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QUALITATIVE PROPERTIES FOR PERONA–MALIK TYPE EQUATIONS

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Abstract

In this paper, we show qualitative results for the solutions of forward-backward parabolic equations, where the forward or backward behaviour depends on the gradient of the solution.

1. Introduction and motivations

In [4], [5], [6] the following parabolic initial boundary value problem is considered:

$$u_t = \varphi''(u_x)u_{xx}, \quad \text{in} \quad (-1, 1) \times [0, T), \tag{1}$$

$$u_x(-1, t) = u_x(1, t) = 0, \quad \forall t \in [0, T), \tag{2}$$

$$u(x, 0) = u_0(x), \quad \forall x \in (-1, 1), \tag{3}$$

where φ is a nonlinear function of class C^2 such that $\varphi'(0) = 0$. Without any constraint on the sign of φ'', (1), (2), (3) is a forward-backward problem, well-posed if $\varphi''(u_x) > 0$ and ill-posed if $\varphi''(u_x) < 0$. Backward problems are in general very difficult to solve. Indeed, the heat equation $u_t = k\,\Delta u$ with $k < 0$ is an ill-posed problem for $t > 0$ in most functional classes, including C^∞ or analytic functions. On the other hand, forward-backward equations appear frequently in many important physical models and this justifies the interest around them. If we choose $\varphi(\sigma) = (1/2)\ln(1 + \sigma^2)$ in (1), we obtain the classical Perona–Malik equation:

$$u_t = \frac{1 - u_x^2}{(1 + u_x^2)^2} u_{xx}. \tag{4}$$

The forward or backward behaviour of (4) is determined respectively by the conditions $|u_x| < 1$ or $|u_x| > 1$. This equation was introduced in 1990 by the engineers P. Perona and J. Malik in [17], as a tool to analyze edge detection and image segmentation problems in computer vision; see also [10] for the connections between the Perona–Malik equation and the Mumford–Shah functional.

The Perona–Malik problem represents a paradox which has not yet been solved, despite the intense research devoted to it in recent years. Indeed, from the analytical

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point of view, there does not exist an acceptable definition of weak solution. Only the following facts are known about classical solutions ([6], [11]):

- if u_0 is *subsonic*, i.e. $|u_{0x}(x)| < 1$, $\forall x \in (-1, 1)$, then the problem has a unique global classical solution, which remains subsonic for all times;
- if u_0 is *transonic*, i.e. $\varphi''(u_{0x})$ changes sign in $[-1, 1]$ and the classical solution exist, it can not be global ([11]). It is proved ([6]) that for a transonic solution we have necessarily

$$T \leq 4 \int_{-1}^{1} \ln(1 + u_{0x}^2(x))\, dx.$$

The paradox lies in the fact that numerical schemes for the equation do not show significant instabilities, despite the expected ill-posedness ([3]); recall also the explicit construction by Höllig [8] of a piecewise affine function φ for which the equation $u_t = [\varphi(u_x)]_x$ has an infinite number of local Lipschitz continuous solutions. In order to explain this situation, Kichenassamy in [12] proposed a notion of *generalized solution* for the initial value problem related to the Perona–Malik equation, for infinitely differentiable data. This definition closely follows the features of numerical solutions (see [14] for more details), but the assumptions on the initial datum are unrealistic in concrete signal processing problems. Therefore, the research on the Perona–Malik equation is still open.

A first result of the present paper concerns the following equations:

$$u_t = \frac{1 - u_x^2}{(1 + u_x^2)^2} u_{xx} \pm F(u_x). \tag{5}$$

This kind of equations appears e.g. in [1] and describes nonlinear diffusion phenomena in hydrology. We consider the initial boundary value problem

$$u_t = \varphi''(u_x)u_{xx} - G(u), \quad \text{in} \quad (-1, 1) \times [0, T), \tag{6}$$

$$u_x(-1, t) = u_x(1, t) = 0, \quad \forall t \in [0, T), \tag{7}$$

$$u(x, 0) = u_0(x), \quad \forall x \in (-1, 1), \tag{8}$$

which is the formal gradient flow associated to the functional

$$H_\varphi(u) = \int_{-1}^{1} [\varphi(u_x) + \Phi(u)]\, dx, \quad \Phi(s) = \int_0^s G(\tau)\, d\tau,$$

and its generalization to an n-dimensional open domain Ω

$$u_t = \operatorname{div}\left[\varphi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right] - G(u), \quad \text{in} \quad \Omega \times [0, T), \tag{9}$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on} \quad \partial\Omega \times [0, T), \tag{10}$$

$$u(x, 0) = u_0(x), \quad \forall x \in \Omega. \tag{11}$$

Equation (9) appears in a reaction-diffusion model proposed by Cohen and Murray in [2], which describes a diffusive mechanism of a population extending the classical Fickian diffusion. We also mention [7], [9], [15], [16], [18], where the following quasilinear diffusive equation is considered:

$$u_t = \Delta\varphi(u).$$

The choices $\varphi(u) = u/(K+u^2)$ or $\varphi(u) = ue^{-u}$ make the equation a forward-backward one. In order to obtain a well-posed problem, some possible regularizations are proposed, for example the well known Cahn–Hilliard equation $u_t = \Delta\varphi(u) - \epsilon\,\Delta^2 u$ or the Sobolev equation $u_t = \Delta\varphi(u) + \epsilon\,\Delta u_t$. Therefore, we consider also the regularized problem:

$$u_t = \varphi''(u_x)u_{xx} - \epsilon u_{xxxx}, \quad \text{in} \quad (-1,1)\times[0,T), \tag{12}$$

$$u_x(-1,t) = u_x(1,t) = 0, \quad \forall t \in [0,T), \tag{13}$$

$$u_{xxx}(-1,t) = u_{xxx}(1,t) = 0, \quad \forall t \in [0,T), \tag{14}$$

$$u(x,0) = u_0(x), \quad \forall x \in (-1,1). \tag{15}$$

Finally, we show the global nonexistence of solutions for the initial boundary value problem related to the equations:

$$u_t = \varphi''(u_x)u_{xx} \pm u^2, \quad \text{in} \quad (-1,1)\times(0,+\infty).$$

2. Preliminaries

For the sake of completeness, we expose a result of existence and uniqueness of the solution for subsonic initial data (due to Kawohl and Kutev [11]).

