Wada, K.
Osaka J. Math.
48 (2011), 895–912

BLOCKS OF CATEGORY $\mathcal{O}$ FOR RATIONAL CHEREDNIK ALGEBRAS AND OF CYCLOTOMIC HECKE ALGEBRAS OF TYPE $G(r, p, n)$

KENTARO WADA

(Received February 12, 2010, revised April 8, 2010)

Abstract

We classify blocks of category $\mathcal{O}$ for rational Cherednik algebras and of cyclotomic Hecke algebras of type $G(r, p, n)$ by using the "residue equivalence" for multi-partitions.

0. Introduction

Let V be a finite dimensional vector space over $\mathbb{C}$, and $W \subset \mathrm{GL}(V)$ be a finite complex reflection group. The rational Cherednik algebra $\mathcal{H} = \mathcal{H}(W)$ over $\mathbb{C}$ associated to W was introduced by [7]. It is known that the category $\mathcal{O}$ of $\mathcal{H}$ is a highest weight category with standard modules $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$, where Λ^+ is an index set of pairwise non-isomorphic simple W-modules over $\mathbb{C}$ ([11], [9]). Let $\mathscr{H} = \mathscr{H}(W)$ be the cyclotomic Hecke algebra associated to W with appropriate parameters. Let KZ: $\mathcal{O} \to \mathscr{H}$-mod be the Knizhnik–Zamolodchikov functor defined in [9]. It is known that $\mathcal{O}$ is a quasi-hereditary cover (highest weight cover) of $\mathscr{H}$ in the sense of [21]. Put $S(\lambda) = \mathrm{KZ}(\Delta(\lambda))$. We see that there exists a one-to-one correspondence between the blocks of $\mathcal{O}$ and of $\mathscr{H}$ thanks to the double centralizer property. Moreover, we see that the classification of blocks of $\mathcal{O}$ and of $\mathscr{H}$ is given by the linkage classes on $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$ or on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ (see §1 for details). Hence, in order to classify the blocks of $\mathcal{O}$ and of $\mathscr{H}$, it is enough to determine the linkage classes on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$.

In the case where W is a complex reflection group of type $G(r, 1, n)$, $\mathscr{H}$ is also called the Ariki–Koike algebra. In this case, Λ^+ is the set of r-partitions of size n, which we denote by $\mathcal{P}_{n,r}$. Then the linkage classes on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ are given by the equivalence relation "$\sim_R$", the so called residue equivalence, on $\mathcal{P}_{n,r}$ by [17]. (Note that the Specht module S^λ ($\lambda \in \Lambda^+$) considered in [17] does not coincide with $S(\lambda)$ in general. However, one sees that the linkage classes on $\{S^\lambda \mid \lambda \in \Lambda^+\}$ coincide with the linkage classes on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$. See §3.)

2000 Mathematics Subject Classification. Primary 20C08; Secondary 20C20, 05E10.
This research was supported by JSPS Research Fellowships for Young Scientists.

Our purpose is to classify the blocks of $\mathcal{O}$ and of $\mathscr{H}$ in the case where W is a complex reflection group of type $G(r, p, n)$. As seen in the above, we should determine the linkage classes on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$. Let $W^\dagger$ be the complex reflection group of type $G(r, 1, n)$, and we denote by adding the superscript † for objects of type $G(r, 1, n)$. It is known that W is a normal subgroup of $W^\dagger$ with the index p, and that $\mathscr{H}$ is a subalgebra of $\mathscr{H}^\dagger$. An index set Λ^+ of pairwise non-isomorphic simple W-modules over $\mathbb{C}$ (thus, Λ^+ is also an index set of standard modules of $\mathcal{O}$) is given as the equivalence classes of $\mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ under a certain equivalence relation "$\sim_*$" on $\mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ (see 4.3 for details). We denote by $\lambda\langle i\rangle \in \Lambda^+$ the equivalence class containing $(\lambda, \bar{i}) \in \mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$.

Some relations between representations of $\mathscr{H}$ and of $\mathscr{H}^\dagger$ have been studied in [2], [8], [12], [13], [14], [15] and [16] by using the Clifford theory. Combining these results with some fundamental properties of quasi-hereditary covers, and with the classification of blocks of $\mathscr{H}^\dagger$ by using the residue equivalence "$\sim_R$", we give the classification of the blocks of $\mathcal{O}$ and of $\mathscr{H}$ by using a certain equivalence relation "$\approx$" on $\mathcal{P}_{n,r}$ as follows.

Let "$\approx$" be the equivalence relation on $\mathcal{P}_{n,r}$ defined by $\lambda \approx \mu$ if $\lambda \sim_R \mu[j]$ for some $j \in \mathbb{Z}$, where $\mu[j] \in \mathcal{P}_{n,r}$ is obtained from $\mu \in \mathcal{P}_{n,r}$ by a certain permutation of components of μ (see 4.3 for the precise definition of $\mu[j]$). Put $\Gamma = \{\lambda \in \mathcal{P}_{n,r} \mid \lambda \not\sim_R \mu$ for any $\mu \in \mathcal{P}_{n,r}$ such that $\mu \neq \lambda\}$. Then our main theorem is the following.

Theorem 4.11 (i) *If $\lambda \in \Gamma$, then $\Delta(\lambda\langle i\rangle)$ (resp. $S(\lambda\langle i\rangle)$) is a simple object of $\mathcal{O}$ (resp. simple $\mathscr{H}$-module) for any $i \in \mathbb{Z}$. Moreover, $\Delta(\lambda\langle i\rangle)$ (resp. $S(\lambda\langle i\rangle)$) is a block of $\mathcal{O}$ (resp. of $\mathscr{H}$) itself.*
(ii) *For $\lambda, \mu \in \mathcal{P}_{n,r} \setminus \Gamma$ and $i, j \in \mathbb{Z}$,*

both of $\Delta(\lambda\langle i\rangle)$ and $\Delta(\mu\langle j\rangle)$ belong to the same block of $\mathcal{O}$

$\Leftrightarrow$ both of $S(\lambda\langle i\rangle)$ and $S(\mu\langle j\rangle)$ belong to the same block of $\mathscr{H}$

$\Leftrightarrow \lambda \approx \mu$.

NOTATIONS. For an algebra $\mathscr{A}$, we denote by $\mathscr{A}$-mod the category of finitely generated $\mathscr{A}$-modules, and denote by $\mathscr{A}$-proj the full subcategory of $\mathscr{A}$-mod consisting of projective objects. Let $K_0(\mathscr{A}\text{-mod})$ be the Grothendieck group of $\mathscr{A}$-mod. We denote by $[M]$ the image of M in the $K_0(\mathscr{A}\text{-mod})$ for $M \in \mathscr{A}$-mod. For $M \in \mathscr{A}$-mod and simple object L of $\mathscr{A}$-mod, we denote by $[M : L]_{\mathscr{A}}$ the multiplicity of L in the composition series of M. We also denote by $\mathscr{A}^{\mathrm{opp}}$ the opposite algebra of $\mathscr{A}$.

1. Some properties of quasi-hereditary covers

In this section, we recall some notions of quasi-hereditary covers from [21], and review some fundamental properties.

1.1. Let $\mathscr{A}$ be a quasi-hereditary algebra over a field. Take a projective object P in $\mathscr{A}$-mod, and put $\mathscr{B} = \operatorname{End}_{\mathscr{A}}(P)^{\mathrm{opp}}$. Then we have an exact functor $F = \operatorname{Hom}_{\mathscr{A}}(P, -)\colon \mathscr{A}\text{-mod} \to \mathscr{B}\text{-mod}$. Let X be a progenerator of $\mathscr{A}$-mod such that $X = P \oplus P'$ for some projective object P' in $\mathscr{A}$-mod. Then $\operatorname{End}_{\mathscr{A}}(X)^{\mathrm{opp}}$ is Morita equivalent to $\mathscr{A}$. We may suppose that $\operatorname{End}_{\mathscr{A}}(X)^{\mathrm{opp}} \cong \mathscr{A}$ without loss of generality.

Throughout this section, we assume the following condition.
(A1): The functor F is fully faithful when we restrict to $\mathscr{A}$-proj.
Hence, $\mathscr{A}$ is a quasi-hereditary cover of $\mathscr{B}$ in the sense of [21]. Since $X \in \mathscr{A}$-proj, we have

$$\mathscr{A} \cong \operatorname{End}_{\mathscr{A}}(X)^{\mathrm{opp}} \cong \operatorname{End}_{\mathscr{B}}(F(X))^{\mathrm{opp}}. \tag{1.1.1}$$

Note that $X = P \oplus P'$. Let $\varphi_P^o \in \operatorname{End}_{\mathscr{A}}(X)$ be such that φ_P^o is the identity map on P, and 0-map on P'. We denote by φ_P the element of $\mathscr{A} \cong \operatorname{End}_{\mathscr{A}}(X)^{\mathrm{opp}}$ corresponding to φ_P^o. It is clear that φ_P is an idempotent. Since

$$\begin{aligned} F(X) &\cong \operatorname{Hom}_{\mathscr{A}}(P, P) \oplus \operatorname{Hom}_{\mathscr{A}}(P, P') \\ &\cong \operatorname{End}_{\mathscr{A}}(X)\varphi_P^o \\ &\cong \varphi_P \mathscr{A} \end{aligned}$$

as right $\mathscr{A}$-modules, we have the following isomorphisms of algebras:

$$\begin{aligned} \operatorname{End}_{\mathscr{A}^{\mathrm{opp}}}(F(X)) &\cong \operatorname{End}_{\mathscr{A}^{\mathrm{opp}}}(\varphi_p \mathscr{A}) \\ &\cong \varphi_P \mathscr{A} \varphi_P \\ &\cong (\varphi_P^o \operatorname{End}_{\mathscr{A}}(X)\varphi_P^o)^{\mathrm{opp}} \\ &\cong \operatorname{End}_{\mathscr{A}}(P)^{\mathrm{opp}} \\ &= \mathscr{B}. \end{aligned}$$

Thus, we have the double centralizer property:

$$\mathscr{A} \cong \operatorname{End}_{\mathscr{B}}(F(X))^{\mathrm{opp}}, \quad \mathscr{B} \cong \operatorname{End}_{\mathscr{A}^{\mathrm{opp}}}(F(X)). \tag{1.1.2}$$

This double centralizer property implies the isomorphism $Z(\mathscr{A}) \to Z(\mathscr{B})$, where $Z(\mathscr{A})$ (resp. $Z(\mathscr{B})$) is the center of $\mathscr{A}$ (resp. $\mathscr{B}$). Thus, there exists a bijection between blocks of $\mathscr{A}$ and of $\mathscr{B}$.

