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# AN ANALOGY BETWEEN REPRESENTATIONS OF KNOT GROUPS AND GALOIS GROUPS

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**Abstract**

We discuss an analogy between the deformations of hyperbolic structures on a knot complement and of certain nearly ordinary Galois representations.

## 0. Introduction

Let $p$ be an odd prime number, and let $G_p$ be the Galois group of the maximal algebraic extension $\mathbb{Q}_{\{p,\infty\}}$ of $\mathbb{Q}$ unramified outside $p$ and the infinite place $\infty$. A two-dimensional continuous representation $\varrho$ of $G_p$ is called *nearly $p$-ordinary* if its restriction to a decomposition subgroup $D_p$ at $p$ is conjugate to an upper triangular representation:

$$\varrho|_{D_p} \sim \begin{pmatrix} \varepsilon_\varrho & u_\varrho \\ 0 & \delta_\varrho \end{pmatrix},$$

where $\varepsilon_\varrho$ and $\delta_\varrho$ are characters and $u_\varrho$ is a continuous function of $D_p$. In addition, $\varrho$ is called *$p$-ordinary* if $\varepsilon_\varrho|_{I_p} = 1$, *nearly $p$-extraordinary* if $u_\varrho = 0$, or *$p$-extraordinary* if $\varrho$ is $p$-ordinary and nearly $p$-extraordinary, where $I_p$ is the inertia subgroup at $p$ (also see Definition 1.2 in Subsection 1.1).

Let $\mathbf{k}$ be a finite field of characteristic $p$, and let

$$\bar{\varrho}\colon G_p \to GL_2(\mathbf{k})$$

be a continuous, absolutely irreducible, odd, and $p$-ordinary Galois representation. Then we assume that $\det\bar{\varrho}|_{I_p} \neq 1$ (also see Assumption 1.3 in Subsection 1.1). Moreover, we assume that the deformation theory of $\bar{\varrho}$ is *cleanly unobstructed* (for the definition, see Assumption 2.5 in Subsection 2.2), and the following

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ASSUMPTION 0.1. (1) If $\bar{\varrho}$ is *full*, i.e., the image of $\bar{\varrho}$ contains $SL_2(\mathbf{k})$, then $\det\bar{\varrho}$ is isomorphic to neither $\mu$, $\mu^{-1}$ nor $\mu^{(p-1)/2}$, where $\mu\colon G_p \to \mathbb{F}_p^\times$ is the mod $p$ cyclotomic character.
(2) If $\bar{\varrho}$ is *tame*, i.e., if the order $q$ of the image of $\bar{\varrho}$ is prime to $p$, then $q$ is not divisible by $p-1$.

Here we denote by $\mathbb{C}_p$ a $p$-adic completion of an algebraic closure of $\mathbb{Q}_p$, and throughout this paper, we fix an embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$. Thus the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$ is also considered as a subfield of $\mathbb{C}_p$, and any extension of $\mathbb{Q}_p$ will be considered in $\mathbb{C}_p$. Because $\bar{\varrho}$ is cleanly unobstructed, the *universal ordinary deformation space* $\mathfrak{X}_p^{2,\mathrm{o}}$ of $\bar{\varrho}$ is locally $p$-adic bianalytic to $\mathbb{C}_p$. If $\bar{\varrho}$ is also $p$-extraordinary, then we see that the *universal nearly extraordinary deformation space* $\mathfrak{X}_p^{2,\mathrm{neo}}$ of $\bar{\varrho}$ is locally $p$-adic bianalytic to $\mathbb{C}_p$ by applying some results in [20] (see Subsection 2.3). Let $\mathfrak{X}_p$ be either $\mathfrak{X}_p^{2,\mathrm{o}}$ or $\mathfrak{X}_p^{2,\mathrm{neo}}$. We denote by $x_{\boldsymbol{\varrho}}$ the point corresponding to a deformation $\boldsymbol{\varrho}$ when we identify $\mathfrak{X}_p$ with a $p$-adic analytic space.

Our purpose in this paper is to show the following

**Theorem 0.2.** *Let $x_0$ in $\mathfrak{X}_p$ be a $p$-ordinary point. Then there exists an element $\tau$ of $I_p$ such that the map*

$$\mathfrak{X}_p \to \mathbb{C}_p;\ x_{\boldsymbol{\varrho}} \mapsto \operatorname{tr}\boldsymbol{\varrho}(\tau)$$

*is $p$-adic bianalytic in a neighborhood of $x_0$, where $\tau$ is an element of $I_p$ that is called "monodromy over $p$" (see* Subsection 2.1).

Theorem 0.2 is an arithmetic analogue of a theorem (Theorem 0.3 below) on the deformations of hyperbolic structures on a knot complement.

Before we introduce the statement of Theorem 0.3, we explain the background of this paper. Arithmetic topology is a study that views 3-dimensional topology and algebraic number theory as analogies from the viewpoint of group theory and Galois theory, which have appeared recently in the classification of mathematics. It was first described in some works by B. Mazur in 1960s, and has been developed by M. Kapranov, M. Morishita and A. Reznikov etc. since the latter half of 1990s. That fundamental concept is based on analogies between knots and prime numbers. We recall a part of basic analogies (for a precise account, see Morishita [18]):

| 3-dimensional topology | | algebraic number theory |
|---|---|---|
| $K$: *knot* | $\longleftrightarrow$ | $p$: *prime* |
| $S^1 \hookrightarrow \mathbb{R}^3 \cup \{\infty\} = S^3$ | | $\mathrm{Spec}\,\mathbb{F}_p \hookrightarrow \mathrm{Spec}\,\mathbb{Z} \cup \{\infty\}$ |
| $\pi_1(S^1) = \langle l \rangle \simeq \mathbb{Z}$ | $\longleftrightarrow$ | $\pi_1^{\text{ét}}(\mathrm{Spec}\,\mathbb{F}_p) \simeq \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) = \langle \sigma \rangle \simeq \hat{\mathbb{Z}}$ |
| $\pi_1(S^3) = \{1\}$ | $\longleftrightarrow$ | $\pi_1^{\text{ét}}(\mathrm{Spec}\,\mathbb{Z}) \simeq \mathrm{Gal}(\mathbb{Q}^{\mathrm{unr}}/\mathbb{Q}) = \{1\}$ |
| $G_K := \pi_1(S^3 \setminus \mathrm{Int}(V_K)) = \pi_1(S^3 \setminus K)$ ($V_K$: a *tubular neighborhood* of $K$) | $\longleftrightarrow$ | $G_p := \mathrm{Gal}(\mathbb{Q}_{\{p,\infty\}}/\mathbb{Q}) \simeq \pi_1^{\text{ét}}(\mathrm{Spec}\,\mathbb{Z} \setminus \{(p)\})$ |
| $\partial V_K \hookrightarrow V_K$ | $\longleftrightarrow$ | $\mathrm{Spec}\,\mathbb{Q}_p \hookrightarrow \mathrm{Spec}\,\mathbb{Z}_p$ |
| $D_K := \pi_1(\partial V_K) \simeq \langle l, m \mid [m, l] = 1 \rangle$ | $\longleftrightarrow$ | $D_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \simeq \pi_1^{\text{ét}}(\mathrm{Spec}\,\mathbb{Q}_p)$ |
| $\pi_1(V_K) \simeq \pi_1(S^1) = \langle l \rangle \simeq \mathbb{Z}$ | $\longleftrightarrow$ | $\pi_1^{\text{ét}}(\mathrm{Spec}\,\mathbb{Z}_p) \simeq \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{unr}}/\mathbb{Q}_p) \simeq \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \simeq \pi_1^{\text{ét}}(\mathrm{Spec}\,\mathbb{F}_p) = \langle \sigma \rangle \simeq \hat{\mathbb{Z}}$ |
| $\{1\} \to I_K := \langle m \rangle \to D_K \to \langle l \rangle \to \{1\}$ | $\longleftrightarrow$ | $\{1\} \to I_p \to D_p \to \langle \sigma \rangle \to \{1\}$ |

