

# ON STOCHASTIC LOGISTIC EQUATION WITH MARKOVIAN SWITCHING AND WHITE NOISE

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## Abstract

This paper mainly concerns with the stochastic logistic equation with Markovian switching and white noise. We represent the unique global positive solution and show some properties such as the stochastic permanence, global attractivity. The main contribution of this paper is to demonstrate that the solution is asymptotically stable in distribution. Moreover, the similar results for the stochastic Gilpin–Ayala equation with Markovian switching and white noise are yielded by the same way. Finally, we give the definition of the stochastic equilibrium solution and prove the existence and uniqueness of such equilibrium solution of stochastic logistic equation.

## 1. Introduction

Stochastic modelling has come to play an important role in many branches of science and industry. Stochastic stability has been one of the most active areas in stochastic analysis and many mathematicians have devoted their great interest to it. There are many types of stochastic stability. X. Mao [28] investigated three types of stochastic stability for non-linear stochastic differential equation: stability in probability,  $p$ -th moment stability and almost sure stability. However such stabilities are too strong and the solution will sometimes not converge to the equilibrium state but converge to a random variable in distribution. This is the concept of another important stochastic stability: stability in distribution (see e.g., [29], [30]). J. Bao, et al. [3] investigated stability in distribution of neutral stochastic differential delay equations with Markovian switching.

There are many articles on the stability of stochastic differential equation, but up to the authors' best knowledge there are few works on stability in distribution. Especially, there are no articles on stability in distribution of stochastic logistic equations with Markovian switching. Logistic equation is of course the most important and most widely used mathematical model to describe the growth of biological species. To demonstrate that the theories of stochastic stability are applicable in population dynamics, we discussed the autonomous stochastic logistic equation with Markovian switching

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and white noise of the form,

$$(1) \quad dX(t) = r(\xi(t))X(t) \left[ 1 - \frac{X(t)}{K(\xi(t))} \right] dt + \sigma(\xi(t))X(t) dB(t)$$

on  $t \geq 0$  with initial conditions  $X(0) = x$  and  $\xi(0) = i$ , where  $\xi(t)$  be a right-continuous Markov chain taking values in a finite state space  $\mathbb{S} = \{1, 2, \dots, N\}$  and  $B(t)$  be the 1-dimensional standard Brownian motion and we always suppose that the Markov chain  $\xi(t)$  is  $\mathcal{F}_t$ -adapted but independent of the Brownian motion  $B(t)$ . For Equation (1),  $X(t)$  denotes the density of the resource population at time  $t$ ,  $r(\cdot)$  is called the intrinsic growth rate,  $K(\cdot)$  is usually referred as the environmental carrying capacity, or saturation level, and  $\sigma^2(\cdot)$  representing the intensity of the noises. Equation (1) can be regarded as the results of the following  $N$  equations

$$dX(t) = r(i)X(t) \left[ 1 - \frac{X(t)}{K(i)} \right] dt + \sigma(i)X(t) dB(t), \quad i \in \mathbb{S}$$

switching from one to the others according to the movement of the Markov chain.

Jiang et al. [15] studied the logistic equation with random perturbation, they mainly investigated the existence and uniqueness, global attractivity of the positive solution and the maximum likelihood parameter estimation. In [16] Jiang et al. discussed non-autonomous logistic equation with random perturbation which is described by the Itô equation

$$dN(t) = N(t)[a(t) - b(t)N(t)] dt + \alpha(t) dB(t), \quad t \geq 0.$$

In [17] Jiang et al. gave the global stability in probability and stochastic permanence of a non-autonomous logistic equation with random perturbation. For the logistic equation with random perturbation, Jiang et al. gave these very important and interesting results, so the contributions of their result for logistic equation is clear. In our paper, we considered the stochastic logistic equation with Markovian switching and white noise, that is we considered the color noise, say telegraph noise besides the white noise in logistic model. Differing from the exists results, we give the stability in distribution and prove the existence and uniqueness of the stochastic equilibrium solution for the stochastic biological model for the first time. The stochastic equilibrium solution plays the same role as the deterministic stationary solution.

When ignoring the perturbation of Markov chain and white noise, we get the autonomous logistic model  $dX(t) = rX(t)(1 - X(t)/K) dt$  from Equation (1). It is well known that this equation has two stationary solutions  $x = 0$  and  $x = K$ . When introducing uncertainty (color noise and white noise) into this model by the way of Equation (1), we transform the deterministic problem into corresponding stochastic problem and obtain only the stationary solution  $x = 0$ , which is the same as in the deterministic case, but can not obtain another stationary solution corresponding  $x = K$  in deterministic model. But in the real world, it is this equilibrium that has great significance in

the sense of bionomics. So from the view of biology, the following problem is important and instructive, i.e., under the perturbation of the noises, whether or not the probability distribution of  $X(t)$  will converge weakly to a stationary distribution which is analogous to  $K$ . When only taking white noise into account, Polansky [33] answered this question. The author showed that there exists a Gamma distribution which plays the same role as in the deterministic stationary solution  $x = K$  under some conditions (see also [32]). If we refer the stationary solution  $x = K$  as a Dirac delta distribution, then this Dirac delta distribution will degenerate into a Gamma distribution under the perturbation of the white noise. Then under the perturbation of both Markov chain and white noise, what does this Dirac delta distribution become? This is the main problem that we will investigate and resolve in our paper.

We note that the coefficients of Equation (1) do not satisfy the linear growth condition, though they are local Lipschitz continuous. Moreover, the state space of our model is not the whole space but  $(0, +\infty)$ . Therefore many well known results on stochastic boundedness and stability for stochastic differential equations can not be used for Equation (1) directly, this results in many difficulties to deal with it. So the significant contribution of our paper is therefore clear.

The rest of this paper is arranged as follows. In Section 2, we represent an explicit expression of the unique solution of Equation (1) and show that it is positive,  $p$ -th moment bounded, stochastic permanence and global attractivity. In Section 3, we give the main results of this paper: Stochastic logistic equation with Markovian switching and white noise is asymptotically stable in distribution. In Section 4, we devote our interest to the Gilpin–Ayala system and give some similar results as in Section 2 and Section 3. In Section 5, we prove the existence and uniqueness of the globally asymptotically stable stochastic equilibrium solution for Equation (1). Our main results demonstrate that there is a trend that the distribution of the state of the system will be gradually stable.

## 2. Stochastic permanence and global attractivity of Equation (1)

Let  $\xi(t)$  be a right-continuous Markov chain taking values in a finite state space  $\mathbb{S} = \{1, 2, \dots, N\}$  with generator  $\Gamma = (\gamma_{ij})_{N \times N}$  given by

$$\mathbb{P}\{\xi(t + \delta) = j \mid \xi(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta) & \text{if } i = j, \end{cases}$$

where  $\delta > 0$ . Here  $\gamma_{ij} \geq 0$  is the transition rate from  $i$  to  $j$  and  $\gamma_{ij} > 0$  if  $i \neq j$  while

$$\gamma_{ii} = - \sum_{j \neq i} \gamma_{ij}.$$

We note that almost every sample path of  $\xi(\cdot)$  is a right continuous step function with a finite number of sample jumps in any finite subinterval of  $\mathbb{R}_+ := [0, \infty)$ . Throughout

this paper we assume that the Markov chain  $\xi(\cdot)$  is irreducible. Under this condition, the Markov chain has a unique stationary (probability) distribution  $\pi = (\pi_1, \pi_2, \dots, \pi_N) \in R^{1 \times N}$  which can be determined by solving the following linear equation

$$\pi \Gamma = 0$$

subject to

$$\sum_{k=1}^N \pi_k = 1 \text{ and } \pi_k > 0, \quad \forall k \in \mathbb{S}.$$

If the Markov chain  $\xi(\cdot)$  is irreducible, then the system will switch from any regime to any other regime. Denote by  $X^{x,i}(t), t \geq 0$  the solution of Equation (1) with initial conditions  $X(0) = x > 0$  and  $\xi(0) = i \in \mathbb{S}$ , Let  $\xi_i(t)$  be the Markov chain starting from state  $i \in \mathbb{S}$  at  $t = 0$ . Since  $X^{x,i}(t), t \geq 0$  denotes the population size, so only positive solutions are of interest.

In this paper, for Equation (1) we let

$$\begin{aligned} r^* &= \max_{1 \leq i \leq N} r(i), & K^* &= \max_{1 \leq i \leq N} K(i), & \sigma^* &= \max_{1 \leq i \leq N} \sigma(i), \\ r_* &= \min_{1 \leq i \leq N} r(i), & K_* &= \min_{1 \leq i \leq N} K(i), & \sigma_* &= \min_{1 \leq i \leq N} \sigma(i). \end{aligned}$$

We also assume that

(H<sub>1</sub>)  $r(i) > 0, K(i) > 0, i \in \mathbb{S}$ ;

(H<sub>2</sub>)  $r_* - (\sigma^*)^2 > 0$ .

DEFINITION 2.1. The Equation (1) is said to be stochastically permanent if for any  $\varepsilon \in (0, 1)$ , there exist positive constants  $\delta = \delta(\varepsilon), \chi = \chi(\varepsilon)$ , such that

$$\liminf_{t \rightarrow +\infty} P\{|X(t)| \leq \chi\} \geq 1 - \varepsilon, \quad \liminf_{t \rightarrow +\infty} P\{|X(t)| \geq \delta\} \leq 1 - \varepsilon$$

where  $X(t)$  is the solution of the equation with the initial conditions  $X(0) = x$  and  $\xi(0) = i$ .