1.2. Recall that $\mathscr{A}$ is a quasi-hereditary algebra. Let $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$ be the set of standard modules, and $\{\nabla(\lambda) \mid \lambda \in \Lambda^+\}$ be the set of costandard modules of $\mathscr{A}$. For $\lambda \in \Lambda^+$, let $L(\lambda)$ be the unique simple top of $\Delta(\lambda)$, and $P(\lambda)$ be the projective cover of $L(\lambda)$. Then $\{L(\lambda) \mid \lambda \in \Lambda^+\}$ gives a complete set of non-isomorphic simple $\mathscr{A}$-modules.

For $\lambda \in \Lambda^+$, put $S(\lambda) = F(\Delta(\lambda))$, $D(\lambda) = F(L(\lambda))$ and $\Lambda_0^+ = \{\lambda \in \Lambda^+ \mid D(\lambda) \neq 0\}$. Since $\mathscr{B} \cong \varphi_P \mathscr{A} \varphi_P$ and $F = \mathrm{Hom}_{\mathscr{A}}(P, -) = \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}\varphi_P, -)$, the following lemma is standard (see e.g. [6, Appendix]).

Lemma 1.3. (i) *For $\lambda \in \Lambda_0^+$, we have $D(\lambda) \cong \mathrm{Top}\, F(P(\lambda)) \cong \mathrm{Top}\, S(\lambda)$.*
(ii) *$\{F(P(\lambda)) \mid \lambda \in \Lambda_0^+\}$ gives a complete set of non-isomorphic indecomposable projective $\mathscr{B}$-modules.*
(iii) *$\{D(\lambda) \mid \lambda \in \Lambda_0^+\}$ gives a complete set of non-isomorphic simple $\mathscr{B}$-modules.*

1.4. For $\lambda, \mu \in \Lambda^+$, we denote by $P(\lambda) \sim P(\mu)$ if there exists a sequence $\lambda = \lambda_1, \lambda_2, \dots, \lambda_{k+1} = \mu$ ($\lambda_i \in \Lambda^+$) such that $P(\lambda_i)$ and $P(\lambda_{i+1})$ have a common composition factor for any $i = 1, \dots, k$. Then "$\sim$" gives an equivalence relation on $\{P(\lambda) \mid \lambda \in \Lambda^+\}$. It is well-known that $P(\lambda) \sim P(\mu)$ if and only if $P(\lambda)$ and $P(\mu)$ belong to the same block of $\mathscr{A}$. Similarly, we define an equivalence relation "$\sim$" on $\{F(P(\lambda)) \mid \lambda \in \Lambda_0^+\}$, and we have $F(P(\lambda)) \sim F(P(\mu))$ if and only if $F(P(\lambda))$ and $F(P(\mu))$ belong to the same block of $\mathscr{B}$. Then the double centralizer property (1.1.2) implies the following lemma.

Lemma 1.5. *For $\lambda, \mu \in \Lambda_0^+$, we have*

$$P(\lambda) \sim P(\mu) \quad \text{if and only if} \quad F(P(\lambda)) \sim F(P(\mu)).$$

Note that all the composition factors of $\Delta(\lambda)$ belong to the same block of $\mathscr{A}$ since $\Delta(\lambda)$ is indecomposable. Then, the exact functor F combined with Lemma 1.5 implies the following corollary.

Corollary 1.6. *For each $\lambda \in \Lambda^+$, all the composition factors of $S(\lambda)$ belong to the same block of $\mathscr{B}$.*

1.7. From now on, we assume the following additional condition:
(A2): $[\Delta(\lambda)] = [\nabla(\lambda)]$ in $K_0(\mathscr{A}\text{-mod})$ for any $\lambda \in \Lambda^+$.

By the general theory of quasi-hereditary algebras, for $\lambda \in \Lambda^+$, $P(\lambda)$ has a Δ-filtration such that $(P(\lambda) : \Delta(\mu)) = [\nabla(\mu) : L(\lambda)]_{\mathscr{A}}$, where $(P(\lambda) : \Delta(\mu))$ is the multiplicity of $\Delta(\mu)$ in a Δ-filtration of $P(\lambda)$. Combining with the assumption (A2), we have

$$(P(\lambda) : \Delta(\mu)) = [\Delta(\mu) : L(\lambda)]_{\mathscr{A}}. \tag{1.7.1}$$

This implies the following lemma.

Lemma 1.8. *For $\lambda, \mu \in \Lambda_0^+$, we have*

$$[F(P(\lambda)) : D(\mu)]_{\mathscr{B}} = \sum_{\nu \in \Lambda^+} [S(\nu) : D(\lambda)]_{\mathscr{B}} [S(\nu) : D(\mu)]_{\mathscr{B}}.$$

1.9. For $\lambda, \mu \in \Lambda^+$, we denote by $S(\lambda) \sim S(\mu)$ if there exists a sequence $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_{k+1} = \mu$ ($\lambda_i \in \Lambda^+$) such that $S(\lambda_i)$ and $S(\lambda_{i+1})$ have a common composition factor for any $i = 1, \ldots, k$. Then "$\sim$" gives an equivalence relation on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$. Similarly, we define an equivalence relation "$\sim$" on $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$.

Corollary 1.6 and Lemma 1.8 imply the following proposition.

Proposition 1.10. *For $\lambda, \mu \in \Lambda^+$ we have the following.*
(i) $S(\lambda) \sim S(\mu)$ *if and only if* $S(\lambda)$ *and* $S(\mu)$ *belong to the same block of* $\mathscr{B}$.
(ii) $\Delta(\lambda) \sim \Delta(\mu)$ *if and only if* $\Delta(\lambda)$ *and* $\Delta(\mu)$ *belong to the same block of* $\mathscr{A}$.

1.11. Finally, we assume the following additional condition:
(A3): $S(\lambda) = F(\Delta(\lambda)) \neq 0$ for any $\lambda \in \Lambda^+$.

Thanks to Proposition 1.10, we can classify blocks of $\mathscr{B}$ (resp. blocks of $\mathscr{A}$) by equivalence classes of $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ (resp. $\{\Delta(\lambda) \mid \lambda \in \Lambda^+\}$) with respect to the relation "$\sim$". Then Lemma 1.5 and Proposition 1.10 (under the assumption (A3)) imply the following proposition which gives a relation between blocks of $\mathscr{A}$ and of $\mathscr{B}$.

Proposition 1.12. *For $\lambda, \mu \in \Lambda^+$, we have*

$$\Delta(\lambda) \sim \Delta(\mu) \quad \text{if and only if} \quad S(\lambda) \sim S(\mu).$$

2. Rational Cherednik algebras

2.1. Let V be a finite dimensional vector space over $\mathbb{C}$, and $W \subset \mathrm{GL}(V)$ be a finite complex reflection group. Let $\mathcal{A}$ be the set of reflecting hyperplanes of W, and $\mathcal{A}/W$ be the set of W-orbits of $\mathcal{A}$. For $H \in \mathcal{A}$, let W_H be the subgroup of W fixing H pointwise, and put $e_H = |W_H|$. Take a set

$$\Omega = \{k_{H,i} \in \mathbb{C} \mid H \in \mathcal{A}/W,\ 0 \le i \le e_H \text{ such that } k_{H,0} = k_{H,e_H} = 0\}.$$

Let $\mathcal{H}$ be the rational Cherednik algebra associated to W with parameters Ω (see [9, 3.1] for definitions). By [7], it is known that $\mathcal{H}$ has the triangular decomposition

$$\mathcal{H} \cong S(V^*) \otimes_{\mathbb{C}} \mathbb{C}W \otimes_{\mathbb{C}} S(V) \quad \text{as vector spaces,}$$

where $S(V)$ (resp. $S(V^*)$) is the symmetric algebra of V (resp. the dual space V^*), and $\mathbb{C}W$ is the group ring of W over $\mathbb{C}$.