It is known that the *knot group* $G_K$ of $K$ reflects the tangled condition of $K$. On the other hand, the "tangled condition" of $p$ is reflected in the Galois group $G_p$. Under such an analogy, the following question arises: is there an analogy between the theories of representations of groups $G_K$ and $G_p$? In the case of one-dimensional representations, an answer of the question is an analogy between the classical Alexander–Fox theory and the classical Iwasawa theory (see Morishita [18] etc.). As a non-abelian generalization of this analogy, K. Fujiwara first pointed out some analogies between deformation theory of hyperbolic structures on a knot complement and of $p$-ordinary Galois representations that were mainly developed by H. Hida and B. Mazur in [5] (see also Mazur [16]). Next, M. Morishita concretely formulated such analogies in [17] (see also Morishita–Terashima [19] and Morishita [18]).

In order to explain his works in detail, we recall some basic results in Culler–Shalen [4]. Let $K$ in $S^3$ be a hyperbolic knot. Then we have the holonomy representation

$$\varrho_h \colon G_K \to PSL_2(\mathbb{C})$$

associated to the hyperbolic structure on $S^3 \setminus \mathrm{Int}(V_K)$. It is known that $\varrho_h$ can be lifted to a representation in $SL_2(\mathbb{C})$. The character space $\mathfrak{X}_K := \mathrm{Hom}(G_K, SL_2(\mathbb{C}))//SL_2(\mathbb{C})$ is an affine complex algebraic set. We consider $\mathfrak{X}_K$ as a deformation space of $\varrho_h$. Here, for a tubular neighborhood $V_K$ of $K$, we call $D_K = \pi_1(\partial V_K)$ the *peripheral subgroup* of $G_K$, which is a free abelian group generated by a *meridian* $m$ and a *longitude* $l$ of $K$, and the subgroup $I_K$ generated by only $m$ the *inertia subgroup* of $G_K$. The triple $(G_K, m, l)$ is called the *peripheral system* of $K$ (see Burde–Zieschang [3]). It should be noted that the restriction of a $SL_2(\mathbb{C})$-representation $\varrho$ of $G_K$ to $D_K$ is conjugate to an upper triangular representation because $D_K$ is abelian.

We denote by $\chi_\varrho$ the character of $\varrho$. M. Morishita gave an arithmetic analogue (see Theorem 1.4 in Subsection 1.2) of the following

**Theorem 0.3** (cf. Thurston [23], Zhou [25]). *If $\chi_0$ in $\mathfrak{X}_K$ is the character of a lift of $\varrho_h$, then the map*

$$\mathfrak{X}_K \to \mathbb{C};\ \chi_\varrho \mapsto \operatorname{tr}\varrho(m)$$

*is bianalytic in a neighborhood of $\chi_0$.*

On the other hand, we studied deformation theory of nearly $p$-extraordinary representations of $G_p$ in our previous work [20]. In this paper, we focus on the fact that $\varrho_h$ is locally abelian, and give another arithmetic analogue of Theorem 0.3 not only in the case where $\bar{\varrho}$ is $p$-ordinary but also in the case where $\bar{\varrho}$ is nearly $p$-extraordinary by applying some results in [20] (see Theorems 2.6 and 2.8 in Subsection 2.3). In the proof, we give an arithmetic analogue of the peripheral system of $K$. Theorem 0.2 follows from Theorems 2.6 and 2.8.

The remainder of this paper is organized as follows. Section 1 presents some preliminaries. In Subsection 1.1, we recall some basic results on deformation theory of Galois representations, and in Subsection 1.2, we review an arithmetic analogue of Theorem 0.3 given by Morishita [17]. Section 2 presents the proof of Theorem 0.2. In Subsection 2.1, we introduce two elements of a certain quotient of $D_p$, which give an arithmetic analogue of a meridian $m$ and a longitude $l$ of $K$. In Subsection 2.2, we study the universal nearly ordinary deformation space $\mathfrak{X}_p^{2,\mathrm{no}}$ of $\bar{\varrho}$. Our space $\mathfrak{X}_p$ is a subspace of $\mathfrak{X}_p^{2,\mathrm{no}}$. Next, we construct a $p$-adic analytic map $\Phi$ from $\mathfrak{X}_p^{2,\mathrm{no}}$ to $\mathbb{C}_p$. In Subsection 2.3, we give the proof by using this map. In Section 3, we introduce some examples of $\bar{\varrho}$ that satisfies all assumptions in Theorem 0.2.

## 1. Preliminaries

**1.1. Review of Mazur's deformation theory.** Let $p$ be an odd prime number, and let $\mathbf{k}$ be a finite field of characteristic $p$. Let $n$ be a positive integer, and let

$$\bar{\varrho}\colon G_p \to GL_n(\mathbf{k})$$

be a continuous homomorphism. Throughout this subsection, we fix $p$, $\mathbf{k}$, and $\bar{\varrho}$.

Let $\mathcal{C}$ be the category of complete noetherian local rings with residue field $\mathbf{k}$. A morphism of $\mathcal{C}$ is a homomorphism of complete noetherian local rings inducing the identity on residue fields. Two liftings $\varrho$ and $\varrho'$ of $\bar{\varrho}$ to an object $R$ in $\mathcal{C}$ are called *strictly equivalent* if they are conjugate by an element of the kernel of the homomorphism $GL_n(R) \to GL_n(\mathbf{k})$. A strict equivalence class of liftings of $\bar{\varrho}$ to $R$ is called a *deformation* of $\bar{\varrho}$ to $R$. By Mazur [13, Proposition 1], if $\bar{\varrho}$ is absolutely irreducible, then there exist a *universal* *deformation ring* $R^n$ in $\mathcal{C}$ and a *universal deformation*

$$\boldsymbol{\varrho}^n : G_p \to GL_n(R^n)$$

of $\bar{\varrho}$ that satisfy the following universal property: for any given object $R$ in $\mathcal{C}$ and any deformation $\boldsymbol{\varrho}$ to $R$ of $\bar{\varrho}$, there exists a unique morphism $\varphi\colon R^n \to R$ such that the composition of $\boldsymbol{\varrho}^n$ with the induced homomorphism $\varphi\colon GL_n(R^n) \to GL_n(R)$ is equal to $\boldsymbol{\varrho}$ as a deformation. The ring $R^n$ is uniquely determined up to canonical isomorphism. We put

$$\mathfrak{X}_p^n := \mathrm{Hom}^{\mathrm{cont}}_{W(\mathbf{k})\text{-alg}}(R^n, \mathbb{C}_p),$$

and call it the *universal deformation space* of $\bar{\varrho}$, where $W(\mathbf{k})$ is the ring of Witt vectors of $\mathbf{k}$.

