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# ASYMPTOTICS OF POLYBALANCED METRICS UNDER RELATIVE STABILITY CONSTRAINTS

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**Abstract**

Under the assumption of asymptotic relative Chow-stability for polarized algebraic manifolds $(M, L)$, a series of weighted balanced metrics $\omega_m$, $m \gg 1$, called polybalanced metrics, are obtained from complete linear systems $|L^m|$ on $M$. Then the asymptotic behavior of the weights as $m \to \infty$ will be studied.

## 1. Introduction

In this paper, we shall study relative Chow-stability (cf. [5]; see also [11]) for polarized algebraic manifolds $(M, L)$ from the viewpoints of the existence problem of extremal Kähler metrics. As balanced metrics are obtained from Chow-stability on polarized algebraic manifolds, our relative Chow-stability similarly provides us with a special type of weighted balanced metrics called *polybalanced metrics*. As a crucial step in the program of [7], we here study the asymptotic behavior of the weights for such polybalanced metrics.

By a *polarized algebraic manifold* $(M, L)$, we mean a pair of a connected projective algebraic manifold $M$ and a very ample holomorphic line bundle $L$ over $M$. For a maximal connected linear algebraic subgroup $G$ of the group $\mathrm{Aut}(M)$ of all holomorphic automorphisms of $M$, let $\mathfrak{g} := \mathrm{Lie}\, G$ denote its Lie algebra. Since the infinitesimal $\mathfrak{g}$-action on $M$ lifts to an infinitesimal bundle $\mathfrak{g}$-action on $L$, by setting

$$V_m := H^0(M, L^m), \quad m = 1, 2, \dots,$$

we view $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{sl}(V_m)$. We now define a symmetric bilinear form $\langle\ ,\ \rangle_m$ on $\mathfrak{sl}(V_m)$ by

$$\langle X, Y\rangle_m = \mathrm{Tr}(XY)/m^{n+2}, \quad X, Y \in \mathfrak{sl}(V_m),$$

where the asymptotic limit of $\langle\ ,\ \rangle_m$ as $m \to \infty$ often plays an important role in the

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study of K-stability. In fact one can show

$$\langle X, Y\rangle_m = O(1)$$

by using the equivariant Riemann–Roch formula, see [1] and [11]. Let $T$ be an algebraic torus in $\mathrm{SL}(V_m)$ such that the corresponding Lie algebra $\mathfrak{t} := \mathrm{Lie}\, T$ satisfies

$$\mathfrak{t} \subset \mathfrak{g}.$$

Then by the $T$-action on $V_m$, we can write the vector space $V_m$ as a direct sum of $\mathfrak{t}$-eigenspaces:

$$V_m = \bigoplus_{k=1}^{\nu_m} V(\chi_k),$$

where $V(\chi_k) := \{v \in V_m\,;\ g \cdot v = \chi_k(g)v \text{ for all } g \in T\}$ for mutually distinct multiplicative characters $\chi_k \in \mathrm{Hom}(T, \mathbb{C}^*)$, $k = 1, 2, \dots, \nu_m$.

To study $V_m$, let $\omega_m$ be a Kähler metric in the class $c_1(L)_{\mathbb{R}}$, and choose a Hermitian metric $h_m$ for $L$ such that $\omega_m = c_1(L; h_m)_{\mathbb{R}}$. We now endow $V_m$ with the Hermitian $L^2$ inner product on $V_m$ defined by

$$(u, v)_{L^2} := \int_M (u, v)_{h_m} \omega_m^n, \quad u, v \in V_m, \tag{1.1}$$

where $(u, v)_{h_m}$ denotes the pointwise Hermitian pairing of $u$, $v$ in terms of $h_m$. Then by this $L^2$ inner product, we have $V(\chi_k) \perp V(\chi_{k'})$, $k \neq k'$. Put $N_m := \dim V_m$ and $n_k := \dim_{\mathbb{C}} V(\chi_k)$. For each $k$, by choosing an orthonormal basis $\{\sigma_{k,i}\,; i = 1, 2, \dots, n_k\}$ for $V(\chi_k)$, we put

$$B_{m,k}(\omega_m) := \sum_{i=1}^{n_k} |\sigma_{k,i}|^2_{h_m},$$

where $|u|^2_{h_m} := (u, u)_{h_m}$ for each $u \in V_m$. Then $\omega_m$ is called a *polybalanced metric*, if there exist real constants $\gamma_{m,k} > 0$ such that

