Grasselli, M., Mola, G. and Yagi, A.
Osaka J. Math.
48 (2011), 987–1004

ON THE LONGTIME BEHAVIOR OF SOLUTIONS TO A MODEL FOR EPITAXIAL GROWTH

MAURIZIO GRASSELLI, GIANLUCA MOLA and ATSUSHI YAGI

(Received May 7, 2010)

Abstract

We consider a fourth-order nonlinear parabolic type equation on a two-dimensional bounded domain Ω. This equation governs the evolution of the height profile of a thin film in an epitaxial growth process. We show that such equation endowed with no-flux boundary conditions generates a dissipative dynamical system under very general assumptions on $\partial\Omega$ on a phase-space of L^2-type. This system possesses a global as well as an exponential attractor. In addition, if $\partial\Omega$ is smooth enough, we show that every trajectory converges to a single equilibrium by means of a suitable Łojasiewicz–Simon inequality. An estimate of the convergence rate is also obtained.

1. Introduction

A well-known and relatively simple model to describe the epitaxial growth process leads to the formulation of the following fourth-order nonlinear equation

$$\partial_t u + \Delta^2 u = -\mu \nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2} \right) \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.1}$$

in a two-dimensional bounded domain Ω, μ being a (positive) constant called surface roughening coefficient. Here u denotes the height profile, measured in a co-moving frame, of a thin film in epitaxial growth. The biharmonic operator accounts for the surface diffusion (the diffusion coefficient has been set equal to one), while the divergence type term was firstly proposed in [9] to model the behavior of adatoms (i.e., adsorbed atoms). We refer the reader to [12] and references therein for further details on equation (1.1) as well as for an analysis of its qualitative properties (see also [11, 15]). We also mention that a similar equation where the divergence type term has a rather general form has been considered in [10] (cf. also references therein). However, the present nonlinearity does not satisfy the coercivity assumption [10, (H2b)] which

---

2000 Mathematics Subject Classification. 35B40, 35B41, 35K55.

This paper originated from a visit of the first author to the Department of Applied Physics at Osaka University. The first author was partially supported by the Italian MIUR-PRIN Research Project 2008 *Transizioni di fase, isteresi e scale multiple*. The third author is supported by Grant-in-Aid for Scientific Research (No. 20340035) of JSPS.

is needed to prove the existence of a weak solution. Thus the present equation is not a particular case of the one studied in [10].

More recently, equation (1.1) has been investigate within the theory of dissipative dynamical systems in a series of papers [5, 6, 7] where further references on (1.1) can also be found. More precisely, the authors have considered the equation subject to the initial condition

$$u(0) = u_0 \quad \text{in} \quad \Omega, \tag{1.2}$$

and to the boundary conditions

$$\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} \Delta u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \tag{1.3}$$

where $\partial_{\mathbf{n}}$ stands for the (outward) normal derivative to $\partial\Omega$. In [5] well-posedness and regularity results for (1.1)–(1.3) have been established (see also [11, Section 3] for the periodic case). Such results lead to the definition of a suitable dynamical system which possesses the global attractor. Existence of exponential attractors and the analysis of ω-limit sets have been the subject of [6]. Then, in [7], the stability properties of the null solution with respect to μ has been analyzed in order to find a lower bound for the dimension of the global attractor. All these results have been obtained by assuming $\partial\Omega$ of class C^4 and working with rather smooth solutions. However, from the physical viewpoint, $\partial\Omega$ can be nonsmooth (for instance, a polygon). Thus it seems necessary to extend the analysis of the longterm behavior to more general spatial domains and to weaker solutions. This is our first goal, namely, to provide a rather general and simple proof of the existence of a global and an exponential attractor which allows to take nonsmooth $\partial\Omega$. In addition, we show that each solution converges to a single stationary state, provided that $\partial\Omega$ is smooth enough. This is done by means of a suitable version of the Łojasiewicz–Simon inequality. An estimate of the convergence rate is also obtained.

2. The dynamical system in $L^2(\Omega)$

Let H be the (real) Hilbert space $L^2(\Omega)$ endowed with the usual scalar product $\langle \cdot, \cdot \rangle$ and the related norm $\| \cdot \|$. Then, we consider the Hilbert triplet $V = H^1(\Omega) \hookrightarrow H \equiv H^* \hookrightarrow V^*$ and we consider $-\Delta \colon W \to H$ where

$$W = \{w \in V \colon \partial_{\mathbf{n}} w = 0,\ \Delta w \in H\} \tag{2.1}$$

endowed with the graph norm $(\|w\|^2 + \|\Delta w\|^2)^{1/2}$. We recall that $W \hookrightarrow H^{3/2-\epsilon}(\Omega)$ for all $\epsilon \in (0, 1/2)$, when $\partial\Omega$ is only Lipschitz (see [14, Theorem 4]). Otherwise, if Ω is a polygonal domain, then we have $W \hookrightarrow H^{3/2}(\Omega)$. Moreover, if $\partial\Omega$ is of class $C^{1,1}$, then $W \hookrightarrow H^2(\Omega)$. Here and by, for the sake of convenience, we replace $\| \cdot \|_{X^2}$ with the shorter notation $\| \cdot \|_X$, for any space $X^2 = X \times X$, X being a Banach space. Besides $\langle \cdot, \cdot \rangle_{X^*,X}$ denotes the duality coupling.