Theorem 2.1. *Let us consider the following problem*:

$$u_t = \varphi''(u_x)u_{xx}, \quad \text{in} \quad (\sigma_1,\sigma_2)\times[0,T), \tag{16}$$

$$u_x(\sigma_1,t) = u_x(\sigma_2,t) = 0, \quad \forall t \in [0,T), \tag{17}$$

$$u(x,0) = u_0(x), \quad \forall x \in (\sigma_1,\sigma_2), \tag{18}$$

where φ is a nonlinear function of class C^2 such that $\varphi'(0) = 0$, convex for $|u_x| < K$ and concave for $|u_x| > K$. Let us assume that $u_0 \in C^{2,\alpha}([\sigma_1,\sigma_2])$, $\alpha \in (0,1)$, satisfies $|u_{0x}| < K$ in $[\sigma_1,\sigma_2]$. Then, there exists $T > 0$ such that the problem (16), (17), (18) *admits a unique classical solution u. Furthermore, u, u_t, u_x, u_{xx} are Hölder-continuous with exponent α in x and $\alpha/2$ in t.*

Proof. In order to prove the existence, we modify the function φ outside the interval $[\sigma_1, \sigma_2]$ in order to obtain a uniformly parabolic problem:

$$\psi(\sigma) = \begin{cases} \varphi(\sigma), & \text{if} \quad \sigma \in [\sigma_1, \sigma_2], \\ \varphi(\sigma_2) + (\sigma - \sigma_2)\varphi'(\sigma_2) + \dfrac{(\sigma - \sigma_2)^2}{2}\varphi''(\sigma_2), & \text{if} \quad \sigma > \sigma_2, \\ \varphi(\sigma_1) + (\sigma - \sigma_1)\varphi'(\sigma_1) + \dfrac{(\sigma - \sigma_1)^2}{2}\varphi''(\sigma_1), & \text{if} \quad \sigma < \sigma_1. \end{cases}$$

In this way, $\psi'' \geq c > 0$ and ψ' is increasing. Then, the new problem:

$$w_t = \psi''(w_x)w_{xx}, \quad \text{in} \quad (\sigma_1, \sigma_2) \times [0, T), \tag{19}$$

$$w_x(\sigma_1, t) = w_x(\sigma_2, t) = 0, \quad \forall t \in [0, T), \tag{20}$$

$$w(x, 0) = u_0(x), \quad \forall x \in (\sigma_1, \sigma_2), \tag{21}$$

admits a classical solution w ([13]). Let us set $w_x = v$; the problem (19), (20), (21) becomes:

$$v_t = [\psi''(v)v_x]_x, \quad \text{in} \quad (\sigma_1, \sigma_2) \times [0, T),$$
$$v(\sigma_1, t) = v(\sigma_2, t) = 0, \quad \forall t \in [0, T),$$
$$v(x, 0) = u_{0x}(x), \quad \forall x \in (\sigma_1, \sigma_2).$$

We can apply the weak maximum principle, therefore:

$$\sup_{(\sigma_1,\sigma_2)\times[0,T)} |w_x(x, t)| = \sup_{(\sigma_1,\sigma_2)\times[0,T)} |v(x, t)| \leq \sup_{(\sigma_1,\sigma_2)} |u_{0x}(x)| < K.$$

The function w is not only the solution of the problem (19), (20), (21), but also of (16), (17), (18). Concerning the uniqueness, let us assume that u and v are two different solutions of the problem. We can write:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\sigma_1}^{\sigma_2}(u - v)^2\, dx &= \int_{\sigma_1}^{\sigma_2}(u - v)(u_t - v_t)\, dx \\ &= \int_{\sigma_1}^{\sigma_2}(u - v)[\psi''(u_x)u_{xx} - \psi''(v_x)v_{xx}]\, dx \\ &= \int_{\sigma_1}^{\sigma_2}(u - v)[\psi'(u_x) - \psi'(v_x)]_x\, dx \\ &= -\int_{\sigma_1}^{\sigma_2}(u_x - v_x)[\psi'(u_x) - \psi'(v_x)]\, dx, \end{aligned}$$

due to the boundary condition. Since ψ is monotonically increasing, we have:

$$(u_x - v_x)[\psi'(u_x) - \psi'(v_x)] \geq 0.$$

Therefore:

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}(u-v)^2\,dx \leq 0 \Longrightarrow u = v. \qquad \square$$

3. Statements

3.1. The one-dimensional case. The following result is an extension of Theorem 3.1 in [6].

Theorem 3.1. *Let us suppose that $u\colon \mathbb{R}^2 \to \mathbb{R}$ is a C^1 solution of* (5), *where $F\colon \mathbb{R} \to \mathbb{R}$ satisfies the hypothesis $F(1/\sigma) = F(\sigma)/\sigma$. Let us assume that there exists a positive constant C such that*:

$$u_x(x,t) \geq C, \quad \forall (x,t) \in \mathbb{R}^2. \tag{22}$$

Then, the function $w\colon \mathbb{R}^2 \to \mathbb{R}$, uniquely defined by:

$$u(w(x,t), -t) = x, \quad \forall (x,t) \in \mathbb{R}^2, \tag{23}$$

is a C^1 solution of (5).

The following results are an extension of the classical estimates and maximum-minimum principles obtained by Ghisi and Gobbino in [5].

Theorem 3.2. *Let us suppose that $u\colon [-1,1] \times [0,T) \to \mathbb{R}$ is a C^2 solution of* (6), (7), (8), *with G a non-negative function. Let φ be an even non-negative function of class C^2, such that $\varphi'(0) = 0$. Then, the function $t \to H_\varphi(u(x,t))$ is non-increasing and, for every $t_1, t_2 \in [0,T)$ such that $t_1 \leq t_2$, we obtain*:

$$H_\varphi(u(x,t_1)) - H_\varphi(u(x,t_2)) = \int_{t_1}^{t_2}\int_{-1}^{1}[u_t]^2\,dx\,dt. \tag{24}$$

Additionally:

$$\|u(x,t_1) - u(x,t_2)\|_{L^2((-1,1))} \leq \{H_\varphi(u_0)\}^{1/2} \cdot |t_1 - t_2|^{1/2}. \tag{25}$$

Let us assume that φ satisfies also $\sigma \cdot \varphi'(\sigma) \geq 0$, $\forall \sigma \in \mathbb{R}$. Then, for every $(x,t) \in [-1,1] \times [0,T)$, we have:

$$u(x,t) \leq \max\{u_0(x)\colon x \in [-1,1]\}; \tag{26}$$

$$u(x,t) \geq \min\{u_0(x)\colon x \in [-1,1]\}. \tag{27}$$

Furthermore, for every $p \in [1, +\infty]$ *and every* $t \in [0, T)$:

$$\|u(x, t)\|_{L^p((-1,1))} \le \|u_0(x)\|_{L^p((-1,1))}. \tag{28}$$

Corollary 3.3. *Let* $u\colon [-1, 1] \times [0, T) \to \mathbb{R}$ *be a* C^2 *solution of* (6), (7), (8), *with* G *a non-negative function. Let us assume that* φ *is an even non-negative function of class* C^2, *such that* $\sigma \cdot \varphi'(\sigma) \ge 0$, $\forall \sigma \in \mathbb{R}$ *and* $\varphi'(0) = 0$.