$$B^\circ_m(\omega_m) = \sum_{k=1}^{\nu_m} \gamma_{m,k} B_{m,k}(\omega_m) \tag{1.2}$$

is a constant function on $M$. Here $\gamma_{m,k}$ are called the *weights* of the polybalanced metric $\omega_m$. On the other hand,

$$B^\bullet_m(\omega_m) := \sum_{k=1}^{\nu_m} B_{m,k}(\omega_m)$$

is called the *m-th asymptotic Bergman kernel* of $\omega_m$. A smooth real-valued function $f \in C^\infty(M)_{\mathbb{R}}$ on the Kähler manifold $(M, \omega_m)$ is said to be *Hamiltonian* if there exists a holomorphic vector field $X \in \mathfrak{g}$ on $M$ such that $i_X\omega_m = \sqrt{-1}\,\bar\partial f$. Put $N'_m := N_m/c_1(L)^n[M]$. In this paper, as the first step in [7], we shall show the following:

**Theorem A.** *For a polarized algebraic manifold* $(M, L)$ *and an algebraic torus* $T$ *as above, assume that* $(M, L)$ *is asymptotically Chow-stable relative to* $T$. *Then for each* $m \gg 1$, *there exists a polybalanced metric* $\omega_m$ *in the class* $c_1(L)_{\mathbb{R}}$ *such that* $\gamma_{m,k} = 1 + O(1/m)$, *i.e.*,

$$|\gamma_{m,k} - 1| \le \frac{C_1}{m}, \quad k = 1, 2, \ldots, \nu_m;\ m \gg 1, \tag{1.3}$$

*for some positive constant* $C_1$ *independent of* $k$ *and* $m$. *Moreover, there exist uniformly* $C^0$*-bounded functions* $f_m \in C^\infty(M)_{\mathbb{R}}$ *on* $M$ *such that*

$$B_m^\bullet(\omega_m) = N'_m + f_m m^{n-1} + O(m^{n-2}) \tag{1.4}$$

*and that each* $f_m$ *is a Hamiltonian function on* $(M, \omega_m)$ *satisfying* $i_{X_m}\omega_m = \sqrt{-1}\,\bar\partial f_m$ *for some holomorphic vector field* $X_m \in \mathfrak{t}$ *on* $M$.

In view of [8], this theorem and the result of Catlin–Lu–Tian–Yau–Zelditch ([3], [12], [13]) allow us to obtain an approach (cf. [7]) to an extremal Kähler version of Donaldson–Tian–Yau's conjecture. On the other hand, as a corollary to Theorem A, we obtain the following:

**Corollary B.** *Under the same assumption as in* Theorem A, *suppose further that the classical Futaki character* $\mathcal{F}_1 \colon \mathfrak{g} \to \mathbb{C}$ *for* $M$ *vanishes on* $\mathfrak{t}$. *Then for each* $m \gg 1$, *there exists a polybalanced metric* $\omega_m$ *in the class* $c_1(L)_{\mathbb{R}}$ *such that* $\gamma_{m,k} = 1 + O(1/m^2)$. *In particular*

$$B_m^\bullet(\omega_m) = N'_m + O(m^{n-2}).$$

## 2. Asymptotic relative Chow-stability

By the same notation as in the introduction, we consider the algebraic subgroup $S_m$ of $\mathrm{SL}(V_m)$ defined by

$$S_m := \prod_{k=1}^{\nu_m} \mathrm{SL}(V(\chi_k)),$$

where the action of each $\mathrm{SL}(V(\chi_k))$ on $V_m$ fixes $V(\chi_i)$ if $i \ne k$. Then the centralizer $H_m$ of $S_m$ in $\mathrm{SL}(V_m)$ consists of all diagonal matrices in $\mathrm{SL}(V_m)$ acting on each

$V(\chi_k)$ by constant scalar multiplication. Hence the centralizer $Z_m(T)$ of $T$ in $\mathrm{SL}(V_m)$ is $H_m \cdot S_m$ with Lie algebra

$$\mathfrak{z}_m(\mathfrak{t}) = \mathfrak{h}_m + \mathfrak{s}_m,$$

where $\mathfrak{s}_m := \mathrm{Lie}\, S_m$ and $\mathfrak{h}_m := \mathrm{Lie}\, H_m$. For the exponential map defined by $\mathfrak{h}_m \ni X \mapsto \exp(2\pi\sqrt{-1}X) \in H_m$, let $(\mathfrak{h}_m)_{\mathbb{Z}}$ denote its kernel. Regarding $(\mathfrak{h}_m)_{\mathbb{R}} := (\mathfrak{h}_m)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ as a subspace of $\mathfrak{h}_m$, we have a real structure on $\mathfrak{h}_m$, i.e., an involution