Let $u_0 \in H$. Our definition of weak solution to is the following (cf. also [11, Definition 3.1])

DEFINITION 2.1. A function $u \in C([0,\infty); H) \cap L^2_{loc}((0,+\infty); W)$ is a weak solution to (1.1)–(1.3) if

$$\langle \partial_t w, z\rangle_{W^*,W} + \langle \Delta w, \Delta z\rangle = \mu\langle (1+|\nabla u|^2)^{-1}\nabla u, \nabla z\rangle, \quad \forall z \in W, \text{ a.e. in } (0,\infty), \tag{2.2}$$

$$u(0) = u_0, \quad \text{a.e. in } \Omega. \tag{2.3}$$

As a consequence, the total mass of $u(t)$ is conserved, that is,

$$\langle u(t), 1\rangle = \langle u_0, 1\rangle, \quad \forall t \geq 0. \tag{2.4}$$

We first prove the following continuous dependence estimate (compare with [5, Proposition 4.3])

Theorem 2.2. *Let $u_0, v_0 \in H$ and denote by u and v the corresponding weak solutions to problem* (1.1)–(1.3). *Then, for any time $T > 0$, there exists a positive constant C, also depending on Ω and μ, such that the following continuous dependence estimate holds*

$$\|(u-v)(t)\|^2 + \int_0^t \|\Delta(u-v)(\tau)\|_H^2 \, d\tau \leq Ce^{CT}\|u_0 - v_0\|^2, \tag{2.5}$$

for any $t \in [0, T]$.

Proof. Set $w = u - v$ and observe that (cf. (2.2))

$$\langle \partial_t w, z\rangle_{W^*,W} + \langle \Delta w, \Delta z\rangle = \mu\langle \mathcal{F}(u,v,w), \nabla z\rangle, \quad \forall z \in W, \text{ a.e. in } (0,\infty), \tag{2.6}$$

where

$$\mathcal{F}(u,v,w) = \frac{\nabla w - (\nabla w \cdot \nabla u)\nabla u - (\nabla w \cdot \nabla v)\nabla u - |\nabla u|^2 \nabla w}{(1+|\nabla u|^2)(1+|\nabla v|^2)}. \tag{2.7}$$

Taking $w(t)$ as test function, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \|\Delta w\|^2 = \mu\langle \mathcal{F}(u,v,w), \nabla w\rangle.$$

It is immediate to realize that

$$\|\mathcal{F}(u,v,w)\|^2 \leq \int_\Omega \frac{1 + 4|\nabla u|^4 + 4|\nabla v|^4}{(1+|\nabla u|^2)^2(1+|\nabla v|^2)^2}|\nabla w|^2 \, d\Omega \leq c\|\nabla w\|^2, \tag{2.8}$$

since the function $(x, y) \mapsto (1 + 4x^4 + 4y^4)(1 + x^2)^{-2}(1 + y^2)^{-2}$ is globally bounded. Therefore, from (2.6), we deduce

$$\frac{d}{dt}\|w\|^2 + \|\Delta w\|^2 \leq \|\nabla w\|^2 \leq \frac{1}{2}\|\Delta w\|^2 + c\|w\|^2, \tag{2.9}$$

for some $c > 0$ depending on Ω and μ. The thesis follows from the standard Gronwall lemma. □

It is now standard to prove the existence of a weak solution. This can be done through a Galerkin scheme (see, e.g., [11]). From now the use of such an approximation scheme will be tacitly assumed.

Then we can summarize the consequences of Theorem 2.2 with the following

Theorem 2.3. *Problem* (1.1)–(1.3) *generates a strongly continuous semigroup* $S(t)$ *on the phase-space* H.

Property (2.4) lead us to define, for all $\alpha \geq 0$, the bounded-average (complete metric) spaces

$$H_\alpha = \{u \in H : |\langle u, 1\rangle| \leq \alpha\}, \quad V_\alpha = V \cap H_\alpha, \quad W_\alpha = W \cap H_\alpha.$$

Accordingly, from now on we set $\hat{u} = u - \langle u, 1\rangle$ (H_0-projection of $u \in H$). On account of (1.3), we have $u(t) = S(t)u_0 \in H_\alpha$ for all times $t > 0$, if $u_0 \in H_\alpha$, i.e., the metric space H_α is invariant under the action of $S(t)$. Moreover, the dynamical system $(H_\alpha, S(t))$ is dissipative. Indeed, recalling the proof of [5, Corollary 4.1], we have

Theorem 2.4. *Let* $u_0 \in H_\alpha$. *Then, for all* $R > 0$ *there exists positive constants* C_0 *and* κ_0, *depending on* μ, $|\Omega|$ *and* α *but independent of* R, *such that*

$$\sup_{\|u_0\| \leq R} \|u(t)\|^2 \leq C_0(e^{-\kappa_0 t}\|u_0\|^2 + 1), \tag{2.10}$$

and

$$\sup_{\|u_0\| \leq R} \int_t^{t+1} \|\Delta u(\tau)\|^2 \, d\tau \leq C_0, \tag{2.11}$$

for all $t \geq 0$.

Therefore the semigroup $S(t)$ can be restricted to a *dissipative* semigroup on the phase-space H_α. In addition, we have

Theorem 2.5. *Let $B_{R_0} \subset H_\alpha$ be a bounded absorbing set for the dynamical system $(H_\alpha, S(t))$. Then, there exists $t_1 = t_1(R_0) > 1$ and $C_1 = C_1(R_0) > 0$ such that*

$$\|u(t)\|_V \leq C_1, \quad \forall t \geq t_1. \tag{2.12}$$

Therefore, $(H_\alpha, S(t))$ has a global attractor $\mathcal{A}_\alpha$ bounded in V_α. Moreover, there holds

$$\int_t^{t+1} \|\nabla \Delta u(\tau)\|^2 \, d\tau \leq C_1, \tag{2.13}$$

for all $t \geq t_1$.