- *If* $u_0(x) \ge 0$, $\forall x \in [-1, 1]$, *then* $u(x, t) \ge 0$, $\forall (x, t) \in [-1, 1] \times [0, T)$.
- *If* u_0 *is bounded, then* u *is bounded.*

Corollary 3.4. *With the same hypothesis of* Corollary 3.3 *and assuming* $u_0 \ge 0$, *the* L^2*-norm of* $u(\,\cdot\,, t)$ *is monotonically decreasing for* $t \ge 0$.

Theorem 3.5. *Let* $u\colon [-1, 1] \times [0, T) \to \mathbb{R}$ *be a* C^2 *solution of* (6), (7), (8), *with* G *a non-negative increasing function of class* C^1. *Let us suppose that* φ *satisfies all the hypothesis of the* Theorem 3.2 *and additionally* φ *is convex in a neighborhood of* 0. *Then, for every* $t \in [0, T)$:

$$\|u_x(x, t)\|_{L^1((-1,1))} \le \|u_{0x}(x)\|_{L^1((-1,1))}.$$

Theorem 3.6. *Let us suppose that* $u\colon [-1, 1] \times [0, T) \to \mathbb{R}$ *is a* C^2 *solution of* (6), (7), (8), *with* G *a non-negative increasing function of class* C^1. *Let us set*:

$$M(t) := \max\{u_x(x, t)\colon x \in [-1, 1]\}, \quad m(t) := \min\{u_x(x, t)\colon x \in [-1, 1]\}.$$

If φ *is convex in an interval* $[\sigma_1, \sigma_2]$, *with* $\sigma_1 < \sigma_2$, *then*:

$$M(0) \le \sigma_2 \Rightarrow M(t) \le \sigma_2, \quad m(0) \ge \sigma_1 \Rightarrow m(t) \ge \sigma_1, \quad \forall t \in [0, T). \tag{29}$$

Similarly, if φ *is concave in the interval* $[\sigma_1, \sigma_2]$, *then*:

$$M(0) \ge \sigma_2 \Rightarrow M(t) \ge \sigma_2, \quad m(0) \le \sigma_1 \Rightarrow m(t) \le \sigma_1, \quad \forall t \in [0, T). \tag{30}$$

Theorem 3.7. *Let us suppose that* $u\colon [-1, 1] \times [0, T) \to \mathbb{R}$ *be a* C^5 *solution of* (12), (13), (14), (15) *and let* φ *be an even non-negative function of class* C^2, *such that* $\sigma \cdot \varphi'(\sigma) \ge 0$, $\forall \sigma \in \mathbb{R}$ *and* $\varphi'(0) = 0$. *We have the following results.*

- *For every* $(x, t) \in [-1, 1] \times [0, T)$:

$$u(x, t) \le \max\{u_0(x)\colon x \in [-1, 1]\}; \tag{31}$$

$$u(x, t) \ge \min\{u_0(x)\colon x \in [-1, 1]\}. \tag{32}$$

- *For every* $t \in [0, T)$:

$$\|u(x, t)\|_{L^\infty((-1,1))} \le \|u_0(x)\|_{L^\infty((-1,1))}. \tag{33}$$

3.2. The *n*-dimensional case. In the following, Ω is an open set of $\mathbb{R}^n$ with piecewise C^1 boundary and exterior normal n. The results are an extension of Theorems 2.14 and 2.15 of [5].

Theorem 3.8. *Let* $u\colon \bar\Omega \times [0, T) \to \mathbb{R}$ *be a* C^2 *solution of* (9), (10), (11), *with* G *a non-negative function*; *let us suppose that* φ *is an even non-negative function of class* C^2, *such that* $\sigma \cdot \varphi'(\sigma) \geq 0$, $\forall \sigma \in \mathbb{R}$ *and* $\varphi'(0) = 0$. *We have the following results.*

- *For every* $(x, t) \in \bar\Omega \times [0, T)$:

$$u(x, t) \leq \max\{u_0(x)\colon x \in \bar\Omega\}; \tag{34}$$

$$u(x, t) \geq \min\{u_0(x)\colon x \in \bar\Omega\}. \tag{35}$$

- *For every* $p \in [1, +\infty]$ *and for every* $t \in [0, T)$:

$$\|u(x, t)\|_{L^p(\Omega)} \leq \|u_0(x)\|_{L^p(\Omega)}. \tag{36}$$

In the n-dimensional case, the total variation estimate of u is true only in the case of radial solutions.

Theorem 3.9. *Let* Ω *be an open disc in* $\mathbb{R}^n$. *Let* $u\colon \bar\Omega \times [0, T) \to \mathbb{R}$ *be a* C^2 *radial solution of* (9), (10), (11), *with* G *an increasing non-negative function. Let us suppose that* φ *is an even non-negative function of class* C^2, *such that* $\sigma \cdot \varphi'(\sigma) \geq 0$, $\forall \sigma \in \mathbb{R}$ *and* $\varphi'(0) = 0$, *convex in a neighborhood of* 0. *Then*:

$$\|\nabla u(x, t)\|_{L^1(\Omega)} \leq \|\nabla u_0(x)\|_{L^1(\Omega)}, \quad \forall t \in [0, T). \tag{37}$$

4. Remarks

1. As it is well known, the Cauchy problem for the backward heat equation is ill-posed in the backward direction $t < 0$ in most function spaces. However, we can prove a result of local existence provided the initial data are in $\gamma^{1/2}$, the space of Gevrey functions with exponent $1/2$; this can be easily seen e.g. using the Fourier transform. In this case, also the solution belongs to $\gamma^{1/2}$. However, this Gevrey class is not stable for products and it seems that a similar result in the nonlinear case can not be true.
2. We notice that all the above results apply to the classical Perona–Malik equation, which corresponds to the choice $\varphi(\sigma) = (1/2)\log(1 + \sigma^2)$. Another interesting case corresponds to the choice $\varphi(\sigma) = (\sigma^2 - 1)^2$, which appears in several applications including nonlinear elasticity and phase transition models.
3. An explicit example of function F satisfying the assumptions in Theorem 3.1 is $F(\sigma) = c\sqrt{\sigma}$, with $c \in \mathbb{R}$.
4. Theorem 3.6 is our main result. It asserts that if the datum $u_x(x,0)$ takes on values inside the interval of convexity of φ, then $u_x(x, t)$ assumes values in this interval for all times; that is, if the initial datum is subsonic, the solution remains subsonic. In the

same way, if $u_x(x,0)$ is outside the interval where φ is concave, then $u_x(x,t)$ remains in this interval for all times. This means that if u_0 is transonic, also the solution will be transonic.
5. Corollary 3.4 shows the L^2-stability of the solution. Corollary 3.3 and 3.4 for the problem (12), (13), (14), (15) are still true.

5. Proofs

In this Section, we give the proofs of our results. Let us prove Theorem 3.1.

