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# THE FUNDAMENTAL GROUP OF A GENERALIZED TRIGONAL CURVE

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**Abstract**

We develop a modification of the Zariski–van Kampen approach for the computation of the fundamental group of a trigonal curve with improper fibers. As an application, we list the deformation families and compute the fundamental groups of all irreducible maximizing simple sextics with a type **D** singular point.

## 1. Introduction

**1.1. Principal results.** We attempt to develop a modification of the classical Zariski–van Kampen approach [18] suitable to compute the fundamental group of a generalized trigonal curve, i.e., a trigonal curve with *improper fibers*, at which the curve meets the exceptional section. A similar question was addressed in [15], where the only improper fiber was 'hidden' at infinity. Here, we consider the case of arbitrarily many improper fibers (up to two in the applications).

The basic tool used in Zariski–van Kampen's method is the braid monodromy related to an appropriate pencil. This concept was introduced by O. Chisini [4], [5], O. Zariski [27], and E.R. van Kampen [18], and the term itself is probably due to B. Moishezon [22], who has also introduced explicitly such notions as the monodromy at infinity, braid monodromy factorization, and Hurwitz equivalence. For more details on the braid monodromy techniques in general and its usage in the computation of the fundamental group and other related invariants, as well as for the recent developments in the subject, we refer to the excellent recent surveys by Vik.S. Kulikov [20] and A. Libgober [21]. Note though that in this paper we are not concerned with the Hurwitz equivalence and merely use a certain modification (see next paragraph) of the braid monodromy as a computational tool. The Hurwitz equivalence of braid monodromy factorizations of a given element, even $\mathbb{B}_3$-valued and even those of algebro-geometric origin, seems to be a rather delicate subject; for some new results and further references, see [16].

In order to keep the braid monodromy well defined, $\mathbb{B}_3$-valued, and easily computable via skeletons (see Subsection 3.6), we pass to the associated genuine trigonal curve and introduce the concept of *slopes*, which compensate for the improper fibers. We compute local slopes (Subsection 3.5), study their properties, and discuss

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the modifications that should be made to the braid relations (3.4.4) and relation at infinity (3.4.6) in the Zariski–van Kampen presentation of the fundamental group, see Corollary 3.4.7.

As a simple application, in Subsection 3.7 we recompute the fundamental groups of irreducible plane quintics with a double point. (These groups were originally found in [7] and [2], but the computation via trigonal curves is much simpler and more straightforward; it could easily be computerized.)

**1.2. Plane sextics.** A more advanced example is the case of irreducible plane sextics with a type **D** singular point.

Recall that a plane sextic $C \subset \mathbb{P}^2$ is called *simple* if all its singular points are simple, i.e., those of types $\mathbf{A}_p$, $\mathbf{D}_q$, $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$ (see e.g. [1] for the notation). The total Milnor number $\mu(C)$ of a simple sextic $C$ does not exceed 19; if $\mu(C) = 19$, the sextic is called *maximizing*. Maximizing sextics are rigid: if two such sextics are equisingular deformation equivalent, they are related by a projective transformation. Each maximizing sextic is defined over an algebraic number field.

A sextic is said to be of *torus type* if its equation can be represented in the form $f_2^3 + f_3^2 = 0$, where $f_2$ and $f_3$ are certain homogeneous polynomials of degree 2 and 3, respectively. Alternatively, $C$ is of torus type if it is the ramification locus of a projection to $\mathbb{P}^2$ of a cubic surface $V \subset \mathbb{P}^3$. This property is invariant under equisingular deformations. Each sextic $C$ of torus type can be perturbed to a six cuspidal sextic, see [27], hence the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ factors to the reduced braid group $\bar{\mathbb{B}}_3 := \mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong \mathbb{Z}_2 * \mathbb{Z}_3 \cong PSL(2, \mathbb{Z})$; in particular, this group is never abelian or finite.

In this paper, we study irreducible maximizing simple sextics with a type **D** singular point and without type **E** singular points. (Sextics with type **E** points are the subject of [11], [12], and [13].) We list the equisingular deformation families of such sextics (Theorem 1.2.1) and compute their fundamental groups (Theorem 1.2.2). As in the previous papers, the principal tool is the reduction of a sextic with a triple singular point to a generalized trigonal curve in $\Sigma_1$.

**Theorem 1.2.1.** *There are* 38 *deformation families of irreducible maximizing simple sextics with a type* **D** *singular point and without type* **E** *singular points, realizing* 25 *sets of singularities* (*see* Tables 1 *and* 2 *in* Section 4). *One of the families is of torus type* (*the set of singularities* $\mathbf{D}_5 \oplus (\mathbf{A}_8 \oplus 3\mathbf{A}_2)$, *no.* 27 *in* Table 2); *the others are not.*

Theorem 1.2.1 is proved in Section 4. In principle, the statement can be obtained by comparing the results of J.-G. Yang [26] (a list of all sets of singularities that can be realized by an irreducible maximizing simple sextic) and I. Shimada [25] (a list of sets of singularities represented by several deformation families), using the global Torelli theorem for $K3$-surfaces. The advantage of our approach is an explicit construction of each sextic, which can further be used in the study of its geometry.

**Theorem 1.2.2.** *Let $C \subset \mathbb{P}^2$ be an irreducible maximizing simple sextic with a type* **D** *singular point. If $C$ is of torus type, then $\pi_1(\mathbb{P}^2 \setminus C)$ is the reduced braid group $\bar{\mathbb{B}}_3 = \mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong \mathbb{Z}_2 * \mathbb{Z}_3$; otherwise, $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}_6$.*

If $C$ has a type **E** point, the statement follows from [11], [12], and [13]. Other sextics as in Theorem 1.2.2 are considered in Section 5, using the models constructed in Section 4 and the approach developed in Section 3. As an immediate consequence, one obtains the following corollary.

**Corollary 1.2.3.** *Let $C'$ be a perturbation of a sextic $C$ as in* Theorem 1.2.2. *If $C'$ is of torus type, then $\pi_1(\mathbb{P}^2 \setminus C') = \bar{\mathbb{B}}_3$; otherwise, $\pi_1(\mathbb{P}^2 \setminus C') = \mathbb{Z}_6$.*

Recall that any induced subgraph of the combined Dynkin graph of a simple sextic $C$ can be realized by a perturbation of $C$.

We do not treat systematically reducible curves, as that would require an enormous amount of work. However, as a simple by-product, we do compute the groups of a few maximizing deformation families and their perturbations, see Table 3 in Subsection 5.2 and Table 4 in Subsection 5.7. Perturbing, one obtains more irreducible sextics with abelian groups, see Proposition 5.7.9. Altogether, the results of this and a few previous papers suggest the following conjecture.

CONJECTURE 1.2.4. With the exception of the maximizing sextics realizing the following three sets of singularities:

- $2\mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3$ (two curves; $\pi_1 = SL(2, \mathbb{F}_5) \rtimes \mathbb{Z}_6$, see [13]),
- $\mathbf{E}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$ (one curve; $\pi_1 = SL(2, \mathbb{F}_{19}) \rtimes \mathbb{Z}_6$, see [11]),
- $\mathbf{E}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$ (one curve; $\pi_1 = SL(2, \mathbb{F}_5) \odot \mathbb{Z}_{12}$, see [12]),

the fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus C)$ of an irreducible simple sextic $C \subset \mathbb{P}^2$ that is not of torus type and has a triple singular point is abelian.

(In the description of the groups, $\rtimes$ stands for a semi-direct product and $\odot$ stands for a central product: $SL(2, \mathbb{F}_5) \odot \mathbb{Z}_{12}$ is the quotient of $SL(2, \mathbb{F}_5) \times \mathbb{Z}_{12}$ by the diagonal subgroup $\mathbb{Z}_2 \subset \operatorname{Center} SL(2, \mathbb{F}_5) \times \mathbb{Z}_2$.) A proof of this conjecture would require a detailed analysis of the degenerations, which would probably lead to reducible sextics, and a computation of the groups of (some) reducible maximizing sextics with a type **D** or type $\mathbf{E}_7$ singular point. Then, it would remain to apply Zariski's epimorphism theorem [27]. Even if the group of the degenerate curve is non-abelian, its presentation arising from the skeleton is very transparent and one can easily compute the extra relations resulting from the perturbation, cf. Proposition 5.7.9 below and similar computation in [11], [12], and [13].

At this point, it is worth mentioning that the study of the degenerations of plane sextics with simple singularities only reduces to a purely arithmetical problem about adjacencies of their homological types: one needs to extend the lattice embedding $\Sigma \oplus \mathbb{Z}h \subset L$, $h^2 = 2$, corresponding to a given curve to an embedding $\Sigma' \oplus \mathbb{Z}h \subset L$, where $L$ is a unimodular even lattice of signature $(3, 19)$ and $\Sigma'$ is a negative definite root system of rank 19. The precise statement and a detailed proof are found in [24]. According to I. Shimada (private communication), one can expect a complete (computer aided) list of all such degenerations in the nearest future. The understanding of adjacencies of simple sextics is of a certain independent interest as well: there do exist sextics not admitting a degeneration to a maximizing one (irreducible or not), the only known example being the set of singularities $9\mathbf{A}_2$. (The fundamental group of this latter curve is known.)