$$\mathfrak{h}_m \ni X \mapsto \bar{X} \in \mathfrak{h}_m$$

defined as the associated complex conjugate of $\mathfrak{h}_m$ fixing $(\mathfrak{h}_m)_{\mathbb{R}}$. We then have a Hermitian metric $(\ ,\ )_m$ on $\mathfrak{h}_m$ by setting

$$(X, Y)_m = \langle X, \bar{Y} \rangle_m, \quad X, Y \in \mathfrak{h}_m. \tag{2.1}$$

For the orthogonal complement $\mathfrak{t}^{\perp}$ of $\mathfrak{t}$ in $\mathfrak{h}_m$ in terms of this Hermitian metric, let $T^{\perp}$ denote the corresponding algebraic torus in $H_m$. We now define an algebraic subgroup $G_m$ of $Z_m(T)$ by

$$G_m := T^{\perp} \cdot S_m. \tag{2.2}$$

For the $T$-equivariant Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$ associated to the complete linear system $|L^m|$ on $M$, let $d(m)$ denote the degree of the image $\Phi_m(M)$ in the projective space $\mathbb{P}^*(V_m)$. For the dual space $W_m^*$ of $W_m := S^{d(m)}(V_m)^{\otimes n+1}$, we have the Chow form

$$0 \neq \hat{M}_m \in W_m^*$$

for the irreducible reduced algebraic cycle $\Phi_m(M)$ on $\mathbb{P}^*(V_m)$, so that the corresponding element $[\hat{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point for the cycle $\Phi_m(M)$. Consider the natural action of $\mathrm{SL}(V_m)$ on $W_m^*$ induced by the action of $\mathrm{SL}(V_m)$ on $V_m$.

DEFINITION 2.3. (1) $(M, L^m)$ is said to be *Chow-stable relative to $T$* if the orbit $G_m \cdot \hat{M}_m$ is closed in $W_m^*$.
(2) $(M, L)$ is said to be *asymptotically Chow-stable relative to $T$* if $(M, L^m)$ is Chow-stable relative to $T$ for each integer $m \gg 1$.

## 3. Relative Chow-stability for each fixed *m*

In this section, we consider a polarized algebraic manifold $(M, L)$ under the assumption that $(M, L^m)$ is Chow-stable relative to $T$ for a fixed positive integer $m$. Then we shall show that a polybalanced metric $\omega_m$ exists in the class $c_1(L)_{\mathbb{R}}$.

The space $\Lambda_m := \left\{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_m}) \in \mathbb{C}^{\nu_m};\ \sum_{k=1}^{\nu_m} n_k \lambda_k = 0\right\}$ and the Lie algebra $\mathfrak{h}_m$ are identified by an isomorphism

$$\Lambda_m \cong \mathfrak{h}_m, \quad \lambda \leftrightarrow X_\lambda, \tag{3.1}$$

with $(\mathfrak{h}_m)_{\mathbb{R}}$ corresponding to the set $(\Lambda_m)_{\mathbb{R}}$ of the real points in $\Lambda_m$, where $X_\lambda$ is the endomorphism of $V_m$ defined by

$$X_\lambda := \bigoplus_{k=1}^{\nu_m} \lambda_k \mathrm{id}_{V(\chi_k)} \in \bigoplus_{k=1}^{\nu_m} \mathrm{End}(V(\chi_k)) \quad (\subset \mathrm{End}(V_m)).$$

In terms of the identification (3.1), we can write the Hermitian metric $(\ ,\ )_m$ on $\mathfrak{h}_m$ in (2.1) in the form

$$(\lambda, \mu)_m := \sum_{k=1}^{\nu_m} \frac{n_k \lambda_k \bar{\mu}_k}{m^{n+2}},$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_m})$ and $\mu = (\mu_1, \mu_2, \dots, \mu_{\nu_m})$ are in $\mathbb{C}^{\nu_m}$. By the identification (3.1), corresponding to the decomposition $\mathfrak{h}_m = \mathfrak{t} \oplus \mathfrak{t}^\perp$, we have the orthogonal direct sum