Proof of Theorem 3.1. We show that the function w, defined by (23), satisfies the equation (5). Hypothesis (22) assures that the function $x \to u(x,-t)$ is bijective for every $t \in \mathbb{R}$ and its inverse function $x \to w(x,t)$ is of class C^1. By derivation of (23), we obtain:

$$\begin{gathered}
u_x(w(x,t),-t)\cdot w_x(x,t) = 1 \Rightarrow w_x(x,t) = \frac{1}{u_x(w(x,t),-t)};\\
-u_t(w(x,t),-t) + u_x(w(x,t),-t)\cdot w_t(x,t) = 0\\
\Rightarrow w_t(x,t) = \frac{u_t(w(x,t),-t)}{u_x(w(x,t),-t)}.
\end{gathered}$$

Assumption (22) guarantees that the denominators of w_x and w_t are not zero, therefore their expressions are well defined. Thus:

$$\begin{aligned}
\varphi''(w_x(x,t))w_{xx} = [\varphi'(w_x(x,t))]_x &= \left[\frac{w_x(x,t)}{1+w_x^2(x,t)}\right]_x\\
&= \left[\frac{u_x(w(x,t),-t)}{1+u_x^2(w(x,t),-t)}\right]_x = [\varphi'(u_x(w(x,t),-t))]_x.
\end{aligned}$$

Hence:

$$[\varphi'(w_x)]_x \pm F(w_x) = [\varphi'(u_x(w,-t))]_x \cdot w_x \pm F(w_x).$$

From the hypothesis $F(1/\sigma) = F(\sigma)/\sigma$ and from the expressions of w_x and w_t, it follows:

$$\begin{aligned}
[\varphi'(w_x)]_x \pm F(w_x) &= [\varphi'(u_x)]_x \cdot \frac{1}{u_x} \pm \frac{F(u_x)}{u_x}\\
&= \frac{[\varphi'(u_x)]_x \pm F(u_x)}{u_x} = \frac{u_t}{u_x} = w_t,
\end{aligned}$$

that is w solves (5). □

In order to establish Theorem 3.2, we need the following result.

Proposition 5.1. *Let* $u\colon [-1,1]\times[0,T)\to\mathbb{R}$ *be a solution of class* C^2 *of* (6), (7), (8), *with* $G\colon\mathbb{R}\to\mathbb{R}$ *of class* C^1. *Let us suppose that* φ *and* ψ *are functions of class* C^2 *satisfying the following hypothesis*:

- φ *is an even non-negative function such that* $\varphi'(0)=0$;
- $\psi'(0)=0$.

Then:

$$\frac{d}{dt}\int_{-1}^{1}\psi(u)\,dx = -\int_{-1}^{1}\psi''(u)\cdot u_x\cdot\varphi'(u_x)\,dx - \int_{-1}^{1}\psi'(u)\cdot G(u)\,dx, \tag{38}$$

$$\begin{aligned}&\frac{d}{dt}\int_{-1}^{1}[\psi(u_x)+\Phi(u)]\,dx\\ &= \frac{d}{dt}\int_{-1}^{1}\Phi(u)\,dx - \int_{-1}^{1}\psi''(u_x)\cdot u_{xx}\cdot[\varphi''(u_x)u_{xx}-G(u)]\,dx,\end{aligned} \tag{39}$$

Proof. The identities (38) and (39) follow immediately from integration by parts.

$$\begin{aligned}\frac{d}{dt}\int_{-1}^{1}\psi(u)\,dx &= \int_{-1}^{1}\psi'(u)u_t\,dx = \int_{-1}^{1}\psi'(u)[\varphi''(u_x)u_{xx}-G(u)]\,dx\\ &= [\psi'(u)\varphi'(u_x)]\big|_{x=-1}^{x=1} - \int_{-1}^{1}\psi''(u)u_x\varphi'(u_x)\,dx - \int_{-1}^{1}\psi'(u)G(u)\,dx.\end{aligned}$$

From (7) and from the assumption on φ', the boundary terms are zero and the identity (38) follows. Analogously, since $u\in C^2$, we have:

$$\begin{aligned}&\frac{d}{dt}\int_{-1}^{1}[\psi(u_x)+\Phi(u)]\,dx\\ &= \int_{-1}^{1}\psi'(u_x)u_{xt}\,dx + \frac{d}{dt}\int_{-1}^{1}\Phi(u)\,dx\\ &= \int_{-1}^{1}\psi'(u_x)[\varphi''(u_x)u_{xx}-G(u)]_x\,dx + \frac{d}{dt}\int_{-1}^{1}\Phi(u)\,dx\\ &= \int_{-1}^{1}\psi'(u_x)[\varphi''(u_x)u_{xx}]_x\,dx - \int_{-1}^{1}\psi'(u_x)[G(u)]_x\,dx + \frac{d}{dt}\int_{-1}^{1}\Phi(u)\,dx.\end{aligned}$$

Let us integrate by parts the previous integrals; using the boundary condition (7) and the hypothesis, we have:

$$\int_{-1}^{1}\psi'(u_x)[\varphi''(u_x)u_{xx}]_x\,dx = -\int_{-1}^{1}[\psi''(u_x)u_{xx}][\varphi''(u_x)u_{xx}]\,dx \tag{40}$$

and

$$\int_{-1}^{1}\psi'(u_x)[G(u)]_x\,dx = -\int_{-1}^{1}[\psi''(u_x)u_{xx}]G(u)\,dx. \tag{41}$$

Therefore, by (40) and (41), we obtain:

$$\begin{aligned}&\frac{d}{dt}\int_{-1}^{1}[\psi(u_x)+\Phi(u)]\,dx\\&=-\int_{-1}^{1}\psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx+\int_{-1}^{1}\psi''(u_x)u_{xx}G(u)\,dx+\frac{d}{dt}\int_{-1}^{1}\Phi(u)\,dx,\end{aligned}$$

which proves (39). □

The proof of Theorem 3.2 will be obtained by a suitable choice of the function ψ in the previous identities.