Let $\mathcal{O}$ be the category of finitely generated $\mathcal{H}$-modules which are locally nilpotent for the action of $S(V) \setminus \mathbb{C}$. Let $\mathrm{Irr} W = \{E^\lambda \mid \lambda \in \Lambda^+\}$ be a complete set of non-isomorphic simple $\mathbb{C}W$-modules. For $\lambda \in \Lambda^+$, put

$$\Delta(\lambda) = \mathcal{H} \otimes_{S(V) \rtimes W} E^\lambda,$$

where $S(V)\rtimes W\cong S(V)\otimes_{\mathbb{C}}\mathbb{C}W$ is a subalgebra of $\mathcal{H}$, and we regard E^{λ} as a $S(V)\rtimes W$-module through the natural surjection $S(V)\rtimes W\to\mathbb{C}W$. It is known that $\mathcal{O}$ turns out to be a highest weight category with standard modules $\{\Delta(\lambda)\mid\lambda\in\Lambda^{+}\}$ ([9], [11]). Let $L(\lambda)$ be the unique simple top of $\Delta(\lambda)$, then $\{L(\lambda)\mid\lambda\in\Lambda^{+}\}$ is a complete set of non-isomorphic simple objects in $\mathcal{O}$. For $\lambda\in\Lambda^{+}$, we denote by $P(\lambda)$ the projective cover of $L(\lambda)$.

2.2. Let $\mathscr{H}$ be the cyclotomic Hecke algebra of W corresponding $\mathcal{H}$ (see [9, 5.2.5] for the choice of parameters). Then the Knizhnik–Zamolodchikov functor (simply, KZ functor) KZ: $\mathcal{O}\to\mathscr{H}$-mod is defined in [9, 5.3]. KZ functor is a exact functor, and represented by a projective object

$$P_{\mathrm{KZ}}=\bigoplus_{\lambda\in\Lambda^{+}}P(\lambda)^{\oplus\dim\mathrm{KZ}(L(\lambda))}\in\mathcal{O},$$

namely, we have $\mathrm{KZ}=\mathrm{Hom}_{\mathcal{O}}(P_{\mathrm{KZ}},-)$. Moreover, by [9, Theorem 5.15], we have

$$\mathscr{H}\cong(\mathrm{End}_{\mathcal{O}}(P_{\mathrm{KZ}}))^{\mathrm{opp}}.$$

By [9, Theorem 5.16], KZ functor is fully faithful when we restrict to projective objects in $\mathcal{O}$. Thus, $\mathcal{O}$ is a quasi-hereditary cover of $\mathscr{H}$.

Put $\mathscr{A}=\mathrm{End}_{\mathcal{O}}(X)$, $\mathscr{B}=\mathscr{H}$ and $F=\mathrm{KZ}$, where X is a progenerator of $\mathcal{O}$ such that $X=P_{\mathrm{KZ}}\oplus P'$ for some projective object P' in $\mathcal{O}$. Then, these satisfy assumptions (A1), (A2), (A3) by [9]. Thus, all results in §1 hold for this setting. In particular, we put $S(\lambda)=\mathrm{KZ}(\Delta(\lambda))$ and $D(\lambda)=\mathrm{KZ}(L(\lambda))$ for $\lambda\in\Lambda^{+}$. Let $\Lambda_0^{+}=\{\lambda\in\Lambda^{+}\mid D(\lambda)\neq 0\}$, then $\{D(\lambda)\mid\lambda\in\Lambda_0^{+}\}$ gives a complete set of non-isomorphic simple $\mathscr{H}$-modules.

2.3. In the rest of this section, we recall a modular system and a decomposition map described in [9]. Let $\mathbb{C}[\hat{\Omega}]$ be the polynomial ring over $\mathbb{C}$ with indeterminates $\hat{\Omega}=\{\mathbf{k}_{H,i}\mid H\in\mathcal{A}/W,\ 1\le i\le e_H-1\}$. We have a homomorphism $\varphi\colon\mathbb{C}[\hat{\Omega}]\to\mathbb{C}$ of $\mathbb{C}$-algebras such that $\mathbf{k}_{H,i}\mapsto k_{H,i}$. Put $\mathfrak{m}=\mathrm{Ker}\varphi$. Let R be the completion of $\mathbb{C}[\hat{\Omega}]$ at the maximal ideal $\mathfrak{m}$. Then R is a regular local ring with the unique maximal ideal $\hat{\mathfrak{m}}=((\mathbf{k}_{H,i}-k_{H,i})_{H\in\mathcal{A}/W,\,1\le i\le e_H-1})$. We have the canonical homomorphism $R\to\mathbb{C}$ such that $\mathbf{k}_{H,i}\mapsto k_{H,i}$. Let K be the quotient field of R.

Let $\mathcal{H}_R$ be the rational Cherednik algebra of W over R with parameters $\hat{\Omega}$ (put $\mathbf{k}_{H,0}=\mathbf{k}_{H,e_H}=0$), and $\mathscr{H}_R$ be the cyclotomic Hecke algebra over R associated to $\mathcal{H}_R$. Then we have $\mathcal{H}=\mathbb{C}\otimes_R\mathcal{H}_R$ and $\mathscr{H}=\mathbb{C}\otimes_R\mathscr{H}_R$. Put $\mathcal{H}_K=K\otimes_R\mathcal{H}_K$ and $\mathscr{H}_K=K\otimes_R\mathscr{H}_R$. We denote objects over $X=R$ or K by adding subscript X, e.g. $\mathcal{O}_X$, $\Delta(\lambda)_X$, KZ_X, $S(\lambda)_X,\dots$.

Under the modular system $(K,R,\mathbb{C})$, we can define the decomposition map

$$d_{K,\mathbb{C}}\colon K_0(\mathscr{H}_K\text{-mod})\to K_0(\mathscr{H}\text{-mod})$$

by $[M] \mapsto [\mathbb{C} \otimes_R N]$, where N is an $\mathscr{H}_R$-lattice of M. Thanks to [9, Theorem 5.13], we have the following lemma.

Lemma 2.4. *For $\lambda \in \Lambda^+$, we have*

$$d_{K,\mathbb{C}}([S_K(\lambda)]) = [S(\lambda)].$$

3. Case of type $G(r, 1, n)$

In this section, we consider the complex reflection group W of type $G(r, 1, n)$, i.e. $W = \mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$. In this case, $\mathscr{H}$ is often called the Ariki–Koike algebra, and many results for representations of $\mathscr{H}$ are known by several authors.

3.1. In this section, we use the modular system $(K, R, \mathbb{C})$ given in the previous section, and we take parameters as follows.

Let V be an n dimensional vector space over $\mathbb{C}$ with a basis $\{e_1, \dots, e_n\}$. Then we have $W \subset \mathrm{GL}(V)$. Let $s_1, t_1 \in W$ be reflections such that

(3.1.1)

$$s_1(e_k) = \begin{cases} e_2 & \text{if} \quad k = 1, \\ e_1 & \text{if} \quad k = 2, \\ e_k & \text{otherwise}, \end{cases} \quad t_1(e_k) = \begin{cases} \zeta e_1 & \text{if} \quad k = 1, \\ e_k & \text{otherwise}, \end{cases} \quad (\zeta = \exp(2\pi\sqrt{-1}/r)),$$

and H_{s_1} (resp. H_{t_1}) be the reflecting hyperplane corresponding to s_1 (resp. t_1). Then $\{H_{s_1}, H_{t_1}\}$ gives a complete set of representatives of W-orbits of $\mathcal{A}$, and we have $e_{H_{s_1}} = 2$ and $e_{H_{t_1}} = r$. Hence, we can take parameters $\{h, k_1, \dots, k_{r-1}\}$ (resp. $\{\mathbf{h}, \mathbf{k}_1, \dots, \mathbf{k}_{r-1}\}$) of $\mathcal{H}$ (resp. $\mathcal{H}_X$ ($X = R$ or K)) such that $h = k_{H_{s_1},1}$ (resp. $\mathbf{h} = \mathbf{k}_{H_{s_1},1}$) and $k_j = k_{H_{t_1},j}$ (resp. $\mathbf{k}_j = \mathbf{k}_{H_{t_1},j}$) for $1 \le j \le r-1$. Then $\mathscr{H}$ (resp. $\mathscr{H}_R$, $\mathscr{H}_K$) is the associative algebra over $\mathbb{C}$ (resp. R, K) defined by generators $T_0, T_1, \dots, T_{n-1}$ with defining relations:

(3.1.2)

$$\begin{aligned} &(T_0 - 1)(T_0 - Q_1) \cdots (T_0 - Q_{r-1}) = 0, \\ &(T_0 - 1)(T_0 + q) = 0, \\ &T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, \\ &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \le i \le n-1), \\ &T_i T_j = T_j T_i \quad (|i-j| > 1), \end{aligned}$$

where $Q_i = \exp(2\pi\sqrt{-1}(k_i + i/r))$, $q = \exp(2\pi\sqrt{-1}h)$ (resp. $Q_i = \exp(2\pi\sqrt{-1}(\mathbf{k}_i + i/r))$, $q = \exp(2\pi\sqrt{-1}\mathbf{h})$).

3.2. Let

$$\mathcal{P}_{n,r} = \left\{ \lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \;\middle|\; \begin{array}{l} \lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)}, \dots) \in \mathbb{Z}_{\geq 0}^n \text{ with } \lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots \\ \sum_{k=1}^{r} \sum_{i \geq 1} \lambda_i^{(k)} = n \end{array} \right\}$$

be the set of r-partitions of size n. It is well-known that the isomorphism classes of simple $\mathbb{C}W$-modules are indexed by $\mathcal{P}_{n,r}$, thus we have $\Lambda^+ = \mathcal{P}_{n,r}$.

3.3. By [5], it is known that $\mathscr{H}_X$ ($X = K$, R or $\mathbb{C}$, we may omit the subscript X when $X = \mathbb{C}$) is a cellular algebra with respect to a poset $(\Lambda^+, \trianglerighteq)$, where "$\trianglerighteq$" is the dominance order on Λ^+. We denote by S_X^λ the Specht (cell) module for $\lambda \in \Lambda^+$ constructed by using the cellular basis in [5].