DEFINITION 2.2. Let  $X(t), Y(t)$  be two arbitrary solutions of Equation (1) with initial values  $X(0), Y(0)$  respectively. If

$$\lim_{t \rightarrow \infty} |X(t) - Y(t)| = 0 \quad \text{a.s.}$$

then we say that Equation (1) is globally attractive.

The two definitions above can also be found in [4], [19].

**Lemma 2.1** (see e.g. [17] or [4]). *Let  $f(t)$  be a nonnegative function defined on  $[0, +\infty)$  such that  $f(t)$  is integrable on  $[0, +\infty)$  and is uniformly continuous on  $[0, +\infty)$ , then  $\lim_{t \rightarrow \infty} f(t) = 0$ .*

**Theorem 2.2.** *Suppose  $(H_1)$  holds. Then there exists a unique continuous positive solution  $X^{x,i}(t)$  to Equation (1) which can be expressed as*

$$(2) \quad X^{x,i}(t) = \left\{ \frac{1}{x} \exp \left\{ - \int_0^t A(\xi(s)) ds + \int_0^t \sigma(\xi(s)) dB(s) \right\} + \int_0^t \frac{r(\xi(s))}{K(\xi(s))} \exp \left\{ - \int_s^t A(\xi(\tau)) d\tau + \int_s^t \sigma(\xi(\tau)) dB(\tau) \right\} ds \right\}^{-1}, \quad t \geq 0$$

for any initial conditions  $X(0) = x > 0$  and  $\xi(0) = i \in \mathbb{S}$ , where  $A(\cdot) = r(\cdot) - (1/2)\sigma^2(\cdot)$ .

*Proof.* Let  $\tau_1 > 0$  be the first jumping time of  $\xi(t)$ , at this stopping time  $\tau_1$ ,  $\xi(t)$  jumps from the state  $i$  to  $j$ , while  $\tau_2 > \tau_1$  is the next jumping time. Without loss of generality, we only need to show that for  $t \in [\tau_1, \tau_2)$ , the solution satisfies

$$(3) \quad X^{x,i}(t) = X^{X^{x,i}(\tau_1),j}(t).$$

In fact for  $t \in [0, \tau_1]$ , in the view of [16] (namely, the result of Theorem 2.2) Equation (2) holds, so

$$(4) \quad X^{x,i}(\tau_1) = \left[ \frac{1}{x} \exp \left\{ - \int_0^{\tau_1} A(\xi(s)) ds + \int_0^{\tau_1} \sigma(\xi(s)) dB(s) \right\} + \int_0^{\tau_1} \frac{r(\xi(s))}{K(\xi(s))} \exp \left\{ - \int_s^{\tau_1} A(\xi(v)) dv + \int_s^{\tau_1} \sigma(\xi(v)) dB(v) \right\} ds \right]^{-1}.$$

Then for  $t \in [\tau_1, \tau_2)$ , we observe from Equation (4) that

$$\begin{aligned} \frac{1}{X^{x,i}(t)} &= \frac{1}{x} \exp \left\{ - \int_0^{\tau_1} A(\xi(s)) ds + \int_0^{\tau_1} \sigma(\xi(s)) dB(s) \right\} \\ &\quad \times \exp \left\{ - \int_{\tau_1}^t A(\xi(s)) ds + \int_{\tau_1}^t \sigma(\xi(s)) dB(s) \right\} \\ &\quad + \int_0^{\tau_1} \frac{r(\xi(s))}{K(\xi(s))} \exp \left\{ - \int_s^{\tau_1} A(\xi(v)) dv + \int_s^{\tau_1} \sigma(\xi(v)) dB(v) \right\} \\ &\quad \times \exp \left\{ - \int_{\tau_1}^t A(\xi(v)) dv + \int_{\tau_1}^t \sigma(\xi(v)) dB(v) \right\} ds \\ &\quad + \int_{\tau_1}^t \frac{r(\xi(s))}{K(\xi(s))} \exp \left\{ - \int_s^t A(\xi(v)) dv + \int_s^t \sigma(\xi(v)) dB(v) \right\} ds \end{aligned}$$

$$\begin{aligned}
 &= \left[ \frac{1}{x} \exp \left\{ - \int_0^{\tau_1} A(\xi(s)) ds + \int_0^{\tau_1} \sigma(\xi(s)) dB(s) \right\} \right. \\
 &\quad \left. + \int_0^{\tau_1} \frac{r(\xi(s))}{K(\xi(s))} \exp \left\{ - \int_s^{\tau_1} A(\xi(v)) dv - \int_s^{\tau_1} \sigma(\xi(v)) dB(v) \right\} ds \right] \\
 &\quad \times \exp \left\{ - \int_{\tau_1}^t A(\xi(v)) dv + \int_{\tau_1}^t \sigma(\xi(v)) dB(v) \right\} \\
 &\quad + \int_{\tau_1}^t \frac{r(\xi(s))}{K(\xi(s))} \exp \left\{ - \int_s^t A(\xi(v)) dv + \int_s^t \sigma(\xi(v)) dB(v) \right\} ds \\
 &= \frac{1}{X^{x,i}(\tau_1)} \exp \left\{ - \int_{\tau_1}^t A(\xi(v)) dv + \int_{\tau_1}^t \sigma(\xi(v)) dB(v) \right\} \\
 &\quad + \int_{\tau_1}^t \frac{r(\xi(s))}{K(\xi(s))} \exp \left\{ - \int_s^t A(\xi(v)) dv + \int_s^t \sigma(\xi(v)) dB(v) \right\} ds \\
 &= (X^{X^{x,i}(\tau_1),j}(t))^{-1}
 \end{aligned}$$

which is the required assertion Equation (3). □

REMARK 1. The conclusion of Theorem 2.2 in this paper can not be obtained directly from Theorem 2.2 of [16]. Because Theorem 2.2 of [16] demands every coefficient function is a bounded continuous function but the coefficient function here is stochastic and not continuous.

REMARK 2 (see e.g. [17], [18], [27]). Suppose that a stochastic process  $X(t)$  on  $t \geq 0$  satisfies the condition

$$(5) \quad E|X(t) - X(s)|^\alpha \leq c|t - s|^{1+\beta}, \quad 0 \leq s, t < \infty$$

for some positive constants  $\alpha, \beta$  and  $c$ . Then there exists a continuous modification  $\tilde{X}(t)$  of  $X(t)$ , which has the property that for every  $\gamma \in (0, \beta/\alpha)$ , there exists a random variable  $h(\omega) > 0$  such that

$$P \left\{ \omega : \sup_{0 \leq s, t < \infty, 0 < |t-s| < h(\omega)} \frac{|\tilde{X}(t, \omega) - \tilde{X}(s, \omega)|}{|t - s|^\gamma} \leq \frac{2}{1 - 2^{-\gamma}} \right\} = 1.$$

That is almost all sample path of  $X(t)$  is locally but uniformly Hölder-continuous with exponent  $\gamma$ .

REMARK 3. If the equation defined on  $[t_0, \infty)$ , then the initial time can be  $t_0$ , that is

$$\begin{aligned}
 X^{x,i}(t) &= \left\{ \frac{1}{x} \exp \left\{ - \int_{t_0}^t A(\xi(s)) ds - \int_{t_0}^t \sigma(\xi(s)) dB(s) \right\} \right. \\
 &\quad \left. + \int_{t_0}^t \frac{r(\xi(s))}{K(\xi(s))} \exp \left\{ - \int_s^t A(\xi(\tau)) d\tau - \int_s^t \sigma(\xi(\tau)) dB(\tau) \right\} ds \right\}^{-1}, \quad t \geq t_0.
 \end{aligned}$$

REMARK 4. If we replace  $x \in R^+$  by a positive random variable in Equation (2), the formula still holds.

Following [17], we can prove Lemma 2.3, Lemma 2.4 and Theorem 2.5.

**Lemma 2.3.** *Let  $X(t)$  be a solution of Equation (1) with initial value  $X^{x,i}(0) = x > 0$ , then*

$$E(X^p(t)) \leq \left[ K^* \left( r^* + \frac{(1/2)(p-1)(\sigma^*)^2}{r_*} \right) \right]^p := M(p).$$

Proof. By the generalised Itô formula, we can easily know

$$\begin{aligned} dX^p(t) &= pX^{p-1}(t) dX(t) + \frac{1}{2}p(p-1)X^{p-2}(t)(dX(t))^2 \\ &= pr(\xi(t))X^p(t) \left[ \left( 1 - \frac{X(t)}{K(\xi(t))} \right) dt + \sigma(\xi(t)) dB(t) \right] \\ &\quad + \frac{1}{2}p(p-1)X^p(t)\sigma^2(\xi(t)) dt. \end{aligned}$$

Integrating from 0 to  $t$  and taking expectation yields

$$\begin{aligned} EX^p(t) &= x^p + \int_0^t pE \left( r(\xi(s))X^p(s) \left( 1 - \frac{X(s)}{K(\xi(s))} \right) \right) ds \\ &\quad + \int_0^t \frac{1}{2}p(p-1)E[X^p(s)\sigma^2(\xi(s))] ds, \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{dEX^p(t)}{dt} &= Er(\xi(t))X^p(t) \left( 1 - \frac{X(t)}{K(\xi(t))} \right) \\ &\quad + \frac{1}{2}p(p-1)E[X^p(t)\sigma^2(\xi(t))], \quad EX^{x,i}(0) = x^p. \end{aligned}$$