Proof of Theorem 3.2. Let us show (24). With the choice $\psi=\varphi$ in (39), we have:

$$\begin{aligned}\frac{d}{dt}\int_{-1}^{1}[\varphi(u_x)+\Phi(u)]\,dx&=-\int_{-1}^{1}\varphi''(u_x)u_{xx}u_t\,dx+\frac{d}{dt}\int_{-1}^{1}\Phi(u)\,dx\\&=-\int_{-1}^{1}[u_t+G(u)]u_t\,dx+\frac{d}{dt}\int_{-1}^{1}\Phi(u)\,dx\\&=-\int_{-1}^{1}[u_t]^2\,dx-\int_{-1}^{1}G(u)u_t\,dx+\frac{d}{dt}\int_{-1}^{1}\Phi(u)\,dx\\&=-\int_{-1}^{1}[u_t]^2\,dx.\end{aligned}$$

Therefore:

$$\begin{aligned}&H_\varphi(u(x,t_1))-H_\varphi(u(x,t_2))\\&=\int_{-1}^{1}[\varphi(u_x(x,t_1))+\Phi(u(x,t_1))]\,dx-\int_{-1}^{1}[\varphi(u_x(x,t_2))+\Phi(u(x,t_2))]\,dx\\&=\int_{t_1}^{t_2}\int_{-1}^{1}[u_t]^2\,dx\,dt.\end{aligned}$$

Using (24) and the Hölder inequality we obtain that:

$$\begin{aligned}\|u(x,t_1)-u(x,t_2)\|_{L^2((-1,1))}&=\left\|\int_{t_1}^{t_2}u_t(x,\tau)\,d\tau\right\|_{L^2((-1,1))}\\&\le\int_{t_1}^{t_2}\|u_t(x,\tau)\|_{L^2((-1,1))}\,d\tau\\&\le\left(\int_{t_1}^{t_2}\|u_t(x,\tau)\|^2_{L^2((-1,1))}\,d\tau\right)^{1/2}\cdot|t_1-t_2|^{1/2}\\&=[H_\varphi(u(x,t_1))-H_\varphi(u(x,t_2))]^{1/2}\cdot|t_1-t_2|^{1/2}\\&\le\{H_\varphi(u_0)\}^{1/2}\cdot|t_1-t_2|^{1/2}.\end{aligned}$$

that is (24).

Now we prove the maximum and minimum principles, respectively (26) and (27) for the solution. Let us set:

$$\psi(\sigma) = \begin{cases} 0, & \text{if} \quad \sigma \le K, \\ (\sigma - K)^4, & \text{if} \quad \sigma > K, \end{cases} \quad K = \max\{u_0(x) \colon x \in [-1, 1]\}.$$

We notice that ψ is a non-negative convex function of class C^2. Moreover:

$$\int_{-1}^{1} \psi(u(x,0))\,dx = \int_{-1}^{1} \psi(u_0(x))\,dx = 0, \quad \forall t \in [0, T),$$

since $u_0(x) \le K$. Applying (38) with this choice of ψ and having $\psi''(\sigma)\cdot\sigma\cdot\varphi'(\sigma) \ge 0$, $\psi'(\sigma)\cdot G(\sigma) \ge 0$, the function

$$t \to \int_{-1}^{1} \psi(u(x,t))\,dx$$

is non-negative and non-increasing. Then:

$$\int_{-1}^{1} \psi(u(x,t))\,dx = 0 \Rightarrow \psi(u(x,t)) = 0, \ \forall t \in [0,T) \Rightarrow u(x,t) \le K$$

which proves (26). The proof of (27) may be easily obtained defining

$$\psi(\sigma) = \begin{cases} 0, & \text{if} \quad \sigma \ge K, \\ (\sigma - K)^4, & \text{if} \quad \sigma < K, \end{cases} \quad K = \min\{u_0(x) \colon x \in [-1, 1]\}$$

and arguing as above.

Let us prove the L^p estimates on u.

We remark that, in the case when $p = +\infty$, the estimate follows immediately from (26) and (27), which ensure the boundedness of u. If $p < +\infty$, let us set the function:

$$\psi(\sigma) = |\sigma|^p.$$

For $p \in [2, +\infty)$, ψ is a non-negative convex function of class C^2. Thus, by (38), the function $t \to \int_{-1}^{1} |u(x,t)|^p\,dx$ is non-increasing in time and (28) follows. In the case when $p \in [1,2)$, ψ is not of class C^2, although it is convex. However, we can approximate it with a family $\{\psi_\epsilon(\sigma)\}_{\epsilon>0}$ of functions satisfying the following conditions:

- for every $\epsilon > 0$, ψ_ϵ is a convex function of class C^2, such that $\psi_\epsilon'(\sigma)\cdot G(\sigma) \ge 0$ and $\psi_\epsilon'(0) = 0$;
- $\psi_\epsilon(\sigma) \to |\sigma|$ uniformly on $\mathbb{R}$, for $\epsilon \to 0^+$.

Applying (38) to ψ_ϵ and using the assumptions, we obtain that the function $t \to \int_{-1}^{1} \psi_\epsilon(u(x,t))$ is non-increasing, thus:

$$\int_{-1}^{1} \psi_\epsilon(u(x,t))\,dx \le \int_{-1}^{1} \psi_\epsilon(u_0(x))\,dx.$$

Letting $\epsilon \to 0^+$, (28) is established also in this case. $\square$

The proof of Corollary 3.3 is a direct consequence of Theorem 3.2. In particular, the non-negativity of the solution u yields immediately from (27) while the boundedness is obtained applying (28) with $p = +\infty$.

Concerning the proof of Corollary 3.4, from the Corollary 3.3 we obtain that the solution u and then $u \cdot G(u)$ are non-negative.

$$\begin{aligned} \frac{d}{dt}\left[\frac{1}{2}|u|^2_{L^2((-1,1))}\right] &= \frac{1}{2}\frac{d}{dt}\int_{-1}^{1} u^2\,dx = \int_{-1}^{1} uu_t\,dx \\ &= \int_{-1}^{1} u[\varphi''(u_x)u_{xx} - G(u)]\,dx. \end{aligned}$$

Integrating by parts, we obtain:

$$\frac{d}{dt}\left[\frac{1}{2}|u|^2_{L^2((-1,1))}\right] = -\int_{-1}^{1} \varphi'(u_x)u_x\,dx - \int_{-1}^{1} uG(u)\,dx.$$

The integrands at the right-hand side of the previous are both non-negative, therefore:

$$\frac{d}{dt}\left[\frac{1}{2}|u|^2_{L^2((-1,1))}\right] \le 0.$$

We prove Theorem 3.5.

Proof of Theorem 3.5. Let us consider a family $\{\psi_\epsilon(\sigma)\}_{\epsilon>0}$ of functions satisfying the following conditions:

- for every $\epsilon > 0$, ψ_ϵ is a convex function of class C^2, such that $\sigma \cdot \psi'_\epsilon(\sigma) \ge 0$, $\psi'_\epsilon(0) = 0$ and $\psi''_\epsilon(\sigma) = 0$, if $|\sigma| \ge \sigma_0$;
- $\psi_\epsilon(\sigma) \to |\sigma|$ uniformly on $\mathbb{R}$, per $\epsilon \to 0^+$.