After Theorem 1.2.2, there still remain five maximizing simple sextics of torus type with unknown fundamental groups; their sets of singularities are

$$(\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_3, \quad (\mathbf{A}_{14} \oplus \mathbf{A}_2) \oplus \mathbf{A}_2 \oplus \mathbf{A}_1, \quad (\mathbf{A}_{11} \oplus 2\mathbf{A}_2) \oplus \mathbf{A}_4,$$
$$(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4, \quad (\mathbf{A}_8 \oplus 3\mathbf{A}_2) \oplus \mathbf{A}_4 \oplus \mathbf{A}_1.$$

(We use the list of irreducible sextics of torus type found in [23]; maximizing sets of singularities can also be extracted from [26]. Due to [25], $(\mathbf{A}_8 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2) \oplus \mathbf{A}_4$ is realized by a pair of complex conjugate curves, whereas the four remaining sets of singularities define a single deformation family each.) Assuming that, up to complex conjugation, each *non*-maximizing set of singularities is realized by at most one connected deformation family of sextics of torus type (which is probably true, but proof is still pending), the groups of all such sextics are known. For details and further references, see recent survey [14].

**1.3. Contents of the paper.** In Section 2, we introduce the terminology and remind a few known results related to generalized trigonal curves. Section 3 deals with the fundamental groups: we remind the general approach, due to Zariski and van Kampen [18], specialize it to genuine trigonal curves (following [8]), and introduce slopes for generalized trigonal curves. Then, we explain how the slopes and the global monodromy can be computed and consider an example, applying the approach to irreducible plane quintics. In Section 4, we enumerate the deformation families of sextics as in Theorem 1.2.1 by describing the skeletons of their trigonal models; this description is used in Section 5 in the computation of the fundamental groups.

## 2. Generalized trigonal curves

In this section, we mainly introduce the terminology and cite a few known results related to (generalized) trigonal curves in Hirzebruch surfaces. Principal references are [10] and [11].

**2.1. Hirzebruch surfaces.** Recall that the *Hirzebruch surface* $\Sigma_k$, $k \geqslant 0$, is a rational geometrically ruled surface with an *exceptional section* $E = E_k$ of self-intersection $-k$. The fibers of the ruling are referred to as the fibers of $\Sigma_k$. The semigroup of classes of effective divisors on $\Sigma_k$ is generated by the classes of the exceptional section $E$ and a fiber $F$; one has $E^2 = -k$, $F^2 = 0$, and $E \cdot F = 1$.

Fix a Hirzebruch surface $\Sigma_k$, $k \geqslant 1$. Denote by $p\colon \Sigma_k \to \mathbb{P}^1$ the ruling, and let $E \subset \Sigma_k$ be the exceptional section, $E^2 = -k$. Given a point $b$ in the base $\mathbb{P}^1$, we denote by $F_b$ the fiber $p^{-1}(b)$. (With a certain abuse of the language, the points in the base $\mathbb{P}^1$ of the ruling are also referred to as fibers of $\Sigma_k$.) Let $F_b^\circ$ be the 'open fiber' $F_b \setminus E$. Observe that $F_b^\circ$ is a dimension 1 affine space over $\mathbb{C}$; hence, one can speak about lines, circles, angles, convexity, etc. in $F_b^\circ$. In particular, one can define the *convex hull* $\operatorname{conv} S$ of a subset $S \subset \Sigma_k \setminus E$ as the union of its fiberwise convex hulls:

$$\operatorname{conv} S = \bigcup_{b \in \mathbb{P}^1} \operatorname{conv}(S \cap F_b^\circ).$$

**2.2. Trigonal curves.** A *generalized trigonal curve* on a Hirzebruch surface $\Sigma_k$ is a reduced curve $B$ not containing the exceptional section $E$ and intersecting each generic fiber at three points. In this paper, we assume in addition that a trigonal curve does not contain a fiber of $\Sigma_k$ as a component.

A *singular fiber* of a generalized trigonal curve $B \subset \Sigma_k$ is a fiber $F$ of $\Sigma_k$ that is not transversal to the union $B \cup E$. Thus, $F$ is either the fiber over a critical value of the restriction to $B$ of the ruling $\Sigma_k \to \mathbb{P}^1$ or the fiber through a point of intersection of $B$ and $E$. In the former case, the fiber is called *proper*; in the latter case, the fiber is called *improper* and the points of intersection of $B$ and $E$ are called *points at infinity*. In general, the local branches of $B$ that intersect a fiber $F$ outside of $E$ are called *proper* at $F$.

A (*genuine*) *trigonal curve* is a generalized trigonal curve $B \subset \Sigma_k$ disjoint from the exceptional section. One has $B \in |3E + 3kF|$; conversely, any reduced curve $B \in |3E + 3kF|$ not containing $E$ as a component is a trigonal curve.

We use the following notation for the topological types of proper fibers:

- $\tilde{\mathbf{A}}_0$: a nonsingular fiber;
- $\tilde{\mathbf{A}}_0^*$: a simple vertical tangent;
- $\tilde{\mathbf{A}}_0^{**}$: a vertical inflection tangent;
- $\tilde{\mathbf{A}}_1^*$: a node of $B$ with one of the branches vertical;
- $\tilde{\mathbf{A}}_2^*$: a cusp of $B$ with vertical tangent;
- $\tilde{\mathbf{A}}_p$, $p \geqslant 2$, $\tilde{\mathbf{D}}_q$, $q \geqslant 4$, $\tilde{\mathbf{E}}_{6r+\epsilon}$, $r \geqslant 1$, $\epsilon = 0, 1, 2$, $\tilde{\mathbf{J}}_{r,p}$, $r \geqslant 2$, $p \geqslant 0$: a singular point of $B$ of the same type (see [1] for the notation) with minimal possible local intersection index with the fiber.

For 'simple' fibers of types $\tilde{\mathbf{A}}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{E}}_6$, $\tilde{\mathbf{E}}_7$, and $\tilde{\mathbf{E}}_8$, this notation refers to the incidence graph of $(-2)$-curves in the corresponding singular elliptic fiber; this graph is an affine Dynkin diagram.

REMARK 2.2.1. The topological classification of singular fibers of trigonal curves is close to that for elliptic surfaces, see [19], except that in this paper we admit curves with non-simple singularities. It would probably be more convenient (but slightly less transparent) to use an appropriate extension of Kodaira's notation, for example $\mathrm{I}_p^r$, $\mathrm{II}^r$, $\mathrm{III}^r$, and $\mathrm{IV}^r$, with $r = 0$ and 1 referring, respectively, to the empty subscript and $^*$ in [19]. Another alternative would be to extend the series $\tilde{\mathbf{J}}_{r,p}$ and $\tilde{\mathbf{E}}_{6r+\epsilon}$ to the values $r = 0$ and 1. Among other advantages, in both cases an elementary transformation (see Subsection 2.3 below) would merely increase the value of $r$ by 1. However, I chose to retain the commonly accepted notation for the types of simple singularities.

The fibers of types $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^*$, and $\tilde{\mathbf{A}}_2^*$ are called *unstable*; all other singular fibers are called *stable*. A trigonal curve $B$ is *stable* if so are all its singular fibers. (This notion of stability differs from the one accepted in algebraic geometry; we refer to the topological stability under equisingular deformations of $B$. An unstable fiber may split as follows: $\tilde{\mathbf{A}}_0^{**} \to 2\tilde{\mathbf{A}}_0^*$, $\tilde{\mathbf{A}}_1^* \to \tilde{\mathbf{A}}_1 \oplus \tilde{\mathbf{A}}_0^*$, or $\tilde{\mathbf{A}}_2^* \to \tilde{\mathbf{A}}_2 \oplus \tilde{\mathbf{A}}_0^*$, the splitting not changing the topology of the pair $(\Sigma_k, B)$.)

The *multiplicity* $\operatorname{mult} F$ of a singular fiber $F$ of a trigonal curve $B$ is the number of simplest (i.e., type $\tilde{\mathbf{A}}_0^*$) fibers into which $F$ splits under deformations of $B$. For the $\tilde{\mathbf{A}}$ type fibers, one has $\operatorname{mult}\tilde{\mathbf{A}}_0 = 0$, $\operatorname{mult}\tilde{\mathbf{A}}_0^* = 1$, $\operatorname{mult}\tilde{\mathbf{A}}_0^{**} = 2$, $\operatorname{mult}\tilde{\mathbf{A}}_1^* = 3$, $\operatorname{mult}\tilde{\mathbf{A}}_2^* = 4$, and $\operatorname{mult}\tilde{\mathbf{A}}_p = p + 1$ for $p > 0$. Each elementary transformation (see Subsection 2.3 below) contracting $F$ increases $\operatorname{mult} F$ by 6. The sum of the multiplicities of all singular fibers of a trigonal curve $B \subset \Sigma_k$ equals $12k$.

**2.3. Elementary transformations.** An *elementary transformation* of $\Sigma_k$ is a birational transformation $\Sigma_k \dashrightarrow \Sigma_{k+1}$ consisting in blowing up a point $P$ in the exceptional section of $\Sigma_k$ followed by blowing down the fiber $F$ through $P$. The inverse transformation $\Sigma_{k+1} \dashrightarrow \Sigma_k$ blows up a point $P'$ *not* in the exceptional section of $\Sigma_{k+1}$ and blows down the fiber $F'$ through $P'$.

