alpha + 2\rho\alpha - \rho)s^2 - 4\rho\alpha s + \rho^2}{2s^2(s-1)^2(s-\rho)^2}\, ds^2 = Q^\alpha_\rho(s)\, ds^2.\end{aligned} \tag{8}$$

In a neighborhood of each of the four punctures (a) the monodromy is parabolic and (b) its developing map is injective in a neighborhood of these: in a suitable coordinate z around a puncture, $\mathcal{D}^\flat$ is $\log(z)$ [13]. This formula gives every projective structure on Σ_ρ satisfying (a) and (b).

Whenever $\mathcal{D}^\flat$ is injective, $\mathcal{D}^\flat(\tilde{\Sigma}_\rho)$ is, by formula (4), invariant under the action of the Kleinian group $\mu^\flat$. In the terminology of Kleinian groups $\tilde{\Sigma}_\rho$ is an *invariant component* for the Kleinian group and $\mu^\flat$ is a *B-group*. Much is known about these groups and their relations to projective structures and Teichmüller spaces. We will now give a rough glimpse of the situation based on [3, 15, 16], where details and more references may be found (see also the Introduction of [17]).

Via the *Bers embedding*, the Teichmüller space of Σ_ρ is embedded into the space of projective structures on Σ_ρ that, at the punctures, have parabolic monodromy and whose developing map is locally injective. This is, the Bers embedding takes values in the space of quadratic differentials $\{Q^\alpha_\rho\}$ of Σ_ρ of formula (8). The image of the Bers embedding is a bounded open set $\mathcal{B}_\rho \subset \mathbf{C}$.

There is a distinguished projective structure in Σ_ρ, the one given by the uniformization theorem: there exists some u_ρ in the interior of $\mathcal{B}_\rho$ such that the developing map of $\Sigma_\rho[u_\rho]$ is injective (and thus the corresponding vector field is semicomplete). Its image is (up to a fractional linear transformation) the upper half plane $\mathbf{H} = \{z;\ \Im(z) > 0\}$. The image of the monodromy $\mu_\rho[u_\rho]$ is a *Fuchsian* group, this is, it preserves $\mathbf{H}$ (the function $\rho \to u_\rho$ is real-analytic though not holomorphic). For every $\alpha \in \mathcal{B}_\rho$, $\alpha \neq u_\rho$ the group $\mu_\rho[\alpha]$ is a *quasifuchsian* group: there is a quasiconformal (and not conformal) homeomorphism $f\colon \mathbf{CP}^1 \to \mathbf{CP}^1$ such that $\mu_\rho[\alpha] = f \circ \mu_\rho[\alpha_\rho] \circ f^{-1}$. The developing map of $\Sigma_\rho[\alpha]$ is injective and as explained before, the corresponding vector field is semicomplete. Its image is a *quasidisk*, a simply connected open set bounded by a Jordan curve in $\mathbf{CP}^1$. The Hausdorff dimension of this curve is strictly greater than one [4].

The boundary of $\mathcal{B}_\rho$ has received a lot of attention since it gives a compactification of the Teichmüller space of Σ_ρ. By continuity, if $\{\alpha_i\}_{i\in\mathbf{N}} \subset \mathcal{B}_\rho$ is s sequence such that $\lim_{i\to\infty} \alpha_i = \beta$ and $\beta \in \partial\mathcal{B}_\rho$ then, since the developing map of $\Sigma_\rho[\alpha_i]$ is injective, the developing map of $\Sigma_\rho[\beta]$ will be injective as well (and the corresponding vector field will be semicomplete). It will have some simply-connected open set Δ as image. We have the following exclusive dichotomy for Δ:

- The group $\mu^\flat_\rho[\beta]$ is *totally degenerate*. The set Δ is its discontinuity domain and is a dense subset of $\mathbf{CP}^1$. Its complement, the limit set, is a closed connected set that

is not locally connected [1].

• The group $\mu_\rho^\flat[\beta]$ is a *cusp group*. The set Δ is the only invariant component of the discontinuity set of $\mu_\rho^\flat[\beta]$. The complement of the closure of Δ is a countable union of round disks that are, by pairs, either disjoint or tangent. The union of the boundaries of these disks is contained in $\partial\Delta$, which is not a Jordan curve. The quotient of the union of these disks is the union of two triply-punctured spheres (see [15] for more on these groups).

Both situations effectively appear within $\partial\mathcal{B}_\rho$ [3]. Theorem 1 is now proved.