From (39), for every $t \in [0, T)$:

$$\begin{aligned} \frac{d}{dt}\int_{-1}^{1} \psi(u_x)\,dx = &-\int_{-1}^{1} \psi''(u_x)\varphi''(u_x)u_{xx}^2\,dx \\ &+\int_{-1}^{1} \psi''(u_x)u_{xx}G(u)\,dx. \end{aligned} \tag{42}$$

Integrating by parts the last term and applying this identity to ψ_ϵ, we obtain:

$$\begin{aligned} \frac{d}{dt}\int_{-1}^{1} \psi_\epsilon(u_x)\,dx = &-\int_{-1}^{1} \psi''_\epsilon(u_x)\varphi''(u_x)u_{xx}^2\,dx \\ &-\int_{-1}^{1} \psi'_\epsilon(u_x)G'(u)u_x\,dx. \end{aligned} \tag{43}$$

The integrands in the right hand side of (43) are non-negative, therefore $t \to \int_{-1}^{1} \psi_\epsilon(u_x(x,t))\,dx$ is non-increasing:

$$\int_{-1}^{1} \psi_\epsilon(u_x(x,t))\,dx \le \int_{-1}^{1} \psi_\epsilon(u_{0x}(x))\,dx.$$

The conclusion follows passing to the limit as $\epsilon \to 0^+$. $\square$

Let us establish Theorem 3.6.

Proof of Theorem 3.6. We prove (29); the boundary condition (7) implies $M(0) \ge 0$, thus $\sigma_2 \ge 0$. We analyze separately the cases $\sigma_2 > 0$ and $\sigma_2 = 0$. For the first one, we introduce the function:

$$\psi(\sigma) = \begin{cases} \sigma - \sigma_2, & \text{if} \quad \sigma > \sigma_2, \\ 0, & \text{if} \quad \sigma \le \sigma_2. \end{cases}$$

Then, we consider a family $\{\psi_\epsilon(\sigma)\}_{\epsilon>0}$ satisfying:
1. for ever $\epsilon > 0$, ψ_ϵ is a convex function of class C^2, with $\sigma \cdot \psi_\epsilon'(\sigma) \ge 0$ and $\psi_\epsilon''(\sigma) = 0$ if $\sigma \notin [\sigma_1, \sigma_2]$;
2. $\psi_\epsilon \to \psi$ uniformly on $\mathbb{R}$, for $\epsilon \to 0^+$;
3. $\psi_\epsilon'(0) = 0$, for every $\epsilon > 0$ (here we use the assumption $\sigma_2 > 0$).

Applying (42) to ψ_ϵ and letting $\epsilon \to 0^+$, we have:

$$\frac{d}{dt}\int_{-1}^{1} \psi(u_x)\,dx \le 0.$$

Hence:

$$0 \le \int_{-1}^{1} \psi(u_x(x,t))\,dx \le \int_{-1}^{1} \psi(u_{0x}(x))\,dx.$$

Since $M(0) \le \sigma_2$, we find:

$$\int_{-1}^{1} \psi(u_{0x}(x)) = 0 \Longrightarrow \psi(u_x(x,t)) = 0,$$

therefore $u_x(x,t) \le \sigma_2$ for every $(x,t) \in [-1,1]\times[0,T)$. If $\sigma_2 = 0$, we need to distinguish two cases: $\varphi''(0) = 0$ and $\varphi''(0) > 0$. In the first one, we can find a family $\{\psi_\epsilon(\sigma)\}_{\epsilon>0}$ that satisfies the first two requests of the previous case, but not the third one and we can conclude in the same way of the case $\sigma_2 > 0$. In the second case, there exists an interval $[\sigma_1, \bar\sigma]$, with $\bar\sigma > 0$, where $\varphi''(\sigma) > 0$. Arguing as in the case $\sigma_2 > 0$, we have $M(t) \le \bar\sigma$, $\forall t \in [0,T)$ and the conclusion follows by taking the limit for $\bar\sigma \to 0$. The other implication in (29) can be analogously proved defining:

$$\psi(\sigma) = \begin{cases} \sigma_1 - \sigma, & \text{if} \quad \sigma < \sigma_1, \\ 0, & \text{if} \quad \sigma \geq \sigma_1. \end{cases}$$

We prove (30); the boundary condition (7) gives immediately the conclusion in the case $\sigma_2 \leq 0$. Thus, we can suppose $\sigma_2 > 0$ and $\epsilon > 0$, in such a way that $\sigma_2 - \epsilon \geq \max\{0, \sigma_1\}$. Let us choose the following function:

$$\psi_\epsilon(\sigma) = \begin{cases} 0, & \text{if} \quad \sigma \leq \sigma_2 - \epsilon, \\ k(\sigma), & \text{if} \quad \sigma > \sigma_2 - \epsilon, \end{cases}$$

where k is a positive function for which ψ_ϵ is convex, of class C^2, such $\sigma \cdot \psi_\epsilon'(\sigma) \leq 0$ and $\psi_\epsilon''(\sigma) = 0$, $\forall \sigma \geq \sigma_2$. With these requests, using (42) we find:

$$\frac{d}{dt} \int_{-1}^{1} \psi_\epsilon(u_x)\, dx \geq 0.$$

Hence:

$$\int_{-1}^{1} \psi_\epsilon(u_x(x, t))\, dx \geq \int_{-1}^{1} \psi_\epsilon(u_{0x}(x))\, dx.$$

The hypothesis $M(0) \geq \sigma_2$ yields

$$\int_{-1}^{1} \psi(u_{0x}(x))\, dx > 0,$$

that is $M(t) \geq \sigma_2 - \epsilon$, $\forall t \in [0, T)$. We can conclude by calculating the limit for $\epsilon \to 0^+$. In a similar way we can prove the other implication. $\square$

In order to prove Theorem 3.7, we need the following result.

Proposition 5.2. *Let* $u\colon [-1, 1] \times [0, T) \to \mathbb{R}$ *be a* C^5 *solution of* (12), (13), (14), (15); *let us assume that* φ *and* ψ *are functions of class* C^2 *and* C^3 *respectively, satisfying the next requests*:

- φ *is an even non-negative function such that* $\varphi'(0) = 0$;
- $\psi'(0) = 0$.

Then:

(44)

$$\frac{d}{dt} \int_{-1}^{1} \psi(u)\, dx = -\int_{-1}^{1} \psi''(u) \cdot u_x \cdot \varphi'(u_x)\, dx \\ - \epsilon \int_{-1}^{1} \psi'''(u) \cdot u_x^2 \cdot u_{xx}\, dx - \epsilon \int_{-1}^{1} \psi''(u) \cdot u_{xx}^2\, dx, \quad \forall t \in [0, T).$$

Proof. We can argue as in the proof of (38). Integration by parts yields:

$$\begin{aligned}\frac{d}{dt}\int_{-1}^{1}\psi(u)\,dx &= \int_{-1}^{1}\psi'(u)u_t\,dx = \int_{-1}^{1}\psi'(u)[\varphi''(u_x)u_{xx} - \epsilon u_{xxxx}]\,dx\\ &= \psi'(u)\varphi'(u_x)|_{x=-1}^{x=1} - \int_{-1}^{1}\psi''(u)u_x\varphi'(u_x)\,dx\\ &\quad - \epsilon[\psi'(u)u_{xxx}]|_{x=-1}^{x=1} + \epsilon\int_{-1}^{1}\psi''(u)u_x u_{xxx}\,dx.\end{aligned}$$