Throughout this paper, we denote by $\Gamma$ the maximal $p$-profinite abelian quotient of $G_p$, and fix a topological generator $\gamma$ of $\Gamma$. Let $\Lambda_{\mathbf{k}}$ be the Iwasawa algebra $W(\mathbf{k})[[\Gamma]]$. Let $g_p$ be the image of an element $g$ in $G_p$ by the natural surjection $G_p$ onto $\Gamma$, and let $[g_p]$ be the image of $g_p$ by the natural injection of $\Gamma$ into $\Lambda_{\mathbf{k}}^{\times}$.

When $n = 1$, the universal deformation ring $R^1$ and the universal deformation $\boldsymbol{\varrho}^1$ of $\bar{\varrho}$ are described explicitly in the following

**Lemma 1.1** (Mazur [13], §1.4). *Let $\bar{\varrho}\colon G_p \to \mathbf{k}^{\times}$ be a character, and let $\tilde{\varrho}\colon G_p \to W(\mathbf{k})^{\times}$ be the Teichmüller lifting of $\bar{\varrho}$. Then the universal deformation ring $R^1$ of $\bar{\varrho}$ is isomorphic to $\Lambda_{\mathbf{k}}$, and the universal deformation $\boldsymbol{\varrho}^1$ of $\bar{\varrho}$ is given by*

$$\boldsymbol{\varrho}^1(g) = \tilde{\varrho}(g) \cdot [g_p]$$

*for an element $g$ in $G_p$.*

Next, in order to discuss the case where $n = 2$, we restrict the deformation theory of $\bar{\varrho}$ to deformations of $\bar{\varrho}$ that satisfy prescribed conditions.

DEFINITION 1.2. When $L$ is an arbitrary Galois extension over $\mathbb{Q}$, we denote by $D_p$ and $I_p$ a decomposition subgroup and the inertia subgroup at $p$ in $\mathrm{Gal}(L/\mathbb{Q})$ respectively. Let $R$ be an object in $\mathcal{C}$. Then a two-dimensional representation

$$\varrho\colon \mathrm{Gal}(L/\mathbb{Q}) \to GL_2(R)$$

or its representation space $V$ is called *nearly $p$-ordinary* if there exists a non-trivial $D_p$-stable sub-$R$-module $V_1$ that is a direct summand of $V$ of $R$-rank 1. Moreover, if there exists a non-trivial $D_p$-stable sub-$R$-module $V_2$ of $R$-rank 1 such that $V = V_1 \oplus V_2$, then we say that $\varrho$ (or $V$) is *nearly $p$-extraordinary*. In addition, if $V_1$ is exactly equal to the sub-$R$-module $V^{I_p}$ of $I_p$-invariants of $V$, then we say that $\varrho$ (or $V$)

is *p-ordinary*, *p-extraordinary* respectively. If $p$ is understood, we refer to $p$-ordinary (resp. $p$-extraordinary) as simply ordinary (resp. extraordinary).

If $\varrho$ is nearly $p$-ordinary, then, up to conjugation, the restriction of $\varrho$ to $D_p$ has the form

$$\begin{pmatrix} \varepsilon_\varrho & u_\varrho \\ 0 & \delta_\varrho \end{pmatrix},$$

for characters $\varepsilon_\varrho$, $\delta_\varrho$: $D_p \to R^\times$, and a continuous function $u_\varrho$: $D_p \to R$. If $\varrho$ is nearly $p$-extraordinary, then $u_\varrho$ is a zero function. In addition, if $\varrho$ is $p$-ordinary, then $\varepsilon_\varrho$ is trivial on $I_p$.

Let $\mathcal{D}$ be the condition being either nearly $p$-ordinary, $p$-ordinary, nearly $p$-extraordinary or $p$-extraordinary. Then we assume that $\bar{\varrho}$ satisfies the condition $\mathcal{D}$, and the following

ASSUMPTION 1.3. $\varepsilon_{\bar{\varrho}}$ is not equal to $\delta_{\bar{\varrho}}$ on $I_p$.

If $\bar{\varrho}$ is absolutely irreducible and satisfies Assumption 1.3, then there exist a *universal $\mathcal{D}$-deformation ring* $R^{2,\mathcal{D}}$ in $\mathcal{C}$ and a *universal $\mathcal{D}$-deformation* $\boldsymbol{\varrho}^{2,\mathcal{D}}$ of $\bar{\varrho}$ such that any $\mathcal{D}$-deformation $\boldsymbol{\varrho}$ of $\bar{\varrho}$ to $R$ in $\mathcal{C}$ is induced from $\boldsymbol{\varrho}^{2,\mathcal{D}}$ by a unique morphism $R^{2,\mathcal{D}} \to R$ by Hida [11, Proposition 5.38], Mazur [13, Proposition 3], and [20, Theorem 0.3]. We put

$$\mathfrak{X}_p^{2,\mathcal{D}} := \mathrm{Hom}^{\mathrm{cont}}_{W(\mathbf{k})\text{-alg}}(R^{2,\mathcal{D}}, \mathbb{C}_p),$$

and call it the *universal $\mathcal{D}$-deformation space* of $\bar{\varrho}$.

**1.2. Review of Morishita's analogy.** Let $p$ be an odd prime number, and let

$$\bar{\varrho}\colon G_p \to GL_2(\mathbb{F}_p)$$

be a continuous, absolutely irreducible, odd, and nearly $p$-ordinary Galois representation. Then we assume that $\bar{\varrho}$ satisfies Assumption 1.3, and fix a continuous character $\delta$: $G_p \to \mathbb{Z}_p^\times$ so that the mod $p$ reduction of $\delta$ is $\det\bar{\varrho}$. By Hida [11, §5.4.4], we have the *universal nearly ordinary deformation ring* $R^{2,\mathrm{no},\delta}$ *of* $\bar{\varrho}$ *with fixed determinant* $\delta$.

Recall that the restriction of a $SL_2(\mathbb{C})$-representation $\varrho$ of a knot group $G_K$ to its peripheral subgroup $D_K$ is conjugate to an upper triangular representation. Morishita's concept in [17] is that $\mathrm{Hom}^{\mathrm{cont}}_{W(\mathbf{k})\text{-alg}}(R^{2,\mathrm{no},\delta}, \mathbb{C}_p)$ may be thought of as an arithmetic analogue of the character space $\mathfrak{X}_K$ in Theorem 0.3 in the introduction. Because $R^{2,\mathrm{no},\delta}$ is isomorphic to the universal ordinary deformation ring $R^{2,\mathrm{o}}$ of $\bar{\varrho}$, the space $\mathrm{Hom}^{\mathrm{cont}}_{W(\mathbf{k})\text{-alg}}(R^{2,\mathrm{no},\delta}, \mathbb{C}_p)$ can be identified with the universal ordinary deformation space $\mathfrak{X}_p^{2,\mathrm{o}}$ of $\bar{\varrho}$ if $\bar{\varrho}$ is $p$-ordinary by Hida [11, Proposition 5.43 (ii)].