An elementary transformation converts a proper fiber as follows:

(1) $\tilde{\mathbf{A}}_0 \to \tilde{\mathbf{D}}_4 \to \tilde{\mathbf{J}}_{2,0} \to \cdots \to \tilde{\mathbf{J}}_{r,0} \to \cdots$ (not detected by the $j$-invariant);

(2) $\tilde{\mathbf{A}}_0^* \to \tilde{\mathbf{D}}_5 \to \tilde{\mathbf{J}}_{2,1} \to \cdots \to \tilde{\mathbf{J}}_{r,1} \to \cdots$ ($j = \infty$, $\operatorname{ord} j = 1$);

(3) $\tilde{\mathbf{A}}_{p-1} \to \tilde{\mathbf{D}}_{p+4} \to \tilde{\mathbf{J}}_{2,p} \to \cdots \to \tilde{\mathbf{J}}_{r,p} \to \cdots$ ($p \geqslant 2$; $j = \infty$, $\operatorname{ord} j = p$);

(4) $\tilde{\mathbf{A}}_0^{**} \to \tilde{\mathbf{E}}_6 \to \tilde{\mathbf{E}}_{12} \to \cdots \to \tilde{\mathbf{E}}_{6r} \to \cdots$ ($j = 0$, $\operatorname{ord} j = 1 \bmod 3$);

(5) $\tilde{\mathbf{A}}_1^* \to \tilde{\mathbf{E}}_7 \to \tilde{\mathbf{E}}_{13} \to \cdots \to \tilde{\mathbf{E}}_{6r+1} \to \cdots$ ($j = 1$, $\operatorname{ord} j = 1 \bmod 2$);

(6) $\tilde{\mathbf{A}}_2^* \to \tilde{\mathbf{E}}_8 \to \tilde{\mathbf{E}}_{14} \to \cdots \to \tilde{\mathbf{E}}_{6r+2} \to \cdots$ ($j = 0$, $\operatorname{ord} j = 2 \bmod 3$).

For the reader's convenience, we also indicate the value $j = v$ and the ramification index $\operatorname{ord} j$ of the $j$-invariant, see Subsection 2.4 below, which is invariant under elementary transformations. In a neighborhood of the fiber, the $j$-invariant has the form $v + t^{\operatorname{ord} j}$ if $v = 0$ or 1 or $1/t^{\operatorname{ord} j}$ if $v = \infty$.

Let $\tilde{B} \subset \Sigma_{\tilde{k}}$ be a generalized trigonal curve. Then, by a sequence of elementary transformations, one can resolve the points of intersection of $\tilde{B}$ and $E$ and obtain a genuine trigonal curve $B \subset \Sigma_k$, $k \geqslant \tilde{k}$, birationally equivalent to $\tilde{B}$. The trigonal curve $B$ obtained from $\tilde{B}$ by a minimal number of elementary transformations is called the *trigonal model* of $\tilde{B}$.

REMARK 2.3.1. Alternatively, given a trigonal curve $B \subset \Sigma_k$ with triple singular points, one can apply a sequence of inverse elementary transformations to obtain a trigonal curve $B' \subset \Sigma_{k'}$, $k' \leqslant k$, birationally equivalent to $B$ and with $\tilde{\mathbf{A}}$ type singular fibers only. This curve $B'$ is called in [10] the *simplified model* of $B$.

**2.4. The *j*-invariant.** The (*functional*) *j-invariant* $j_B\colon \mathbb{P}^1 \to \mathbb{P}^1$ of a generalized trigonal curve $B \subset \Sigma_k$ is defined as the analytic continuation of the function sending a point $b$ in the base $\mathbb{P}^1$ of $\Sigma_k$ representing a nonsingular fiber $F_b$ of $B$ to the $j$-invariant (divided by $12^3$) of the elliptic curve covering $F_b$ and ramified at the four points of intersection of $F_b$ and $B + E$. The curve $B$ is called *isotrivial* if $j_B = \text{const}$. Such curves can easily be enumerated, see e.g. [10].

By definition, $j_B$ is invariant under elementary transformations. The values of $j_B$ at the singular fibers of $B$ are listed in Subsection 2.3. The points $b \subset \mathbb{P}^1$ with $j_B(b) = 0$ and $\operatorname{ord}_b j_B = 0 \bmod 3$ or $j_B(b) = 1$ and $\operatorname{ord}_b j_B = 0 \bmod 2$ correspond to fibers $F_b$ admitting extra symmetries. Assuming $F_b$ proper (hence nonsingular), consider the three points of intersection of $B$ and $F_b^\circ$. Then

– the three points form an equilateral triangle if $j_B(b) = 0$, $\operatorname{ord}_b j_B = 0 \bmod 3$;

– one of the points is at the center of the segment connecting the two others if $j_B(b) = 1$, $\operatorname{ord}_b j_B = 0 \bmod 2$.

DEFINITION 2.4.1. A non-isotrivial trigonal curve $B$ is called *maximal* if it has the following properties:

(1) $B$ has no singular fibers of type $\tilde{\mathbf{D}}_4$ or $\tilde{\mathbf{J}}_{r,0}$, $r \geqslant 2$;

(2) $j = j_B$ has no critical values other than 0, 1, and $\infty$;

(3) each point in the pull-back $j^{-1}(0)$ has ramification index at most 3;

(4) each point in the pull-back $j^{-1}(1)$ has ramification index at most 2.

An important property of maximal trigonal curves is their rigidity, see [10]: any small fiberwise equisingular deformation of such a curve $B \subset \Sigma_k$ is isomorphic to $B$. Any maximal trigonal curve is defined over an algebraic number field. Such curves are classified by their skeletons, see Theorem 2.6.1 below.

A maximal trigonal curve $B$ with simple singularities only can be characterized in terms of its total Milnor number $\mu(B)$ (i.e., the sum of the Milnor numbers of all singular points of $B$). The following criterion is proved in [11].

**Theorem 2.4.2.** *For a non-isotrivial genuine trigonal curve $B \subset \Sigma_k$ with simple singularities only one has*

$$\mu(B) \leqslant 5k - 2 - \#\{\textit{unstable fibers of } B\}, \tag{2.4.3}$$

*the equality holding if and only if $B$ is maximal.*

REMARK 2.4.4. The inequality in Theorem 2.4.2 may not hold is $B$ has non-simple singular points, as each elementary transformation producing a non-simple singular point increases $\mu$ by 6 while increasing $k$ by 1.

**2.5. Skeletons.** The *skeleton* $\mathrm{Sk} = \mathrm{Sk}_B$ of a trigonal curve $B \subset \Sigma_k$ is defined as Grothendieck's *dessin d'enfants* of its $j$-invariant $j_B$. More precisely, Sk is the planar map $j_B^{-1}([0, 1]) \subset S^2 \cong \mathbb{P}^1$. The pull-backs of 0 are called $\bullet$-*vertices*, and the pull-backs of 1 are called $\circ$-*vertices*. The $\bullet$- and $\circ$-vertices are called *essential*; the other vertices that Sk may have (due to the critical values of $j_B$ in the interval $(0, 1)$) are called *unessential*.

By definition, Sk is a graph in the base of the ruling $\Sigma_k \to \mathbb{P}^1$, so that one can speak about the fibers of $\Sigma_k$ represented by points of Sk. On the other hand, for the classification statements, see e.g. Theorem 2.6.1 below, it is important that Sk is regarded as a graph in the *topological* sphere $S^2$; the analytic structure is given by the skeleton itself via Riemann's existence theorem.

The $\bullet$-vertices of valency 1 mod 3 or 2 mod 3 and $\circ$-vertices of valency 1 mod 2 are called *singular*; they correspond to singular fibers of the curve of one of the types 2.3 (4)–(6). All other $\bullet$- and $\circ$-vertices are called *nonsingular*.

After a small fiberwise equisingular deformation of a trigonal curve $B$ one can assume that its skeleton $\mathrm{Sk}_B$ has the following properties:
(1) all vertices of $\mathrm{Sk}_B$ are essential;
(2) each $\bullet$-vertex has valency at most 3;
(3) each $\circ$-vertex has valency at most 2.
A skeleton satisfying these conditions is called *generic*. Note that any skeleton satisfying condition (1) is a bipartite graph. For this reason, in the drawings below we omit bivalent $\circ$-vertices, assuming that such a vertex is to be inserted in the middle of each edge connecting two $\bullet$-vertices. In particular, for a generic skeleton, only singular monovalent $\circ$-vertices are drawn.

A *region* of a skeleton $\mathrm{Sk} \subset \mathbb{P}^1$ is a connected component of the complement $\mathbb{P}^1 \setminus \mathrm{Sk}$. One can also speak about *closed regions*, which are connected components of the manifold theoretical cut of $\mathbb{P}^1$ along Sk. (In general, a closed region $\bar{R}$ is *not* the same as the closure of the corresponding open region $R$.) We say that a region $R$ is an *$m$-gon* (or an *$m$-gonal region*) if the boundary of the corresponding closed region $\bar{R}$ contains $m$ $\bullet$-vertices. For example, in Fig. 4 (b) below, the three regions marked with $\alpha$, $\beta$, and $\bar{\beta}$ are monogons, whereas the outer region is a nonagon. In Fig. 4 (c), there are two monogons (marked with $\alpha$ and $\beta$) and two pentagons.

Each region $R$ of $\mathrm{Sk}_B$ contains a finite number of singular fibers of $B$, which can be of one of the types 2.3 (1)–(3) (excluding $\tilde{\mathbf{A}}_0$, which is not singular). One can use a sequence of inverse elementary transformations and convert these fibers to the $\tilde{\mathbf{A}}$ type fibers starting the series. If $R$ is an $m$-gonal region, the total multiplicity of these $\tilde{\mathbf{A}}$ type fibers equals $m$.