So we can see that

$$\begin{aligned} \frac{dEX^p(t)}{dt} &= E(r(\xi(t))X^p(t)) + \frac{1}{2}p(p-1)E[X^p(t)\sigma^2(\xi(t))] - E \left[ \frac{r(\xi(t))}{K(\xi(t))} (X^{p+1}(t)) \right] \\ &\leq r^* EX^p(t) + \frac{1}{2}p(p-1)(\sigma^*)^2 E[X^p(t)] - \frac{r_*}{K^*} E(X^{p+1}(t)) \\ &\leq pEX^p(t) \left\{ \left[ r^* + \frac{1}{2}(p-1)(\sigma^*)^2 \right] - \frac{r_*}{K^*} [EX^p(t)]^{1/p} \right\}. \end{aligned}$$

Let  $p \geq 1$  be chosen such that the initial value

$$0 < x < K^* \frac{r^* + (1/2)(p-1)(\sigma^*)^2}{r_*}.$$

Then by the standard comparison theorem we know that

$$[E(X^p(t))]^{1/p} \leq K^* \frac{r^* + (1/2)(p-1)(\sigma^*)^2}{r^*}.$$

So

$$E(X^p(t)) \leq M(p).$$

This completes the proof of Lemma 2.3. □

**Lemma 2.4.** *Let  $X(t)$  be a solution of Equation (1) with the initial value  $X^{x,i}(0) = x > 0$ , then almost every sample path of  $X(t)$  is uniformly continuous on  $t \geq 0$ .*

*Proof.* We rewrite Equation (1) as

$$X(t) = x + \int_0^t f(s, \xi(s), X(s)) ds + \int_0^t g(s, \xi(s), X(s)) dB(s),$$

where  $f(s, \xi(s), X(s)) = r(\xi(s))X(s)[1 - X(s)/K(\xi(s))]$  and  $g(s, \xi(s), X(s)) = \sigma(\xi(s))X(s)$ . Then

$$\begin{aligned} E(|f(s, \xi(s), X(s))|^p) &= E\left(r^p(\xi(s))X^p(s)\left|1 - \frac{X(s)}{K(\xi(s))}\right|^p\right) \\ &\leq \frac{1}{2}(Er^{2p}(\xi(s))X^{2p}(s) + \frac{1}{2}E\left[\left(1 - \frac{X(s)}{K(\xi(s))}\right)^{2p}\right]) \\ &\leq \frac{1}{2}(Er^{2p}(\xi(s))X^{2p}(s) + 2^{2p-2}E\left[1 + \frac{X^{2p}(s)}{K^{2p}(\xi(s))}\right]) \\ &\leq \frac{1}{2}(r^*)^{2p}M(2p) + 2^{2p-2}\left[1 + \frac{M(2p)}{(K^*)^{2p}}\right] \\ &=: F(p), \end{aligned}$$

and

$$E(|g(s, \xi(s), X(s))|^p) = E(\sigma^2(\xi(s))X^p(s)) \leq (\sigma^*)^p M(p) =: G(p).$$

By the moment inequality (cf. Friedman [8] or Mao [28]) and Hölder inequality, we have that, for  $t_1 < t_2 < \infty$  and  $p > 2$ ,

$$\begin{aligned} E|X(t_2) - X(t_1)|^p &\leq 2^{p-1}E\left(\int_{t_1}^{t_2} |f(s, \xi(s), X(s))| ds\right)^p \\ &\quad + 2^{p-1}E\left|\int_{t_1}^{t_2} g(s, \xi(s), X(s)) dB(s)\right|^p \end{aligned}$$

$$\begin{aligned} &\leq 2^{p-1}(t_2 - t_1)^{p-1} E \left( \int_{t_1}^{t_2} |f(s, \xi(s), X(s))|^p ds \right) \\ &\quad + 2^{p-1} \left[ \frac{p(p-1)}{2} \right]^{p/2} (t_2 - t_1)^{(p-2)/2} E \int_{t_1}^{t_2} |g(s, \xi(s), X(s))|^p ds \\ &\leq 2^{p-1}(t_2 - t_1)^p F(p) + 2^{p-1} \left[ \frac{p(p-1)}{2} \right]^{p/2} (t_2 - t_1)^{p/2} G(p) \\ &\leq 2^{p-1}(t_2 - t_1)^{p/2} \left\{ 1 + \left[ \frac{p(p-1)}{2} \right]^{p/2} \right\} [F(p) + G(p)], \end{aligned}$$

We see from Remark 2 that almost every sample path of  $X(t)$  is locally but uniformly Hölder-continuous with exponent  $\gamma$  for every  $\gamma \in (0, p - 2/2p)$  and therefore almost every sample path of  $X(t)$  is uniformly continuous on  $t \geq 0$ . This completes the proof of Lemma 2.4. □

**Theorem 2.5.** *Suppose (H<sub>1</sub>) and (H<sub>2</sub>) hold,  $X^{x,i}(t)$  and  $X^{y,i}(t)$  are solutions of Equation (1) with initial values  $X^{x,i}(0) = x > 0$  and  $X^{y,i}(t) = y > 0$ . Then*

$$\lim_{t \rightarrow +\infty} |X^{x,i}(t) - X^{y,i}(t)| = 0 \quad \text{for almost all } \omega \in \Omega.$$

*Proof.* Consider a Lyapunov function  $V(t)$  which is defined by

$$V(t) = |\log X^{x,i}(t) - \log X^{y,i}(t)|, \quad t \geq 0.$$

Then the generalised Itô formula implies that

$$d(\log X^{x,i}(t) - \log X^{y,i}(t)) = -\frac{r(\xi(t))}{K(\xi(t))} (X^{x,i}(t) - X^{y,i}(t)) dt.$$

Thus, a direct calculation of the right differential  $d^+V(t)$  of  $V(t)$  along the solutions yields

$$\begin{aligned} (6) \quad d^+V(t) &= \operatorname{sgn}(X^{x,i}(t) - X^{y,i}(t)) d(\log X^{x,i}(t) - \log X^{y,i}(t)) \\ &= -\frac{r(\xi(t))}{K(\xi(t))} |X^{x,i}(t) - X^{y,i}(t)| dt \\ &\leq -\frac{r^*}{K^*} |X^{x,i}(t) - X^{y,i}(t)| dt. \end{aligned}$$

Integrating (6) from 0 to  $t$  we obtain

$$V(t) + \frac{r^*}{K^*} \int_0^t |X^{x,i}(s) - X^{y,i}(s)| ds \leq V(0) < \infty.$$

So,

$$|X^{x,i}(t) - X^{y,i}(t)| \in L^1[0, +\infty).$$

Therefore from Lemma 2.1 and Lemma 2.4 we get

$$\lim_{t \rightarrow +\infty} |X^{x,i}(t) - X^{y,i}(t)| = 0 \quad \text{for almost all } \omega \in \Omega.$$

This completes the proof of Theorem 2.5. □

From Theorem 2.5 we can see that the positive solutions of Equation (1) attract each other. In fact, any positive solution can be regarded as an attractor of Equation (1).

**Lemma 2.6.** *Let  $\xi(t)$  be a right-continuous Markov chain taking values in a finite state space  $S = \{1, 2, \dots, N\}$ , then*

$$E \left[ \exp \left\{ \int_{t_0}^t \sigma(\xi(s)) dB(s) \right\} \right] \leq \exp \left\{ \frac{(\sigma^*)^2}{2} (t - t_0) \right\}, \quad 0 \leq t_0 \leq t.$$

*Proof.* Applying the generalised Itô formula, we obtain

$$\begin{aligned} \exp \left\{ \int_{t_0}^t \sigma(\xi(s)) dB(s) \right\} &= 1 + \int_{t_0}^t \sigma(\xi(s)) \exp \left\{ \int_{t_0}^s \sigma(\xi(\tau)) dB(\tau) \right\} dB(s) \\ &\quad + \frac{1}{2} \int_{t_0}^t \sigma^2(\xi(s)) \exp \left\{ \int_{t_0}^s \sigma(\xi(\tau)) d\tau \right\} ds \\ &\leq 1 + \int_{t_0}^t \sigma(\xi(s)) \exp \left\{ \int_{t_0}^s \sigma(\xi(\tau)) dB(\tau) \right\} dB(s) \\ &\quad + \frac{(\sigma^*)^2}{2} \int_{t_0}^t \exp \left\{ \int_{t_0}^s \sigma(\xi(\tau)) d\tau \right\} ds. \end{aligned}$$

Taking expectation on both sides yields

$$\begin{aligned} E \exp \left\{ \int_{t_0}^t \sigma(\xi(s)) dB(s) \right\} &= 1 + \frac{1}{2} E \int_{t_0}^t \sigma^2(\xi(s)) \exp \left\{ \int_{t_0}^s \sigma(\xi(\tau)) d\tau \right\} ds \\ &\leq 1 + \frac{(\sigma^*)^2}{2} \int_{t_0}^t E \exp \left\{ \int_{t_0}^s \sigma(\xi(\tau)) d\tau \right\} ds. \end{aligned}$$

Then the Gronwall's inequality implies that

$$E \left[ \exp \left\{ \int_{t_0}^t \sigma(\xi(s)) dB_s \right\} \right] \leq \exp \left\{ \frac{(\sigma^*)^2}{2} (t - t_0) \right\}$$

which is the required assertion. □

Lemma 2.6 is different from Lemma 2.1 of [16] since here the integrand has a Markov chain and the result is a inequality. The following Lemma implies that Equation (1) is stochastic uniform permanence.