On the other hand, $\bar{\varrho}$ is *modular*, i.e., isomorphic to the mod $p$ reduction of a $p$-adic representation $\varrho_f$ associated to a $p$-ordinary Hecke eigenform $f$ by Khare [12, Theorem 1.1]. Let $\mathbb{T}^{2,\mathrm{o}}$ be the *universal ordinary Hecke algebra* for $\bar{\varrho}$ (for the definition, see Mazur [14, §6, p. 125, Definition]). Then $\mathbb{T}^{2,\mathrm{o}}$ is a finite flat $\Lambda_{\mathbf{k}}$-algebra. Moreover, from the construction of $\mathbb{T}^{2,\mathrm{o}}$, we have a mapping from $R^{2,\mathrm{o}}$ to $\mathbb{T}^{2,\mathrm{o}}$, and it is known that the mapping is surjective. Here we assume that $R^{2,\mathrm{o}}$ is isomorphic to $\mathbb{T}^{2,\mathrm{o}}$. Note that it is conjectured that this isomorphism always holds (see Mazur [14, §6, Conjecture]). In fact, many cases of this conjecture have now been proved thanks to the work of Wiles [24] (and others). For example, if

- $p \geq 5$,
- $\bar{\varrho}$ is absolutely irreducible when restricted to $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$, and
- Assumption 1.3 is satisfied on $\bar{\varrho}$,

then the conjecture is true by Hida [11, Theorem 5.29].

Let $\mathbb{Q}_\infty$ be the subextension of the maximal algebraic extension of $\mathbb{Q}$ unramified outside $p$ and the infinite place $\infty$ with Galois group being isomorphic to $\mathbb{Z}_p$. Then we have the following

**Theorem 1.4** (Morishita [17], (8.4)). *Let $\bar{\varrho}$ be as above, and let $x_0$ be an arithmetic point of $\mathfrak{X}_p^{2,\mathrm{o}}$. Take an element $\gamma$ in $I_p$ that is mapped to a generator of* $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. *Then the map*

$$\mathfrak{X}_p^{2,\mathrm{o}} \to \mathbb{C}_p;\ x_{\boldsymbol{\varrho}} \mapsto \operatorname{tr} \boldsymbol{\varrho}(\gamma)$$

*is $p$-adic bianalytic in a neighborhood of $x_0$.*

## 2. Proof of the main theorem

**2.1. Arithmetic analogue of the peripheral system of a knot.** Let $p$ be an odd prime number, let $\mathbf{k}$ be a finite field of characteristic $p$. Let

$$\bar{\varrho}\colon G_p \to GL_2(\mathbf{k})$$

be a continuous, absolutely irreducible, odd, and $p$-ordinary Galois representation. Then we assume that $\bar{\varrho}$ satisfies Assumption 1.3. Let $L$ be the splitting field of $\bar{\varrho}$ and let $L^{(p)}$ be the maximal pro-$p$ extension of $L$ unramified outside $p$ and $\infty$. We put $P := \mathrm{Gal}(L^{(p)}/L)$, $C := \mathrm{Gal}(L/\mathbb{Q})$, and $\Pi := \mathrm{Gal}(L^{(p)}/\mathbb{Q})$. Then we have the short exact sequence

$$1 \to P \to \Pi \to C \to 1.$$

The quotient $\Pi$ of $G_p$ is called the *$p$-completion of $G_p$ relative to $\bar{\varrho}$*. It is known that all liftings $\varrho\colon G_p \to GL_2(R)$ of $\bar{\varrho}$ to $R$ in $\mathcal{C}$, factor through $\Pi$ (see Boston [1, p. 182]). Throughout this subsection, we regard $\bar{\varrho}$ as a homomorphism from $\Pi$.

We consider the associated inertia subgroup $I$ and decomposition subgroup $D$ in $\Pi$ relative to the fixed embedding $v\colon \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, so that $I$ is the image of the inertia subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ in $\Pi$ (via the map induced by $v$) and $D$ is the image of the full group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $I^0$ and $D^0$ be pro-$p$ Sylow subgroups in $I$ and $D$ respectively. Because $\bar{\varrho}$ is $p$-ordinary, the images of $I$ and $D$ by $\bar{\varrho}$ are contained in the subgroup $B_2'(\mathbf{k})$ of upper triangular matrices of the form $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ and in the subgroup $B_2(\mathbf{k})$ of upper triangular matrices, respectively. The subgroup $U_2(\mathbf{k})$ of matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is the normal $p$-Sylow subgroup in $B_2'(\mathbf{k})$ and $B_2(\mathbf{k})$. We denote by $\pi'$ the natural homomorphism from $B_2'(\mathbf{k})$ to $B_2'(\mathbf{k})/U_2(\mathbf{k})$ and by $\pi$ the one from $B_2(\mathbf{k})$ to $B_2(\mathbf{k})/U_2(\mathbf{k})$. Then $I^0$ is isomorphic to the kernel of $\pi' \circ \bar{\varrho}|_I$ and $D^0$ is isomorphic to the kernel of $\pi \circ \bar{\varrho}|_D$. Thus we see that $I^0$ is normal in $I$ and $D^0$ is normal in $D$. Moreover, $A := I/I^0$ is a cyclic group of order prime to $p$, and $B := D/D^0$ is an abelian group of order prime to $p$. The natural inclusion $I \subset D$ induces an injection $A \hookrightarrow B$. By Schur–Zassenhaus' theorem (see Boston [1, Proposition (2.1)]), we have a lifting $A \hookrightarrow I$ and a compatible lifting $B \hookrightarrow D$. Fix such liftings, and identify $A$ (resp. $B$) with its image in $I$ (resp. $D$). Then we have semi-direct product decompositions $I = A \ltimes I^0$ and $D = B \ltimes D^0$.

Let $F_v$ be the intermediate field in the extension $\bar{\mathbb{Q}}_p/\mathbb{Q}_p$ that is fixed by the kernel of the natural mapping from $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ to $B$ such that $\mathrm{Gal}(F_v/\mathbb{Q}_p)$ is isomorphic to $B$. Then we have the following

**Lemma 2.1** (Mazur [14], §8, Lemma). *There exist elements $\tau$ and $\nu$ in $I^0$ and $\sigma$ in $D^0$ with the following properties*:

(1) *The subgroup $B \hookrightarrow D$ is in the centralizer of $\sigma$.*

(2) *If $F_v$ contains no primitive $p$-th root of* 1, *then the element $\nu$ is trivial. Otherwise, it satisfies the following commutation relation with elements of $B$: for $g \in B$,*

$$g\nu g^{-1} = \nu^{\tilde{\mu}(g)},$$

*where $\tilde{\mu}\colon B \to \mathbb{Z}_p^\times$ is the Teichmüller lifting of the mod $p$ cyclotomic character $\mu\colon B \to \mathbb{F}_p^\times$, which defines the natural action of $B$ on the subgroup of $p$-th roots of unity in $F_v$. Exponentiation above refers to the operation of raising an element in a pro-$p$-group to a $p$-adic unit power.*

(3) *The elements $\{g\tau g^{-1}$ $(g \in B)$, $\nu$, and $\sigma\}$ generate $D^0$ as a pro-$p$ group.*

(4) *The closed normal subgroup generated by the elements $\{g\tau g^{-1}$ $(g \in B)$, and $\nu\}$ is equal to $I^0$.*