**2.6. Skeletons and maximal curves.** The skeleton $\mathrm{Sk}_B$ of a maximal trigonal curve $B \subset \Sigma_k$ is necessarily generic and connected. (It follows that each region of $\mathrm{Sk}_B$ is a topological disk.) Each $m$-gonal region $R$ of $\mathrm{Sk}_B$ contains a single singular fiber $F_R$ of $B$; its type is one of Section 2.3 (2) if $m = 1$ or one of Section 2.3 (3) with $p = m$ if $m \geqslant 2$. Thus, the type of $F_R$ is determined by its multiplicity. The other singular fibers of $B$ are over the singular vertices of $\mathrm{Sk}_B$; the type of such a singular fiber $F_v$ is also determined by its multiplicity (and the type and the valency of $v$).

The function $\mathrm{ts}_B$ sending each region $R$ to the multiplicity $\operatorname{mult} F_R$ and each singular vertex $v$ to the multiplicity $\operatorname{mult} F_v$ is called the *type specification*. It has the following properties:

(1) $\mathrm{ts}_B(m\text{-gonal region } R) = m + 6s,\ s \in \mathbb{Z}_{\geqslant 0}$;
(2) $\mathrm{ts}_B(\text{singular } \bullet\text{-vertex } v) = 2(\text{valency of } v) + 6s,\ s \in \mathbb{Z}_{\geqslant 0}$;
(3) $\mathrm{ts}_B(\text{singular } \circ\text{-vertex}) = 3 + 6s,\ s \in \mathbb{Z}_{\geqslant 0}$;
(4) the sum of all values of $\mathrm{ts}_B$ equals $12k$.

The following statement is essentially contained in [10].

**Theorem 2.6.1.** *The map* $B \mapsto (\mathrm{Sk}_B, \mathrm{ts}_B)$ *establishes a bijection between the set of isomorphism classes* (*equivalently, fiberwise equisingular deformation classes*) *of maximal trigonal curves in* $\Sigma_k$ *and the set of orientation preserving diffeomorphism classes of pairs* $(\mathrm{Sk}, \mathrm{ts})$, *where* $\mathrm{Sk} \subset S^2$ *is a connected generic skeleton and* ts *is a function on the set of regions and singular vertices of* Sk *satisfying conditions* (1)–(4) *above.*

REMARK 2.6.2. Often it is more convenient to replace $\mathrm{ts}_B$ with the $\mathbb{Z}_{\geqslant 0}$-valued function $\mathrm{td}_B$ sending each region and singular vertex to the integer $s$ appearing in (1)–(3). In term of $\mathrm{ts}_B$, the index $k$ of the Hirzebruch surface $\Sigma_k$ containing $B$ is given as follows, cf. [11]:

$$\#_\bullet + \#_\circ(1) + \#_\bullet(2) = 2\Big(k - \sum \mathrm{td}_B\Big),$$

where $\#_\bullet$ is the total number of $\bullet$-vertices, $\#_*(i)$ is the number of $*$-vertices of valency $i$, and $\sum \mathrm{td}_B$ is the sum of all values of $\mathrm{td}_B$. The singular points of $B$ are simple if and only if $\mathrm{td}_B$ takes values in $\{0, 1\}$; in this case, $\sum \mathrm{td}_B$ is merely the number of triple singular points of $B$.

## 3. The Zariski–van Kampen method

In Subsections 3.1–3.3, we briefly remind the classical Zariski–van Kampen approach [18] to the computation of the fundamental group of an algebraic curve and the construction of [8], which makes the braid monodromy of a genuine trigonal curve almost canonically defined. In Subsection 3.4, we introduce the concept of *slope* which lets one treat a generalized trigonal curve in terms of its trigonal model and, in particular, keep the braid monodromy $\mathbb{B}_3$-valued and easily computable. In Subsections 3.5 and 3.6, we compute the local slopes and cite the results of [10] related to the global braid monodromy of a trigonal curve in terms of its skeleton. Finally, in Subsection 3.7, we consider a simple example, computing the groups of irreducible quintics.

**3.1. Proper sections and braid monodromy.** Fix a Hirzebruch surface $\Sigma_k$, $k \geqslant 1$, and a genuine trigonal curve $B \subset \Sigma_k$. The term 'section' below stands for a continuous section of (an appropriate restriction of) the fibration $p\colon \Sigma_k \to \mathbb{P}^1$.

DEFINITION 3.1.1. Let $\Delta \subset \mathbb{P}^1$ be a closed (topological) disk. A partial section $s\colon \Delta \to \Sigma_k$ of $p$ is called *proper* if its image is disjoint from both $E$ and $\operatorname{conv} B$.

The following statement is found in [8]; it is an immediate consequence of the fact that the restriction $p\colon p^{-1}(\Delta) \setminus (E \cup \operatorname{conv} B) \to \Delta$ is a locally trivial fibration with connected fibers and contractible base.

**Lemma 3.1.2.** *Any disk $\Delta \subset \mathbb{P}^1$ admits a proper section $s\colon \Delta \to \Sigma_k$. Any two proper sections over $\Delta$ are homotopic in the class of proper sections*; *furthermore*, *any homotopy over a fixed point $b \in \Delta$ extends to a homotopy over $\Delta$.*

Fix a disk $\Delta \subset \mathbb{P}^1$ and let $b_1, \ldots, b_r \in \Delta$ be all singular and, possibly, some nonsingular fibers of $B$ that belong to $\Delta$. Denote $F_i = p^{-1}(b_i)$. We assume that all these fibers are in the interior of $\Delta$. Denote $\Delta^\sharp = \Delta \setminus \{b_1, \ldots, b_l\}$ and fix a point $b \in \Delta^\sharp$. The restriction $p^\sharp\colon p^{-1}(\Delta^\sharp) \setminus (B \cup E) \to \Delta^\sharp$ is a locally trivial fibration with a typical fiber $F_b^\circ \setminus B$, and any proper section $s\colon \Delta \to \Sigma_k$ restricts to a section of $p^\sharp$. Hence, given a proper section $s$, one can define the group $\pi_F := \pi_1(F_b^\circ \setminus B, s(b))$ and the *braid monodromy* $\mathfrak{m}\colon \pi_1(\Delta^\sharp, b) \to \operatorname{Aut} \pi_F$. More generally, given a path $\gamma\colon [0,1] \to \Delta^\sharp$ with $\gamma(0) = b$, one can define the *translation homomorphism* $\mathfrak{m}_\gamma\colon \pi_F \to \pi_1\bigl(F_{\gamma(1)}^\circ \setminus B, s(b)\bigr)$.

Denote by $\rho_b \in \pi_F$ the 'counterclockwise' generator of the abelian subgroup $\mathbb{Z} \cong \pi_1(F_b^\circ \setminus \operatorname{conv} B) \subset \pi_F$. (In other words, $\rho_b$ is the class of a large circle in $F_b^\circ$ encompassing $\operatorname{conv} B \cap F_b^\circ$.) Since the fibration $p^{-1}(\Delta) \setminus (\operatorname{conv} B \cup E) \to \Delta$ is trivial, hence 1-simple, $\rho_b$ is invariant under the braid monodromy and is preserved by the translation homomorphisms. Thus, there is a canonical identification of the elements $\rho_{b'}$, $\rho_{b''}$ in the fibers over any two points $b', b'' \in \Delta^\sharp$; for this reason, we will omit the subscript $b$ in the sequel.

In this paper, we reserve the terms 'braid monodromy' and 'translation homomorphism' for the homomorphisms $\mathfrak{m}$ constructed above using a *proper* section $s$. Under this convention, next lemma follows from Lemma 3.1.2 and the obvious fact that the braid monodromy is homotopy invariant.

**Lemma 3.1.3.** *The braid monodromy* $\mathfrak{m}\colon \pi_1(\Delta^\sharp, b) \to \operatorname{Aut} \pi_F$ *is well defined and independent of the choice of a proper section over* $\Delta$ *passing through* $s(b)$*. Given a path* $\gamma$ *in* $\Delta^\sharp$*, the translation homomorphism* $\mathfrak{m}_\gamma$ *is independent of the choice of a proper section passing through* $s(\gamma(0))$ *and* $s(\gamma(1))$ *up to conjugation by* $\rho$*.*

**3.2. The Zariski–van Kampen theorem.** Pick a basis $\{\alpha_1, \alpha_2, \alpha_3\}$ for $\pi_F$ and a basis $\{\gamma_1, \dots, \gamma_r\}$ for $\pi_1(\Delta^\sharp, b)$. Both $F_b^\circ \setminus B$ and $\Delta^\sharp$ are oriented punctured planes, and we usually assume that the bases are standard: each basis element is represented by the loop formed by the counterclockwise boundary of a small disk centered at a puncture and a simple arc connecting this disk to the base point; all disks and arcs are disjoint except at the common base point. With a certain abuse of the language, we will refer to $\gamma_i$ (respectively, $\alpha_j$) as the generator about the $i$-th singular fiber (respectively, about the $j$-th branch) of $B$. We also assume that the basis elements are numbered so that $\alpha_1\alpha_2\alpha_3 = \rho$ and $\gamma_1 \cdots \gamma_r$ is freely homotopic to the boundary $\partial\Delta$. Under this convention on the basis $\{\alpha_1, \alpha_2, \alpha_3\}$, the braid monodromy does indeed take values in the braid group $\mathbb{B}_3 \subset \operatorname{Aut} \pi_F$.

Using a proper section $s$, we can identify each generator $\gamma_i$ with a certain element of the group $\pi_1(p^{-1}(\Delta^\sharp) \setminus (B \cup E), s(b))$; this element does not depend on the choice of a section. The following presentation of the latter group is the essence of Zariski–van Kampen's method for computing the fundamental group of a plane algebraic curve, see [18] for the proof and further details.