It is known that $\mathscr{H}_K$ is semi-simple, and $\{S_K^\lambda \mid \lambda \in \Lambda^+\}$ gives a complete set of non-isomorphic simple $\mathscr{H}_K$-modules.

By the general theory of cellular algebras (see [10] or [19]), we can define the canonical bilinear form $\langle\ ,\ \rangle \colon S^\lambda \times S^\lambda \to \mathbb{C}$ by using the cellular basis. Put $\operatorname{Rad} S^\lambda = \{x \in S^\lambda \mid \langle x, y\rangle = 0 \text{ for any } y \in S^\lambda\}$ and $D^\lambda = S^\lambda / \operatorname{Rad} S^\lambda$. Let $\mathcal{K}_{n,r}$ be the set of Kleshchev multi-partitions containing in Λ^+ (see e.g. [3] and [18] for the definition). Then it is known that $\{D^\lambda \mid \lambda \in \mathcal{K}_{n,r}\}$ gives a complete set of non-isomorphic simple $\mathscr{H}$-modules by [3].

It is known that all composition factor of S^λ belong to the same block of $\mathscr{H}$. Let "$\sim$" be an equivalence relation on $\{S^\lambda \mid \lambda \in \Lambda^+\}$ defined in a similar way as the equivalence relation "$\sim$" on $\{S(\lambda) \mid \lambda \in \Lambda^+\}$ in the previous section. Then it is known that

$$S^\lambda \sim S^\mu \quad \text{if and only if} \quad S^\lambda \text{ and } S^\mu \text{ belong to the same block of } \mathscr{H}. \tag{3.3.1}$$

By (3.3.1), we can classify the blocks of $\mathscr{H}$ by the equivalence classes of $\{S^\lambda \mid \lambda \in \Lambda^+\}$ with respect to "$\sim$", and such equivalence classes are described by using some combinatorics in [17] as follows. For $\lambda \in \Lambda^+$, put

$$[\lambda] = \{(i, j, k) \in \mathbb{Z}_{>0}^3 \mid 1 \leq j \leq \lambda_i^{(k)},\ 1 \leq k \leq r\}.$$

For $x = (i, j, k) \in [\lambda]$, we define

$$\operatorname{res}(x) = \begin{cases} q^{j-i} Q_{k-1} & \text{if} \quad q \neq 1 \text{ and } Q_{k-1} \neq 0, \\ (j - i, Q_{k-1}) & \text{if} \quad q = 1 \text{ and } Q_{l-1} \neq Q_{k-1} \text{ for } k \neq l, \\ Q_{k-1} & \text{otherwise,} \end{cases}$$

where we put $Q_0 = 1$. Put $\operatorname{Res}(\Lambda^+) = \{\operatorname{res}(x) \mid x \in [\lambda] \text{ for some } \lambda \in \Lambda^+\}$. Then, we define an equivalence relation (called residue equivalence) "$\sim_R$" on Λ^+ by

$$\lambda \sim_R \mu \quad \text{if} \quad \#\{x \in [\lambda] \mid \operatorname{res}(x) = a\} = \#\{y \in [\mu] \mid \operatorname{res}(y) = a\} \quad \text{for all} \quad a \in \operatorname{Res}(\Lambda^+).$$

Theorem 3.4 ([17, Theorem 2.11]). *For* $\lambda, \mu \in \Lambda^+$, *we have*

$$S^\lambda \sim S^\mu \quad \textit{if and only if} \quad \lambda \sim_R \mu.$$

3.5. We take $\operatorname{Irr} W = \{E^\lambda \mid \lambda \in \Lambda^+\}$ such that $K \otimes_{\mathbb{C}} E^\lambda \cong S_K^\lambda$ via the isomorphism $\mathscr{H}_K \cong K \otimes_{\mathbb{C}} \mathbb{C}W$. Since $S_K^\lambda = K \otimes_R S_R^\lambda$ and $S^\lambda = \mathbb{C} \otimes_R S_R^\lambda$, we have

$$d_{K,\mathbb{C}}([S_K^\lambda]) = [S^\lambda]. \tag{3.5.1}$$

It is also well-known that $K \otimes_{\mathbb{C}} E^\lambda \cong S_K(\lambda)$ via the isomorphism $\mathscr{H}_K \cong K \otimes_{\mathbb{C}} \mathbb{C}W$ (see before [9, Theorem 5.13]). Thus, we have $S_K^\lambda \cong S_K(\lambda)$ as $\mathscr{H}_K$-modules. Then Lemma 2.4 together with (3.5.1) implies that

$$[S(\lambda)] = [S^\lambda] \quad \text{in} \quad K_0(\mathscr{H}\text{-mod}) \quad \text{for} \quad \lambda \in \Lambda^+. \tag{3.5.2}$$

Note that $S(\lambda) \ncong S^\lambda$ as $\mathscr{H}$-modules in general. Hence, $\operatorname{Top} S(\lambda) \ncong \operatorname{Top} S^\lambda$ in general. Moreover, $\Lambda_0^+ \neq \mathcal{K}_{n,r}$ in general. Let

$$\theta : \Lambda_0^+ \to \mathcal{K}_{n,r}$$

be the bijection such that $D(\lambda) \cong D^{\theta(\lambda)}$ as $\mathscr{H}$-modules. Then we have the following proposition.

Proposition 3.6. *For* $\lambda \in \Lambda^+$ *and* $\mu \in \Lambda_0^+$, *we have*

$$[\Delta(\lambda) : L(\mu)]_{\mathcal{O}} = [S(\lambda) : D(\mu)]_{\mathscr{H}} = [S^\lambda : D^{\theta(\mu)}]_{\mathscr{H}}.$$

Proof. The first equality is clear since the KZ functor is exact. By (3.5.2), we have $[S(\lambda)] = [S^\lambda]$ in $K_0(\mathscr{H}\text{-mod})$, and $D(\mu) \cong D^{\theta(\mu)}$. Thus, we have the second equality. □

The following theorem gives a relation between blocks of $\mathcal{O}$ and blocks of $\mathscr{H}$. In particular, we obtain the classification of blocks of $\mathcal{O}$ by using the residue equivalence.

Theorem 3.7. *For* $\lambda, \mu \in \Lambda^+$, *we have*

$$\Delta(\lambda) \sim \Delta(\mu) \Leftrightarrow S(\lambda) \sim S(\mu) \Leftrightarrow S^\lambda \sim S^\mu \Leftrightarrow \lambda \sim_R \mu.$$

Proof. The first equivalence is Proposition 1.12. The second equivalence follows from (3.5.2). The third equivalence is Theorem 3.4. □

REMARK 3.8. By [21], under a certain condition for parameters, it is known that $\mathcal{O}$ is equivalent to $\mathscr{S}_{n,r}$-mod as highest weight categories, where $\mathscr{S}_{n,r}$ is the cyclotomic

q-Schur algebra associated to $\mathscr{H}$ defined in [5]. In this case, we have $S(\lambda) \cong S^{\lambda}$, and θ is the identity map (in particular, $\Lambda_0^+ = \mathcal{K}_{n,r}$). So, the above theorem is known by [17]. However, the above theorem claim that a classification of blocks of $\mathcal{O}$ is given by the equivalence relation "$\sim_R$" on Λ^+ (residue equivalence) even if the case where $\mathcal{O}$ is not equivalent to $\mathscr{S}_{n,r}$-mod.

4. Case of type $G(r, p, n)$

In this section, we consider the case where W is the complex reflection group of type $G(r, p, n)$, where $r = pd$ for some $d \geq 1$. It is well-known that the complex reflection group of type $G(r, p, n)$ is a normal subgroup of the complex reflection group of type $G(r, 1, n)$ with the index p, and we will study some relations between type $G(r, 1, n)$ and type $G(r, p, n)$. Hence, we denote by $W^{\dagger}$ the complex reflection group of type $G(r, 1, n)$, and we use the results in the previous section for $W^{\dagger}$. In this section, we use the notations in §2 for corresponding objects of type $G(r, p, n)$, and we denote by adding the superscript † for corresponding objects of type $G(r, 1, n)$, e.g. $\mathcal{H}^{\dagger}$, $\mathscr{H}^{\dagger}$, $\Delta^{\dagger}(\lambda)$, $KZ^{\dagger}$, $S^{\dagger}(\lambda)$, ... Let $\operatorname{Irr} W^{\dagger} = \{E^{\dagger\lambda} \mid \lambda \in \mathcal{P}_{n,r}\}$ be a complete set of non-isomorphic simple $\mathbb{C}W^{\dagger}$-modules considered in the previous section.

4.1. Let V be an n dimensional vector space over $\mathbb{C}$ with a basis $\{e_1, \ldots, e_n\}$. Then we have $W \subset \mathrm{GL}(V)$. Recall that $s_1, t_1 \in \mathrm{GL}(V)$ is a reflection defined in (3.1.1). Then s_1 (resp. t_1^p in the case where $p \neq r$) is a reflection contained in W, and let H_{s_1} (resp. $H_{t_1^p}$) be the reflecting hyperplane corresponding to s_1 (resp. t_1^p). In the case where $p \neq r$, $\{H_{s_1}, H_{t_1^p}\}$ gives a complete set of representatives of W-orbits of $\mathcal{A}$, and we have $e_{H_{s_1}} = 2$ and $e_{H_{t_1^p}} = d$. Hence, we can take parameters $\{h, k_1, \ldots, k_{d-1}\}$ (resp. $\{\mathbf{h}, \mathbf{k}_1, \ldots, \mathbf{k}_{d-1}\}$) of $\mathcal{H}$ (resp. $\mathcal{H}_X$ ($X = R$ or K)) such that $h = k_{H_{s_1},1}$ (resp. $\mathbf{h} = \mathbf{k}_{H_{s_1},1}$) and $k_j = k_{H_{t_1^p},j}$ (resp. $\mathbf{k}_j = \mathbf{k}_{H_{t_1^p},j}$) for $1 \leq j \leq d-1$. On the other hand, in the case where $r = p$, $\mathcal{A}$ is the W-orbit of $\mathcal{A}$ itself. Hence $\mathcal{H}$ (resp. $\mathcal{H}_X$ ($X = R$ or K)) has a parameter $\{h\}$ (resp. $\{\mathbf{h}\}$).