**Lemma 2.7.** *Suppose (H<sub>1</sub>) and (H<sub>2</sub>) hold. Then the solution  $X^{x,i}(t)$  with the initial conditions  $X(0) = x$  and  $\xi(0) = i$  has the property that*

$$\min\left(x, \frac{K_*(r_* - (\sigma^*)^2)}{r^*}\right) \leq EX^{x,i}(t) \leq \max\left(x, \frac{r^*K^*}{r_*}\right).$$

Proof. We rewrite Equation (1) as

$$\begin{aligned} X^{x,i}(t) &= x + \int_0^t r(\xi(s))X^{x,i}(s) ds - \int_0^t \frac{r(\xi(s))}{K(\xi(s))}(X^{x,i}(s))^2 ds \\ &\quad + \int_0^t \sigma(\xi(s))X^{x,i}(s) dB(s). \end{aligned}$$

Taking expectation yields

$$EX^{x,i}(t) = x + \int_0^t E(r(\xi(s))X^{x,i}(s)) ds - \int_0^t E\left[\frac{r(\xi(s))}{K(\xi(s))}(X^{x,i}(s))^2\right] ds,$$

i.e.,

$$(7) \quad \frac{dEX^{x,i}(t)}{dt} = E(r(\xi(t))X^{x,i}(t)) - E\left[\frac{r(\xi(t))}{K(\xi(t))}(X^{x,i}(t))^2\right], \quad EX^{x,i}(0) = x.$$

By Jensen’s inequality we show that

$$\begin{aligned} \frac{dEX^{x,i}(t)}{dt} &= E(r(\xi(t))X^{x,i}(t)) - E\left[\frac{r(\xi(t))}{K(\xi(t))}(X^{x,i}(t))^2\right] \\ &\leq r^*EX^{x,i}(t) - \frac{r_*}{K^*}E(X^{x,i}(t))^2 \\ &\leq r^*EX^{x,i}(t) - \frac{r_*}{K^*}(EX^{x,i}(t))^2. \end{aligned}$$

Considering the following logistic equation, based on ordinary differential equation, which is described by

$$(8) \quad y' = r^*y - \frac{r_*}{K^*}y^2, \quad y(0) = x.$$

The solution of Equation (8) satisfies

$$y(t) \leq \max\left(x, \frac{r^*K^*}{r_*}\right) \quad \text{when } t \geq 0.$$

So, making use of the comparison principle for Equation (7) and Equation (8), we get

$$EX^{x,i}(t) \leq \max\left(x, \frac{r^*K^*}{r_*}\right).$$

Let  $Y^{1/x,i}(t) = 1/X^{x,i}(t)$ ,  $x > 0$ , then by Itô formula one can get

$$dY^{1/x,i}(t) = \left[ (\sigma^2(\xi(t)) - r(\xi(t)))Y^{1/x,i}(t) + \frac{r(\xi(t))}{K(\xi(t))} \right] dt - \sigma(\xi(t))Y^{1/x,i}(t) dB(t).$$

That is

$$Y^{1/x,i}(t) = \frac{1}{x} + \int_0^t (\sigma^2(\xi(s)) - r(\xi(s)))(Y^{1/x,i}(s)) + \frac{r(\xi(s))}{K(\xi(s))} ds - \int_0^t \sigma(\xi(s))(Y^{1/x,i}(s)) dB(s).$$

Taking expectation one can get

$$EY^{1/x,i}(t) = \frac{1}{x} + \int_0^t E \left[ (\sigma^2(\xi(s)) - r(\xi(s)))(Y^{1/x,i}(s)) + \frac{r(\xi(s))}{K(\xi(s))} \right] ds,$$

i.e.,

$$\begin{aligned} \frac{dEY^{1/x,i}(t)}{dt} &= E \left[ (\sigma^2(\xi(t)) - r(\xi(t)))(Y^{1/x,i}(t)) + \frac{r(\xi(t))}{K(\xi(t))} \right] \\ &\leq ((\sigma^*)^2 - r_*)EY^{1/x,i}(t) + \frac{r^*}{K_*}. \end{aligned}$$

From the comparison principle we get

$$EY^{1/x,i}(t) \leq \max\left(\frac{1}{x}, \frac{r^*}{K_*(r_* - (\sigma^*)^2)}\right).$$

Consequently, by Jensen's inequality,

$$\frac{1}{E} X^{x,i}(t) \leq E \frac{1}{X^{x,i}(t)} = EY^{1/x,i}(t) \leq \max\left(\frac{1}{x}, \frac{r^*}{K_*(r_* - (\sigma^*)^2)}\right).$$

Then we can derive directly that

$$EX^{x,i}(t) \geq \min\left(x, \frac{K_*(r_* - (\sigma^*)^2)}{r^*}\right).$$

This completes the proof of Lemma 2.7. □

**Lemma 2.8.** *Suppose (H<sub>1</sub>) and (H<sub>1</sub>) hold. Then the solution of Equation (1) obeys*

$$\lim_{t \rightarrow \infty} E \left| \frac{1}{X^{x,i}(t)} - \frac{1}{X^{y,i}(t)} \right| = 0 \quad \text{uniformly in } (x, y, i) \in M \times M \times \mathbb{S},$$

for any compact subset  $M$  of  $(0, +\infty)$ .

Proof. Theorem 2.2 implies that

$$\begin{aligned} & \left| \frac{1}{X^{x,i}(t)} - \frac{1}{X^{y,i}(t)} \right| \\ &= \left| \frac{1}{x} - \frac{1}{y} \right| \exp \left\{ - \int_0^t \left[ r(\xi(s)) - \frac{\sigma^2(\xi(s))}{2} \right] ds \right\} \cdot \exp \left\{ - \int_0^t \sigma(s) dB(s) \right\} \\ &\leq \left| \frac{1}{x} - \frac{1}{y} \right| \exp \left\{ - \left[ r_* - \frac{(\sigma^*)^2}{2} \right] t \right\} \cdot \exp \left\{ - \int_0^t \sigma(s) dB(s) \right\}. \end{aligned}$$

Noting that  $-B(t)$  is also a Brownian motion if  $B(t)$  is. So by Lemma 2.4 and condition  $(H_2)$ , we get

$$\begin{aligned} E \left| \frac{1}{X^{x,i}(t)} - \frac{1}{X^{y,i}(t)} \right| &\leq \left| \frac{1}{x} - \frac{1}{y} \right| \exp \left\{ - \left( r_* - \frac{(\sigma^*)^2}{2} \right) t \right\} \cdot E \exp \left\{ - \int_0^t \sigma(s) dB(s) \right\} \\ &\leq \left| \frac{1}{x} - \frac{1}{y} \right| \exp \left\{ - (r_* - (\sigma^*)^2) t \right\}. \end{aligned}$$

Then letting  $t \rightarrow +\infty$  produce

$$\lim_{t \rightarrow \infty} E \left| \frac{1}{X^{x,i}(t)} - \frac{1}{X^{y,i}(t)} \right| = 0.$$

It is obvious that this limit is uniformly in any compact subset  $M$  of  $(0, +\infty)$  for  $x, y \in M$ . The proof of Lemma 2.8 is complete.  $\square$

**Lemma 2.9.** *The solutions of Equation (1) satisfy*

$$E(X^{x,i}(t)X^{y,i}(t)) \leq \max \left( xy, \frac{\alpha^2}{\beta^2} \right),$$

where  $\alpha = 2r_* + (\sigma^*)^2$ ,  $\beta = (2r_*)/K^*$ .

Proof. By the generalized Itô formula we obtain

$$\begin{aligned} & d(X^{x,i}(t)X^{y,i}(t)) \\ &= X^{y,i}(t) dX^{x,i}(t) + X^{x,i}(t) dX^{y,i}(t) + dX^{x,i}(t) \cdot dX^{y,i}(t) \\ &= \left[ 2r(\xi(t))X^{x,i}(t)X^{y,i}(t) \right. \\ &\quad \left. - \frac{r(\xi(t))}{K(\xi(t))} X^{x,i}(t)X^{y,i}(t)(X^{x,i}(t) + X^{y,i}(t)) + (\sigma(\xi(t)))^2 X^{y,i}(t)X^{x,i}(t) \right] dt \\ &\quad + 2\sigma(\xi(t))X^{y,i}(t)X^{x,i}(t) dB(t) \end{aligned}$$

$$\begin{aligned}
 &= \left[ (2r(\xi(t)) + (\sigma(\xi(t)))^2)X^{x,i}(t)X^{y,i}(t) \right. \\
 &\quad \left. - \frac{r(\xi(t))}{K(\xi(t))}X^{x,i}(t)X^{y,i}(t)(X^{x,i}(t) + X^{y,i}(t)) \right] dt \\
 &\quad + 2\sigma(\xi(t))X^{y,i}(t)X^{x,i}(t) dB(t).
 \end{aligned}$$

So

$$\begin{aligned}
 &X^{x,i}(t)X^{y,i}(t) \\
 &= xy + \int_0^t \left\{ (2r(\xi(s)) + (\sigma(\xi(s)))^2)X^{x,i}(s)X^{y,i}(s) \right. \\
 &\quad \left. - \frac{r(\xi(s))}{K(\xi(s))}X^{x,i}(s)X^{y,i}(s)(X^{x,i}(s) + X^{y,i}(s)) \right\} ds \\
 &\quad + \int_0^t 2\sigma(\xi(s))X^{y,i}(s)X^{x,i}(s) dB(s).
 \end{aligned}$$