After conjugating $\bar{\varrho}$, if necessary, we may assume that the image of $B \hookrightarrow D$ by $\bar{\varrho}$ is a subgroup of diagonal matrices in $GL_2(\mathbf{k})$ and that $A \subset B$ maps to matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$. Let $R^2$ be the universal deformation ring of $\bar{\varrho}$. Then we may regard the universal deformation of $\bar{\varrho}$ as a homomorphism

$$\varrho^2\colon \Pi \to GL_2(R^2),$$

that is determined only up to strict equivalence. We can choose $\varrho^2$ in its strict equivalence class so that the image of $B$ by $\varrho^2$ lies in the image in $GL_2(R^2)$ of the subgroup of diagonal matrices of $GL_2(W(\mathbf{k}))$, where the mapping $GL_2(W(\mathbf{k})) \to GL_2(R^2)$ is induced from the natural homomorphism $W(\mathbf{k}) \to R^2$. Moreover, we may arrange it, so that the image of $A$ by $\varrho^2$ lies in the image in $GL_2(R^2)$ of the subgroup of diagonal matrices of $GL_2(W(\mathbf{k}))$ of the form $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$.

We now assume Assumption 0.1. Let $\tau$, $\nu$ and $\sigma$ be elements in $D^0$ having the properties stipulated in Lemma 2.1. First, we see that $\varrho^2(\sigma)$ is a diagonal matrix in $GL_2(R^2)$ by Lemma 2.1 (1) and Assumption 1.3. Second, we obtain the following

CLAIM 2.2. $\varrho^2(\nu) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. If $\bar{\varrho}$ is full, this claim is the same as Mazur [14, §9, Claim]. If $\bar{\varrho}$ is tame, it follows from Lemma 2.1 (2) directly because we see that $F_\upsilon$ contains no primitive $p$-th root of 1 by Assumption 0.1 (ii). Hence $\nu$ is equal to 1. □

Finally, we consider $\varrho^2(\tau)$. We put

$$\varrho^2(\tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $R^{2,\mathrm{o}}$ be the universal ordinary deformation ring of $\bar{\varrho}$. We denote by $\pi^{\mathrm{o}}$ the natural epimorphism $R^2 \twoheadrightarrow R^{2,\mathrm{o}}$, and by $\varrho^{2,\mathrm{o}}$ the induced homomorphism $\pi^{\mathrm{o}}_* \circ \varrho^2$. Because $\varrho^{2,\mathrm{o}}(\tau)$ is contained in the subgroup of matrices of the form $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ in $GL_2(R^{2,\mathrm{o}})$, the epimorphism $\pi^{\mathrm{o}}$ factors through $R' := R^2/(a-1, c)$. We denote by $\pi'$ the natural projection $R^2 \twoheadrightarrow R'$, and by $\pi_1$ the mapping $R' \to R^{2,\mathrm{o}}$ induced by $\pi^{\mathrm{o}}$. Let

$$\varrho' \colon \Pi \to GL_2(R')$$

be the induced homomorphism $\pi'_* \circ \varrho^2$. Then we see that $\varrho'$ is $p$-ordinary by Lemma 2.1 (3). Therefore, by the universality of $R^{2,\mathrm{o}}$, there exists a unique homomorphism $\pi_2\colon R^{2,\mathrm{o}} \to R'$ with $\pi' = \pi_2 \circ \pi^{\mathrm{o}}$. Thus we obtain an endomorphism $\pi_1 \circ \pi_2$ on $R^{2,\mathrm{o}}$. Because $R^{2,\mathrm{o}}$ is noetherian, and $\pi_1 \circ \pi_2$ is surjective, we see that it is an isomorphism. Hence $(\pi_1 \circ \pi_2)_* \circ \varrho^{2,\mathrm{o}}$ is equal to $\varrho^{2,\mathrm{o}}$ up to conjugation. Then we have $(\pi_1 \circ \pi_2)_* \circ \operatorname{tr} \varrho^{2,\mathrm{o}} = \operatorname{tr} \varrho^{2,\mathrm{o}}$. Because $\bar{\varrho}$ is absolutely irreducible, we see that $(\pi_1 \circ \pi_2)_*$ is the identity by Mazur [15, §25, Proposition 1]. Hence we see that $R^{2,\mathrm{o}}$ is equal to $R^2/(a-1, c)$.

Similarly, we let $R^{2,\mathrm{eo}}$ (resp. $R^{2,\mathrm{neo}}$) be the universal (resp. nearly) extraordinary deformation ring of $\bar{\varrho}$ when $\bar{\varrho}$ is $p$-extraordinary. Then we see that $R^{2,\mathrm{neo}}$ is equal to $R^2/(b, c)$, and that $R^{2,\mathrm{eo}}$ is equal to $R^2/(a-1, b, c)$. Hence we obtain the following

**Proposition 2.3** (cf. [20], Corollary 2.2). *If $\bar{\varrho}$ is $p$-extraordinary and satisfies* Assumptions 1.3 *and* 0.1, *then the kernel of the natural epimorphism from $R^{2,\mathrm{o}}$ to $R^{2,\mathrm{eo}}$ of $\bar{\varrho}$ is a principal ideal.*

Thus, only $\tau$ and $\sigma$ are essential generators of a decomposition subgroup in the deformation theory of $\bar{\varrho}$. As we have stated in the introduction, the generator $\sigma$ that comes from Frobenius element over $p$ is an arithmetic analogue of a longitude $l$ of a knot $K$. Another generator $\tau$ in a quotient of $I_p$ may be thought of as an arithmetic analogue of a meridian $m$ of $K$. In particular, if $\bar{\varrho}$ is $p$-extraordinary, then $\varrho^{2,\mathrm{neo}}([\tau,\sigma])$ is trivial in $GL_2(R^2)$, where $\varrho^{2,\mathrm{neo}}$ is the homomorphism $\pi_*^{\mathrm{neo}}\circ\varrho^2$ induced by $\pi^{\mathrm{neo}}\colon R^2 \twoheadrightarrow R^{2,\mathrm{neo}}$. Recall that, in knot theory, the triple $(G_K, m, l)$ consisting of the knot group $G_K$, a meridian $m$ and a longitude $l$ of a knot $K$ is called the peripheral system of $K$. We consider the triple $(\Pi, \tau, \sigma)$ to be an arithmetic analogue of the peripheral system of $K$. Here, we call an element of $I_p$ (resp. $D_p$) that maps to $\tau$ (resp. $\sigma$) by the natural mapping *monodromy over* $p$ (resp. *Frobenius over* $p$).