Proof. Take $-\Delta u(t)$ as a test function in the weak formulation of (1.1). This yields,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\nabla \Delta u\|^2 = -\mu \left\langle \frac{\nabla u}{1+|\nabla u|^2}, \nabla \Delta u \right\rangle.$$

Therefore, we infer

$$\frac{d}{dt}\|\nabla u\|^2 + \|\nabla \Delta u\|^2 \leq c\|\nabla u\|^2. \tag{2.14}$$

Recalling (2.11), thanks to the uniform Gronwall lemma, we find $t_1 = t_1(R_0)$ and $C_0 = C_0(R_0)$ such that (2.12) holds. Then we integrate (2.14) from t to $t+1$ for $t \geq t_1$ and we deduce (2.13). The existence of the global attractor is a straightforward consequence of (2.12). □

It is also easy to prove the so-called smoothing property (see [3])

Theorem 2.6. *For every $u_0, v_0 \in B_{R_0}$, there exists $t_2 = t_2(R_0) > 1$ and $C_2 = C_2(R_0) > 0$ such that the following estimate holds*

$$\|S(t)u_0 - S(t)v_0\|_V \leq C_2\|u_0 - v_0\|, \tag{2.15}$$

for any $t \geq t_2$.

Proof. Take $-\Delta w(t)$ as test function in (2.6). This yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2 + \|\nabla \Delta w\|^2 = -\mu \langle \mathcal{F}(u, v, w), \nabla \Delta w \rangle.$$

By the Young inequality and (2.8), we deduce

$$\frac{d}{dt}\|\nabla w\|^2 \leq c\|\nabla w\|^2, \tag{2.16}$$

for some $c > 0$ depending only on Ω and μ. The assertion is then achieved by invoking the uniform Gronwall lemma and (2.5). □

In order to establish the existence of an exponential attractor, we also need to establish the Hölder continuity of $(t, u_0) \mapsto S(t)u_0$. This follows from (2.5) and

Lemma 2.7. *Let $B_{R_0} \subset H_\alpha$ be a bounded absorbing set for the dynamical system $(H_\alpha, S(t))$. Then, there exists $C_2 = C_2(R_0) > 0$ such that*

$$\|S(t)u_0 - S(\tilde{t})u_0\| \le C_2|t - \tilde{t}|^{1/4}, \tag{2.17}$$

for all $t, \tilde{t} \in [t_1, t_1 + 1]$, t_1 being given by Theorem 2.5,

Proof. Observe first that, on account of (2.11) and (2.13), we have

$$\int_t^{t+1} \|\partial_t u(\tau)\|_{V^*}^2 \, d\tau \le C(R_0), \tag{2.18}$$

for all $t \ge t_1$. Therefore, for all $t, \tilde{t} \in [t_1, t_1 + 1]$ such that $\tilde{t} \ge t$ there holds

$$\begin{aligned} \|u(\tilde{t}) - u(t)\|^2 &\le C\|u(\tilde{t}) - u(t)\|_V \|u(\tilde{t}) - u(t)\|_{V^*} \\ &\le C(R_0) \int_t^{\tilde{t}} \|\partial_t u(\tau)\|_{V^*} \, d\tau \le C(R_0)\|\tilde{t} - t\|^{1/2}, \end{aligned}$$

whence the thesis. □

Collecting the above results, on account of [3], we deduce

Theorem 2.8. *$(H_\alpha, S(t))$ possesses an exponential attractor $\mathcal{E}_\alpha$ bounded in V_α. As a consequence, $\mathcal{A}_\alpha$ has finite fractal dimension.*

REMARK 2.9. Note that the eigenfunctions used in a Galerkin scheme need only to belong to W (see (2.1)).

3. The dynamical system in $H^1(\Omega)$

We recall that (see (2.12)–(2.13), cf. also [5, Corollary 4.1])

Theorem 3.1. *Let $u_0 \in V_\alpha$. Then, for all $R > 0$ there exists positive constants C_3 and κ_1, depending on μ, $|\Omega|$ and α but independent of R, such that*

$$\sup_{\|u_0\|_V \le \rho} \|u(t)\|_V^2 \le C_3(e^{-\kappa_1 t}\|u_0\|_V^2 + 1),$$

and

$$\sup_{\|u_0\|_V \le R} \int_t^{t+1} \|\nabla \Delta u(\tau)\|^2 \, d\tau \le C_3,$$

for all $t \ge 0$.

On the other hand, we have

Theorem 3.2. *For every* $u_0, v_0 \in V$ *there exists a positive constant* C, *depending on* Ω *and* μ, *such that, denoting by* u, v *the respective solutions to* (1.1)–(1.3), *the following continuous dependence estimate holds*

$$\|(u - v)(t)\|_V^2 + \int_0^t \|\Delta[(u - v)(\tau)]\|_V^2 \, d\tau \le C e^{CT} \|u_0 - v_0\|_V^2, \tag{3.1}$$

for any $t \in [0, T]$, $T > 0$.

Proof. Set $w = u - v$ and take $(w - \Delta w)(t)$ as test function in (2.6). We get

$$\frac{1}{2} \frac{d}{dt} [\|w\|^2 + \|\nabla w\|^2] + \|\Delta w\|^2 + \|\nabla \Delta w\|^2 = \mu \langle \mathcal{F}(u, v, w), \nabla w - \nabla \Delta w \rangle. \tag{3.2}$$

By the Young inequality, we infer

$$\mu \langle \mathcal{F}(u, v, w), \nabla w - \nabla \Delta w \rangle \le \mu^2 |\mathcal{F}(u, v, w)|^2 + \|\nabla w\|^2 + \frac{1}{2} \|\nabla \Delta w\|^2.$$

On the other hand (cf. (2.8)),

$$\|\mathcal{F}(u, v, w)\|^2 \le \int_\Omega \frac{1 + 4|\nabla u|^4 + 4|\nabla v|^4}{(1 + |\nabla u|^2)^2 (1 + |\nabla v|^2)^2} |\nabla w|^2 \, d\Omega \le c \|w\|_V^2. \tag{3.3}$$

Therefore, from (3.2) and (3.3) we deduce

$$\frac{d}{dt} \|w\|_V^2 + \|\Delta w\|_V^2 \le c \|w\|_V^2,$$

and the thesis follows from the standard Gronwall lemma. □

As a consequence, the semigroup $S(t)$ restricted to V_α is strongly continuous and dissipative.