Taking expectation we derive that

$$\begin{aligned}
 E(X^{x,i}(t)X^{y,i}(t)) &= xy + \int_0^t E \left\{ (2r(\xi(s)) + (\sigma(\xi(s)))^2)X^{x,i}(s)X^{y,i}(s) \right. \\
 &\quad \left. - \frac{r(\xi(s))}{K(\xi(s))}X^{x,i}(s)X^{y,i}(s)(X^{x,i}(s) + X^{y,i}(s)) \right\} ds,
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 (9) \quad \frac{dE[X^{x,i}(t)X^{y,i}(t)]}{dt} &= E \left\{ (2r(\xi(t)) + (\sigma(\xi(t)))^2)X^{x,i}(t)X^{y,i}(t) \right. \\
 &\quad \left. - \frac{r(\xi(t))}{K(\xi(t))}X^{x,i}(t)X^{y,i}(t)(X^{x,i}(t) + X^{y,i}(t)) \right\}
 \end{aligned}$$

with the initial value  $E(X^{x,i}(0)X^{y,i}(0)) = xy$ . Furthermore, Jensen’s inequality implies that

$$\begin{aligned}
 \frac{dE(X^{x,i}(t)X^{y,i}(t))}{dt} &= E \left\{ (2r(\xi(t)) + (\sigma(\xi(t)))^2)X^{x,i}(t)X^{y,i}(t) \right. \\
 &\quad \left. - \frac{r(\xi(t))}{K(\xi(t))}X^{x,i}(t)X^{y,i}(t)(X^{x,i}(t) + X^{y,i}(t)) \right\} \\
 &\leq E \left[ (2r^* + (\sigma^*)^2)X^{x,i}(t)X^{y,i}(t) - \frac{2r^*}{K^*(X^{x,i}(t)X^{y,i}(t))^{3/2}} \right] \\
 &= (2r^* + (\sigma^*)^2)E(X^{x,i}(t)X^{y,i}(t)) - \frac{2r^*}{K^*E(X^{x,i}(t)X^{y,i}(t))^{3/2}}
 \end{aligned}$$

$$\begin{aligned} &\leq (2r^* + (\sigma^*)^2)E(X^{x,i}(t)X^{y,i}(t)) - \frac{2r_*}{K^*(E(X^{x,i}(t)X^{y,i}(t)))^{3/2}} \\ &= \alpha E(X^{x,i}(t)X^{y,i}(t)) - \beta(E(X^{x,i}(t)X^{y,i}(t)))^{3/2}. \end{aligned}$$

Considering the ordinary differential equation

$$(10) \quad z' = \alpha z - \beta z^{3/2}, \quad z(0) = xy.$$

Clearly, Equation (10) is a Bernoulli type equation and its' solution is given by

$$z(t) = \left\{ \frac{1}{(1/\sqrt{xy} - \beta/\alpha) \exp(-(\alpha/2)t) + \beta/\alpha} \right\}^2, \quad t \geq 0.$$

Furthermore, we can show that

$$z(t) \leq \max\left(xy, \frac{\alpha^2}{\beta^2}\right).$$

The required assertion follows from the comparison principle for Equation (9) and Equation (10). □

**Lemma 2.10.**  $E|X^{x,i}(t)|^{1/2} \leq \{\max[x, (r^* K^*)/r_*]\}^{1/2}, \forall t \geq 0.$

Proof. It is easy to prove by Lemma 2.7 and Cauchy inequality. □

REMARK 5. Lemma 2.10 guarantees that for any  $(x, i) \in (0, +\infty) \times \mathbb{S}$ , the family of transition probabilities  $\{p(t, x, i, dy \times \{j\}) : t \geq 0\}$  is tight. That is, for any  $\varepsilon > 0$  there is a compact subset  $M = M(\varepsilon, x, i)$  of  $(0, +\infty)$  such that

$$p(t, x, i, M \times \mathbb{S}) \geq 1 - \varepsilon, \quad \forall t \geq 0.$$

**Lemma 2.11.** *Let  $M$  be a compact subset of  $(0, +\infty)$ . Then the solution of Equation (1) obeys*

$$(11) \quad \lim_{t \rightarrow +\infty} E|X^{x,i}(t) - X^{y,i}(t)|^{1/2} = 0 \quad \text{uniformly in } (x, y, i) \in M \times M \times \mathbb{S}.$$

Proof. By Cauchy inequality, Lemma 2.6 and Lemma 2.7, we can check that

$$\begin{aligned} E|X^{x,i}(t) - X^{y,i}(t)|^{1/2} &= E\left[(X^{x,i}(t)X^{y,i}(t))^{1/2} \left| \frac{1}{X^{x,i}(t)} - \frac{1}{X^{y,i}(t)} \right|^{1/2}\right] \\ &\leq (E(X^{x,i}(t)X^{y,i}(t)))^{1/2} \left(E\left| \frac{1}{X^{x,i}(t)} - \frac{1}{X^{y,i}(t)} \right|\right)^{1/2} \\ &\leq \left[\max\left(xy, \frac{\beta^2}{\alpha^2}\right)\right]^{1/2} \left|\frac{1}{x} - \frac{1}{y}\right| \exp\{-(r_* - (\sigma^*)^2)t\}. \end{aligned}$$

Letting  $t \rightarrow +\infty$  yields the assertion (11). □

REMARK 6. The explicit expression of the solution is not necessary in the proof of this paper although itself is interesting and meaningful. In fact, it is only used in the proof of Lemma 2.8. However, we can prove it with the following method: Let  $Y^{1/x,i}(t) = 1/X^{x,i}(t)$ ,  $x > 0$ , then by Itô formula one can get

$$dY^{1/x,i}(t) = \left[ (\sigma^2(\xi(t)) - r(\xi(t)))Y^{1/x,i}(t) + \frac{r(\xi(t))}{K(\xi(t))} \right] dt - \sigma(\xi(t))Y^{1/x,i}(t) dB(t).$$

So

$$Y^{1/x,i}(t) - Y^{1/y,i}(t) = \frac{1}{x} - \frac{1}{y} + \int_0^t (\sigma^2(\xi(s)) - r(\xi(s)))(Y^{1/x,i}(s) - Y^{1/y,i}(s)) ds - \int_0^t \sigma(\xi(s))(Y^{1/x,i}(s) - Y^{1/y,i}(s)) dB(s).$$

Taking expectation we get

$$E|Y^{1/x,i}(t) - Y^{1/y,i}(t)| \leq \left| \frac{1}{x} - \frac{1}{y} \right| + \int_0^t ((\sigma^*)^2 - r_*)E|Y^{1/x,i}(s) - Y^{1/y,i}(s)| ds.$$

Then Gronwall's inequality implies that

$$E|Y^{1/x,i}(t) - Y^{1/y,i}(t)| \leq \left| \frac{1}{x} - \frac{1}{y} \right| \exp\{-(r_* - (\sigma^*)^2)t\}.$$

Letting  $t \rightarrow \infty$  yields the result of Lemma 2.8.

Of course the explicit solution has another use, that is it can reveal the existence of the unique global positive solution. But this can be seen in [30] on p. 353 (namely, Theorem 10.2).

### 3. Stability in distribution

Based on the results of Section 2 of Equation (1), we are going to prove the main result of our paper: Equation (1) is stable in distribution. First we introduce some notations. Let  $y(t)$  denote the  $(0, +\infty) \times \mathbb{S}$ -valued process  $(X(t), \xi(t))$ . Then  $y(t)$  is a time homogeneous Markov process. Let  $p(t, x, i, dy \times \{j\})$  denote the transition probability density of the process  $y(t)$ . Let  $P(t, x, i, A \times D)$  denote the transition probability of of the event  $\{y(t) \in A \times D\}$  with the initial value  $y(0) = (x, i)$ , where  $A$  is a Borel measurable set of  $(0, +\infty)$ ,  $D$  is a subset of  $\mathbb{S}$ , and

$$P(t, x, i, A \times D) = \sum_{j \in D} \int_A p(t, x, i, dy \times \{j\}).$$

Denote by  $\mathcal{P}((0, \infty) \times \mathbb{S})$  the space of all probability measures on  $(0, \infty) \times \mathbb{S}$ . For  $p_1, p_2 \in \mathcal{P}((0, \infty) \times \mathbb{S})$  define the metric  $d_{\mathbb{L}}$  as follows:

$$d_{\mathbb{L}}(p_1, p_2) = \sup_{f \in \mathbb{L}} \left| \sum_{i=1}^N \int_{(0, \infty)} f(x, i) p_1(dx, i) - \sum_{i=1}^N \int_{(0, \infty)} f(x, i) p_2(dx, i) \right|,$$

where

$$\mathbb{L} = \{f : (0, \infty) \times \mathbb{S} \rightarrow \mathbb{R} : |f(x, i) - f(y, j)| \leq |x - y| + |i - j|, |f(\cdot \times \cdot)| \leq 1\}.$$

**DEFINITION 3.1.** The process  $y(t)$  is said to be asymptotically stable in distribution if there exists a probability measure  $\pi(\cdot \times \cdot)$  on  $(0, \infty) \times \mathbb{S}$  such that the transition probability density  $p(t, x, i, dy \times \{j\})$  of  $y(t)$  converges weakly to  $\pi(dy \times \{j\})$  as  $t \rightarrow \infty$  for every  $(x, i) \in (0, \infty) \times \mathbb{S}$ .

Equation (1) is asymptotically stable in distribution if the process  $y(t)$  is asymptotically stable in distribution.