Then $\mathscr{H}$ (resp. $\mathscr{H}_R$, $\mathscr{H}_K$) is the associative algebra over $\mathbb{C}$ (resp. R, K) defined by generators $a_0, a_1', a_1, a_2, \ldots, a_{n-1}$ with defining relations:

$$
\begin{aligned}
&(a_0 - 1)(a_0 - x_1)\cdots(a_0 - x_{d-1}) = 0,\\
&(a_1' - 1)(a_1' + q) = 0, \quad (a_i - 1)(a_i + q) = 0 \quad (1 \leq i \leq n-1),\\
&a_0 a_1' a_1 = a_1' a_1 a_0, \quad a_1' a_2 a_1' = a_2 a_1' a_2, \quad (a_2 a_1' a_1)^2 = (a_1' a_1 a_2)^2,\\
&a_0 a_i = a_i a_0 \quad (2 \leq i \leq n-1), \quad a_1' a_j = a_j a_1' \quad (3 \leq j \leq n-1),\\
&\underbrace{a_1 a_0 a_1' a_1 a_1' a_1 a_1' \cdots}_{p+1 \text{ factors}} = \underbrace{a_0 a_1' a_0 a_1' a_0 a_1' a_0 \cdots}_{p+1 \text{ factors}},\\
&a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad (1 \leq i \leq n-2),\\
&a_i a_j = a_j a_i \quad (1 \leq i < j-1 \leq n-2),
\end{aligned}
$$

where $x_i = \exp(2\pi\sqrt{-1}(k_i + i/d)$, $q = \exp(2\pi\sqrt{-1}h)$ (resp. $x_i = \exp(2\pi\sqrt{-1}(\mathbf{k}_i + i/d)$, $q = \exp(2\pi\sqrt{-1}\mathbf{h}))$ (see [4] or [2] for braid relations).

4.2. Put

$$k^{\dagger}_{c\cdot p+j} = k_c + \frac{c}{d} + \frac{j}{p} - \frac{c\cdot p + j}{r} \quad (0 \le c \le d-1,\ 0 \le j \le p-1),$$
$$\mathbf{k}^{\dagger}_{c\cdot p+j} = \mathbf{k}_c + \frac{c}{d} + \frac{j}{p} - \frac{c\cdot p + j}{r} \quad (0 \le c \le d-1,\ 0 \le j \le p-1),$$

where we set $k_0 = \mathbf{k}_0 = 0$.

Throughout this section, let $\mathcal{H}^{\dagger}$ (resp. $\mathcal{H}^{\dagger}_X$ ($X = R$ or K)) be the rational Cherednik algebra associated to $W^{\dagger}$ with parameters $\{h, k^{\dagger}_1, \dots, k^{\dagger}_{r-1}\}$ (resp. $\{\mathbf{h}, \mathbf{k}^{\dagger}_1, \dots, \mathbf{k}^{\dagger}_{r-1}\}$) such that $h = k^{\dagger}_{H_{s_1},1}$ (resp. $\mathbf{h} = \mathbf{k}^{\dagger}_{H_{s_1},1}$) and $k^{\dagger}_j = k^{\dagger}_{H_{t_1},j}$ (resp. $\mathbf{k}^{\dagger}_j = \mathbf{k}^{\dagger}_{H_{t_1},j}$) for $1 \le j \le r-1$. Since

$$\exp\left(2\pi\sqrt{-1}\left(k^{\dagger}_{c\cdot p+j} + \frac{c\cdot p+j}{r}\right)\right) = \exp\left(2\pi\sqrt{-1}\left(k_c + \frac{c}{d} + \frac{j}{p}\right)\right)$$
$$= x_c \xi^j \quad (\xi = \exp(2\pi\sqrt{-1}/p)),$$

where we put $x_0 = 1$ (similar for $\mathbf{k}^{\dagger}_{c\cdot p+j}$), the defining relation (3.1.2) of $\mathscr{H}^{\dagger}$ (resp. $\mathscr{H}^{\dagger}_X$) replaced by

$$(T_0^p - 1)(T_0^p - x_1^p)\cdots(T_0^p - x_{d-1}^p) = 0.$$

Since $\mathscr{H}^{\dagger}_K$ is semi-simple (thus, $\mathcal{O}^{\dagger}_K$ is also semi-simple) by [1], we can obtain any results for type $G(r,1,n)$ in the previous sections even if the case of these parameters.

By [2, Proposition 1.6], there is the injective algebra homomorphism $\varphi\colon \mathscr{H}_X \to \mathscr{H}^{\dagger}_X$ ($X = \mathbb{C}$, R or K) such that $\varphi(a_0) = T_0^p$, $\varphi(a_1') = T_0^{-1}T_1T_0$, $\varphi(a_i) = T_i$ ($1 \le i \le n-1$). Under this injective homomorphism φ, we regard $\mathscr{H}_X$ as a subalgebra of $\mathscr{H}^{\dagger}_X$.

4.3. For $M^{\dagger} \in \mathbb{C}W^{\dagger}$-mod, we denote by $M^{\dagger}\!\downarrow$ the restriction of the action to $\mathbb{C}W$. For $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}) \in \mathcal{P}_{n,r}$ and $i \in \mathbb{Z}$, we define $\lambda[i] = (\lambda[i]^{(1)}, \dots, \lambda[i]^{(r)}) \in \mathcal{P}_{n,r}$ by

$$\lambda[i]^{(c\cdot p+j)} = \lambda^{(c\cdot p+k)} \quad (0 \le c \le d-1,\ 1 \le j \le p),$$

where $c\cdot p < c\cdot p + k \le (c+1)\cdot p$ such that $k \equiv j + i \mod p$. For an example, if $r = 6$ and $p = 3$, we have

$$\lambda[1] = (\lambda^{(2)}, \lambda^{(3)}, \lambda^{(1)}, \vdots\, \lambda^{(5)}, \lambda^{(6)}, \lambda^{(4)}),$$
$$\lambda[2] = (\lambda^{(3)}, \lambda^{(1)}, \lambda^{(2)}, \vdots\, \lambda^{(6)}, \lambda^{(4)}, \lambda^{(5)}),$$
$$\lambda[3] = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \vdots\, \lambda^{(4)}, \lambda^{(5)}, \lambda^{(6)}) = \lambda.$$

Let $\mathfrak{k}_\lambda$ be the minimum positive integer such that $\lambda[\mathfrak{k}_\lambda] = \lambda$. It is clear that $\mathfrak{k}_\lambda \mid p$. Put $\mathfrak{d}_\lambda = p/\mathfrak{k}_\lambda$. Then we have $\lambda[i + \mathfrak{k}_\lambda] = \lambda[i]$. Let $\sim_*$ be the equivalence relation on $\mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ defined by

$$(\lambda, \bar{j}) \sim_* (\lambda[i], \overline{c \cdot \mathfrak{d}_\lambda + j}) \quad (i, c \in \mathbb{Z}),$$

where we denote by $\bar{m}$ the image of $m \in \mathbb{Z}$ in $\mathbb{Z}/p\mathbb{Z}$. Let Λ^+ be the set of equivalence classes of $\mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ with respect to the relation $\sim_*$, and we denote by $\lambda\langle j\rangle \in \Lambda^+$ the equivalence class containing $(\lambda, \bar{j}) \in \mathcal{P}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$. Thus we have $\lambda\langle j\rangle = \lambda[i]\langle j\rangle = \lambda\langle c \cdot \mathfrak{d}_\lambda + j\rangle = \lambda[i]\langle c \cdot \mathfrak{d}_\lambda + j\rangle$ for $i, c \in \mathbb{Z}$. Then it is known that,

$$E^{\dagger\lambda}\downarrow \cong E^{\dagger\lambda[i]}\downarrow \cong E^{\lambda\langle 1\rangle} \oplus \cdots \oplus E^{\lambda\langle \mathfrak{d}_\lambda\rangle} \quad (i \in \mathbb{Z}) \text{ as } \mathbb{C}W\text{-modules}, \tag{4.3.1}$$

for some simple $\mathbb{C}W$-modules $E^{\lambda\langle j\rangle}$ $(1 \le j \le \mathfrak{d}_\lambda)$, and $\{E^{\lambda\langle j\rangle} | \lambda\langle j\rangle \in \Lambda^+\}$ gives a complete set of pairwise non-isomorphic simple $\mathbb{C}W$-modules. Hence, we have

$$\mathrm{Irr}\mathbb{C}W = \{E^{\lambda\langle j\rangle} \mid \lambda\langle j\rangle \in \Lambda^+\}.$$

Moreover, we have

$$E^{\lambda\langle j\rangle}\uparrow \cong E^{\dagger\lambda[1]} \oplus \cdots \oplus E^{\dagger\lambda[\mathfrak{k}_\lambda]} \quad \text{as } W^\dagger\text{-modules}(1 \le j \le \mathfrak{d}_\lambda). \tag{4.3.2}$$

4.4. For $M^\dagger \in \mathscr{H}_X^\dagger$-mod, we denote by $M^\dagger\downarrow$ the restriction of the action to $\mathscr{H}_X$. On the other hand, for $N \in \mathscr{H}_X$-mod, we denote by $N\uparrow$ the induced module $\mathscr{H}_X^\dagger \otimes_{\mathscr{H}_X} N$.