**Theorem 3.2.1.** *In the notation above, one has*

$$\begin{aligned}&\pi_1(p^{-1}(\Delta^\sharp) \setminus (B \cup E), s(b))\\ &\quad = \langle \alpha_1, \alpha_2, \alpha_3, \gamma_1, \dots, \gamma_r \mid \gamma_i^{-1}\alpha_j\gamma_i = \mathfrak{m}_i(\alpha_j),\ i = 1, \dots, r,\ j = 1, 2, 3\rangle,\end{aligned}$$

*where* $\mathfrak{m}_i = \mathfrak{m}(\gamma_i)$, $i = 1, \dots, r$.

**3.3. The monodromy at infinity and relation at infinity.** Let $\Delta \subset \mathbb{P}^1$ be a disk as above. Connecting $\partial\Delta$ with the base point $b$ by a path in $\Delta^\sharp$ and traversing it in the counterclockwise direction (with respect to the canonical complex orientation of $\Delta$), one obtains a certain element $[\partial\Delta] \in \pi_1(\Delta^\sharp, b)$ (which depends on the choice of the path above). The following two statements are proved in [8].

**Lemma 3.3.1.** *Assume that the interior of $\Delta$ contains all singular fibers of $B$. Then, for any $\alpha \in \pi_F$, one has $\mathfrak{m}([\partial\Delta])(\alpha) = \rho^k \alpha \rho^{-k}$. In particular, the image $\mathfrak{m}([\partial\Delta]) \in \operatorname{Aut} \pi_F$ does not depend on the choices in the definition of* $[\partial\Delta]$; *it is called the monodromy at infinity.*

**Lemma 3.3.2.** *Assume that the interior of $\Delta$ contains all singular fibers of $B$. Then a presentation for the group*

$$\pi_1\left(\Sigma_k \setminus \left(B \cup E \cup \bigcup_{i=1}^{r} F_i\right), s(b)\right)$$

*is obtained from that given by* Lemma 3.2.1 *by adding the so called relation at infinity* $\gamma_1 \cdots \gamma_r \rho^k = 1$.

It remains to remind that patching back in a singular fiber $F_i$ results in an extra relation $\gamma_i = 1$. Hence, for a *genuine* trigonal curve $B$, one has

$$\pi_1(\Sigma_k \setminus (B \cup E)) = \langle \alpha_1, \alpha_2, \alpha_3 \mid \mathfrak{m}_i = \mathrm{id},\ i = 1, \ldots, r,\ \rho^k = 1 \rangle, \tag{3.3.3}$$

where each *braid relation* $\mathfrak{m}_i = \mathrm{id}$, $i = 1, \ldots, r$, should be understood as a triple of relations $\mathfrak{m}_i(\alpha_j) = \alpha_j$, $j = 1, 2, 3$.

**3.4. Slopes.** Now, let $\tilde{B} \subset \Sigma_{\tilde{k}}$ be a *generalized* trigonal curve, and let $B \subset \Sigma_k$ be its trigonal model. Denote by $F_1, \ldots, F_r$ the singular fibers of $\tilde{B}$ and let $b_i \in \mathbb{P}^1$ be the projection of $F_i$, $i = 1, \ldots, r$. The birational transformation between $\tilde{B}$ and $B$ establishes a diffeomorphism

$$\Sigma_{\tilde{k}} \setminus \left(\tilde{B} \cup E \cup \bigcup_{i=1}^{r} F_i\right) \cong \Sigma_k \setminus \left(B \cup E \cup \bigcup_{i=1}^{r} F_i\right);$$

hence, it establishes an isomorphism of the fundamental groups. Let $\tilde{\Gamma}_i$ be a small analytic disk in $\Sigma_{\tilde{k}} \setminus E$ transversal to $F_i$ and disjoint from $\tilde{B}$ and from the other singular fibers of $\tilde{B}$, and let $\Gamma_i$ be the transform of $\tilde{\Gamma}_i$ in $\Sigma_k$. We will call $\Gamma_i$ a *geometric slope* of $\tilde{B}$ at $F_i$. According to van Kampen's theorem [18], patching back in the fiber $F_i$ results in an extra relation $[\partial\tilde{\Gamma}_i] = 1$ or, equivalently, $[\partial\Gamma_i] = 1$.

Fix a proper (with respect to the genuine trigonal curve $B$) section $s$ over a disk $\Delta \subset \mathbb{P}^1$ containing the projection $p(\Gamma_i)$. Pick a base point $b_i' \in p(\partial\Gamma_i)$ and denote $F_i' = p^{-1}(b_i')$ and $\gamma_i' = [p(\partial\Gamma_i)]$. As above, we can regard $\gamma_i'$ both as an element of $\pi_1(\Delta^\sharp, b_i')$ and, via $s$, as an element of $\pi_1(p^{-1}(\Delta^\sharp) \setminus (B \cup E), s(b_i'))$. Furthermore, we can assume that the basis element $\gamma_i \subset \pi_1(\Delta^\sharp, b)$ introduced in Subsection 3.2 has the form $\gamma_i = \zeta_i \cdot \gamma_i' \cdot \zeta_i^{-1}$, where $\zeta_i$ is a simple arc in $\Delta^\sharp$ connecting $b$ to $b_i'$.

Dragging the nonsingular fiber $F_i'$ along $\gamma_i'$ and keeping two points in the image of $s$ and in $\Gamma_i$, one can define the *relative braid monodromy*

$$\mathfrak{m}_i^{\mathrm{rel}} \in \operatorname{Aut} \pi_1((F_i')^\circ \setminus B,\, F_i' \cap \Gamma_i,\, s(b_i')).$$

DEFINITION 3.4.1. The *local slope* of a generalized trigonal curve $\tilde{B}$ at its singular fiber $F_i$ is the element $\varkappa_i' := \mathfrak{m}_i^{\mathrm{rel}}(\xi) \cdot \xi^{-1} \in \pi_1((F_i')^\circ \setminus B, s(b_i'))$, where $\xi_i$ is any path in $(F_i')^\circ \setminus B$ connecting $s(b_i')$ and $F_i' \cap \Gamma_i$. The (*global*) *slope* of $\tilde{B}$ at $F_i$ (defined by a standard basis element $\gamma_i$ or, more precisely, by a path $\zeta_i$ connecting the base point $b$ to a point $b_i'$ 'close' to $b_i$) is the image $\varkappa_i := \mathfrak{m}_{\zeta_i}^{-1}(\varkappa_i') \in \pi_F$.

The following two statements are immediate consequences of the definition.

**Lemma 3.4.2.** *The slope $\varkappa_i$ is defined by the curve $\tilde{B}$ and generator $\gamma_i$ up to conjugation by $\rho$* (*due to the indeterminacy of the translation homomorphism, see* Lemma 3.1.3) *and the transformation $\varkappa_i \mapsto \mathfrak{m}_i(\beta)\varkappa_i\beta^{-1}$, $\beta \in \pi_F$* (*due to the choice of path $\xi_i$ in the definition*).

**Lemma 3.4.3.** *In the fundamental group $\pi_1(p^{-1}p(\partial\Gamma_i) \setminus (B \cup E), s(b_i'))$, the conjugacy class containing $[\partial\Gamma_i]$ consists of all elements of the form $\gamma_i'\varkappa_i'$, where $\varkappa_i'$ is a local slope of $\tilde{B}$ at $F_i$.*

Note that, in view of Lemma 3.4.2 and the relation $(\gamma_i')^{-1}\beta\gamma_i' = \mathfrak{m}(\gamma_i')(\beta)$, cf. Lemma 3.2.1, the elements $\gamma_i'\varkappa_i'$ do indeed form a conjugacy class.

As a consequence, in terms of the basis $\{\alpha_1,\alpha_2,\alpha_3,\gamma_1,\dots,\gamma_r\}$, the relation $[\partial\Gamma_i] = 1$ resulting from patching the singular fiber $F_i$ *in the original surface* $\Sigma_{\tilde{k}}$ becomes $\gamma_i = \varkappa_i^{-1}$. Eliminating $\gamma_i$, the relations $\gamma_i^{-1}\alpha_j\gamma_i = \mathfrak{m}_i(\alpha_j)$, $j = 1, 2, 3$, cf. Lemma 3.2.1, turn into the *braid relations*

$$\varkappa_i\alpha_j\varkappa_i^{-1} = \mathfrak{m}_i(\alpha_j),\ \ j = 1, 2, 3, \quad \text{or} \quad \tilde{\mathfrak{m}}_i = \mathrm{id}, \tag{3.4.4}$$

where $\tilde{\mathfrak{m}}_i \colon \alpha \mapsto \varkappa_i^{-1}\mathfrak{m}_i(\alpha)\varkappa_i$ is the *twisted braid monodromy*.

Clearly, if $F_i$ is a proper fiber, then $\Gamma_i = \tilde{\Gamma}_i$ and the path $\xi_i$ in the definition can be chosen so that $\varkappa_i = 1$. In this case $\tilde{\mathfrak{m}}_i = \mathfrak{m}_i$ is still a braid.