**Theorem 3.1.** Equation (1) is asymptotically stable in distribution.

**Lemma 3.2.** For any compact subset  $M$  of  $(0, \infty)$ ,

$$\lim_{t \rightarrow \infty} d_{\mathbb{L}}(p(t, x, i, \cdot \times \cdot), p(t, y, j, \cdot \times \cdot)) = 0,$$

uniformly holds in  $x, y \in M$  and  $i, j \in \mathbb{S}$ .

**Lemma 3.3.** For any  $(x, i) \in (0, \infty) \times \mathbb{S}$ ,  $\{p(t, x, i, \cdot \times \cdot) : t \geq 0\}$  is Cauchy in the space  $\mathcal{P}((0, \infty) \times \mathbb{S})$  with metric  $d_{\mathbb{L}}$ .

The proofs of Lemma 3.2 and Lemma 3.3 repeated the proofs of [30] on pp.212–216, so we omit the proofs here.

**Proof of Theorem 3.1.** By Lemma 3.3,  $\{p(t, 0, 1, \cdot \times \cdot) : t \geq 0\}$  is Cauchy in the space  $\mathcal{P}((0, \infty) \times \mathbb{S})$  with metric  $d_{\mathbb{L}}$ . So there is a unique  $\pi(\cdot \times \cdot) \in \mathcal{P}((0, \infty) \times \mathbb{S})$  such that

$$\lim_{t \rightarrow \infty} d_{\mathbb{L}}(p(t, 0, 1, \cdot \times \cdot), \pi(\cdot \times \cdot)) = 0.$$

Furthermore, by Lemma 3.2, for any  $(x, i) \in (0, \infty) \times \mathbb{S}$ ,

$$\begin{aligned} & \lim_{t \rightarrow \infty} d_{\mathbb{L}}(p(t, x, i, \cdot \times \cdot), \pi(\cdot \times \cdot)) \\ (12) \quad & \leq \lim_{t \rightarrow \infty} [d_{\mathbb{L}}(p(t, 0, 1, \cdot \times \cdot), \pi(\cdot \times \cdot)) + d_{\mathbb{L}}(p(t, x, i, \cdot \times \cdot), p(t, 0, 1, \cdot \times \cdot))] \\ & = 0. \end{aligned}$$

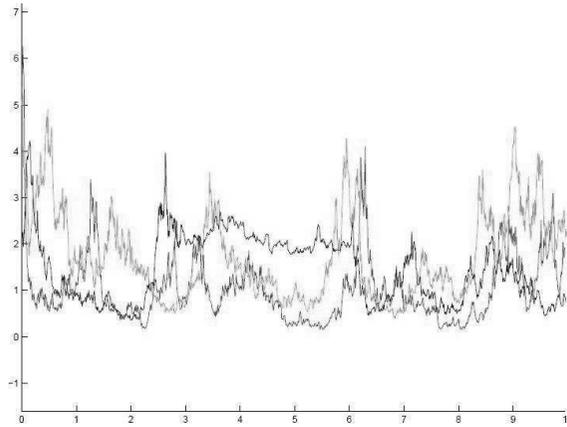


Fig. 1. Sample path of logistic equation (13), (14) and (15).

In view of the fact that the weak convergence of probability measures is a metric concept (see [14]), Equation (12) indicates that for any  $(x, i) \in (0, \infty) \times \mathbb{S}$ , the transition probabilities  $\{p(t, x, i, \cdot \times \cdot) : t \geq 0\}$  converge weakly to the probability measure  $\pi(\cdot \times \cdot) \in \mathcal{P}((0, \infty) \times \mathbb{S})$ . By the definition of asymptotically stable in distribution we complete the proof. □

REMARK 7. We note that the state space of Equation (1) is not the whole space but  $(0, +\infty)$ . Therefore the results of stability in distribution in [30] can not be used for Equation (1) directly, this results in some difficulties to deal with it.

EXAMPLE. Consider the following logistic system with regime switching

$$(13) \quad dX(t) = r(\xi(t))X(t) \left[ 1 - \frac{X(t)}{K(\xi(t))} \right] dt + \sigma(\xi(t))X(t) dB(t)$$

where  $B(t)$  is a standard Brownian motion,  $\xi(t)$  is a right-continuous Markov chain taking value in a two-state space  $\mathbb{S} = \{1, 2\}$ . So we may regard this system as the result of the following two equations switching from one to another according to the movement of the Markov chain

$$(14) \quad dX(t) = r(1)X(t) \left[ 1 - \frac{X(t)}{K(1)} \right] dt + \sigma(1)X(t) dB(t),$$

where  $r(1) = 2.5$ ,  $K(1) = 3$ ,  $\sigma(1) = 1$ ;

$$(15) \quad dX(t) = r(2)X(t) \left[ 1 - \frac{X(t)}{K(2)} \right] dt + \sigma(2)X(t) dB(t),$$

where  $r(2) = 3$ ,  $K(2) = 9$ ,  $\sigma(2) = 1.5$ . Let the the jump matrix of the Markov chain  $\xi(t)$  be

$$\begin{pmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{pmatrix},$$

We observe that the SDE (14) (green) and (15) (red) are all stable in distribution, after switching, the whole system (13) (blue) is sill stable in distribution. Because it is only a statistical result, it can not be accurate as the deterministic model.

**4. Generalized results**

The Gilpin–Ayala equation, based on ordinary differential equation, is usually denoted by

$$(16) \quad \dot{N}(t) = N(t)[a(t) - b(t)N^\theta(t)] \quad (\theta > 0)$$

where  $\theta$  is a parameter to modify the classical logistic model. When establishing mathematical model to describe the growth of the population, we can make the model more effective by adjusting the value of  $\theta > 0$ . Some detailed studies about this model may be found in [9, 10]. Jiang et al. [17] considered the randomized model (17) based on (16) with intensity  $\alpha^2(t)$

$$(17) \quad dN(t) = N(t)[a(t) - b(t)N^\theta(t)]dt + \alpha(t) dB(t) \quad (t \geq 0)$$

with the initial value  $N(0) = N_0$  ( $N_0$  is a positive random variable), where  $\theta > 0$  is an odd number,  $B(t)$  is the 1-dimensional standard Brownian motion. Here  $a(t)$ ,  $b(t)$  and  $\alpha(t)$  are bounded continuous functions defined on  $[0, \infty)$ ,  $a(t) > 0$ ,  $b(t) > 0$  and  $N_0$  is independent of  $B(t)$ .

For the stochastic Gilpin–Ayala equation with Markovian switching,

$$(18) \quad dX(t) = r(\xi(t))X(t) \left[ 1 - \frac{X^\theta(t)}{K(\xi(t))} \right] dt + \sigma(\xi(t))X(t) dB(t)$$

with initial conditions  $X(0) = x$  and  $\xi(0) = i$ , where  $\xi(t)$  be a right-continuous Markov chain taking values in a finite state space  $\mathbb{S} = \{1, 2, \dots, N\}$ ,  $\theta > 1$  and  $B(t)$  is the 1-dimensional standard Brownian motion. Let  $X(t)$  be a solution of Equation (18), then the generalised Itô formula implies that

$$(19) \quad dX^\theta(t) = X^\theta(t) \left[ \left( \theta r(\xi(t)) + \frac{\theta(\theta - 1)}{2} \sigma^2(\xi(t)) - \theta \frac{r(\xi(t))}{K(\xi(t))} X^\theta(t) \right) dt + \theta \sigma(\xi(t)) dB(t) \right]$$

Let  $Y(t) = X^\theta(t)$  then Equation (19) is equivalent to

$$(20) \quad dY(t) = Y(t) \left[ \left( \theta r(\xi(t)) + \frac{\theta(\theta - 1)}{2} \sigma^2(\xi(t)) - \theta \frac{r(\xi(t))}{K(\xi(t))} Y(t) \right) dt + \theta \sigma(\xi(t)) dB(t) \right]$$

with the initial values  $Y(0) = x^\theta > 0$  and  $\xi(0) = i \in \mathbb{S}$ .

For Equation (18), we assume that

(H<sub>1</sub>)<sub>θ</sub>  $r(i) > 0, K(i) > 0, i \in \mathbb{S};$

(H<sub>2</sub>)<sub>θ</sub>  $r_* - [(\theta + 1)/2](\sigma^*)^2 > 0.$

Note that Equation (20) is similar with Equation (1). So by the similar methods as in Section 2, we can give the analogous results for Equation (20) as Equation (1). Then we can directly derive the following results for Equation (18) and here omit the detailed proofs.

**Theorem 4.1.** *Suppose (H<sub>1</sub>)<sub>θ</sub> holds. Then for any initial conditions  $X(0) = x > 0$  and  $\xi(0) = i \in \mathbb{S}$ , Equation (15) has a unique continuous positive solution which is given by*

$$\begin{aligned} & (X^{x,i}(t))^\theta \\ &= \left\{ \frac{1}{x^\theta} \exp \left\{ \theta \left( - \int_0^t A(\xi(s)) ds - \int_0^t \sigma(\xi(s)) dB(s) \right) \right\} \right. \\ & \quad \left. + \theta \int_0^t \frac{r(\xi(s))}{K(\xi(s))} \exp \left\{ \theta \left( - \int_s^t A(\xi(\tau)) d\tau - \int_s^t \sigma(\xi(\tau)) dB(\tau) \right) \right\} ds \right\}^{-1}, \quad t \geq 0 \end{aligned}$$

where  $A(\cdot) = r(\cdot) - (1/2)\sigma^2(\cdot)$ .