Then, by (4.3.1), we have

$$S_K^\dagger(\lambda)\downarrow \cong S_K(\lambda\langle 1\rangle) \oplus \cdots \oplus S_K(\lambda\langle \mathfrak{d}_\lambda\rangle) \quad \text{for} \quad \lambda \in \mathcal{P}_{n,r} \tag{4.4.1}$$

and, by (4.3.2), we have

$$S_K(\lambda\langle j\rangle)\uparrow \cong S_K^\dagger(\lambda[1]) \oplus \cdots \oplus S_K^\dagger(\lambda[\mathfrak{k}_\lambda]) \quad \text{for} \quad \lambda\langle j\rangle \in \Lambda^+. \tag{4.4.2}$$

We define the group homomorphism $\mathrm{Res}_X\colon K_0(\mathscr{H}_X^\dagger\text{-mod}) \to K_0(\mathscr{H}_X\text{-mod})$ by $[M^\dagger] \mapsto [M^\dagger\downarrow]$. We also define the group homomorphism $\mathrm{Ind}_X\colon K_0(\mathscr{H}_X\text{-mod}) \to K_0(\mathscr{H}_X^\dagger\text{-mod})$ by $[N] \mapsto [N\uparrow]$. Since $\mathscr{H}_X^\dagger$ is a free right $\mathscr{H}_X$-module, induced functor from $\mathscr{H}_X$-mod to $\mathscr{H}_X^\dagger$-mod is exact. Thus Ind_X is well-defined. Then we have the following lemma.

Lemma 4.5. (i) *For $\lambda \in \mathcal{P}_{n,r}$, we have*

$$[S^\dagger(\lambda)\downarrow] = [S(\lambda\langle 1\rangle)] + \cdots + [S(\lambda\langle \mathfrak{d}_\lambda\rangle)] \quad \textit{in} \quad K_0(\mathscr{H}\textit{-mod}).$$

(ii) *For $\lambda\langle j\rangle \in \Lambda^+$, we have*

$$[S(\lambda\langle j\rangle)\uparrow] = [S^\dagger(\lambda[1])] + \cdots + [S^\dagger(\lambda[\mathfrak{k}_\lambda])] \quad \textit{in} \quad K_0(\mathscr{H}^\dagger\textit{-mod}).$$

Proof. (i) By Lemma 2.4 and (4.4.1), we have

$$\begin{aligned} d_{K,\mathbb{C}}([S_K^{\dagger}(\lambda)\downarrow]) &= d_{K,\mathbb{C}}([S_K(\lambda\langle 1\rangle)] + \cdots + [S_K(\lambda\langle \mathfrak{d}_\lambda\rangle)]) \\ &= [S(\lambda\langle 1\rangle)] + \cdots + [S(\lambda\langle \mathfrak{d}_\lambda\rangle)]. \end{aligned}$$

On the other hand, by the definition of decomposition maps, we have

$$d_{K,\mathbb{C}}([S_K^{\dagger}(\lambda)\downarrow]) = [S^{\dagger}(\lambda)\downarrow].$$

Then (i) was proven. By using (4.4.2) together with Lemma 2.4, we have (ii) in a similar way as in (i). □

4.6. We recall some relations between simple $\mathscr{H}$-modules and simple $\mathscr{H}^{\dagger}$-modules which have been studied in [8] and [16]

Let $\{S^{\dagger\lambda} \mid \lambda \in \mathcal{P}_{n,r}\}$ be the set of Specht modules of $\mathscr{H}^{\dagger}$ constructed in [5] as seen in the previous section. Then $\{D^{\dagger\lambda} \mid \lambda \in \mathcal{K}_{n,r}\}$ is a complete set of simple $\mathscr{H}^{\dagger}$-modules.

Let σ be the algebra automorphism of $\mathscr{H}^{\dagger}$ defined by $\sigma(T_0) = \xi T_0$ ($\xi = \exp(2\pi\sqrt{-1}/p)$), $\sigma(T_i) = T_i$ for $i = 1, \ldots, n-1$. Then we see that the restriction $\sigma|_{\mathscr{H}}$ of σ to $\mathscr{H}$ is the identity map on $\mathscr{H}$. We also define the algebra automorphism τ of $\mathscr{H}^{\dagger}$ by $\tau(x) = T_0^{-1} x T_0$ for $x \in \mathscr{H}^{\dagger}$. Then we have $\tau(\mathscr{H}) = \mathscr{H}$.

For $M^{\dagger} \in \mathscr{H}^{\dagger}$-mod, let $(M^{\dagger})^{\sigma}$ be the twisted $\mathscr{H}^{\dagger}$-module of M via σ. Since $\sigma|_{\mathscr{H}}$ is identity map, we have $(M^{\dagger})^{\sigma}\downarrow \cong M^{\dagger}\downarrow$ as $\mathscr{H}$-modules. Similarly, for $N \in \mathscr{H}$-mod, let N^{τ} be the twisted $\mathscr{H}$-module of N via τ.

For $\lambda \in \mathcal{K}_{n,r}$ and $i \in \mathbb{Z}$, we define $\lambda[i]^{\flat}$ by $(D^{\dagger\lambda})^{\sigma^i} \cong D^{\dagger\lambda[i]^{\flat}}$. Let $\mathfrak{k}_\lambda^{\flat}$ be the minimum positive integer such that $\lambda[\mathfrak{k}_\lambda^{\flat}]^{\flat} = \lambda$ (thus $(D^{\dagger\lambda})^{\sigma^{\mathfrak{k}_\lambda^{\flat}}} \cong D^{\dagger\lambda}$), and put $\mathfrak{d}_\lambda^{\flat} = p/\mathfrak{k}_\lambda^{\flat}$. Let D be a simple $\mathscr{H}$-submodule of $D^{\dagger\lambda}\downarrow$. Then by [8, Lemma 2.2], $\mathfrak{d}_\lambda^{\flat}$ is the minimum positive integer such that $D^{\tau^{\mathfrak{d}_\lambda^{\flat}}} \cong D$. Moreover we have, for $\lambda \in \mathcal{K}_{n,r}$ and $i = 1, \ldots, \mathfrak{k}_\lambda^{\flat}$,

$$D^{\dagger\lambda}\downarrow \cong D^{\dagger\lambda[i]^{\flat}}\downarrow \cong D \oplus D^{\tau} \oplus \cdots \oplus D^{\tau^{\mathfrak{d}_\lambda^{\flat}-1}} \quad \text{as } \mathscr{H}\text{-modules.} \tag{4.6.1}$$

Let $\sim_{\star}$ be the equivalence relation on $\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ defined by

$$(\lambda, \bar{j}) \sim_{\star} \left(\lambda[i]^{\flat}, \overline{c \cdot \mathfrak{d}_\lambda^{\flat} + j}\right) \quad (i, c \in \mathbb{Z}).$$

We denote by $(\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_{\star}$ the set of equivalence classes of $\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$ with respect to the relation $\sim_{\star}$, and we denote by $\lambda\langle j\rangle^{\flat} \in (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_{\star}$ the equivalence class containing $(\lambda, \bar{j}) \in \mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z}$. Then, by [8, Lemma 2.2] (see also [16, Proposition 2.4]),

$$\{D^{\lambda\langle j\rangle^{\flat}} \mid \lambda\langle j\rangle^{\flat} \in (\mathcal{K}_{n,r} \times \mathbb{Z}/p\mathbb{Z})/\sim_{\star}\}$$

gives a complete set of pairwise non-isomorphic simple $\mathscr{H}$-modules, where we put $D^{\lambda\langle j\rangle^\flat} = D^{\tau^j}$ for some simple $\mathscr{H}$-submodule D of $D^{\dagger\lambda}\downarrow$ (see (4.6.1)).

By [8, Lemma 2.2], we also have, for $\lambda\langle j\rangle^\flat \in (\mathcal{K}_{n,r}\times\mathbb{Z}/p\mathbb{Z})/\sim_\star$,

$$D^{\lambda\langle j\rangle^\flat}\uparrow \cong D^{\dagger\lambda[1]^\flat}\oplus\cdots\oplus D^{\dagger\lambda[\mathfrak{k}^\flat_\lambda]^\flat} \quad \text{as } \mathscr{H}^\dagger\text{-modules.} \tag{4.6.2}$$

REMARKS 4.7. (i) For $\lambda\in\mathcal{K}_{n,r}$, $\lambda[i]^\flat$ ($1\le i\le \mathfrak{k}^\flat_\lambda$) is described in [13] (the case of type D), [15] (the case of type $G(r,r,n)$) and [8], [16] (general case).
(ii) Recall that $\{D(\lambda\langle i\rangle')\mid \lambda\langle i\rangle'\in\Lambda^+_0\}$ gives a complete set of non-isomorphic simple $\mathscr{H}$-modules (Lemma 1.3). Hence, there exists the bijection η: $\Lambda^+_0\to(\mathcal{K}_{n,r}\times\mathbb{Z}/p\mathbb{Z})/\sim_\star$ such that $D(\lambda\langle i\rangle')\cong D^{\eta(\lambda\langle i\rangle')}$.

Now we have the following proposition.