**2.2. Universal nearly ordinary deformation space.** Let $p \geq 5$ be a prime number, and let $\mathbf{k}$ be a finite field of characteristic $p$. Let

$$\bar{\varrho}\colon G_p \to GL_2(\mathbf{k})$$

be a continuous, absolutely irreducible, odd, and $p$-ordinary Galois representation. Then we assume that $\bar{\varrho}$ satisfies Assumption 1.3. Recall that the restriction of $\bar{\varrho}$ to a decomposition subgroup $D_p$ at $p$ of $G_p$ is isomorphic to an upper triangular representation with diagonal characters $\varepsilon_{\bar{\varrho}}$ and $\delta_{\bar{\varrho}}$, and that the character $\varepsilon_{\bar{\varrho}}$ is trivial and $\delta_{\bar{\varrho}}$ is nontrivial on the inertia subgroup $I_p$ at $p$. Note that the restriction of a character $\varepsilon$ of $D_p$ to $I_p$ has a unique extension $\varepsilon^{G_p}$ to $G_p$. Let $\mathfrak{X}_p^1$ be the universal deformation space of $\varepsilon_{\bar{\varrho}}^{G_p}$, and let $\mathfrak{X}_p^{2,\mathrm{o}}$ (resp. $\mathfrak{X}_p^{2,\mathrm{no}}$) be the universal (resp. nearly) ordinary deformation space of $\bar{\varrho}$. Then we have the following

**Lemma 2.4** (cf. Hida [11], p. 310). *There is a one-to-one correspondence*

$$\mathfrak{X}_p^{2,\mathrm{no}} \simeq \mathfrak{X}_p^{2,\mathrm{o}} \times \mathfrak{X}_p^1; \quad \boldsymbol{\varrho} \mapsto \left(\boldsymbol{\varrho} \otimes \left(\varepsilon_{\boldsymbol{\varrho}}^{G_p}\right)^{-1}, \varepsilon_{\boldsymbol{\varrho}}^{G_p}\right).$$

Here we assume the following

ASSUMPTION 2.5. The universal ordinary deformation ring $R^{2,\mathrm{o}}$ of $\bar{\varrho}$ is isomorphic to $\Lambda_{\mathbf{k}}$, and the universal deformation ring $R^2$ of $\bar{\varrho}$ is isomorphic to a power series ring in two variables over $\Lambda_{\mathbf{k}} = W(\mathbf{k})[\![\Gamma]\!]$. Then, we say that the deformation theory of $\bar{\varrho}$ is *cleanly unobstructed* (see Mazur [14, p. 120, Definition]).

We see that $\mathfrak{X}_p^{2,\mathrm{o}}$ and $\mathfrak{X}_p^1$ are isomorphic to $\mathrm{Hom}^{\mathrm{cont}}_{W(\mathbf{k})\text{-alg}}(\Lambda_{\mathbf{k}}, \mathbb{C}_p)$ by Assumption 2.5 and Lemma 1.1, respectively. On the other hand, $\mathrm{Hom}^{\mathrm{cont}}_{W(\mathbf{k})\text{-alg}}(\Lambda_{\mathbf{k}}, \mathbb{C}_p)$ is identified with the $p$-adic unit disc $\mathfrak{D}_p := \{t \in \mathbb{C}_p \mid |t|_p < 1\}$ by sending $\varphi$ to $\varphi(\gamma) - 1$, where $|*|_p$ means the $p$-adic absolute value, and $\gamma$ is a fixed topological generator of $\Gamma$. Hence we can identify $\mathfrak{X}_p^{2,\mathrm{no}}$ with the $p$-adic analytic space $\mathfrak{D}_p \times \mathfrak{D}_p$. Under this identification, we regard $\mathfrak{X}_p^{2,\mathrm{no}}$ as a $p$-adic analytic space, and we denote by $x_{\boldsymbol{\varrho}} := (s_{\boldsymbol{\varrho}}, t_{\boldsymbol{\varrho}})$ the point in $\mathfrak{D}_p \times \mathfrak{D}_p$ corresponding to a deformation $\boldsymbol{\varrho}$ in $\mathfrak{X}_p^{2,\mathrm{no}}$.

Moreover, we assume that $\bar{\varrho}$ satisfies Assumption 0.1, and let $\tau$ be a monodromy over $p$. Then we define $\Phi$ to be the $p$-adic analytic map

$$\Phi\colon \mathfrak{X}_p^{2,\mathrm{no}} \to \mathbb{C}_p; \quad x_{\boldsymbol{\varrho}} \mapsto \mathrm{tr}\, \boldsymbol{\varrho}(\tau).$$

We describe $\Phi$ explicitly. By the above argument, we have

$$\begin{aligned}\mathrm{tr}\, \boldsymbol{\varrho}(\tau) &= \varepsilon_{\boldsymbol{\varrho}}(\tau) + \delta_{\boldsymbol{\varrho}}(\tau) \\ &= (1 + \delta_{\boldsymbol{\varrho}} \cdot \varepsilon_{\boldsymbol{\varrho}}^{-1}(\tau)) \cdot \varepsilon_{\boldsymbol{\varrho}}(\tau).\end{aligned}$$

Because $\boldsymbol{\varrho} \otimes \left(\varepsilon_{\boldsymbol{\varrho}}^{G_p}\right)^{-1}$ is $p$-ordinary, we have the unique homomorphism $\varphi_0\colon R^{2,\mathrm{o}} \to \mathbb{C}_p$ corresponding to $\boldsymbol{\varrho} \otimes \left(\varepsilon_{\boldsymbol{\varrho}}^{G_p}\right)^{-1}$. On the other hand, the universal deformation ring of the character $\det \bar{\varrho}$ is isomorphic to $\Lambda_{\mathbf{k}}$ by Lemma 1.1. Let $\boldsymbol{\varrho}^1$ be the universal deformation of $\det \bar{\varrho}$, and let $\iota$ be an isomorphism such that $\det \boldsymbol{\varrho}^{2,\mathrm{o}}$ is equal to $\iota \circ \boldsymbol{\varrho}^1$ as a deformation. Then we have

$$\begin{aligned}\delta_{\boldsymbol{\varrho}} \cdot \varepsilon_{\boldsymbol{\varrho}}^{-1}(\tau) &= \det(\boldsymbol{\varrho} \otimes (\varepsilon_{\boldsymbol{\varrho}}^{G_p})^{-1})(\tau) \\ &= \varphi_0 \circ \iota \circ \boldsymbol{\varrho}^1(\tau) \\ &= \widetilde{\det \bar{\varrho}}(\tau) \cdot (\varphi_0 \circ \iota)([\tau_p]).\end{aligned}$$

Let $\boldsymbol{\varepsilon}^1$ be the universal deformation of $\varepsilon_{\bar{\varrho}}^{G_p}$, and let $\varphi_1\colon \Lambda_{\mathbf{k}} \to \mathbb{C}_p$ be the homomorphism corresponding to $\varepsilon_{\boldsymbol{\varrho}}^{G_p}$. Then we have

$$\begin{aligned}\varepsilon_{\boldsymbol{\varrho}}(\tau) &= \varphi_1 \circ \boldsymbol{\varepsilon}^1(\tau) \\ &= \widetilde{\varepsilon_{\bar{\varrho}}^{G_p}}(\tau) \cdot \varphi_1([\tau_p]).\end{aligned}$$

Recall that the restriction of $\delta_{\boldsymbol{\varrho}} \cdot \varepsilon_{\boldsymbol{\varrho}}^{-1}$ and $\varepsilon_{\boldsymbol{\varrho}}$ to $I_p$ factor through the group $\Gamma$, and that we have Lemma 2.1 (4) and Claim 2.2. Hence there exist non-zero $p$-adic integers $a$ and $b$ in $\mathbb{Z}_p$ such that $(\varphi_0 \circ \iota)([\tau_p]) = \varphi_0(\gamma)^a$ and $\varphi_1([\tau_p]) = \varphi_1(\gamma)^b$. Thus we obtain

$$\Phi(s_{\boldsymbol{\varrho}}, t_{\boldsymbol{\varrho}}) = \left\{1 + (\widetilde{\det \bar{\varrho}})(\tau)(1 + s_{\boldsymbol{\varrho}})^a\right\} \cdot \widetilde{\varepsilon_{\bar{\varrho}}^{G_p}}(\tau)(1 + t_{\boldsymbol{\varrho}})^b.$$

**2.3. Proof of Theorem 0.2.** Let $p \geq 5$ be a prime number, and let $\mathbf{k}$ be a finite field of characteristic $p$. Let

$$\bar{\varrho}\colon G_p \to GL_2(\mathbf{k})$$

be a continuous, absolutely irreducible, odd, and $p$-ordinary Galois representation. Then we assume that $\bar{\varrho}$ satisfies Assumptions 1.3, 2.5, and 0.1.