The existence of a (compact) absorbing set is given by

Theorem 3.3. *Let $B_{R_1} \subset V_\alpha$ be a bounded absorbing set for $(V_\alpha, S(t))$. Then, there exists $t_2 = t_2(R_1) > 1$ and $C_4 = C_4(R_1) > 0$ such that*

$$\|u(t)\|_W \leq C_4. \tag{3.4}$$

Moreover, there holds

$$\int_t^{t+1} \|\Delta^2 u(\tau)\|^2 \, d\tau \leq C_4, \tag{3.5}$$

for all $t \geq t_2$.

Proof. Let us take $\Delta^2 u(t)$ as a test function in (2.6). Thus, we get

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \|\Delta^2 u\|^2 = -\mu\left\langle \nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right), \Delta^2 u\right\rangle. \tag{3.6}$$

Observe that

$$\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right) = \frac{\Delta u}{1+|\nabla u|^2} - 2\frac{(\mathrm{Hess}_u \nabla u)\cdot \nabla u}{(1+|\nabla u|^2)^2},$$

where Hess_u denotes the hessian matrix of u. Then, we have

$$\begin{aligned}
&\mu\left\langle \nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right), \Delta^2 u\right\rangle \\
&\leq \frac{\mu^2}{2}\int_\Omega \left[\frac{|\Delta u|^2}{(1+|\nabla u|^2)^2} + 2\frac{|\mathrm{Hess}_u|^2\,|\nabla u|^4}{(1+|\nabla u|^2)^4}\right] d\Omega + \frac{1}{2}\|\Delta^2 u\|^2 \\
&\leq \frac{\mu^2}{2}\|\Delta u\|^2 + \mu^2\|\mathrm{Hess}_u\|^2 + \frac{1}{2}\|\Delta^2 u\|^2,
\end{aligned}$$

since the functions $x \mapsto (1+x^2)^{-2}$ and $x \mapsto x^4(1+x^2)^{-4}$ are globally bounded. Therefore, we infer from (3.6) that

$$\frac{d}{dt}\|\Delta u\|^2 + \|\Delta^2 u\|^2 \leq c(1 + \|u\|_W^2).$$

Recalling (2.10) and exploiting the uniform Gronwall lemma, we obtain (3.4). Bound (3.5) can be easily deduced by integrating both members of the differential inequality above on $(t, t+1)$, for $t \geq t_2$, and using the uniform bound on $\|u\|_W$. □

On account of the above results, we have

Corollary 3.4. *The global attractor $\mathcal{A}_\alpha$ of $(H_\alpha, S(t))$ is bounded in W_α and attracts any bounded set in V_α in the V-metric.*

REMARK 3.5. For instance, if Ω is a polygonal domain, then $\mathcal{A}_\alpha$ is bounded in $H^{3/2}(\Omega)$. Instead, if $\partial\Omega$ is of class $C^{1,1}$, then $\mathcal{A}_\alpha$ is bounded in $H^2(\Omega)$. Note that, thanks to Theorem 3.3, we can also construct an exponential attractor $\mathcal{E}_\alpha$ which is bounded in W_α. Also, we can prove that $\mathcal{E}_\alpha$ attracts any bounded set in V_α in the V-metric and $\mathcal{A}_\alpha$ has finite fractal dimension in the V-metric. In [5] further regularity results of invariant sets are proven under stronger assumptions on $\partial\Omega$.

4. Convergence to equilibrium

In this section we shall prove the convergence to equilibrium of single trajectories. Let us set

$$Z = \{u \in H^3(\Omega)\colon \partial_{\mathbf{n}} u = 0 \text{ a.e. on } \partial\Omega\},$$

endowed with its (natural) norm

$$\|\cdot\|_Z^2 = \|\cdot\|_W^2 + \|\nabla\Delta\cdot\|^2.$$

We also define $Z_\alpha = Z \cap H_\alpha$. By using the techniques described above (see also [5]) it is not difficult to prove the following

Proposition 4.1. *Let $\partial\Omega$ be of class $C^{2,1}$. For every $u_0 \in H_\alpha$, we have*

$$\bigcup_{t\geq 1}\{S(t)u_0\} \subset Z_\alpha.$$

Consider now the set $\mathcal{S}_\alpha$ of all steady states of problem (1.1)–(1.3) with average bounded by α, namely any $u_\infty \in Z_\alpha$ such that

$$\langle \nabla\Delta u_\infty + \mu(1+|\nabla u_\infty|^2)^{-1}\nabla u_\infty, \nabla z\rangle = 0, \quad \forall z \in V_\alpha. \tag{4.1}$$

REMARK 4.2. To the best of our knowledge it is not clear whether the set of the nonconstant stationary states is a continuum. However, it has been proven that there are (at least) infinitely many equilibria in the case of periodic boundary conditions (see [12, Section 4]).

The main result of this section is

Theorem 4.3. *Let $\partial\Omega$ be of class $C^{2,1}$. For every $u_0 \in H_\alpha$ there exists $u_\infty \in \mathcal{S}_\alpha$ such that*

$$u(t) = S(t)u_0 \to u_\infty \quad \text{in} \quad H^2(\Omega), \tag{4.2}$$

as $t \to \infty$. *Moreover, there exists* $t_1 > 0$ *and a positive constant* $\bar{c}$ *such that*

$$\|u(t) - u_\infty\|_W \le \bar{c}(1+t)^{-\vartheta/(2(1-2\vartheta))}, \quad \forall t \ge t_1, \tag{4.3}$$

$\vartheta \in (0, 1/2)$ *being the same constant as in the Łojasiewicz–Simon inequality* (*see* Lemma 4.4).