The boundary terms are null due to (13) and (14), then:

$$\frac{d}{dt}\int_{-1}^{1}\psi(u)\,dx = -\int_{-1}^{1}\psi''(u)u_x\varphi'(u_x)\,dx + \epsilon\int_{-1}^{1}\psi''(u)u_x u_{xxx}\,dx. \tag{45}$$

Let us integrate by parts the second integral in the right-hand side of (45):

$$\begin{aligned}\int_{-1}^{1}\psi''(u)u_x u_{xxx}\,dx &= [\psi''(u)u_x u_{xx}]_{x=-1}^{x=1} - \int_{-1}^{1}[\psi''(u)u_x]_x u_{xx}\,dx\\ &= -\int_{-1}^{1}[\psi'''(u)u_x^2 + \psi''(u)u_{xx}]u_{xx}\,dx\\ &= -\int_{-1}^{1}\psi'''(u)u_x^2 u_{xx}\,dx - \int_{-1}^{1}\psi''(u)u_{xx}^2\,dx.\end{aligned}$$

Substituting the previous expression in (45), we obtain (44). □

We prove Theorem 3.7.

Proof of Theorem 3.7. We choose

$$\psi(\sigma) = \begin{cases} 0, & \text{if } \sigma \le K,\\ (\sigma - K)^2, & \text{if } \sigma > K,\end{cases} \qquad K = \max\{u_0(x) : x \in [-1, 1]\}.$$

The C^2 function ψ is convex and non-negative. Using (44) and arguing as in the proof of Theorem 3.2, we have (31). From (31) and (32), we have (33). □

As in the one-dimensional case, we need the following result in order to prove Theorem 3.8.

Proposition 5.3. *Let* $u : \bar{\Omega} \times [0, T) \to \mathbb{R}$ *be a* C^2 *solution of* (9), (10), (11), *with* $G : \mathbb{R} \to \mathbb{R}$; *let us assume that* φ *and* ψ *are functions of class* C^2, *such that*:

- *φ is an even non-negative function such that $\varphi'(0) = 0$;*
- $\psi'(0) = 0$.

Then:

$$\begin{aligned} \frac{d}{dt}\int_\Omega \psi(u)\,dx = &-\int_\Omega \psi''(u)|\nabla u|\varphi'(|\nabla u|)\,dx \\ &-\int_\Omega \psi'(u)G(u)\,dx, \quad \forall t \in [0,T). \end{aligned} \tag{46}$$

The identity (46) can be easily established as (38) via integration by parts and Theorem 3.8 can be proved arguing as in Theorem 3.2.

Concerning the n-dimensional problem, we can take $\Omega := \{x \in \mathbb{R}^n : |x| < R\}$; if we consider $u = u(r,t)$ with $r = |x|$, the problem (9), (10), (11) becomes:

$$u_t = \varphi''(u_r)u_{rr} + (n-1)\frac{\varphi'(u_r)}{r} - G(u), \quad \forall (r,t) \in (0,R)\times[0,T); \tag{47}$$

$$u_r(0,t) = u_r(R,t) = 0, \quad \forall t \in [0,T). \tag{48}$$

In order to prove Theorem 3.9, we need the following result.

Proposition 5.4. *Let u: $[0,R]\times[0,T)\to\mathbb{R}$ be a C^2 solution of* (47), (48), *with G of class C^1; let us assume that φ and ψ are C^2 functions verifying the next requests*:

- *φ is an even non-negative function of class C^2 satisfying $\varphi'(0) = 0$;*
- $\psi'(0) = 0$.

Then, $\forall t \in [0,T)$:

$$\begin{aligned} \frac{d}{dt}\int_0^R r^{n-1}\psi(u_r)\,dr = &-\int_0^R r^{n-1}\psi''(u_r)\varphi''(u_r)u_{rr}^2\,dr \\ &-(n-1)\int_0^R \psi'(u_r)\varphi'(u_r)r^{n-3}\,dr \\ &-\int_0^R r^{n-1}\psi'(u_r)G'(u)u_r\,dr, \quad \forall t \in [0,T). \end{aligned} \tag{49}$$

Proof. Integrating by parts, we have:

$$\begin{aligned} &\frac{d}{dt}\int_0^R r^{n-1}\psi(u_r)\,dr \\ &= \int_0^R r^{n-1}\psi'(u_r)u_{rt}\,dr \\ &= \int_0^R r^{n-1}\psi'(u_r)\left[\varphi''(u_r)u_{rr} + (n-1)\frac{\varphi'(u_r)}{r} - G(u)\right]_r dr \end{aligned}$$

$$
\begin{aligned}
&= \int_0^R r^{n-1}\psi'(u_r)[\varphi''(u_r)u_{rr}]_r\,dr + (n-1)\int_0^R r^{n-1}\psi'(u_r)\left[\frac{\varphi'(u_r)}{r}\right]_r dr\\
&\quad - \int_0^R r^{n-1}\psi'(u_r)[G(u)]_r\,dr\\
&= -\int_0^R [r^{n-1}\psi'(u_r)]_r\varphi''(u_r)u_{rr}\,dr + (n-1)\int_0^R r^{n-1}\psi'(u_r)\left(\frac{\varphi'(u_r)}{r}\right)_r dr\\
&\quad - \int_0^R r^{n-1}\psi'(u_r)[G(u)]_r\,dr\\
&= -\int_0^R [(n-1)r^{n-2}\psi'(u_r) + r^{n-1}\psi''(u_r)u_{rr}]\varphi''(u_r)u_{rr}\,dr\\
&\quad + (n-1)\int_0^R r^{n-1}\psi'(u_r)\frac{r\varphi''(u_r)u_{rr} - \varphi'(u_r)}{r^2}\,dr - \int_0^R r^{n-1}\psi'(u_r)G'(u)u_r\,dr.
\end{aligned}
$$

and (49) follows. □

Indicating with ω_{n-1} the $(n-1)$-dimensional Hausdorff measure of the unit sphere in $\mathbb{R}^n$, we have:

$$
\|\nabla u(x,t)\|_{L^1(\Omega)} = \omega_{n-1}\int_0^R r^{n-1}|u_r(r,t)|\,dr.
$$

We establish Theorem 3.9.

Proof of Theorem 3.9. We can apply (49) to a family $\{\psi_\epsilon(\sigma)\}_{\epsilon>0}$ of functions verifying the following requests:

- for every $\epsilon > 0$, ψ_ϵ is a convex function of class C^2, such that $\psi'_\epsilon(0) = 0$ and $\psi''_\epsilon(\sigma) = 0$, if $|\sigma| \geq \sigma_0$;
- $\psi_\epsilon(\sigma) \to |\sigma|$ uniformly on $\mathbb{R}$, for $\epsilon \to 0^+$;
- $\sigma \cdot \psi'_\epsilon(\sigma) \geq 0$, for every $\sigma \in \mathbb{R}$ and every $\epsilon > 0$.