$$\Lambda_m = \Lambda(\mathfrak{t}) \oplus \Lambda(\mathfrak{t}^\perp),$$

where $\Lambda(\mathfrak{t})$ and $\Lambda(\mathfrak{t}^\perp)$ are the subspace of $\Lambda_m$ associated to $\mathfrak{t}$ and $\mathfrak{t}^\perp$, respectively. Take a Hermitian metric $\rho_k$ on $V(\chi_k)$, and for the metric

$$\rho := \bigoplus_{k=1}^{\nu_m} \rho_k$$

on $V_m$, we see that $V(\chi_k) \perp V(\chi_{k'})$ whenever $k \neq k'$. By choosing an orthonormal basis $\{s_{k,i};\ i = 1, 2, \dots, n_k\}$ for the Hermitian vector space $(V(\chi_k), \rho_k)$, we now set

$$j(k, i) := i + \sum_{l=1}^{k-1} n_l, \quad i = 1, 2, \dots, n_k;\ k = 1, 2, \dots, \nu_m, \tag{3.2}$$

where the right-hand side denotes $i$ in the special case $k = 1$. By writing $s_{k,i}$ as $s_{j(k,i)}$, we have an orthonormal basis

$$\mathcal{S} := \{s_1, s_2, \dots, s_{N_m}\}$$

for $(V_m, \rho)$. By this basis, the vector space $V_m$ and the algebraic group $\mathrm{SL}(V_m)$ are identified with $\mathbb{C}^{N_m} = \{(z_1, \dots, z_{N_m})\}$ and $\mathrm{SL}(N_m, \mathbb{C})$, respectively. In terms of $\mathcal{S}$, the Kodaira embedding $\Phi_m$ is given by

$$\Phi_m(x) := (s_1(x) : \cdots : s_{N_m}(x)), \quad x \in M.$$

Consider the associated *Chow norm* $W_m^* \ni \xi \mapsto \|\xi\|_{\mathrm{CH}(\rho)} \in \mathbb{R}_{\geq 0}$ as in Zhang [14] (see also [4]). Then by the closedness of $G_m \cdot \hat{M}_m$ in $W_m^*$ (cf. (2.2) and Definition 2.3), the Chow norm on the orbit $G_m \cdot \hat{M}_m$ takes its minimum at $g_m \cdot \hat{M}_m$ for some $g_m \in G_m$. Note that, by complexifying

$$K_m := \prod_{k=1}^{\nu_m} \mathrm{SU}(V(\chi_k)),$$

we obtain the reductive algebraic group $S_m$. For each $\kappa \in K_m$ and each diagonal matrix $\Delta$ in $\mathfrak{sl}(N_m, \mathbb{C})$, we put

$$e(\kappa, \Delta) := \exp\{\mathrm{Ad}(\kappa)\Delta\}.$$

Then $g_m$ is written as $\kappa_1 \cdot e(\kappa_0, D)$ for some $\kappa_0, \kappa_1 \in K_m$ and a diagonal matrix $D = (d_j)_{1 \leq j \leq N_m}$ in $\mathfrak{sl}(N_m, \mathbb{C})$ with the $j$-th diagonal element $d_j$. Put $g'_m := e(\kappa_0, D)$. In view of $\|g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} = \|g_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)}$ (cf. [10], Proposition 4.1), we obtain

$$\|g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} \leq \|e(\kappa_0, t(X_\lambda + A)) \cdot g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)}, \quad t \in \mathbb{C}, \tag{3.3}$$

for all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\nu_m}) \in \Lambda(\mathfrak{t}^\perp)$ and all diagonal matrices $A = (a_j)_{1 \leq j \leq N_m}$ in $\mathfrak{s}_m$, where each $a_j$ denotes the $j$-th diagonal element of $A$. We now write $a_{j(k,i)}$ as $a_{k,i}$ for simplicity. Put

$$s'_j := \kappa_0^{-1} \cdot s_j, \quad b_{k,i} := \lambda_k + a_{k,i}, \quad c_{k,i} := \exp d_{j(k,i)}.$$