Let $\mathfrak{X}_p^{2,\mathrm{o}}$ (resp. $\mathfrak{X}_p^{2,\mathrm{no}}$) be the universal (resp. nearly) ordinary deformation space of $\bar{\varrho}$, which is identified with the $p$-adic analytic space $\mathfrak{D}_p$ (resp. $\mathfrak{D}_p \times \mathfrak{D}_p$). Then we obtain the following

**Theorem 2.6.** *Let $x_0$ be a point in $\mathfrak{X}_p^{2,\mathrm{o}}$. Then the map*

$$\Phi_0 \colon \mathfrak{X}_p^{2,\mathrm{o}} \to \mathbb{C}_p; \quad x_{\boldsymbol{\varrho}} \mapsto \operatorname{tr} \boldsymbol{\varrho}(\tau)$$

*is $p$-adic bianalytic in a neighborhood of $x_0$.*

Proof. By the argument in the previous subsection, for any point $s_{\boldsymbol{\varrho}}$ in $\mathfrak{D}_p$, there exists a non-zero $p$-adic integer $a$ in $\mathbb{Z}_p$ such that

$$\Phi_0(s_{\boldsymbol{\varrho}}) = \Phi(s_{\boldsymbol{\varrho}}, 0) = \widetilde{\varepsilon_{\bar{\varrho}}^{G_p}}(\tau)\{1 + (\widetilde{\det \bar{\varrho}})(\tau)(1 + s_{\boldsymbol{\varrho}})^a\}.$$

Thus $\Phi_0$ is everywhere locally $p$-adic bianalytic. $\square$

Next, we consider the case where $\bar{\varrho}$ is also $p$-extraordinary. Let $R^{2,\mathrm{eo}}$ be the universal extraordinary deformation ring of $\bar{\varrho}$. Then we obtain the following

**Lemma 2.7.** *The Krull dimension of $R^{2,\mathrm{eo}}$ is equal to* 1.

Proof. Remark that $\bar{\varrho}$ is modular, i.e., isomorphic to the mod $p$ reduction of a $p$-adic representation $\varrho_f$ associated to a $p$-ordinary Hecke eigenform $f$ on $SL_2(\mathbb{Z})$ by Khare [12, Theorem 1.1]. Moreover, the universal ordinary deformation $\boldsymbol{\varrho}^{2,\mathrm{o}}$ of $\bar{\varrho}$ is also modular by Assumption 2.5, that is, we can take a primitive ordinary $\Lambda$-adic form $\mathcal{F}$ such that the strict equivalence class of the Galois representation $\varrho_{\mathcal{F}}$ associated to $\mathcal{F}$ is equal to $\boldsymbol{\varrho}^{2,\mathrm{o}}$ (see Hida [9], [10]), where $\Lambda = \Lambda_{\mathbb{F}_p}$.

By our assumptions and Proposition 2.3, we see that there exists a single element $u$ in $\Lambda_{\mathbf{k}}$ such that

$$R^{2,\mathrm{eo}} \simeq \Lambda_{\mathbf{k}}/(u).$$

Because the Krull dimension of $\Lambda_{\mathbf{k}}$ is equal to 2 and $u$ is not unit in $\Lambda_{\mathbf{k}}$, the Krull dimension of $R^{2,\mathrm{eo}}$ is equal to either 1 or 2. We assume that the Krull dimension of $R^{2,\mathrm{eo}}$ is equal to 2. Then $u$ must be equal to 0. Hence $R^{2,\mathrm{eo}}$ is isomorphic to $\Lambda_{\mathbf{k}}$, and the universal extraordinary deformation $\boldsymbol{\varrho}^{2,\mathrm{eo}}$ is equal to $\boldsymbol{\varrho}^{2,\mathrm{o}}$ as a deformation. Because $\boldsymbol{\varrho}^{2,\mathrm{o}}$ is $p$-extraordinary, the restriction of $\varrho_{\mathcal{F}}$ to $D_p$ must be diagonalizable. Then $\mathcal{F}$ must have complex multiplication, i.e., all arithmetic specializations of $\mathcal{F}$ have complex multiplication by Ghate–Vatsal [6, Theorem 0.3]. On the other hand, $f$ does not have complex multiplication because $f$ is of level 1 (see Ribet [21, p. 34]). Hence the arithmetic specialization $f_w$ at weight $w$ of $\mathcal{F}$, which is the corresponding $p$-stabilized form of $f$, does not have complex multiplication. This is a contradiction. $\square$

We denote by $\mathfrak{X}_p^{2,\mathrm{eo}}$ (resp. $\mathfrak{X}_p^{2,\mathrm{neo}}$) the universal (resp. nearly) extraordinary deformation space of $\bar{\varrho}$. We have a one-to-one correspondence

$$\mathfrak{X}_p^{2,\mathrm{neo}} \simeq \mathfrak{X}_p^{2,\mathrm{eo}} \times \mathfrak{X}_p^{1}$$

induced by the correspondence in Lemma 2.4. Thus we may view $\mathfrak{X}_p^{2,\mathrm{eo}}$ and $\mathfrak{X}_p^{2,\mathrm{neo}}$ as subspaces of $\mathfrak{X}_p^{2,\mathrm{no}}$. Then we obtain the following

**Theorem 2.8.** *Let $x_0$ be a $p$-ordinary point in $\mathfrak{X}_p^{2,\mathrm{neo}}$. Then the map*

$$\Phi_1 \colon \mathfrak{X}_p^{2,\mathrm{neo}} \to \mathbb{C}_p; \quad x_{\varrho} \mapsto \mathrm{tr}\, \varrho(\tau)$$

*is $p$-adic bianalytic in a neighborhood of $x_0$.*

Proof. By Lemma 2.7, we see that $\mathfrak{X}_p^{2,\mathrm{eo}}$ is a finite set. If $\mathfrak{X}_p^{2,\mathrm{eo}}$ is a non-empty set and $x_0 = (s_0, 0)$ is an element of $\mathfrak{X}_p^{2,\mathrm{eo}}$, then there exists a neighborhood $\mathcal{U}$ of $x_0$ in $\mathfrak{X}_p^{2,\mathrm{neo}}$ such that the intersection of $\mathcal{U}$ and $\mathfrak{X}_p^{2,\mathrm{eo}}$ consists of only $x_0$. By the argument in the previous subsection, for any point $(s_0, t_{\varrho})$ in $\mathcal{U}$, there exist non-zero $p$-adic integers $a$ and $b$ in $\mathbb{Z}_p$ such that

$$\Phi_1(t_{\varrho}) = \Phi(s_0, t_{\varrho}) = \widetilde{\varepsilon_{\bar{\varrho}}^{G_p}}(\tau)\{1 + \widetilde{(\det \bar{\varrho})}(\tau)(1+s_0)^a\}(1+t_{\varrho})^b.$$

Thus $\Phi_1$ is $p$-adic bianalytic in a neighbourhood of $x_0$. □

Theorem 0.2 follows from Theorems 2.6 and 2.8.