REMARK 3.4.5. In view of Lemma 3.4.2 and the fact that $\rho$ is invariant under $\mathfrak{m}_i$, for each fixed $i = 1, \dots, r$ the normal subgroup of $\pi_F$ defined by the relations $\tilde{\mathfrak{m}}_i = \mathrm{id}$ does not depend on the choice of a particular slope $\varkappa_i$, and the projection of $\varkappa_i$ to the quotient group $\pi_F/\tilde{\mathfrak{m}}_i = \mathrm{id}$ is a well defined element of this group (depending on the curve $\tilde{B}$ and basis element $\gamma_i$ only). In particular, each slope commutes with $\rho$ (in the corresponding quotient), making irrelevant the ambiguity in the definition of the translation homomorphisms, see Lemma 3.1.3.

If all singular fibers are patched, hence all generators $\gamma_i$ are eliminated, the relation at infinity takes the form

$$\rho^k = \varkappa_r \cdots \varkappa_1. \tag{3.4.6}$$

Finally, one obtains the following statement, cf. (3.3.3), expressing the fundamental group $\pi_1(\Sigma_{\tilde{k}} \setminus (\tilde{B} \cup E))$ in terms of the slopes and braid monodromy of the *genuine* trigonal curve $B$.

**Corollary 3.4.7.** *For a generalized trigonal curve $\tilde{B} \subset \Sigma_{\tilde{k}}$ one has*

$$\pi_1(\Sigma_{\tilde{k}} \setminus (\tilde{B} \cup E)) = \langle \alpha_1, \alpha_2, \alpha_3 \mid \tilde{\mathfrak{m}}_i = \mathrm{id},\ i = 1, \ldots, r,\ \rho^k = \varkappa_r \cdots \varkappa_1 \rangle,$$

*where each braid relation $\tilde{\mathfrak{m}}_i = \mathrm{id}$, $i = 1, \ldots, r$, should be understood as a triple of relations $\tilde{\mathfrak{m}}_i(\alpha_j) = \alpha_j$, $j = 1, 2, 3$.*

The following statement simplifies the computation of the groups.

**Proposition 3.4.8.** *In the presentation given by* Corollary 3.4.7, *one can omit* (*any*) *one of the braid relations $\tilde{\mathfrak{m}}_i = \mathrm{id}$.*

Proof. First, show that the first relation $\tilde{\mathfrak{m}}_1 = \mathrm{id}$ can be omitted. Each braid relation $\tilde{\mathfrak{m}}_i = \mathrm{id}$, $i = 1, \ldots, r$, can be rewritten as $\varkappa_i \alpha = \mathfrak{m}_i(\alpha)\varkappa_i$, $\alpha \in \pi_F$. Hence, using all but the first braid relations, one can rewrite the relation at infinity (3.4.6) in the form $\rho^k = \bar{\varkappa}_1 \cdots \bar{\varkappa}_r$, where $\bar{\varkappa}_i = \mathfrak{m}_r \circ \cdots \circ \mathfrak{m}_{i+1}(\varkappa_i)$ for $i = 1, \ldots, r-1$ and $\bar{\varkappa}_r = \varkappa_r$. On the other hand, since $\mathfrak{m}_r \circ \cdots \circ \mathfrak{m}_1$ is the conjugation by $\rho^{-k}$, see Lemma 3.3.1, the product $\tilde{\mathfrak{m}}_r \circ \cdots \circ \tilde{\mathfrak{m}}_1$ is the conjugation by $\rho^{-k}\bar{\varkappa}_1 \cdots \bar{\varkappa}_r = 1$. It is the identity, and the relation $\tilde{\mathfrak{m}}_1 = \mathrm{id}$ follows from $\tilde{\mathfrak{m}}_2 = \cdots = \tilde{\mathfrak{m}}_r = \mathrm{id}$.

Now, assume that the relation to be omitted is $\tilde{\mathfrak{m}}_d = \mathrm{id}$ for some $d = 2, \ldots, r$. Since $\rho$ is invariant under $\mathfrak{m}_1$, the relation $\tilde{\mathfrak{m}}_1 = \mathrm{id}$ implies that $\varkappa_1$ commutes with $\rho$. Then, due to (3.4.6), it also commutes with $\varkappa_r \cdots \varkappa_2$ and (3.4.6) is equivalent to $\rho^k = \varkappa_1\varkappa_r \cdots \varkappa_2$. Proceeding by induction and using only $\tilde{\mathfrak{m}}_1 = \cdots = \tilde{\mathfrak{m}}_{d-1} = \mathrm{id}$, one can rewrite (3.4.6) in the form $\rho^k = \varkappa_{d-1} \cdots \varkappa_1\varkappa_r \cdots \varkappa_d$, the right hand side being a cyclic permutation of $\varkappa_r \cdots \varkappa_1$. On the other hand, the cyclic permutation $\{\gamma_d, \ldots, \gamma_r, \gamma_1, \ldots, \gamma_{d-1}\}$ is another standard basis for $\pi_1(\Delta^\sharp, b)$, with the same (but rearranged) slopes $\varkappa_i$ and braid monodromies $\mathfrak{m}_i$, and with respect to this new basis the braid relation to be omitted is the first one. Hence the statement follows from the first part of the proof. □

REMARK 3.4.9. A by-product of the previous proof is the fact that, modulo (all but one) braid relations the slopes $\varkappa_r, \ldots, \varkappa_1$ cyclically commute, i.e., one has

$$\varkappa_r \cdots \varkappa_2\varkappa_1 = \varkappa_{d-1} \cdots \varkappa_1\varkappa_r \cdots \varkappa_d$$

for each $d = 2, \ldots, r$. In particular, if there are only two nontrivial slopes (which is always the case in Section 5 below), their order in the relation at infinity (3.4.6) is irrelevant.

**3.5. Local slopes and braid relations.** Given two elements $\alpha$, $\beta$ of a group and a nonnegative integer $m$, introduce the notation

$$\{\alpha, \beta\}_m = \begin{cases} (\alpha\beta)^k(\beta\alpha)^{-k}, & \text{if } m = 2k \text{ is even,} \\ ((\alpha\beta)^k\alpha)((\beta\alpha)^k\beta)^{-1}, & \text{if } m = 2k+1 \text{ is odd.} \end{cases}$$

The relation $\{\alpha, \beta\}_m = 1$ is equivalent to $\sigma^m = \mathrm{id}$, where $\sigma$ is the Artin generator of the braid group $\mathbb{B}_2$ acting on the free group $\langle \alpha, \beta \rangle$. Hence,

$$\{\alpha, \beta\}_m = \{\alpha, \beta\}_n = 1 \quad \text{is equivalent to} \quad \{\alpha, \beta\}_{\mathrm{g.c.d.}(m,n)} = 1. \tag{3.5.1}$$

For the small values of $m$, the relation $\{\alpha, \beta\}_m = 1$ takes the following form:
– $m = 0$: tautology;
– $m = 1$: the identification $\alpha = \beta$;
– $m = 2$: the commutativity relation $[\alpha, \beta] = 1$;
– $m = 3$: the braid relation $\alpha\beta\alpha = \beta\alpha\beta$.

Let $F_i$ be a type $\tilde{\mathbf{A}}_p$ (type $\tilde{\mathbf{A}}_0^*$ if $p = 0$) singular fiber of a trigonal curve $B$, and let $b_i = p(F_i) \subset \Delta$ be its projection. Pick a simple arc $\zeta_i \colon [0,1] \to \Delta$ connecting the base point $b$ to $b_i$ and such that $\zeta_i([0,1)) \subset \Delta^\sharp$. We say that two consecutive generators $\alpha_j$, $\alpha_{j+1}$ of $\pi_F$, $j = 1$ or 2, *collide at $F_i$* (along $\zeta_i$) if there is a Milnor ball $M$ about the point of non-transversal intersection of $B$ and $F_i$ such that, for each sufficiently small $\epsilon > 0$, the images of $\alpha_j$ and $\alpha_{j+1}$ under the translation along the restriction $\zeta_i|_{[0,1-\epsilon]}$ are represented by a pair of loops that differ only inside $M$, whereas the image of the third generator is represented by a loop totally outside $M$.

In Paragraphs 3.5.2–3.5.4 below, we pick a standard basis $\{\alpha_1, \alpha_2, \alpha_3\}$ for $\pi_F$ and assume that two consecutive elements of this basis collide at $F_i$ along a certain path $\zeta_i$; then we use this path $\zeta_i$ to construct a generator $\gamma_i$ about $F_i$ as in Subsection 3.2. In other words, it is $\zeta_i$ that is used to define the global braid monodromy $\mathfrak{m}_i$ and the slope $\varkappa_i$. All computations are straightforward, using local normal forms of the singularities involved; we merely state the results.

We denote by $\sigma_1$, $\sigma_2$ the Artin generators of the braid group $\mathbb{B}_3$ acting on the free group $\pi_F = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$, so that $\sigma_i \colon \alpha_i \mapsto \alpha_i\alpha_{i+1}\alpha_i^{-1}$, $\alpha_{i+1} \mapsto \alpha_i$, $i = 1, 2$.

Note that the slope $\varkappa_i$ is only useful if the fiber $F_i$ is to be patched back in (as otherwise the presentation for the group $\pi_1(\Sigma_{\tilde{k}} \setminus (\tilde{B} \cup E \cup F_i \cup \cdots))$ would contain the original generator $\gamma_i$ rather than $\varkappa_i$). For this reason, after a small equisingular deformation of $\tilde{B} + E$, one can assume that $\tilde{B}$ is maximally transversal to $F_i$ at infinity. We do make this assumption below.