The key tool to prove this result is to use a suitable Łojasiewicz–Simon inequality (see, e.g., [8] and references therein). To state it, we consider the functional

$$E(u) = \frac{1}{2}\|\Delta u\|^2 - \frac{\mu}{2}\int_\Omega \ln(1+|\nabla u|^2)\, d\Omega,$$

defined for all $u \in Z$. Clearly $E \in C^2(W)$, with

$$E'(u) = \Delta^2 u + \mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) \colon W \to W^*,$$

and

$$E''(u)v = \Delta^2 v + \mu\nabla\cdot\left[\frac{(1+|\nabla u|)^2\nabla v - 2(\nabla u\cdot\nabla v)\nabla u}{1+|\nabla u|^2}\right], \quad v \in W.$$

Here and below prime denotes the Fréchet derivative. The restriction of E to Z satisfies the following basic property

Lemma 4.4. *The functional* $E\colon Z \to \mathbb{R}$ *is real analytic.*

Then, the inequality we need reads

Lemma 4.5. *Let* $u_\infty \in Z_\alpha$ *be a solution to the stationary equation* (4.1). *Then there exists* $\theta \in (0, 1/2]$, $C > 0$ *and* $\sigma > 0$ *such that, for all* $u \in Z_\alpha$ *satisfying* $\|u - u_\infty\|_Z \le \sigma$, *there holds*

$$|E(u) - E(u_\infty)|^{1-\theta} \le C\left\|\Delta^2 u + \mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right\|_{Z_0^*}. \tag{4.4}$$

Proofs of Lemmas 4.4 and 4.5 are given in Appendix. Let us recall some basic facts before proceeding to the proof of Theorem 4.3.

For all $u \in H_\alpha$, we define the ω-limit as

$$\omega(u_0) = \{u_\infty \in Z_\alpha \colon \exists t_n \to \infty \text{ as } n \to \infty, \text{ s.t. } S(t_n)u_0 \to u_\infty \text{ in } W\}.$$

First notice that, by multiplying equation (1.1) by $\partial_t u$ in H, we have

$$\frac{d}{dt}E(u) = -\|\partial_t u\|^2. \tag{4.5}$$

Note that this can be done when $u_0 \in W$ since equation (1.1) holds almost everywhere (see Theorem 3.3). Therefore, we deduce the following

Proposition 4.6. *The functional E is a Lyapunov functional for $(W, S(t))$.*

Consequently, standard results (cf. [1, Theorems 9.2.3 and 9.2.7]) entail that

Lemma 4.7. *For any $u_0 \in H_\alpha$, the set $\omega(u_0)$ is nonempty, compact, invariant and connected in W and the following inclusion holds $\omega(u_0) \subset \mathcal{S}_\alpha$. Moreover, E is constant on $\omega(u_0)$.*

Proof of Theorem 4.3. In the course of the proof, the following result (see [4, Lemma 7.1]) will play a fundamental role

Lemma 4.8. *Let $\Phi \in L^2(0,\infty)$, with $\|\Phi\|_{L^2(0,\infty)} \le b$, and suppose that there exist $a \in (1, 2)$, $c > 0$ and an open set $\mathcal{P} \subset (0, \infty)$ such that*

$$\left(\int_t^\infty \Phi^2(\tau)\, d\tau\right)^a \le c\Phi^2(t), \quad \text{for} \quad \text{a.e. } t \in \mathcal{P}.$$

Then $\Phi \in L^1(\mathcal{P})$ and there exists a constant $C = C(a,b,c)$, independent of $\mathcal{P}$, such that

$$\int_{\mathcal{P}} \Phi(\tau)\, d\tau \le C.$$

Integrating equation (4.5) on (t, ∞), we deduce

$$\int_t^\infty \|\partial_t u(\tau)\|^2\, d\tau = E(u(t)) - E(u_\infty),$$

for some $u_\infty \in \mathcal{S}_\alpha$. Setting now

$$\mathcal{P} = \{t \in (0, \infty) \colon \|u(t) - u_\infty\|_V < \omega\},$$

Lemma 4.4 yields

$$\begin{aligned} |E(u(t)) - E(u_\infty)| &\le C\left\|\Delta^2 u + \mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right\|_{Z_0^*}^{1/(1-\vartheta)} \\ &\le c\left\|\Delta^2 u + \mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right\|^{1/(1-\vartheta)}. \end{aligned} \tag{4.6}$$

Therefore, thanks to equation (1.1), we have

$$\int_t^\infty \|\partial_t u(\tau)\|^2\, d\tau \le c\|\partial_t u\|^{2/(2-2\vartheta)}.$$

Since $2-2\rho \in (1,2)$, we can apply Lemma 4.8 to the function $\Phi(t) = \|\partial_t u(t)\|$, and conclude that

$$\int_{\mathcal{P}} \|\partial_t u(t)\|\, dt < \infty.$$

Thus, for any $t_1, t_2 \in \mathcal{P}$, with $t_1 < t_2$, we have

$$\|u(t_2) - u(t_1)\| \le \int_{t_1}^{t_2} \|\partial_t u(t)\|\, dt < \frac{r}{4}, \tag{4.7}$$

provided that t_1 is large enough and the whole interval (t_1, t_2) lies in $\mathcal{P}$. Observing that $u_\infty \in \mathcal{S}_\alpha$, and recalling Proposition 4.1, we can then choose $t_0 > 0$ such that

$$\|u(t_0) - u_\infty\| < \frac{r}{4} \tag{4.8}$$

and, consequently, $[t_0, \infty) \subset \mathcal{P}$. Set now

$$T_0 = \inf\{t > t_0 \colon \|u(t) - u_\infty\| \ge r\};$$

clearly we have $T_0 > t_0$. If we assume that $T_0 < \infty$, we also infer

$$\|u(T_0) - u_\infty\| = r.$$

On the other hand, as a consequence of (4.7) and (4.8),

$$\|u(t) - u_\infty\| \le \|u(t) - u(t_0)\| + \|u(t_0) - u_\infty\| < \frac{r}{2},$$

for all $t \in [t_0, T_0)$, which, by contradiction, implies $T_0 = \infty$ and, therefore,

$$u(t) \to u_\infty \quad \text{in} \quad H,$$

as $t \to \infty$. The thesis then follows by Proposition 4.1.