**Lemma 4.2.** *Suppose (H<sub>1</sub>)<sub>θ</sub> and (H<sub>2</sub>)<sub>θ</sub> hold. Then*

$$\lim_{t \rightarrow +\infty} |(X^{x,i}(t))^\theta - (X^{y,i}(t))^\theta| = 0 \quad \text{for almost all } \omega \in \Omega,$$

where  $X^{x,i}(t)$  and  $X^{y,i}(t)$  are solutions of Equation (18) with initial values  $X^{x,i}(0) = x > 0, \xi(0) = i \in \mathbb{S}$  and  $X^{y,i}(0) = y > 0, \xi(0) = i \in \mathbb{S}$  respectively.

**Lemma 4.3.** *Suppose (H<sub>1</sub>)<sub>θ</sub> and (H<sub>2</sub>)<sub>θ</sub> hold. Then the solution  $X^{x,i}(t)$  with the initial conditions  $X(0) = x$  and  $\xi(0) = i$  has the property that*

$$\begin{aligned} \min \left[ x^\theta, \frac{[r_* - ((\theta + 1)/2)(\sigma^*)^2]K_*}{r_*} \right] &\leq E(X^{x,i}(t))^\theta \\ &\leq \max \left[ x^\theta, \frac{[r_* + ((\theta - 1)/2)(\sigma^*)^2]K^*}{r_*} \right]. \end{aligned}$$

**Lemma 4.4.** *Suppose  $(H_1)_\theta$  and  $(H_2)_\theta$  hold. Then*

$$\lim_{t \rightarrow \infty} E \left| \frac{1}{(X^{x,i}(t))^\theta} - \frac{1}{(X^{y,i}(t))^\theta} \right| = 0 \quad \text{uniformly in } (x, y, i) \in M \times M \times \mathbb{S},$$

for any compact subset  $M$  of  $(0, +\infty)$ .

**Lemma 4.5.** *The solution of Equation (18) obeys*

$$E((X^{x,i}(t))^\theta (X^{y,i}(t))^\theta) \leq \max \left[ (xy)^\theta, \frac{(2r^* + (2\theta - 1)(\sigma^*)^2)K^*}{4r_*^2} \right].$$

**Lemma 4.6.** *The solution of Equation (18) satisfies*

$$E|(x^{x,i}(t))^\theta|^{1/2} \leq \left\{ \max \left( x^\theta, \frac{(r^* + [(\theta - 1)/2](\sigma^*)^2)K^*}{r_*} \right) \right\}^{1/2}, \quad \forall t \geq 0.$$

**Lemma 4.7.** *The solution of Equation (18) has the property that*

$$\lim_{t \rightarrow +\infty} E|(X^{x,i}(t))^\theta - (X^{y,i}(t))^\theta|^{1/2} = 0 \quad \text{uniformly in } (x, y, i) \in M \times M \times \mathbb{S}.$$

By the same argument as in Section 3 we have the following theorem.

**Lemma 4.8.** *Suppose  $(H_1)_\theta$  and  $(H_2)_\theta$  hold. Then the solution of Equation (18) satisfies: the transition probability density  $\bar{p}(t, x, i, dy \times \{j\})$  of  $((X^{x,i}(t))^\theta, \xi_i(t))$  converges weakly to  $\pi(dy \times \{j\})$  as  $t \rightarrow \infty$  for every  $(x, i) \in (0, \infty) \times \mathbb{S}$ , where  $\pi(\cdot \times \cdot)$  is a probability measure on  $(0, \infty) \times \mathbb{S}$ .*

Let  $P(t, x, i, A \times D)$  denote the probability of the event  $\{(X^{x,i}(t), \xi_i(t)) \in A \times D\}$ ,  $A$  is a Borel measurable set of  $(0, +\infty)$ ,  $D$  is a subset of  $\mathbb{S}$ . Then

$$P(t, x, i, A \times D) = P(X^{x,i}(t) \leq y, \xi_i(t) \in D) = \sum_{j \in D} \int_0^y p(t, x, i, ds \times \{j\}),$$

where  $y \in (0, +\infty)$ .

**Theorem 4.9.** *Equation (18) is asymptotically stable in distribution.*

**Proof.** Lemma 4.8 implies that

$$\begin{aligned} P(X^{x,i}(t) \leq z, \xi_i(t) \in D) &= P((X^{x,i}(t))^\theta \leq z^\theta, \xi_i(t) \in D) \\ &= \sum_{j \in D} \int_0^{z^\theta} \bar{p}(t, x, i, ds \times \{j\}) \end{aligned}$$

converges weakly to  $\sum_{j \in D} \int_0^{z^\theta} \pi(ds \times \{j\})$  as  $t \rightarrow \infty$  since the transition probability density  $\bar{p}(t, x, i, dy \times \{j\})$  of  $((X^{x,i}(t))^\theta, \xi_i(t))$  converges weakly to  $\pi(dy \times \{j\})$  as  $t \rightarrow \infty$ . Let  $\bar{s} = s^{1/\theta}$ , then

$$\sum_{j \in D} \int_0^{z^\theta} \pi(ds \times \{j\}) = \sum_{j \in D} \int_0^z \theta(\bar{s})^{\theta-1} \pi(d\bar{s} \times \{j\}).$$

So  $P(X^{x^\theta,i}(t) \leq z, \xi_i(t) \in D)$  converges weakly to  $\sum_{j \in D} \int_0^z \theta(\bar{s})^{\theta-1} \pi(d\bar{s} \times \{j\})$ . So there is a probability measure  $\theta(t)^{\theta-1} \pi(dt \times \{j\})$  on  $(0, \infty) \times \mathbb{S}$  such that the transition probability density  $p(t, x, i, dy \times \{j\})$  of  $(X^{x,i}(t), \xi_i(t))$  converges weakly to  $\theta(t)^{\theta-1} \pi(dt \times \{j\})$ . This completes the proof of Theorem 4.9.  $\square$

### 5. Stochastic equilibrium solution

The classical deterministic logistic equation has a globally asymptotically stable equilibrium solution which has the great ecological significance. When considering the random factors, the classical deterministic logistic equation becomes the stochastic logistic equation. One may ask what does the equilibrium solution become? As far as we know, this is an important problem which has not been solved until now. In order to investigate this issue more better. We firstly need to give a reasonable definition of the stochastic equilibrium solution for the stochastic system. Here the stochastic equilibrium solution is a counterpart of the constant solution of the deterministic system. The deterministic constant solution  $x = x(t), t \geq 0$  should satisfy

- For any  $\bar{t} \geq 0, y(t) = x(t + \bar{t}) = x(t), t \geq 0$ .

That is, equilibrium solution should have the time translation invariance. So we have the following definition for stochastic equilibrium solution.

For stochastic differential equation with Markovian switching

$$(21) \quad dX(t) = f(t, \xi(t), X(t)) dt + g(t, \xi(t), X(t)) dB(t).$$

We give the definition as follows:

**DEFINITION 5.1.** The solution  $X = X(t), t \geq 0$  is said to be a (strictly) stochastic equilibrium solution of Equation (21) if for any  $\bar{t} \geq 0$ , the solution process  $Y(t) := X(t + \bar{t})$  and  $X(t)$  have the same finite-dimensional distributions.

That is to say that if the solution  $X(t)$  of Equation (21) is the strictly stochastic process, then  $X(t)$  is said to be a (strictly) stochastic equilibrium solution.

If besides, any of the solution  $Y(t)$  of Equation (21) converges to  $X(t)$  in distribution as  $t \rightarrow \infty$ , then  $X(t)$  is said to be globally asymptotically stable.

Similarly, if the solution  $X(t)$  of Equation (21) is a wide stochastic process, then  $X(t)$  is said to be a wide stochastic equilibrium solution of Equation (21).

DEFINITION 5.2. The solution of Equation (21) is said to be weakly unique if any two solutions  $Y(t)$  and  $X(t)$  have the same distribution provided  $Y(0)$  and  $X(0)$  have the same distribution.

Here we need extend the definition of Brownian motion  $B_t$  to  $-\infty < t < \infty$  as in [2] (p.547) or [37].

**Theorem 5.1.** *Suppose that Markov chain  $\xi(t)$  is irreducible and the solution of Equation (1) is defined on  $R$ . Then Equation (1) has a globally asymptotically stable stochastic equilibrium solution. Moreover, this equilibrium solution is weak unique.*

Proof. Let the asymptotic stationary distribution of Markov  $\xi(\cdot)$  be  $\bar{\pi}$  and  $X_n(t)$ ,  $t \geq -n$  be the solution of Equation (1) with the initial value  $X(-n) = 1$ , then by Remark 3 we have

$$X_n(t) = \left[ e^{-\int_{-n}^t A(\xi(s)) ds + \int_{-n}^t \sigma(\xi(s)) dB_s} + \int_{-n}^t \frac{r(\xi(s))}{K(\xi(s))} e^{-\int_s^t A(\xi(\tau)) d\tau + \int_s^t \sigma(\xi(\tau)) dB_\tau} ds \right]^{-1},$$

where  $A(\cdot) = r(\cdot) - \sigma^2(\cdot)/2$ . Particularly

$$\begin{aligned} X_n(0) &= \left[ e^{-\int_{-n}^0 A(\xi(s)) ds + \int_{-n}^0 \sigma(\xi(s)) dB_s} + \int_{-n}^0 \frac{r(\xi(s))}{K(\xi(s))} e^{-\int_s^0 A(\xi(\tau)) d\tau + \int_s^0 \sigma(\xi(\tau)) dB_\tau} ds \right]^{-1} \\ &= \left[ e^{-\int_0^n A(\bar{\xi}(s)) ds + \int_0^n \sigma(\bar{\xi}(s)) d\bar{B}_s} + \int_0^n \frac{r(\bar{\xi}(s))}{K(\bar{\xi}(s))} e^{-\int_s^n A(\bar{\xi}(\tau)) d\tau + \int_s^n \sigma(\bar{\xi}(\tau)) d\bar{B}_\tau} ds \right]^{-1} \\ &= X^{1, \bar{\pi}}(n), \end{aligned}$$

where

$$\bar{\xi}(t) = \xi(t - n), \quad \bar{B} = B_{t-n} - B_{-n}.$$

By Theorem 3.1, the sequence  $\{X_n(0), n = 0, 1, \dots\}$  of random variables is convergence in distribution. Denote  $\pi(\cdot)$  the limit of  $\{X_n(0), n = 0, 1, \dots\}$  in the sense of distribution. On the other hand, by assumption (H<sub>1</sub>) and (H<sub>2</sub>), it is evident that

$$\lim_{n \rightarrow \infty} e^{-\int_{-n}^0 A(\xi(s)) ds + \int_{-n}^0 \sigma(\xi(s)) dB_s} = 0.$$