Then we shall now identify $V_m$ with $\mathbb{C}^{N_m} = \{(z'_1, \ldots, z'_{N_m})\}$ by the orthonormal basis $\mathcal{S}' := \{s'_1, s'_2, \ldots, s'_{N_m}\}$ for $V_m$. In view (3.2), we rewrite $s'_j$, $z'_j$ as $s'_{k,i}$, $z'_{k,i}$, respectively by

$$s'_{k,i} := s'_{j(k,i)}, \quad z'_{k,i} := z'_{j(k,i)},$$

where $k = 1, 2, \ldots, \nu_m$ and $i = 1, 2, \ldots, n_k$. By writing $b_{k,i}$, $c_{k,i}$ also as $b_{j(k,i)}$, $c_{j(k,i)}$, respectively, we consider the diagonal matrices $B$ and $C$ of order $N_m$ with the $j$-th diagonal elements $b_j$ and $c_j$, respectively. Note that the right-hand side of (3.3) is

$$\|(\exp tB) \cdot C \cdot \kappa_0^{-1} \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)},$$

and its derivative at $t = 0$ vanishes by virtue of the inequality (3.3). Hence, by setting $\Theta := (\sqrt{-1}/2\pi)\, \partial\bar{\partial} \log\big(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} z'_{k,i}|^2\big)$, we obtain the equality (see for instance (4.4) in [4])

$$\int_M \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} b_{k,i} |c_{k,i} s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2} \Phi'^*_m(\Theta^n) = 0 \tag{3.4}$$

for all $\lambda \in \Lambda(\mathfrak{t}^{\perp})$ and all diagonal matrices $A$ in the Lie algebra $\mathfrak{s}_m$, where $\Phi'_m : M \hookrightarrow \mathbb{P}^*(V_m)$ is the Kodaira embedding of $M$ by $\mathcal{S}'$ which sends each $x \in M$ to $\big(s'_1(x) : s'_2(x) : \cdots : s'_{N_m}(x)\big) \in \mathbb{P}^*(V_m)$. Here we regard

$$s'_{k,i} = \Phi'^* z'_{k,i}.$$

Let $k_0 \in \{1, 2, \ldots, \nu_m\}$ and let $i_1, i_2 \in \{1, 2, \ldots, n_{k_0}\}$ with $i_1 \neq i_2$. Using Kronecker's delta, we first specify the real diagonal matrix $B$ by

$$\lambda_k = 0 \quad \text{and} \quad a_{k,i} = \delta_{kk_0}(\delta_{ii_1} - \delta_{ii_2}),$$

where $k = 1, 2, \ldots, \nu_m$; $i = 1, 2, \ldots, n_k$. By (3.4) applied to this $B$, and let $(i_1, i_2)$ run through the set of all pairs of two distinct integers in $\{1, 2, \ldots, n_{k_0}\}$, where positive integer $k_0$ varies from 1 to $\nu_m$. Then there exists a positive constant $\beta_k > 0$ independent of the choice of $i$ in $\{1, 2, \ldots, n_k\}$ such that, for all $i$,

$$\int_M \frac{|c_{k,i} s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2} \Phi'^*_m(\Theta^n) = \beta_k, \quad k = 1, 2, \ldots, \nu_m. \tag{3.5}$$

Let $k_0, i_1, i_2$ be as above, and let $\kappa_2$ be the element in $K_m$ such that

$$\kappa_2 z'_{k_0,i_1} = \frac{1}{\sqrt{2}}(z'_{k_0,i_1} - z'_{k_0,i_2}), \quad \kappa_2 z'_{k_0,i_2} = \frac{1}{\sqrt{2}}(z'_{k_0,i_1} + z'_{k_0,i_2})$$

and that $\kappa_2$ fixes all other $z_{k,i}$'s. Let $\kappa_3$ be the element in $K_m$ such that

$$\kappa_3 z'_{k_0,i_1} = \frac{1}{\sqrt{2}}\big(z'_{k_0,i_1} + \sqrt{-1} z'_{k_0,i_2}\big), \quad \kappa_3 z'_{k_0,i_2} = \frac{1}{\sqrt{2}}\big(\sqrt{-1} z'_{k_0,i_1} + z'_{k_0,i_2}\big)$$

and that $\kappa_3$ fixes all other $z'_{k,i}$'s. Now

$$\|\kappa_2 g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} = \|\kappa_3 g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} = \|g'_m \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)},$$

and note that

$$2 z'_{k_0,i_1} \bar{z}'_{k_0,i_2} = (|\kappa_2 z'_{k_0,i_2}|^2 - |\kappa_2 z'_{k_0,i_1}|^2) - \sqrt{-1}(|\kappa_3 z'_{k_0,i_2}|^2 - |\kappa_3 z'_{k_0,i_1}|^2).$$

Hence replacing $g'_m$ by $\kappa_\alpha g'_m$, $\alpha = 2, 3$, in (3.3), we obtain the case $k' = k''$ of the following by an argument as in deriving (3.5) from (3.3):

$$\int_M \frac{s'_{k',i'} \bar{s}'_{k'',i''}}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2} \Phi'^*_m(\Theta^n) = 0, \quad \text{if} \quad (k', i') \neq (k'', i''). \tag{3.6}$$