4. Final comments

4.1. On the existence of first integrals. Halphen's equations do not have a meromorphic first integral [14] (see also [9]) and our generalizations share this property. Semicomplete vector fields within the family (2) do not have a first integral, in view of the following result [10, Corollary D]: *Let X be a semicomplete meromorphic vector field in $\mathbf{C}^3$ and suppose that the maximal solution with initial condition p is defined in a subset U of $\mathbf{C}$ such that the complement of U is uncountable. Then the orbit of X through p is Zariski dense in $\mathbf{C}^3$ and, in particular, X does not have a meromoprhic first integral.*

4.2. An example. In the cases where $X_\rho(\alpha)$ is semicomplete, $\omega(t)$ is a single-valued function. Our geometric reasoning implied that $X_\rho(\alpha)$ is semicomplete whenever $\omega(t)$ is single-valued. It is legitimate to ask if all this is needed and if the single-valuedness of ω does not imply, in a more direct way, the semicompleteness of X. This is, in general, not true. For example, the vector field $(z_1^2 + z_2^4)\,\partial/\partial z_1 + z_1 z_2\,\partial/\partial z_2 + z_3(2z_2^2 - z_1)\,\partial/\partial z_3$ satisfies relations (3) with respect to the vector fields $Z = \partial/\partial z_1$ and $L = \partial/\partial z_1 + (1/2)\,\partial/\partial z_2 - (1/2)\,\partial/\partial z_3$. The three vector fields are linearly independent in the complement of $\{z_2 z_3 = 0\}$. The fibration corresponding to the right action of P is now given by $\omega = z_3 z_3$ and takes values in $\mathbf{C} \setminus \{0\}$. A solution to this vector field is given by

$$\left(\frac{t}{1-t^2}, \frac{1}{\sqrt{t^2-1}}, \frac{(t-1)^2}{\sqrt{t^2-1}}\right),$$

and hence the vector field is not semicomplete. Despite this, $\omega(t) = (t-1)/(t+1)$ is a single-valued function (it has vanishing Schwarzian).

4.3. Symmetries of the equations. The vector fields $X_\rho(\alpha)$ are invariant under a simultaneous change of sign of z_2 and z_3 and can thus be defined in the quotient of $\mathbf{C}^3$ under this involution. For ω a primitive cubic root of unity, the functions

$$X = z_1, \quad Y = z_1 + z_2^2 - z_2 z_3, \quad Z = z_1 - (\omega+1) z_2 z_3, \quad W = z_1 + \omega z_3^2 - \omega z_2 z_3,$$

generate the ring of polynomials invariant by a simultaneous change of sign of z_2 and z_3 and are bound by the algebraic relation

$$\omega^2(XZ + YW) + \omega(XW + YZ) + (XY + ZW) = 0. \tag{9}$$

This quadratic cone has a vector field induced by $X_\rho(\alpha)$. For example, if (z_1, z_2, z_3) are a solution to $X_{-\omega}((1/3)(1-\omega))$, the above functions satisfy the differential relations

$$\begin{aligned} W' + X' + Y' &= WX + XY + YW, \\ W' + Y' + Z' &= WY + YZ + ZW, \\ W' + X' + Z' &= WX + XZ + ZW, \\ X' + Y' + Z' &= XY + YZ + ZX. \end{aligned}$$

This is exactly the system of equations satisfied by modular forms of level three considered by Ohyama in [19].

4.4. Some questions. Let us end by formulating some questions:

Does there exist a semicomplete polynomial vector field in $\mathbf{C}^3$ having a maximal solution defined in a domain

(1) with uncountable complement, or

(2) whose complement has nonempty interior,

necessarily part of a Lie algebra of rational vector fields? (the answer is negative if we restrict to polynomial vector fields, see [8, Section 3]).

In [5], a reduction of the Yang–Mills equations is shown to produce the system (1). Is there a reduction of these equations yielding the vector fields $X_\rho(\alpha)$? More generally, does there exist an interesting partial differential equation admitting $X_\rho(\alpha)$ as a reduction?

Does there exist a polynomial vector field in $\mathbf{C}^n$ having a single-valued solution defined in an annulus $\{r < |z - q| < R\}$?

ACKNOWLEDGEMENTS. We thank Carlos Cabrera and Petr Makienko for helpful conversations. The author was partially supported by CONACyT grant 58354 and PAPIIT-UNAM grant IN104510.

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