Proposition 4.8. *For $\lambda\in\mathcal{P}_{n,r}$ and $\mu\in\mathcal{K}_{n,r}$, we have the following.*
(i) $\sum_{s=1}^{\mathfrak{d}_\lambda}[S(\lambda\langle s\rangle):D^{\mu\langle i\rangle^\flat}]_{\mathscr{H}} = \sum_{t=1}^{\mathfrak{k}^\flat_\mu}[S^{\dagger\lambda[j]}:D^{\dagger\mu[t]^\flat}]_{\mathscr{H}^\dagger}$ $(1\le i\le\mathfrak{d}^\flat_\mu,\ 1\le j\le\mathfrak{k}_\lambda)$.
(ii) $\sum_{s=1}^{\mathfrak{d}^\flat_\mu}[S(\lambda\langle i\rangle):D^{\mu\langle s\rangle^\flat}]_{\mathscr{H}} = \sum_{t=1}^{\mathfrak{k}_\lambda}[S^{\dagger\lambda[t]}:D^{\dagger\mu[j]^\flat}]_{\mathscr{H}^\dagger}$ $(1\le i\le\mathfrak{d}_\lambda,\ 1\le j\le\mathfrak{k}^\flat_\mu)$.

Proof. Let

$$S^{\dagger\lambda[j]} = M_k\supset M_{k-1}\supset\cdots\supset M_1\supset M_0=0$$

be a composition series of $S^{\dagger\lambda[j]}$ in $\mathscr{H}^\dagger$-mod such that $M_l/M_{l-1}\cong D^{\dagger\mu_l}$. Applying the restriction functor, we have the filtration

$$S^{\dagger\lambda[j]}\downarrow = M_k\downarrow\supset M_{k-1}\downarrow\supset\cdots\supset M_1\downarrow\supset M_0\downarrow=0$$

such that $M_l\downarrow/M_{l-1}\downarrow\cong D^{\dagger\mu_l}\downarrow$ in $\mathscr{H}$-mod. Thus, by (4.6.1), we have

$$[S^{\dagger\lambda[j]}\downarrow : D^{\mu\langle i\rangle^\flat}]_{\mathscr{H}} = \sum_{t=1}^{\mathfrak{k}^\flat_\mu}[S^{\dagger\lambda[j]}:D^{\dagger\mu[t]^\flat}]_{\mathscr{H}^\dagger}. \tag{4.8.1}$$

On the other hand, by (3.5.2) and Lemma 4.5 (i) together with $S^{\dagger\lambda}\downarrow\cong S^{\dagger\lambda[j]}\downarrow$,

$$[S^{\dagger\lambda[j]}\downarrow : D^{\mu\langle i\rangle^\flat}]_{\mathscr{H}} = \sum_{s=1}^{\mathfrak{d}_\lambda}[S(\lambda\langle s\rangle):D^{\mu\langle i\rangle^\flat}]_{\mathscr{H}}. \tag{4.8.2}$$

(4.8.1) and (4.8.2) imply (i). Next we prove (ii). Let

$$S(\lambda\langle i\rangle) = N_k\supset N_{k-1}\supset\cdots\supset N_1\supset N_0=0$$

be a composition series of $S(\lambda\langle i\rangle)$ in $\mathscr{H}$-mod such that $N_l/N_{l-1} \cong D^{\mu_l\langle j_l\rangle^\flat}$. Applying the induced functor, we have the filtration

$$S(\lambda\langle i\rangle)\uparrow = N_k\uparrow \supset N_{k-1}\uparrow \supset \cdots \supset N_1\uparrow \supset N_0\uparrow = 0$$

such that $N_l\uparrow/N_{l-1}\uparrow \cong D^{\mu_l\langle j_l\rangle^\flat}\uparrow$ in $\mathscr{H}^\dagger$-mod. Thus, by (4.6.2), we have

$$[S(\lambda\langle i\rangle)\uparrow : D^{\dagger\mu[j]^\flat}]_{\mathscr{H}^\dagger} = \sum_{s=1}^{\mathfrak{d}_\mu^\flat}[S(\lambda\langle i\rangle) : D^{\mu\langle s\rangle^\flat}]_{\mathscr{H}}. \tag{4.8.3}$$

On the other hand, by (3.5.2) and Lemma 4.5 (ii), we have

$$[S(\lambda\langle i\rangle)\uparrow : D^{\dagger\mu[j]^\flat}]_{\mathscr{H}^\dagger} = \sum_{t=1}^{\mathfrak{k}_\lambda}[S^{\dagger\lambda[t]} : D^{\dagger\mu[j]^\flat}]_{\mathscr{H}^\dagger}. \tag{4.8.4}$$

(4.8.3) and (4.8.4) imply (ii). □

4.9. Recall that "$\sim_R$" is the residue equivalence on $\mathcal{P}_{n,r}$ defined in the previous section. We define an equivalence relation "$\approx$" on $\mathcal{P}_{n,r}$ by $\lambda \approx \mu$ if $\lambda \sim_R \mu[j]$ for some $j \in \mathbb{Z}$. Put

$$\Gamma = \{\lambda \in \mathcal{P}_{n,r} \mid \lambda \not\sim_R \mu \text{ for any } \mu \in \mathcal{P}_{n,r} \text{ such that} \mu \neq \lambda\}.$$

We see easily that $\lambda \sim_R \mu$ if and only if $\lambda[i] \sim_R \mu[i]$ for any $i \in \mathbb{Z}$. Thus, we have $\lambda[i] \in \Gamma$ if $\lambda \in \Gamma$. Then we have the following proposition.

Proposition 4.10. *For $\lambda \in \mathcal{P}_{n,r} \setminus \Gamma$, we have*

$$S(\lambda\langle 1\rangle) \sim S(\lambda\langle 2\rangle) \sim \cdots \sim S(\lambda\langle \mathfrak{d}_\lambda\rangle).$$

Proof. If $\mathfrak{k}_\lambda = p$, there is nothing to prove since $\mathfrak{d}_\lambda = 1$. Hence, we assume that $\mathfrak{k}_\lambda \neq p$. First, we show the following claim.

Claim. *For $\lambda \in \mathcal{P}_{n,r} \setminus \Gamma$ such that $\mathfrak{k}_\lambda \neq p$, we can take $\mu \in \mathcal{P}_{n,r}$ such that $\lambda \sim_R \mu$, and that $\mathfrak{k}_\mu = p$ (thus $\mathfrak{d}_\mu = 1$).*

Since $\lambda \in \mathcal{P}_{n,r} \setminus \Gamma$, we can take $\mu \in \mathcal{P}_{n,r}$ such that $\lambda \sim_R \mu$ and $\mu \neq \lambda$. By [17, Theorem 2.11], it is known that $\lambda \sim_R \mu$ if and only if $\lambda \sim_J \mu$, where "$\sim_J$" is the Jantzen equivalence on $\mathcal{P}_{n,r}$ (see [17, Definition 2.8] for definitions). By the definition of the Jantzen equivalence, we may assume that μ obtained by unwrapping a rim hook r_x^λ from λ, and wrapping another rim hook r_y^μ from $[\lambda] \setminus r_x^\lambda$. Namely, we have

$[\lambda] \setminus r_x^\lambda = [\mu] \setminus r_y^\mu$ (See [17] for notations here). Suppose that $x \in \lambda^{(i)}$ and $y \in \mu^{(j)}$. Then $[\lambda] \setminus r_x^\lambda = [\mu] \setminus r_y^\mu$ implies that

$$\lambda^{(i)} \neq \mu^{(i)},\ \lambda^{(j)} \neq \mu^{(j)} \quad \text{and} \quad \lambda^{(l)} = \mu^{(l)} \text{ for } l \neq i, j. \tag{4.10.1}$$

Note that $\mu^{(i)} \neq \mu^{(j)}$ if $\lambda^{(i)} = \lambda^{(j)}$ and $i \neq j$. Thus, we have $\mu^{(i)} \neq \mu^{(l)}$ for any $l \neq i$ such that $l \equiv i \mod \mathfrak{k}_\lambda$ and $c \cdot p < l \leq (c+1) \cdot p$ when $c \cdot p < i \leq (c+1) \cdot p$. This implies that

$$\mathfrak{k}_\lambda \nmid \mathfrak{k}_\mu \quad \text{unless} \quad \mathfrak{k}_\mu = p. \tag{4.10.2}$$

In the case where p is a prime number, (4.10.2) implies $\mathfrak{k}_\mu = p$ since $\mathfrak{k}_\lambda = 1$ by $\mathfrak{k}_\lambda \mid p$ and $\mathfrak{k}_\lambda \neq p$. In the case where $p = 4$, one can easily check that $\mathfrak{k}_\mu = p$ directly. Let $p \geq 6$ be not a prime number. Assume that $\mathfrak{k}_\mu \neq p$. Then we have $\mathfrak{k}_\lambda \nmid \mathfrak{k}_\mu$ by (4.10.2). In a similar way as in the above arguments, we have $\mathfrak{k}_\mu \nmid \mathfrak{k}_\lambda$ (note that $\mathfrak{k}_\lambda \neq p$). By the conditions $p \geq 6$, $\mathfrak{k}_\lambda \nmid \mathfrak{k}_\mu$ and $\mathfrak{k}_\mu \nmid \mathfrak{k}_\lambda$, one sees that there are at least three integers x_1, x_2, x_3 such that $\lambda^{(x_l)} \neq \mu^{(x_l)}$ $(l = 1, 2, 3)$. However, this contradicts to (4.10.1). Thus we have $\mathfrak{k}_\mu = p$, and the claim was proved.