Here (3.6) holds easily for $k' \neq k''$, since for every element $g$ of the maximal compact subgroup of $T$, we have:

$$\begin{aligned}\text{L.H.S. of (3.6)} &= \int_M g^*\left\{\frac{s'_{k',i'}\bar{s}'_{k'',i''}}{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2}\,\Phi'^*_m(\Theta^n)\right\}\\ &= \frac{\chi_{k''}(g)}{\chi_{k'}(g)}\int_M \frac{s'_{k',i'}\bar{s}'_{k'',i''}}{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2}\,\Phi'^*_m(\Theta^n).\end{aligned}$$

Put $\beta := (\beta_1, \beta_2, \dots, \beta_{\nu_m}) \in \mathbb{R}^{\nu_m}$ and $\beta_0 := \left(\sum_{k=1}^{\nu_m} n_k\beta_k\right)/N_m$, where $\beta_k$ is given in (3.5). In view of $N_m = \sum_{k=1}^{\nu_m} n_k$, by setting $\underline{\beta}_k := \beta_k - \beta_0$, we have

$$\underline{\beta} := \left(\underline{\beta}_1, \underline{\beta}_2, \dots, \underline{\beta}_{\nu_m}\right) \in \Lambda_m.$$

Next for each $\lambda \in \Lambda(\mathfrak{t}^{\perp})$, by setting $a_{k,i} = 0$ for all $(k, i)$, the equality (3.4) above implies $0 = \sum_{k=1}^{\nu_m}(n_k\lambda_k)\beta_k$. From this together with the equality $\sum_{k=1}^{\nu_m} n_k\lambda_k = 0$, we obtain $(\lambda, \underline{\beta})_m = 0$, i.e.,

$$\underline{\beta} \in \Lambda(\mathfrak{t}). \tag{3.7}$$

We now define a Hermitian metric $h_{\mathrm{FS}}$ (cf. [14]) for $L^m$ as follows. Let $u$ be a local section for $L^m$. Then[1]

$$|u|^2_{h_{\mathrm{FS}}} := \frac{|u|^2}{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2}. \tag{3.8}$$

For the Hermitian metric $h_m := (h_{\mathrm{FS}})^{1/m}$ for $L$, we consider the associated Kähler metric $\omega_m := c_1(L; h_m)_{\mathbb{R}}$ on $M$. In view of (3.5),

$$\begin{aligned}\beta_0 &= \frac{\sum_{k=1}^{\nu_m} n_k\beta_k}{N_m} = N_m^{-1}\int_M \Phi^*_m(\Theta^n)\\ &= N_m^{-1}m^n c_1(L)^n[M] = n!\left\{1 + O\left(\frac{1}{m}\right)\right\}.\end{aligned} \tag{3.9}$$

Then for $\gamma_{m,k} := \beta_k/\beta_0$ and $\sigma_{k,i} := c_{k,i}s'_{k,i}(m^n\beta_k^{-1})^{1/2}$, we have

$$\begin{aligned}\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\gamma_{m,k}|\sigma_{k,i}|^2_{h_m} &= \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\gamma_{m,k}|\sigma_{k,i}|^2_{h_{\mathrm{FS}}}\\ &= \frac{m^n}{\beta_0}\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2_{h_{\mathrm{FS}}} = \frac{m^n}{\beta_0}.\end{aligned} \tag{3.10}$$

[1] In view of (3.8), there is some error in [4]. Actually, for the numerator of (5.7) in the paper [4], please read $(N_m + 1)|s|^2$.

By operating $(\sqrt{-1}/2\pi)\,\bar\partial\partial$ log on (3.8), we obtain

$$\Phi_m'^*(\Theta) = c_1(L^m; h_{\mathrm{FS}})_{\mathbb{R}} = mc_1(L; h_m)_{\mathbb{R}} = m\omega_m. \tag{3.11}$$

Then in terms of the Hermitian $L^2$ inner product (1.1), we see from (3.5), (3.6), (3.8) and (3.11) that $\{\sigma_{k,i}\,;\, k = 1, \dots, \nu_m,\ i = 1, \dots, n_k\}$ is an orthonormal basis for $V_m$. Moreover, (3.10) is rewritten as

$$B_m^\circ(\omega_m) = \frac{m^n}{\beta_0}, \tag{3.12}$$

where $B_m^\circ(\omega_m)$ is as in (1.2). Hence $\omega_m$ is a polybalanced metric, and the proof of Theorem A is reduced to showing (1.3) and (1.4). By summing up, we obtain