**3.5.2. A proper fiber.** Assume that $F_i$ is a proper type $\tilde{\mathbf{A}}_p$ (type $\tilde{\mathbf{A}}_0^*$ if $p = 0$) fiber of $\tilde{B} = B$. Then one has:
(1) if $\alpha_1$ and $\alpha_2$ collide at $F_i$, then $\mathfrak{m}_i = \sigma_1^{p+1}$ and $\varkappa_i = 1$, so that the braid relations are $\{\alpha_1, \alpha_2\}_{p+1} = 1$;
(2) if $\alpha_2$ and $\alpha_3$ collide at $F_i$, then $\mathfrak{m}_i = \sigma_2^{p+1}$ and $\varkappa_i = 1$, so that the braid relations are $\{\alpha_2, \alpha_3\}_{p+1} = 1$.

**3.5.3. A nonsingular branch at infinity.** Assume that $F_i$ is a type $\tilde{\mathbf{A}}_p$ singular fiber (type $\tilde{\mathbf{A}}_0^*$ if $p = 0$ or no singularity if $p = -1$) and a single smooth branch of $\tilde{B}$ intersects $E$ at $F_i \cap E$ with multiplicity $q \geqslant 1$. Then $F_i$ is a type $\tilde{\mathbf{A}}_{p+2q}$ singular fiber of $B$ and one has:
(1) if $\alpha_1$ and $\alpha_2$ collide at $F_i$, then $\mathfrak{m}_i = \sigma_1^{p+2q+1}$ and $\varkappa_i = (\alpha_1\alpha_2)^q$, so that the braid relations are $\{\alpha_1, \alpha_2\}_{p+1} = [(\alpha_1\alpha_2)^q, \alpha_3] = 1$;
(2) if $\alpha_2$ and $\alpha_3$ collide at $F_i$, then $\mathfrak{m}_i = \sigma_2^{p+2q+1}$ and $\varkappa_i = (\alpha_2\alpha_3)^q$, so that the braid relations are $\{\alpha_2, \alpha_3\}_{p+1} = [\alpha_1, (\alpha_2\alpha_3)^q] = 1$.

**3.5.4. A double point at infinity.** Assume that $\tilde{B}$ has a type $\mathbf{A}_p$ singular point at $F_i \cap E$ and intersects $E$ at this point with multiplicity $2q$, $1 \leqslant q \leqslant (p+1)/2$. Then $F_i$ is a type $\tilde{\mathbf{A}}_{p-2q}$ singular fiber of $B$ (with the same convention as in Paragraph 3.5.3 for the values $p - 2q = 0$ or $-1$) and one has:
(1) if $\alpha_1$ and $\alpha_2$ collide at $F_i$, then $\mathfrak{m}_i = \sigma_1^{p-2q+1}$ and $\varkappa_i = \alpha_3^q$, so that the braid relations are $\sigma_1^{p-2q+1}(\alpha_j) = \alpha_3^q\alpha_j\alpha_3^{-q}$, $j = 1, 2$;
(2) if $\alpha_2$ and $\alpha_3$ collide at $F_i$, then $\mathfrak{m}_i = \sigma_2^{p-2q+1}$ and $\varkappa_i = \alpha_1^q$, so that the braid relations are $\sigma_2^{p-2q+1}(\alpha_j) = \alpha_1^q\alpha_j\alpha_1^{-q}$, $j = 2, 3$.
(If $p - 2q = -1$, there is no collision and $\mathfrak{m}_i = \mathrm{id}$. In this case, the slope is $\varkappa_i = \alpha_j^q$, where $\alpha_j$ is the generator about the proper branch of the original curve $\tilde{B}$.)

Now, assume that $\tilde{B}$ has a type $\mathbf{A}_{2p-1}$ singular point at $F_i \cap E$ and the two branches at this point intersect $E$ with multiplicities $p$ and $p + q$ for some $q \geqslant 1$. Then $F_i$ is a type $\tilde{\mathbf{A}}_{2q-1}$ singular fiber of $B$, and one of the two branches of $B$ at its type $\mathbf{A}_{2q-1}$ singular point in $F_i$ is distinguished: it is the transform of the proper branch of $\tilde{B}$. Choose generators $\alpha_1, \alpha_2, \alpha_3$ so that $\alpha_1$ and $\alpha_2$ collide at $F_i$ and $\alpha_1$ is the generator about the distinguished branch of $B$. Then
(3) $\mathfrak{m}_i = \sigma_1^{2q}$ and $\varkappa_i = (\alpha_1\alpha_2)^q\alpha_1^p$, so that the braid relations are $[\alpha_1^p, \alpha_2] = 1$ and $[(\alpha_1\alpha_2)^q\alpha_1^p, \alpha_3] = 1$.

Finally, assume that $\tilde{B}$ has a type $\mathbf{A}_{2p}$ singular point at $F_i \cap E$ and intersects $E$ at this point with multiplicity $2p + 1$. Then $F_i$ is a type $\tilde{\mathbf{A}}_1^*$ singular fiber of $B$ and, in an appropriate basis $\{\alpha_1, \alpha_2, \alpha_3\}$, such that $\alpha_2$ is the generator about the branch of $B$ transversal to $F_i$, one has
(4) $\mathfrak{m}_i = \sigma_1\sigma_2\sigma_1$ and $\varkappa_i = \alpha_1\alpha_2^{p+1}$, so that the braid relations are $[\alpha_1, \alpha_2^{p+1}] = 1$ and $\alpha_3 = \alpha_2^p\alpha_1\alpha_2^{-p}$.

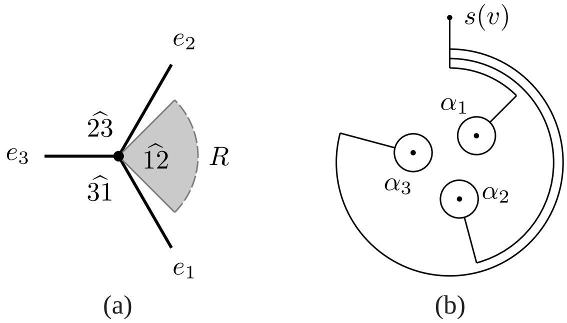


Fig. 1. A marking (a) and a canonical basis (b).

**3.5.5. A triple point at infinity.** Assume that $\tilde{B}$ has a triple point at $F_i \cap E$ and consider the (generalized) trigonal curve $\tilde{B}'$ obtained from $\tilde{B}$ by one elementary transformation centered at this point. Then the transform $\tilde{\Gamma}_i'$ of $\tilde{\Gamma}_i$ is still disjoint from $\tilde{B}'$ and thus can be used to define the slope of $\tilde{B}$; hence the slope of $\tilde{B}$ at $F_i$ equals that of $\tilde{B}'$. As a consequence, one has the following statement.

**Corollary 3.5.6.** *For $\tilde{B} \subset \Sigma_{\tilde{k}}$ and $B'' \subset \Sigma_{\tilde{k}+1}$ as above there is a canonical isomorphism $\pi_1(\Sigma_{\tilde{k}} \setminus (\tilde{B} \cup E)) = \pi_1(\Sigma_{\tilde{k}+1} \setminus (\tilde{B}' \cup E))$.*

**3.6. Braid monodromy via skeletons.** Let $B \subset \Sigma_k$ be a trigonal curve, and let $\mathrm{Sk} \subset \mathbb{P}^1$ be its skeleton. Below, we cite a few results of [10] concerning the braid monodromy of $B$ in terms of Sk. For simplicity, we assume that all $\bullet$-vertices of Sk are trivalent and all its $\circ$-vertices are bivalent (hence omitted). An alternative description, including more general skeletons, is found in [15].

Recall that a *marking* at a trivalent $\bullet$-vertex $v$ of Sk is a counterclockwise order $e_1$, $e_2$, $e_3$ of the three edges adjacent to $v$, see Fig. 1 (a). We consider the indices defined modulo 3, so that $e_{i+3} = e_i$. The three points of intersection of $B$ and the fiber $F_v$ over $v$ form an equilateral triangle. These points are in a canonical one-to-one correspondence with the edges $e_i$ of Sk at $v$. Hence, a marking gives rise to a *canonical basis* $\{\alpha_1, \alpha_2, \alpha_3\}$ of the group $\pi_F = \pi_1(F_v^\circ \setminus B)$, see Fig. 1 (b); this basis is well defined up to simultaneous conjugation by a power of $\rho = \alpha_1\alpha_2\alpha_3$.

As in Subsection 3.5, let $\sigma_1$ and $\sigma_2$ be the Artin generators of the braid group $\mathbb{B}_3$ acting on $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$. Denote $\sigma_3 = \sigma_1^{-1}\sigma_2\sigma_1$ and extend indices to all integers via $\sigma_{i\pm 3} = \sigma_i$. Note that the map $(\sigma_{i-1}, \sigma_i) \mapsto (\sigma_i, \sigma_{i+1})$ is an automorphism of $\mathbb{B}_3$. Recall also that the center of $\mathbb{B}_3$ is the cyclic subgroup generated by $(\sigma_1\sigma_2)^3 = (\sigma_2\sigma_3)^3 = (\sigma_3\sigma_1)^3$.