In this way, $\psi''_\epsilon(\sigma)\varphi''(\sigma) \geq 0$ and $\psi'_\epsilon(\sigma)\varphi'(\sigma) \geq 0$ for every $\sigma \in \mathbb{R}$. Therefore the function $t \to \int_0^R r^{n-1}\psi_\epsilon(u_r)\,dr$ is non-increasing:

$$
\int_0^R r^{n-1}\psi_\epsilon(u_r(r,t))\,dr \leq \int_0^R r^{n-1}\psi_\epsilon(u_{0r}(r))\,dr
$$

and the theorem follows as $\epsilon \to 0^+$. □

6. An example of global non-existence of solution

In this section, we show examples of global non-existence of solution for problems related to the equations:

$$
u_t = \varphi''(u_x)u_{xx} \pm u^2,
$$

where φ is a C^2 non-negative even function, such that $\varphi'(0) = 0$, with suitable hypothesis on the initial datum.

EXAMPLE 6.1. Let us consider the following problem:

$$u_t = \varphi''(u_x)u_{xx} + u^2, \quad \text{in} \quad (-1, 1) \times (0, +\infty); \tag{50}$$

$$u_x(-1, t) = u_x(1, t) = 0, \quad \forall t > 0; \tag{51}$$

$$u(x, 0) = u_0(x), \quad \forall x \in (-1, 1). \tag{52}$$

We prove that it does not exist a global solution of (50), (51), (52) if u_0 is a non-negative function satisfying:

$$\int_{-1}^{1} u_0(x)\, dx > 0.$$

Indeed, integrating the equation with respect to x, we have:

$$\begin{aligned} \frac{d}{dt} \int_{-1}^{1} u\, dx &= \int_{-1}^{1} u^2\, dx + \int_{-1}^{1} \varphi''(u_x)u_{xx}\, dx \\ &= \int_{-1}^{1} u^2\, dx + \varphi'(u_x(1, t)) - \varphi'(u_x(-1, t)) = \int_{-1}^{1} u^2\, dx, \end{aligned}$$

due to the boundary condition (51). Hölder inequality yields:

$$\int_{-1}^{1} u\, dx \leq |[-1, 1]|^{1/2} \cdot \left[\int_{-1}^{1} u^2\, dx\right]^{1/2} \Rightarrow \left(\int_{-1}^{1} u\, dx\right)^2 \leq 2 \int_{-1}^{1} u^2\, dx.$$

Hence

$$\frac{d}{dt} \int_{-1}^{1} u\, dx \geq \frac{1}{2}\left(\int_{-1}^{1} u\, dx\right)^2.$$

Solving the previous inequality with initial datum (52), we find:

$$\int_{-1}^{1} u\, dx \geq \frac{2 \int_{-1}^{1} u_0(x)\, dx}{2 - t \int_{-1}^{1} u_0(x)\, dx}.$$

Therefore, letting $t \to t^* = 2\Big/\left(\int_{-1}^{1} u_0(x)\, dx\right)$, we have:

$$\int_{-1}^{1} u(x, t)\, dx \to +\infty,$$

i.e. the solution blows up in a finite time.

EXAMPLE 6.2. We consider the equation

$$u_t = \varphi''(u_x)u_{xx} - u^2,$$

with the conditions (51) and (52) and $\int_{-1}^{1} u_0(x)\,dx < 0$. Arguing as above, we obtain

$$\int_{-1}^{1} u\,dx \leq \frac{2\int_{-1}^{1} u_0(x)\,dx}{2 + t\int_{-1}^{1} u_0(x)\,dx}.$$

Thus, $t \to t^* = -2\big/\big(\int_{-1}^{1} u_0(x)\,dx\big)$ yields $\int_{-1}^{1} u(x,t)\,dx \to -\infty$.

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References

[1] M. Badii: *Periodic solutions for a doubly nonlinear diffusion equation in hydrology*; in Progress in Partial Differential Equations **1**, Pitman Res. Notes Math. Ser. **383**, Longman, Harlow, 28–39, 1998.

[2] D.S. Cohen and J.D. Murray: *A generalized diffusion model for growth and dispersal in a population*, J. Math. Biol. **12** (1981), 237–249.

[3] S. Esedoḡlu: *Stability properties of the Perona–Malik scheme*, SIAM J. Numer. Anal. **44** (2006), 1297–1313.

[4] M. Ghisi and M. Gobbino: *A class of local classical solutions for the one-dimensional Perona–Malik equation*, Trans. Amer. Math. Soc. **361** (2009), 6429–6446.

[5] M. Ghisi and M. Gobbino: *Gradient estimates for the Perona–Malik equation*, Math. Ann. **337** (2007), 557–590.

[6] M. Gobbino: *Entire solutions of the one-dimensional Perona–Malik equation*, Comm. Partial Differential Equations **32** (2007), 719–743.

[7] V.N. Grebenev: *Interfacial phenomenon for one-dimensional equation of forward-backward parabolic type*, Ann. Mat. Pura Appl. (4) **171** (1996), 379–394.

[8] K. Höllig: *Existence of infinitely many solutions for a forward backward heat equation*, Trans. Amer. Math. Soc. **278** (1983), 299–316.

[9] D. Horstmann, K.J. Painter and H.G. Othmer: *Aggregation under local reinforcement*: *from lattice to continuum*, European J. Appl. Math. **15** (2004), 546–576.

[10] B. Kawohl: *From Mumford–Shah to Perona–Malik in image processing*, Math. Methods Appl. Sci. **27** (2004), 1803–1814.

[11] B. Kawohl and N. Kutev: *Maximum and comparison principle for one-dimensional anisotropic diffusion*, Math. Ann. **311** (1998), 107–123.

[12] S. Kichenassamy: *The Perona–Malik paradox*, SIAM J. Appl. Math. **57** (1997), 1328–1342.

[13] O.A. Ladyženskaja, V.A. Solonnikov and N.N. Ural'ceva: Linear and Quasilinear Equations of Parabolic Type, Nauka, Moscow, 1967.

[14] R. March, M. Rosati, A. Schiaffino: *BV solutions of the one-dimensional Perona–Malik equation*, (2008), preprint.

[15] C. Mascia, A. Terracina and A. Tesei: *Two-phase entropy solutions of a forward-backward parabolic equation*, Arch. Ration. Mech. Anal. **194** (2009), 887–925.

[16] A. Novick-Cohen and R.L. Pego: *Stable patterns in a viscous diffusion equation*, Trans. Amer. Math. Soc. **324** (1991), 331–351.

[17] P. Perona, J. Malik: *Scale space and edge detection using anisotropic diffusion*, IEEE Trans. Pattern Anal. Mach. Intell. **12** (1990), 629–639.

[18] P.I. Plotnikov: *Forward-backward parabolic equations and hysteresis*, J. Math. Sci. (New York) **93** (1999), 747–766.

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