**Theorem C.** *If $(M, L^m)$ is Chow-stable relative to $T$ for a positive integer $m$, then the Kähler class $c_1(L)_{\mathbb{R}}$ admits a polybalanced metric $\omega_m$ with the weights $\gamma_{m,k}$ as above.*

## 4. The asymptotic behavior of the weights $\gamma_{m,k}$

The purpose of this section is to prove (1.3). If $\underline{\beta} = 0$, then we are done. Hence, we may assume that $\underline{\beta} \neq 0$. Consider the sphere

$$\sum := \{X \in \mathfrak{t}_{\mathbb{R}}\,;\, \langle X, X\rangle_0 = 1\}$$

in $\mathfrak{t}_{\mathbb{R}} := \mathfrak{t} \cap (\mathfrak{h}_m)_{\mathbb{R}}$, where $\langle\ ,\ \rangle_0$ denotes the positive definite symmetric bilinear form on $\mathfrak{g}$ as in [2]. Since all components $\underline{\beta}_k$ of $\underline{\beta}$ are real, we see from (3.7) that, in view of (3.1), $\lambda := r_m\underline{\beta}$ satisfies

$$X_\lambda \in \sum \tag{4.1}$$

for some positive real number $r_m$. Hence by writing $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_m})$, we obtain positive constants $C_2$, $C_3$ independent of $k$ and $m$ such that (see for instance [5], Lemma 2.6)

$$-C_2 m \le \lambda_k \le C_3 m. \tag{4.2}$$

Put $g(t) := \exp(tX_\lambda)$ and $\gamma(t) := \log\|g(t)\cdot C\cdot\kappa_0^{-1}\cdot\hat{M}_m\|_{\mathrm{CH}(\rho)}$, $t \in \mathbb{R}$, by using the notation in Section 3. Since $g(t)$ commutes with $C\cdot\kappa_0^{-1}$, we see that $g(t)$ defines a holomorphic automorphism of $\Phi_m'(M)$. In view of Theorem 4.5 in [4], it follows from Remark 4.6 in [4] that (cf. [14])

$$\dot\gamma(t) = \dot\gamma(0) = \sum_{k=1}^{\nu_m} n_k\lambda_k\beta_k, \quad -\infty < t < +\infty. \tag{4.3}$$

Consider the classical Futaki invariant $\mathcal{F}_1(X_\lambda)$ associated to the holomorphic vector field $X_\lambda$ on $(M, L)$. Since $M$ is smooth, this coincides with the corresponding Donaldson–Futaki's invariant for test configurations. Then by applying Lemma 4.8 in [6] to the product configuration of $(M, L)$ associated to the one-parameter group generated by $X_\lambda$ on the central fiber, we obtain (see also [1], [11])

$$\lim_{t\to-\infty} \dot{\gamma}(t) = C_4\left\{\mathcal{F}_1(X_\lambda) + O\left(\frac{1}{m}\right)\right\}m^n, \tag{4.4}$$

where $C_4 := (n+1)!\, c_1(L)^n[M] > 0$. Hence by $\sum_{k=1}^{\nu_m} n_k\lambda_k = 0$ and $\lambda = r_m\underline{\beta}$, it now follows from (4.3) and (4.4) that

$$\begin{aligned} C_4\left\{\mathcal{F}_1(X_\lambda) + O\left(\frac{1}{m}\right)\right\}m^n &= r_m^{-1}\sum_{k=1}^{\nu_m} n_k\lambda_k^2 \\ &= r_m^{-1}m^{n+2}\langle X_\lambda, X_\lambda\rangle_m, \end{aligned} \tag{4.5}$$

where by (4.1) above, (7) in [1] (see also [11]) implies $\langle X_\lambda, X_\lambda\rangle_m \geq C_5$ for some positive real constant $C_5$ independent of $m$. Furthermore $\mathcal{F}_1(X_\lambda) = O(1)$ again by (4.1). Hence from (4.5), we obtain

$$r_m^{-1} = O\left(\frac{1}{m^2}\right). \tag{4.6}$$

In view of (3.9), since $\underline{\beta}_k = r_m^{-1}\lambda_k$, (4.2) and (4.6) imply the required estimate (1.3) as follows:

$$\gamma_{m,k} - 1 = \frac{\beta_k}{\beta_0} - 1 = \frac{\underline{\beta}_k}{\beta_0} = O\left(\frac{1}{m}\right).$$

## 5. Proof of (1.4) and Corollary B

In this section, keeping the same notation as in the preceding sections, we shall prove (1.4) and Corollary B.