It remains to prove inequality (4.3). Let us establish first the inequality in H. Set

$$\Theta(t) = E(u(t)) - E(u_\infty), \quad \forall t \in (0, \infty).$$

Since the map $t \mapsto E(u(t))$ is monotone nonincreasing, $\Theta(t) \ge 0$ for all $t \in [0, \infty)$. Observe that, by means of the convergence result (4.2), combining (4.5) with (4.6), we get

$$\frac{d}{dt}\Theta(t) + c[\Theta(t)]^{1-\vartheta} \le 0, \quad \forall t \ge t_1,$$

for some $t_1 > 0$, $\vartheta \in (0, 1/2)$ being as in Theorem 4.4. This yields

$$\Theta(t) \le c(1+t)^{-1/(2(1-2\rho))}, \quad \forall t \ge t_1. \tag{4.9}$$

On the other hand, we observe that

$$[\Theta(t)]^{1-\vartheta} \leq c\|\partial_t u(t)\|, \quad \forall t \geq t_1,$$

and

$$\frac{d}{dt}[\Theta(t)]^{\vartheta} = \vartheta[\Theta(t)]^{-1+\vartheta}\frac{d}{dt}\Theta(t) \leq 0, \quad \forall t \geq t_1.$$

Therefore, for any $t \geq t_1$, we get

$$\|\partial_t u(t)\| \leq -c\frac{d}{dt}[\Theta(t)]^{\vartheta}.$$

Thus, integrating the above inequality from t to ∞, we obtain

$$\int_t^{\infty} \|\partial_t u(\tau)\|^2 \, d\tau \leq c[\Theta(t)]^{\vartheta}, \quad \forall t \geq t_1,$$

and, on account of (4.9), we immediately infer

$$\int_t^{\infty} \|\partial_t u(\tau)\| \, d\tau \leq c(1+t)^{-\rho/(2(1-2\rho))}, \quad \forall t \geq t_1.$$

Hence, the order estimate has been obtained in H using

$$u(t) - u_\infty = -\int_t^{\infty} \partial_t u(\tau)\, d\tau \quad \text{in} \quad H.$$

In order to achieve the result in V (without any loss in the decay rate), we come back to inequality (2.14), having set

$$w(t) = u(t) - v(t) = S(t)u_0 - u_\infty.$$

On account of (2.4), we have $w(t) \in V_0$ for all $t \geq 0$. Thus, invoking the usual Poincaré inequality, it is immediate to deduce

$$\nu\|w\|_V^2 \leq \|\Delta w\|_V^2,$$

for some suitable constant ν. Therefore, by interpolation, it is easy to get

$$\|w\|_V^2 \leq \|w\|\|w\|_W = \|w\|(\|w\| + \|\Delta w\|) \leq \frac{1}{2}\|\Delta w\|_V^2 + c\|w\|^2,$$

so that (2.14) yields

$$\frac{d}{dt}\|u - u_\infty\|_V^2 + \frac{\nu}{2}\|u - u_\infty\|_V^2 \leq c\|u - u_\infty\|^2.$$

Then, the Gronwall lemma yields (4.3) in the V-norm. The last step is obtained by multiplying equation (1.1) by $\Delta^2 u(t)$. This gives, for some $c_1, c_2 > 0$,

$$\frac{d}{dt}\|\Delta u\|^2 + c_1\|\Delta^2 u\|^2 \le c_2\|\nabla u\|^2, \tag{4.10}$$

and we conclude by the Gronwall lemma combined with the obtained rate control in V-norm. □

5. Appendix

Proof of Lemma 4.4. We recall that, if X and Y are Banach spaces, a functional $\mathcal{H}\colon X \to Y$ is analytic (see [16, Volume I, Definition 8.8]) if and only if for each $x_0 \in X$ there exist a ball B centered in 0 and a continuous mapping $T_n\colon B + \{x_0\} \to \Sigma_n(X, Y)$, for $n \ge 0$, such that

$$x \in B + \{x_0\},\ h \in B \Rightarrow \mathcal{H}(x+h) - \mathcal{H}(x) = \sum_{n=1}^{\infty} \frac{T_n(x)(h, \dots, h)}{n!}.$$

Here $\Sigma_n(X, Y) = \{T \in \mathcal{L}(X^n; Y)\colon T$ is symmetric and n-linear$\}$.

Then, we divide the proof into several steps. First, notice that it is enough to prove the analyticity of $E' \in C(Z; Z^*)$. Thus, it suffices to prove the claim for the nonlinear operator $F \in C(Z; V^*)$ defined by

$$F(u) = \nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2}\right), \quad u \in Z.$$

Indeed, it is immediate to check that $F = F_3 \circ F_2 \circ F_1$, where

$$F_1 \in \mathcal{L}(Z; W^2), \quad F_1(u) = \nabla u,$$
$$F_2 \in C(W^2; L^\infty(\Omega)^2), \quad F_2(\mathbf{v}) = \frac{\mathbf{v}}{1 + |\mathbf{v}|^2},$$
$$F_3 \in \mathcal{L}(L^\infty(\Omega)^2; V^*), \quad F_3(\mathbf{w}) = \nabla \cdot \mathbf{w}.$$

Once again, as F_1 and F_3 are linear and bounded, we are left to prove the claim for F_2 only. As $F_2(\mathbf{v}) = f(\mathbf{v})\mathbf{v}$, it is enough to show the analyticity of the map

$$W^2 \ni \mathbf{v} \mapsto f(\mathbf{v}) = (1 + |\mathbf{v}|^2)^{-1} \in C(\bar{\Omega}).$$