## 3. Examples

In this section, we introduce some examples of $\bar{\varrho}$ which satisfies all assumptions in Theorem 0.2. For a choice of weight $w$ in the set $\{12, 16, 18, 20, 22, 26\}$, we denote by $f_w = \sum a_n q^n$ the unique cuspidal newform on $SL_2(\mathbb{Z})$ of that weight $w$. For any prime number $p$, we denote by

$$\bar{\varrho}_{w,p} \colon G_p \to GL_2(\mathbb{F}_p)$$

the mod $p$ reduction of the unique continuous semi-simple representation associated to $f_w$, which has the property that

$$\mathrm{tr}\, \bar{\varrho}_{w,p}(\sigma_l) \equiv a_l \pmod{p}, \quad \text{and} \quad \det \bar{\varrho}_{w,p}(\sigma_l) \equiv l^{w-1} \pmod{p}$$

for any prime number $l \neq p$, where $\sigma_l$ is the Frobenius element at $l$. Note that $\bar{\varrho}_{w,p}$ is odd, and satisfies Assumption 1.3 if $w < p$. If $a_p \not\equiv 0 \pmod{p}$, then $\bar{\varrho}_{w,p}$ is $p$-ordinary and satisfies Assumption 2.5 by Gouvêa [7, pp. 192–193], Mazur [14, p. 120, Corollary 2], and Boston [14, Proposition (6.3), Example]. We can find many examples of such $\bar{\varrho}_{w,p}$ there. If $\bar{\varrho}_{w,p}$ is full, i.e., the image of $\bar{\varrho}_{w,p}$ contains $SL_2(\mathbb{F}_p)$, then $\bar{\varrho}_{w,p}$ is absolutely irreducible, and if $w \not\equiv 0, 2, (p+1)/2 \pmod{p-1}$, then $\bar{\varrho}_{w,p}$ satisfies Assumption 0.1 (i). For $f_w$, it is known that there exist only finitely many primes $p$ such that $\bar{\varrho}_{w,p}$ is not full by Swinnerton-Dyer [22, Theorem 4]. Moreover, there exist only 2 pairs such that $\bar{\varrho}_{w,p}$ is absolutely irreducible, $p$-ordinary, and tame i.e., the order of the image of $\bar{\varrho}_{w,p}$ is prime to $p$. Such pairs are

$$(w, p) = (12, 23), (16, 31).$$

Therefore we obtain the following explicit examples for $p < 400$ in the case where $\bar{\varrho}_{w,p}$ is full.

EXAMPLE 3.1. If a pair $(w, p)$ appears in the following list:

- $w = 12,\ 17 \le p \le 65063,\ p \ne 23, 691, 2411,$
- $w = 16,\ 19 \le p \le 397,\ p \ne 31, 59,$
- $w = 18,\ 23 \le p \le 397,$
- $w = 20,\ 29 \le p \le 397,\ p \ne 283,$
- $w = 22,\ 29 \le p \le 397,\ p \ne 131,$
- $w = 26,\ 31 \le p \le 397,$

then $\bar{\varrho}_{w,p}$ is absolutely irreducible, odd, $p$-ordinary and full, and satisfies Assumptions 1.3, 2.5 and 0.1 (i).

Next, we introduce some examples of $\bar{\varrho}_{w,p}$ that is also $p$-extraordinary. Let $f_{w,p}$ be the mod $p$ reduction of $f_w$. By Gross [8], if $f_{w,p}$ has *companion form*, then $\bar{\varrho}_{w,p}$ is $p$-extraordinary. A computer search by Elkies, extended by Atkin, showed that there exist pairs $(w, p)$ such that $\bar{\varrho}_{w,p}$ is full and $f_{w,p}$ has companion form. For $p < 3500$, such pairs are

$$(w, p) = (16, 397), (18, 271), (20, 139), (20, 379), (26, 107).$$

On the other hand, $f_{w,p}$ is equal to its own companion form when

$$(w, p) = (12, 23), (16, 31).$$

Then $\bar{\varrho}_{w,p}$ is tame and $p$-extraordinary. By Mazur [13], $\bar{\varrho}_{w,p}$ for these 2 pairs are *special $S_3$-representations*. In particular, the image of $\bar{\varrho}_{w,p}$ is isomorphic to the symmetric group $S_3$. Hence the order of the image of $\bar{\varrho}_{w,p}$ is not divisible by $p-1$. Thus we obtain the following

EXAMPLE 3.2. If a pair $(w, p)$ is one of these 7 pairs, then $\bar{\varrho}_{w,p}$ is absolutely irreducible, odd, and $p$-extraordinary, and satisfies Assumptions 1.3, 2.5 and 0.1.

Remark that we have not known unfortunately whether the universal extraordinary deformation space $\mathfrak{X}_p^{2,\mathrm{eo}}$ of $\bar{\varrho}_{w,p}$ is non-empty whenever $\bar{\varrho}_{w,p}$ is full. However we see that $\mathfrak{X}_p^{2,\mathrm{eo}}$ is non-empty for $(w, p) = (12, 23), (16, 31)$ by the following

**Proposition 3.3.** *For each pair* $(w, p) = (12, 23), (16, 31)$, *the universal extraordinary deformation ring* $R^{2,\mathrm{eo}}$ *of* $\bar{\varrho}_{w,p}$ *is isomorphic to* $\mathbb{Z}_p$.

Proof. By Boston–Mazur [2, p. 13, Computation], we see that $\bar{\varrho}_{w,p}$ is a *generic* special $S_3$-representation (for the definition, see also Boston–Mazur [2, §2.3]). For such a representation, an explicit description of the universal deformation space is given in Boston–Mazur [2, §3.3]. In fact, let $\mathfrak{m}$ be the maximal ideal of $\mathbb{Z}_p[[T_1, T_2, T_3]]$ $(\simeq R^2)$. Then there exist power series $f$ and $g$ in $\mathfrak{m}$ such that

$$f(T_1, T_2, T_3) \equiv T_1 - T_2 + T_3 \pmod{\mathfrak{m}^2},$$
$$g(T_1, T_2, T_3) \equiv 3T_1 - 3T_2 - 3T_3 \pmod{\mathfrak{m}^2},$$

by Boston–Mazur [2, Proposition 12]. Then the universal ordinary deformation ring of $\bar{\varrho}_{w,p}$ is isomorphic to $\mathbb{Z}_p[[T_1, T_2, T_3]]/(g, T_1)$ by Boston–Mazur [13, Proposition 13 (c)]. Moreover, we see that

$$R^{2,\mathrm{eo}} \simeq \mathbb{Z}_p[[T_1, T_2, T_3]]/(g, T_1, f) \simeq \mathbb{Z}_p$$

by [13, Lemmas 6 and 8]. □

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