Proof of (1.4). For $X_\lambda$ in (4.1), the associated Hamiltonian function $f_\lambda \in C^\infty(M)_{\mathbb{R}}$ on the Kähler manifold $(M, \omega_m)$ is

$$f_\lambda = \frac{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\lambda_k\gamma_{m,k}|\sigma_{k,i}|^2_{h_m}}{m\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\gamma_{m,k}|\sigma_{k,i}|^2_{h_m}} = \frac{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\lambda_k|c_{k,i}s'_{k,i}|^2}{m\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}|c_{k,i}s'_{k,i}|^2},$$

where by (4.2), when $m$ runs through the set of all sufficiently large integers, the function $f_\lambda$ is uniformly $C^0$-bounded. Now by (3.10),

$$\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k}\lambda_k\gamma_{m,k}|\sigma_{k,i}|^2_{h_m} = \frac{m^{n+1}f_\lambda}{\beta_0}. \tag{5.1}$$

We now define a function $I_m$ on $M$ by

$$I_m := \frac{r_m^{-1}}{\beta_0} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \{1 - (\gamma_{m,k})^{-1}\} \lambda_k \gamma_{m,k} |\sigma_{k,i}|^2_{h_m}. \tag{5.2}$$

Then by (1.3), (3.9), (3.10), (4.2) and (4.6), we easily see that

$$I_m = O\left( m^{-2} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|^2_{h_m} \right) = O(m^{n-2}). \tag{5.3}$$

By (3.10) together with (5.1) and (5.2), it now follows that

$$\begin{aligned}
\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\sigma_{k,i}|^2_{h_m} &= \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \gamma_{m,k} |\sigma_{k,i}|^2_{h_m} - \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} (\gamma_{m,k} - 1) |\sigma_{k,i}|^2_{h_m} \\
&= \frac{m^n}{\beta_0} - \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \frac{\underline{\beta}_k}{\beta_0} |\sigma_{k,i}|^2_{h_m} \\
&= \frac{m^n}{\beta_0} + I_m - \frac{r_m^{-1}}{\beta_0} \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \lambda_k \gamma_{m,k} |\sigma_{k,i}|^2_{h_m} \\
&= \frac{m^n}{\beta_0} + I_m - \frac{r_m^{-1} m^2}{\beta_0^2} f_\lambda m^{n-1}.
\end{aligned}$$

By (3.9), $m^n / \beta_0 = N'_m$. Moreover, by (4.6), $r_m^{-1} m^2 / \beta_0^2 = O(1)$. Since

$$f_m := -\frac{r_m^{-1} m^2}{\beta_0^2} f_\lambda, \quad m \gg 1,$$

are uniformly $C^0$-bounded Hamiltonian functions on $(M, \omega_m)$ associated to holomorphic vector fields in $\mathfrak{t}$, in view of (5.3), we obtain

$$B_m^\bullet(\omega_m) = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\sigma_{k,i}|^2_{h_m} = N'_m + f_m m^{n-1} + O(m^{n-2}),$$

as required. □

Proof of Corollary B. Since the classical Futaki character $\mathcal{F}_1$ vanishes on $\mathfrak{t}$, we have $\mathcal{F}_1(X_\lambda) = 0$ in (4.5), so that

$$r_m^{-1} = O\left( \frac{1}{m^3} \right).$$

Then from (3.9) and $\underline{\beta}_k = r_m^{-1}\lambda_k$, by looking at (4.2), we obtain the following required estimate:

$$\gamma_{m,k} - 1 = \frac{\underline{\beta}_k}{\beta_0} = O\left(\frac{1}{m^2}\right).$$

Hence $B_m^{\bullet}(\omega_m) = \{1 + O(1/m^2)\} B_m^{\circ}(\omega_m)$. Integrating this over $M$ by the volume form $\omega_m^n$, in view of (3.12), we see that

$$N_m' = \left\{1 + O\left(\frac{1}{m^2}\right)\right\} \frac{m^n}{\beta_0}.$$

Therefore, from (3.12) and $B_m^{\bullet}(\omega_m) = \{1 + O(1/m^2)\} B_m^{\circ}(\omega_m)$, we now conclude that $B_m^{\bullet}(\omega_m) = N_m' + O(m^{n-2})$, as required. □

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