So we get that

$$\lim_{n \rightarrow \infty} X_n(0) = \left[ \int_{-\infty}^0 \frac{r(\xi(s))}{K(\xi(s))} e^{-\int_s^0 A(\xi(\tau)) d\tau + \int_s^0 \sigma(\xi(\tau)) dB_\tau} ds \right]^{-1} := \pi^*.$$

Obviously,  $\pi^*$  has the same distribution as  $\pi(\cdot)$ . So we claim that the solution  $X^{\pi^*}(t)$ ,  $t \geq 0$  of Equation (1) with the initial value  $X(0) = \pi^*$  is the stochastic equilibrium

solution of Equation (1). Following we will prove this assertion. By Theorem 2.2, Remark 3 and Remark 4 we derive that

$$\begin{aligned} X^{\pi^*, \tilde{\pi}}(t) &= \left[ \frac{1}{\pi^*} e^{-\int_0^t A(\xi(\tau)) d\tau + \int_0^t \sigma(\xi(\tau)) dB_\tau} + \int_0^t \frac{r(\xi(s))}{K(\xi(s))} e^{-\int_s^t A(\xi(\tau)) d\tau + \int_s^t \sigma(\xi(\tau)) dB_\tau} ds \right]^{-1} \\ &= \left[ e^{-\int_0^t A(\xi(\tau)) d\tau + \int_0^t \sigma(\xi(\tau)) dB_\tau} \int_{-\infty}^0 \frac{r(\xi(s))}{K(\xi(s))} e^{-\int_s^0 A(\xi(\tau)) d\tau + \int_s^0 \sigma(\xi(\tau)) dB_\tau} ds \right. \\ &\quad \left. + \int_0^t \frac{r(\xi(s))}{K(\xi(s))} e^{-\int_s^t A(\xi(\tau)) d\tau + \int_s^t \sigma(\xi(\tau)) dB_\tau} ds \right]^{-1} \\ &:= [I_1 + I_2]^{-1}, \end{aligned}$$

where

$$\begin{aligned} I_1 &= e^{-\int_0^t A(\xi(\tau)) d\tau + \int_0^t \sigma(\xi(\tau)) dB_\tau} \int_{-\infty}^0 \frac{r(\xi(s))}{K(\xi(s))} e^{-\int_s^0 A(\xi(\tau)) d\tau + \int_s^0 \sigma(\xi(\tau)) dB_\tau} ds \\ &= e^{-\int_0^t A(\xi(\tau)) d\tau + \int_0^t \sigma(\xi(\tau)) dB_\tau} \int_{-\infty}^{-t} \frac{r(\xi(s+t))}{K(\xi(s+t))} e^{-\int_{s+t}^0 A(\xi(\tau)) d\tau + \int_{s+t}^0 \sigma(\xi(\tau)) dB_\tau} ds \\ &= \int_{-\infty}^{-t} \frac{r(\xi(s+t))}{K(\xi(s+t))} e^{-\int_{s+t}^t A(\xi(\tau)) d\tau + \int_{s+t}^t \sigma(\xi(\tau)) dB_\tau} ds \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^t \frac{r(\xi(s))}{K(\xi(s))} e^{-\int_s^t A(\xi(\tau)) d\tau + \int_s^t \sigma(\xi(\tau)) dB_\tau} ds \\ &= \int_{-t}^0 \frac{r(\xi(s+t))}{K(\xi(s+t))} e^{-\int_{s+t}^t A(\xi(\tau)) d\tau + \int_{s+t}^t \sigma(\xi(\tau)) dB_\tau} ds. \end{aligned}$$

So

$$(22) \quad X^{\pi^*, \tilde{\pi}}(t) = [I_1 + I_2]^{-1} = \left[ \int_{-\infty}^0 \frac{r(\xi(s+t))}{K(\xi(s+t))} e^{-\int_{s+t}^t A(\xi(\tau)) d\tau + \int_{s+t}^t \sigma(\xi(\tau)) dB_\tau} ds \right]^{-1}.$$

Therefore for any  $\bar{t} \geq 0$

$$\begin{aligned} (23) \quad X^{\pi^*, \tilde{\pi}}(t + \bar{t}) &= \left[ \int_{-\infty}^0 \frac{r(\xi(s+t+\bar{t}))}{K(\xi(s+t+\bar{t}))} e^{-\int_{s+t+\bar{t}}^{t+\bar{t}} A(\xi(\tau)) d\tau + \int_{s+t+\bar{t}}^{t+\bar{t}} \sigma(\xi(\tau)) dB_\tau} ds \right]^{-1} \\ &= \left[ \int_{-\infty}^0 \frac{r(\tilde{\xi}(s+t))}{K(\tilde{\xi}(s+t))} e^{-\int_{s+t}^t A(\tilde{\xi}(\tau)) d\tau + \int_{s+t}^t \sigma(\tilde{\xi}(\tau)) d\tilde{B}_\tau} ds \right]^{-1}, \end{aligned}$$

where

$$\tilde{\xi}(s) = \xi(s + \bar{t}), \quad \tilde{B}_s = B_{s+\bar{t}} - B_{\bar{t}}, \quad s \geq 0.$$

It can be seen from (22) and (23) that  $X^{\pi^*, \tilde{\pi}}(t + \bar{t})$ ,  $t \geq 0$  and  $X^{\pi^*, \tilde{\pi}}(t)$  have the same finite-dimensional distributions. By the definition of stochastic equilibrium solution,

$X^{\pi^*, \bar{\pi}}(t)$ ,  $t \geq 0$  is the stochastic equilibrium solution of Equation (1). By Theorem 3.1, it is globally asymptotically stable and weak unique. In the proof of this theorem we use the fact that: If two stochastic processes compounded by a number of stochastic processes which have the same finite-dimensional distribution in the same way, then the two stochastic process have the same finite-dimensional distribution.  $\square$

## 6. Discussion and conclusion

Owing to its theoretical and practical significance, the deterministic biological system and its generalization form have been extensively studied and many important results on the global dynamics of solutions have been given. As mentioned above, logistic equation and Gilpin–Ayala equation are the most important basically models to describe the growth of the the single-species. When not taking into the random factors, one can establish the deterministic mathematical models to predict the long-time behavior of the biological population accurately. For the classical autonomic logistic equation, its equilibrium solution called the carrying capacity, is a simple mathematical expression. But in the real world, the population systems are often subject to various types of environmental noises, so it is important to discover whether such noises affect the results or not. Therefore such problem is advanced naturally: as the basically models that describe the growth of the single-species, whether or not stochastic logistic equation and stochastic Gilpin–Ayala equation with Markovian switching can give some prediction of the long run behavior of the biological population in random environments? Obviously, this is an important and meaningful problem in the theoretical and practical biology.

The results in our paper demonstrate that, the population growth law which is described by the stochastic logistic equation or stochastic Gilpin–Ayala equation with Markovian switching is different from the deterministic model because of the impact of randomness or uncertainty. Because of the random factors, we can not get the accurate result as the deterministic system. But we can give the statistics law of the state from the mathematical model. The main result of our paper is that the model in our paper is asymptotically stable in distribution. This is an important result in the ecological theory. Its importance can be seen from the following example, when we throw a coin randomly, although we can not know it appears positive or negative for the next time, we know exactly that the probability it appears positive or negative is  $1/2$ .

Theoretically, the important problem is how to get this statistical law of the long-time behavior of this model. When only white noise is taken into account, it has been showed that there exists a Gamma distribution which can be determined by the corresponding model (see [32]). When the system is perturbed by the color, whether there are white noises or not, what is this statistics law and how to get it from the model are not known up to now. In our paper, we haven't solved this problem absolutely. But we proved the existence and the uniqueness of this statistics law. So our work provided a necessary foundation for solving this problem ulteriorly and this will be one of the most important contributions in the theory of the biology.

We firstly proved that the distribution of the solution of Equation (1) convergence in distribution and then proved that Equation (1) had a unique globally asymptotically stable stochastic equilibrium solution. The important ecological significance is: Because of the randomness of the system, we can not exactly know the state of the system in a specific moment. But when time is long enough, we can know from our results that the distribution of the state of the system will be gradually stable in the sense of statistical.

The result of stochastic equilibrium solution in Section 5 is new and it is an important result for the stochastic system. It needs to be further studied for the general non-linear stochastic differential equation.

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