Thanks to the claim, we can take $\mu \in \mathcal{P}_{n,r}$ such that $\lambda \sim_R \mu$, and that $\mathfrak{d}_\mu = 1$. Then we can take a sequence $\lambda = \lambda_0, \dots, \lambda_k = \mu$ satisfying the following two conditions:

- $S^{\dagger\lambda_{i-1}}$ and $S^{\dagger\lambda_i}$ have a common composition factor $D^{\dagger\nu_i}$.
- There exists an integer l such that $\mathfrak{d}_{\lambda_i} \neq 1$ for any $i < l$, and that $\mathfrak{d}_{\lambda_l} = 1$.

By Proposition 4.8 (i), one sees that $S(\lambda_l\langle 1\rangle)$ has a composition factor $D^{\nu_l\langle i\rangle^\flat}$ for any $i \in \{1, \dots, \mathfrak{d}^\flat_{\nu_l}\}$ (note that $\mathfrak{d}_{\lambda_l} = 1$). On the other hand, by Proposition 4.8 (ii), one sees that $S(\lambda_{l-1}\langle j\rangle)$ $(1 \leq j \leq \mathfrak{d}_{\lambda_{l-1}})$ has a composition factor $D^{\nu_l\langle i\rangle^\flat}$ for some $i \in \{1, \dots, \mathfrak{d}^\flat_{\nu_l}\}$. Thus, we have $S(\lambda_l\langle 1\rangle) \sim S(\lambda_{l-1}\langle j\rangle)$ for any $j = 1, \dots, \mathfrak{d}_{\lambda_{l-1}}$. This implies that $S(\lambda_{l-1}\langle 1\rangle) \sim S(\lambda_{l-1}\langle 2\rangle) \sim \cdots \sim S(\lambda_{l-1}\langle \mathfrak{d}_{\lambda_{l-1}}\rangle)$. By using the (backword) inductive argument combined with Proposition 4.8, we have the proposition. □

Theorem 4.11. (i) *For $\lambda \in \Gamma$ and $i = 1, \dots, \mathfrak{d}_\lambda$, we have $S(\lambda\langle i\rangle)$ (resp. $\Delta(\lambda\langle i\rangle)$) is a simple $\mathscr{H}$-module (resp. a simple object of $\mathcal{O}$). Moreover, $S(\lambda\langle i\rangle)$ (resp. $\Delta(\lambda\langle i\rangle)$) is a block of $\mathscr{H}$ (resp. of $\mathcal{O}$) itself.*
(ii) *For $\lambda, \mu \in \mathcal{P}_{n,r} \setminus \Gamma$ and $i, j \in \mathbb{Z}$, we have*

$$\Delta(\lambda\langle i\rangle) \sim \Delta(\mu\langle j\rangle) \Leftrightarrow S(\lambda\langle i\rangle) \sim S(\mu\langle j\rangle) \Leftrightarrow \lambda \approx \mu.$$

Proof. Suppose that $S(\lambda\langle i\rangle)$ and $S(\mu\langle j\rangle)$ have a common composition factor $D^{\nu\langle k\rangle^\flat}$. Then, by Proposition 4.8 (ii), $S^{\dagger\lambda[i']}$ and $S^{\dagger\mu[j']}$ have a common composition factor $D^{\dagger\mu}$ for some i', j'. This implies that

$$S(\lambda\langle i\rangle) \sim S(\mu\langle j\rangle) \quad \text{only if} \quad \lambda \approx \mu. \tag{4.11.1}$$

(i) Suppose that $\lambda \in \Gamma$, then $S^{\dagger\lambda}$ is a simple $\mathscr{H}^\dagger$-module from the definition of Γ. If $S(\lambda\langle i\rangle) \sim S(\mu\langle j\rangle)$ for some $\mu\langle j\rangle \in \Lambda^+$, we have $\lambda \approx \mu$ by (4.11.1). This implies that there exists an integer l such that $\lambda = \mu[l]$ since $\lambda \in \Gamma$. Thus, we have $\mu\langle j\rangle = \mu[l]\langle j\rangle = \lambda\langle j\rangle$ from the definition of Λ^+. Now we may assume that $S(\lambda\langle i\rangle)$ and $S(\lambda\langle j\rangle)$ have a common composition factor $D^{\mu\langle k\rangle^\flat}$. If $\lambda\langle i\rangle \neq \lambda\langle j\rangle$ (i.e. $i \not\equiv j \bmod \mathfrak{d}_\lambda$), we have $\sum_{s=1}^{\mathfrak{d}_\lambda}[S(\lambda\langle s\rangle) : D^{\mu\langle k\rangle^\flat}]_{\mathscr{H}} \geq 2$. On the other hand, we have $\sum_{t=1}^{\mathfrak{k}_\mu^\flat}[S^{\dagger\lambda} : D^{\dagger\mu[t]^\flat}]_{\mathscr{H}^\dagger} \leq 1$ since $S^{\dagger\lambda}$ is simple. These contradict to Proposition 4.8 (i). Thus we have $\lambda\langle i\rangle = \lambda\langle j\rangle = \mu\langle j\rangle$. This implies (i).

Next we prove (ii). For $\lambda, \mu \in \mathcal{P}_{n,r} \setminus \Gamma$, suppose that $S^{\dagger\lambda}$ and $S^{\dagger\mu}$ have a common composition factor $D^{\dagger\nu}$. Then, by Proposition 4.8 (i), $S(\lambda\langle i\rangle)$ and $S(\mu\langle j\rangle)$ have a common composition factor $D^{\nu\langle l\rangle^\flat}$ for some i, j (and for any l). Thus, $S(\lambda\langle i\rangle) \sim S(\mu\langle j\rangle)$. Combining Proposition 4.10 and (4.11.1), we obtain the theorem. □

ACKNOWLEDGMENT. The author is grateful to Professors S. Ariki, T. Kuwabara, H. Miyachi and T. Shoji for many valuable discussions and comments.

References

[1] S. Ariki: *On the semi-simplicity of the Hecke algebra of* $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$, J. Algebra **169** (1994), 216–225.

[2] S. Ariki: *Representation theory of a Hecke algebra of* $G(r, p, n)$, J. Algebra **177** (1995), 164–185.

[3] S. Ariki: *On the classification of simple modules for cyclotomic Hecke algebras of type* $G(m,1,n)$ *and Kleshchev multipartitions*, Osaka J. Math. **38** (2001), 827–837.

[4] M. Broué, G. Malle and R. Rouquier: *Complex reflection groups*, *braid groups*, *Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127–190.

[5] R. Dipper, G. James and A. Mathas: *Cyclotomic* q*-Schur algebras*, Math. Z. **229** (1998), 385–416.

[6] S. Donkin, The q-Schur Algebra, London Mathematical Society Lecture Note Series **253**, Cambridge Univ. Press, Cambridge, 1998.

[7] P. Etingof and V. Ginzburg: *Symplectic reflection algebras*, *Calogero–Moser space*, *and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2002), 243–348.

[8] G. Genet and N. Jacon: *Modular representations of cyclotomic Hecke algebras of type* $G(r,p,n)$, Int. Math. Res. Not. (2006), 1–18.

[9] V. Ginzburg, N. Guay, E. Opdam and R. Rouquier: *On the category* $\mathcal{O}$ *for rational Cherednik algebras*, Invent. Math. **154** (2003), 617–651.

[10] J.J. Graham and G.I. Lehrer: *Cellular algebras*, Invent. Math. **123** (1996), 1–34.

[11] N. Guay: *Projective modules in the category* $\mathcal{O}$ *for the Cherednik algebra*, J. Pure Appl. Algebra **182** (2003), 209–221.

[12] J. Hu: *A Morita equivalence theorem for Hecke algebra* $\mathcal{H}_q(D_n)$ *when* n *is even*, Manuscripta Math. **108** (2002), 409–430.

[13] J. Hu: *Crystal bases and simple modules for Hecke algebra of type* D_n, J. Algebra **267** (2003), 7–20.

[14] J. Hu: *Modular representations of Hecke algebras of type* $G(p, p, n)$, J. Algebra **274** (2004), 446–490.

[15] J. Hu: *Crystal bases and simple modules for Hecke algebras of type* $G(p, p, n)$, Represent. Theory **11** (2007), 16–44.

[16] J. Hu: *The number of simple modules for the Hecke algebras of type* $G(r, p, n)$, J. Algebra **321** (2009), 3375–3396.

[17] S. Lyle and A. Mathas: *Blocks of cyclotomic Hecke algebras*, Adv. Math. **216** (2007), 854–878.

[18] A. Matahs: *The representation theory of the Ariki–Koike and cyclotomic q-Schur algebras*; In Representation Theory of Algebraic Groups and Quantum Groups, Adv. Stud. Pure Math. **40**, Math. Soc. Japan, Tokyo, 261–320, 2004.

[19] A. Mathas: Iwahori–Hecke Algebras and Schur Algebras of the Symmetric Group, University Lecture Series **15**, Amer. Math. Soc., Providence, RI, 1999.

[20] A. Ram: *Seminormal representations of Weyl groups and Iwahori–Hecke algebras*, Proc. London Math. Soc. (3) **75** (1997), 99–133.

[21] R. Rouquier: *q-Schur algebras and complex reflection groups*, Mosc. Math. J. **8** (2008), 119–158.

Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya, 464-8602
Japan
e-mail: kentaro-wada@math.nagoya-u.ac.jp

Current address:
Department of Mathematics
Faculty of Science
Shinshu University
Asahi 3-1-1, Matsumoto 390-8621
Japan
e-mail: wada@math.shinshu-u.ac.jp