To this purpose, consider the following statement, which is a suitable extension of [13, Lemma 1]

Lemma 5.1. *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be an analytic function and K be a compact subset of $\mathbb{R}^N$. Then the formal series $\sum_\alpha (c_\alpha/\alpha!)x^\alpha$ (here α is a multi-index of length $|\alpha| = n$), with $c_\alpha = \max_{x\in K}|\partial^\alpha f(x)|$, has positive convergence radius.*

Proof. Let A be a complex neighborhood of R^N in which f can be extended to a holomorphic function g. Choose $R > 0$ such that $d(K, C^N \setminus A) > R$. Define $L = \{z \in A\colon d(z, K) \le R\}$, and $M = \max_{z \in L} |g(z)|$. Then, by the Cauchy inequalities, we deduce the bound $c_\alpha \le n!\, M/R^n$. This proves the claim. □

By means of the standard Sobolev inclusion $W \hookrightarrow L^\infty(\Omega)$, we deduce that the set $K = \overline{\mathbf{v}(\Omega)}$ (here the over-line bar denotes the closure in $\mathbb{R}^2$) is compact. Therefore, if we set

$$T_0(\mathbf{v}) = f(\mathbf{v}) \quad \text{and} \quad T_n(\mathbf{v})(\mathbf{h}_1, \ldots, \mathbf{h}_n) = D^n f(\mathbf{v})(\mathbf{h}_1, \ldots, \mathbf{h}_n), \quad n \ge 1,$$

for all $(\mathbf{h}_1, \ldots, \mathbf{h}_n) \in (W^2)^n$ (here D^n denotes the differential of f of order n), it follows that

$$\|T_n(\mathbf{v})\|_{\mathcal{L}((W^2)^n; C(\bar{\Omega}))} \le c_\alpha,$$

being c_α as in Lemma 5.1 (in the case $N = 2$), and

$$f(\mathbf{v} + \mathbf{h}) = \sum_{n=0}^{\infty} \frac{T_n(\mathbf{v})(\mathbf{h}, \ldots, \mathbf{h})}{n!},$$

provided that the series is convergent. This concludes the proof of Lemma 4.4. □

Proof of Lemma 4.5. Lemma 4.5 can be proven arguing as in [2, Section 2]. For the reader's convenience, we outline the argument therein provided. This approach applies when the underlying function set is a Hilbert space, which in our present case is true only when $\alpha = 0$. Nevertheless, as $E(u) = E(\hat{u})$, there is no loss of generality supposing that $u \in Z_0$. We recall that in this case a Poincaré inequality holds, namely,

$$c_P \|\cdot\|_V^2 \le \|\nabla \cdot\|^2, \quad c_P \|\cdot\|_W^2 \le \|\Delta \cdot\|^2 \quad \text{and} \quad c_P \|\cdot\|_Z^2 \le \|\nabla\Delta \cdot\|^2,$$

c_P being the Poincaré constant. Thus, we introduce the Hilbert triplet

$$Z_0 \hookrightarrow V_0 \equiv V_0^* \hookrightarrow Z_0^*,$$

all the injections being compact, and the bilinear form $\mathcal{B}_u\colon Z_0 \times Z_0 \to \mathbb{R}$ given by

$$\mathcal{B}_u(v, w) = \langle \nabla\Delta v, \nabla\Delta w \rangle + \mu \Big\langle \frac{(1 + |\nabla u|)^2}{1 + |\nabla u|^2} \nabla v - 2 \frac{\nabla u \cdot \nabla v}{1 + |\nabla u|^2} \nabla u, \nabla\Delta w \Big\rangle. \tag{5.1}$$

Note that $\langle E''(u)v, w \rangle_{V_0} = \mathcal{B}_u(v, w)$.

We have

Proposition 5.2. *$\mathcal{B}_u$ is symmetric, continuous and (Z_0, Z_0^*)-coercive. Then, for any $u \in Z_0$, $E''(u) \in \mathcal{L}(Z_0, Z_0^*)$ is a Fredholm operator.*

Proof. Symmetry is straightforward. Let us prove continuity and coercivity. In the sequel, we shall use the boundedness (from above) of the function

$$x \mapsto [(1+x)^2 + 2x^2](1+x^2)^{-1}.$$

Concerning the continuity, by means of the injection $Z \hookrightarrow V$, it is easy to see that, for any $v, w \in Z_0$, we get

$$\begin{aligned}\mathcal{B}_u(v, w) &\leq \|\nabla\Delta v\|\,\|\nabla\Delta w\| + \mu \int_\Omega \frac{(1+|\nabla u|)^2 + 2|\nabla u|^2}{1+|\nabla u|^2} |\nabla v|\,|\nabla\Delta w|\, d\Omega \\ &\leq \|\nabla\Delta v\|\,\|\nabla\Delta w\| + c\|\nabla v\|\,\|\nabla\Delta w\| \leq \|v\|_Z\|w\|_Z + c\|v\|_V\|w\|_Z \\ &\leq c\|v\|_Z\|w\|_Z.\end{aligned}$$

In order to prove coercivity, we recall the interpolation inequality

$$\|v\|_V \leq c\|v\|_{Z^*}^{1/2}\|v\|_Z^{1/2}, \quad v \in Z,$$

for some positive constant c. Thus, for any $v \in Z_0$, using also the Poincaré inequality, we obtain

$$\begin{aligned}\mathcal{B}_u(v, v) &= \langle\nabla\Delta v, \nabla\Delta v\rangle + \mu\left\langle \frac{(1+|\nabla u|)^2}{1+|\nabla u|^2}\nabla v, \nabla\Delta v\right\rangle - 2\mu\left\langle\frac{\nabla u\cdot\nabla v}{1+|\nabla u|^2}\nabla u, \nabla\Delta v\right\rangle \\ &\geq \|\nabla\Delta v\|^2 - \mu\int_\Omega \frac{(1+|\nabla u|)^2 + 2|\nabla u|^2}{1+|\nabla u|^2}|\nabla v|\,|\nabla\Delta v|\, d\Omega \\ &\geq c_P\|v|_Z^2 - c\int_\Omega |\nabla v|\,|\nabla\Delta v|\, d\Omega \geq c_P\|v\|_Z^2 - c\|v\|_V\|v\|_Z \\ &\geq \frac{3c_P}{4}\|v\|_Z^2 - c\|v\|_V^2 \geq \frac{3c_P}{4}\|v\|_Z^2 - c\|v\|_{Z^*}\|v\|_Z \\ &= \frac{c_P}{2}\|v\|_Z^2 - \lambda\|v\|_{Z^*}^2,\end{aligned}$$

for some positive λ. □

Let now $P: V_0 \to V_0$ be the orthogonal projection onto $\ker(E''(u))$. As L_u is a Fredholm operator, $\ker(E''(u))$ is finite dimensional. Therefore, by symmetry, it can be extended to a bounded projection in Z_0^*. From now on, we shall suppose $u_\infty \in Z_0$ to be a solution to stationary equation (4.1) (i.e., $E'(u_\infty) = 0$).

The next statement subsumes [2, Lemmas 1 and 2] adapted to the present case.

Lemma 5.3. *The set*

$$\mathcal{S} = \{u \in Z_0 \colon (I - P)E'(u) = 0\}$$

is locally near u_∞ *an analytic manifold satisfying*

$$\dim \mathcal{S} = \dim \ker(E''(u_\infty)).$$

Lemma 5.4. *Assume that the restriction of* $E|_{\mathcal{S}}$ *satisfies the Łojasiewicz–Simon inequality near* u_∞, *i.e., there exists a neighborhood* $U \subset Z_0$ *of* u_∞ *and constants* $\theta \in (0, 1/2]$ *and* $C > 0$ *such that*

$$|E(u) - E(u_\infty)|^{1-\theta} \le C\|E'(u)\|_{Z_0^*}, \quad \forall u \in U \cap \mathcal{S}.$$

Then E *satisfies itself the Łojasiewicz–Simon inequality near* u, *with the same Łojasiewicz exponent* ϑ.

As E is real analytic, its projection on $\mathcal{S}$ is real analytic. Therefore, the thesis of Lemma 4.5 is achieved, as a consequence of Lemmas 5.3 and 5.4. □

References

[1] T. Cazenave and A. Haraux: An Introduction to Semilinear Evolution Equations, Clarendon Press, Oxford, 1998.

[2] R. Chill: *The Łojasiewicz–Simon gradient inequality on Hilbert spaces*; in Proceeding of the 5th European–Maghrebian Workshop on Semigroup Theory, Evolution Equations and Applications, 25–36, 2006.

[3] M. Efendiev, A. Miranville and S. Zelik: *Exponential attractors for a nonlinear reaction-diffusion system in* $\mathbf{R}^3$, C.R. Acad. Sci. Paris Sér. I Math. **330** (2000), 713–718.

[4] E. Feireisl and F. Simondon: *Convergence for semilinear degenerate parabolic equations in several space dimensions*, J. Dynam. Differential Equations **12** (2000), 647–673.

[5] H. Fujimura and A. Yagi: *Dynamical system for a BCF model describing crystal surface growth*, Vestnik Chelyab. Univ. Ser. 3 Mat. Mekh. Inform. **10** (2008), 75–88.

[6] H. Fujimura and A. Yagi: *Asymptotic behavior of solutions for BCF model describing crystal surface growth*, Int. Math. Forum **3** (2008), 1803–1812.

[7] H. Fujimura and A. Yagi: *Homogeneous stationary solution for BCF model describing crystal surface growth*, Sci. Math. Jpn. **69** (2009), 295–302.

[8] M.A. Jendoubi: *A simple unified approach to some convergence theorems of L. Simon*, J. Funct. Anal. **153** (1998), 187–202.

[9] M.D. Johnson, C. Orme, A.W. Hunt, D. Graff, J. Sudijono, L.M. Sander and B.G. Orr: *Stable and unstable growth in molecular beam epitaxy*, Phys. Rev. Lett. **72** (1994), 116–119.

[10] B.B. King, O. Stein and M. Winkler: *A fourth-order parabolic equation modeling epitaxial thin film growth*, J. Math. Anal. Appl. **286** (2003), 459–490.

[11] B. Li and J.-G. Liu: *Thin film epitaxy with or without slope selection*, European J. Appl. Math. **14** (2003), 713–743.

[12] B. Li and J.-G. Liu: *Epitaxial growth without slope selection*: *energetics, coarsening, and dynamic scaling*, J. Nonlinear Sci. **14** (2004), 429–451.

[13] V. Rădulescu: *Finitely many solutions for a class of boundary value problems with superlinear convex nonlinearity*, Arch. Math. (Basel) **84** (2005), 538–550.
[14] G. Savaré: *Regularity results for elliptic equations in Lipschitz domains*, J. Funct. Anal. **152** (1998), 176–201.
[15] C. Wang, X. Wang and S.M. Wise: *Unconditionally stable schemes for equations of thin film epitaxy*, Discrete Contin. Dyn. Syst. **28** (2010), 405–423.
[16] E. Zeidler: Nonlinear Functional Analysis and its Applications, I, Springer, New York, 1988.

Maurizio Grasselli
Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano
Via E. Bonardi 9, 20133 Milano
Italy
e-mail: maurizio.grasselli@polimi.it

Gianluca Mola
Dipartimento di Matematica "F. Enriques"
Università degli Studi di Milano
Via C. Saldini 50, 20133 Milano
Italy
e-mail: gianluca.mola@unimi.it

Atsushi Yagi
Department of Applied Physics
Osaka University
Suita, 565-0871 Osaka
Japan
e-mail: yagi@ap.eng.osaka-u.ac.jp