**3.6.1. The translation homomorphisms.** Let $u$ and $v$ be two marked $\bullet$-vertices of Sk connected by a single edge $e$; to indicate the markings, we use the notation $e = [i, j]$, where $i$ and $j$ are the indices of $e$ at $u$ and $v$, respectively. Choosing a pair of canonical bases defined by the markings, one can identify the groups $\pi_1(F_u^\circ \setminus B)$ and $\pi_1(F_v^\circ \setminus B)$ with the 'standard' free group $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ and thus regard the translation homomorphism $\mathfrak{m}_e\colon \pi_1(F_u^\circ \setminus B) \to \pi_1(F_v^\circ \setminus B)$ as an automorphism of $\langle\alpha_1, \alpha_2, \alpha_3\rangle$. It is a braid. However, since both the bases and the homomorphism $\mathfrak{m}_e$ itself are only defined up to conjugation by $\rho$ (unless a proper section is fixed, see Lemma 3.1.3), this automorphism should be regarded as an element of the reduced braid group $\bar{\mathbb{B}}_3 = \mathbb{B}_3/(\sigma_1\sigma_2)^3 \cong PSL(2, \mathbb{Z})$. On the other hand, this ambiguity does not affect the computation of the fundamental group, cf. Remark 3.4.5.

With the above convention, the translation homomorphism $\mathfrak{m}_{[i,j]} \in \bar{\mathbb{B}}_3$ along an edge $e = [i, j]$ is given as follows:

$$\mathfrak{m}_{[i,i+1]} = \sigma_i, \quad \mathfrak{m}_{[i+1,i]} = \sigma_i^{-1}, \quad \mathfrak{m}_{[i,i]} = \sigma_i\sigma_{i-1}\sigma_i, \quad i \in \mathbb{Z}.$$

The translation homomorphism $\mathfrak{m}_\gamma \in \bar{\mathbb{B}}_3$ along a path $\gamma$ composed by edges of Sk is the composition of the contributions of single edges. If $\gamma$ is a loop, the braid monodromy $\mathfrak{m}_\gamma$ is a well defined element of $\mathbb{B}_3$. It is uniquely recovered from its projection to $\bar{\mathbb{B}}_3$ just described and its *degree* (i.e., the image in $\mathbb{B}_3/[\mathbb{B}_3, \mathbb{B}_3] = \mathbb{Z}$); the latter is determined by the number and the types of the singular fibers of $B$ encompassed by $\gamma$. More precisely, for a disk $\Delta \subset \mathbb{P}^1$ as in Subsection 3.1, the composed homomorphism $\pi_1(\Delta^\sharp) \to \mathbb{B}_3 \to \mathbb{Z}$ sends a generator $\gamma_i$ about a singular fiber $F_i$ to the multiplicity mult $F_i$, see Subsection 2.2.

**3.6.2. The braid relations resulting from a region.** Given a trivalent $\bullet$-vertex $v$ of Sk, one can define three (germs of) angles at $v$, which are represented by the connected components of the intersection of $\mathbb{P}^1 \setminus \mathrm{Sk}$ and a regular neighborhood of $v$ in $\mathbb{P}^1$. If $v$ is marked, we denote these angles $\widehat{12}$, $\widehat{23}$, and $\widehat{31}$, according to the two edges adjacent to an angle, see Fig. 1 (a). The position of a region $R$ adjacent to $v$ with respect to the marking at $v$ can then be described by indicating the angle(s) that belong to $R$; for example, in Fig. 1 (a) one has $\widehat{12} \subset R$. Note that a region may contain two or even all three angles at $v$, see e.g. the outer nonagon and the central vertex in Fig. 4 (b) below.

Let $\Delta \subset \mathbb{P}^1$ be a closed disk as in Subsection 3.1. Assume that $v \in \partial\Delta$ and that $\Delta \setminus v \subset R$, cf. the shaded area in Fig. 1 (a). Then $\Delta$ intersects exactly one of the three angles at $v$ (in the figure this angle is $\widehat{12}$). Take $v = b$ for the base point and let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a canonical basis for $\pi_F = \pi_1(F_v^\circ \setminus B)$ defined by the marking at $v$. The following three statements are straightforward; for details see [10].

**Lemma 3.6.3.** *If a disk $\Delta$ as above intersects angle $\widehat{12}$ (respectively, $\widehat{23}$), then $\alpha_1$ and $\alpha_2$ (respectively, $\alpha_2$ and $\alpha_3$) collide at any type $\tilde{\mathbf{A}}$ singular fiber of $B$ in $\Delta$ along any path contained in $\Delta$.*

**Lemma 3.6.4.** *If a disk $\Delta$ as above intersects angle $\widehat{12}$ (respectively, $\widehat{23}$ or $\widehat{31}$), the braid monodromy $\mathfrak{m}\colon \pi_1(\Delta^\sharp, v) \to \operatorname{Aut}\pi_F$ takes values in the abelian subgroup of $\mathbb{B}_3 \subset \operatorname{Aut}\pi_F$ generated by the central element $(\sigma_1\sigma_2)^3$ and $\sigma_1$ (respectively, $\sigma_2$ or $\sigma_3$). If all singular fibers in $\Delta$ are of type $\tilde{\mathbf{A}}$, then $\mathfrak{m}$ takes values in the cyclic subgroup generated by $\sigma_1$ (respectively, by $\sigma_2$ or $\sigma_3$).*

More precisely, in Lemma 3.6.4, the value of $\mathfrak{m}$ on a type $\tilde{\mathbf{A}}_{p-1}$ fiber (type $\tilde{\mathbf{A}}_0^*$ if $p = 1$) is $\sigma_i^p$ (assuming that $\Delta$ intersects the angle spanned by $e_i$ and $e_{i+1}$), and its value on a type $\tilde{\mathbf{D}}_{q+4}$ fiber is $(\sigma_1\sigma_2)^3\sigma_i^q$. The value at a non-simple singular fiber of type $\tilde{\mathbf{J}}_{r,p}$ is $(\sigma_1\sigma_2)^{3r}\sigma_i^p$.

**Corollary 3.6.5.** *Assume that a region $R$ of* Sk *adjacent to a marked vertex $v$ contains, among others, singular fibers of types $\tilde{\mathbf{A}}_{p_i-1}$ ($\tilde{\mathbf{A}}_0^*$ if $p_i = 1$), $i = 1, \dots, s$. Denote $p = \text{g.c.d.}(p_i)$. Then the braid relations $\mathfrak{m}_i = \mathrm{id}$ resulting from these fibers are equivalent to a single relation as follows*:

– $\{\alpha_1, \alpha_2\}_p = 1$ *if* $\widehat{12} \subset R$;

– $\{\alpha_2, \alpha_3\}_p = 1$ *if* $\widehat{23} \subset R$;

– $\{\alpha_1, \alpha_2\alpha_3\alpha_2^{-1}\}_p = 1$ *if* $\widehat{31} \subset R$.

*In particular, if an $m$-gonal region $R$ contains a single singular fiber, which is of type $\tilde{\mathbf{A}}$, it results in a single braid relation as above with $p = m$.*

REMARK 3.6.6. In Corollary 3.6.5, if $B$ is the trigonal model of a generalized trigonal curve $\tilde{B} \subset \Sigma_{\tilde{k}}$ and it is $\pi_1(\Sigma_{\tilde{k}} \setminus (\tilde{B} \cup E))$ that is computed, one should assume in addition that the fibers considered are proper for $\tilde{B}$.

**3.6.7. An irreducibility criterion.** A *marking* of a skeleton Sk is a collection of markings at all its trivalent $\bullet$-vertices. A marking of a generic skeleton without singular $\bullet$-vertices is called *splitting* if it satisfies the following three conditions:

(1) the types of all edges, cf. 3.6.1, are [1, 1], [2, 3], or [3, 2];

(2) an edge connecting a $\bullet$-vertex $v$ and a singular $\circ$-vertex has index 1 at $v$;

(3) if a region $R$ contains angle $\widehat{12}$ or $\widehat{31}$ at one of its vertices, the multiplicities of all singular fibers inside $R$ are even.

(Note that, given (1) and (2), the last condition holds automatically if $R$ contains a single singular fiber, as $R$ is necessarily a $(2m)$-gon.) The following criterion is essentially contained in [10]; it is obtained by reducing the braid monodromy to the symmetric group $\mathbb{S}_3$.

**Theorem 3.6.8.** *A trigonal curve $B \subset \Sigma_k$ with connected generic skeleton $\mathrm{Sk}_B$ is reducible if and only if $\mathrm{Sk}_B$ has no singular $\bullet$-vertices and admits a splitting marking. Each such marking defines a component of $B$ that is a section of $\Sigma_k$.*

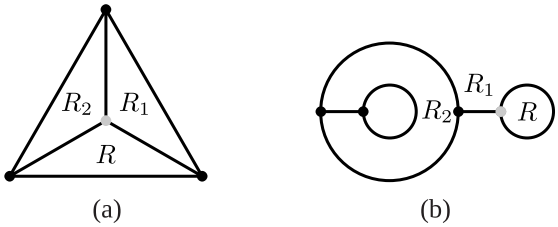


Fig. 2. Skeletons of plane quintics.

REMARK 3.6.9. A splitting marking defines a component of $B$ as follows: over each $\bullet$-vertex $v$, in a canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ defined by the marking, $\alpha_1$ is the generator about the distinguished component.

**3.7. Example: irreducible quintics.** As a simple example of application of the techniques developed in this section, we recompute the non-abelian fundamental groups of irreducible plane quintics, see [7] and [2]. A more advanced example is the contents of Section 5 below.

Let $C \subset \mathbb{P}^2$ be a quintic with the set of singularities $\mathbf{A}_6 \oplus 3\mathbf{A}_2$ or $3\mathbf{A}_4$. Blow up the type $\mathbf{A}_6$ point (respectively, one of the type $\mathbf{A}_4$ points) to obtain a generalized trigonal curve $\tilde{B} \subset \Sigma_1$ and let $B \subset \Sigma_2$ be the trigonal model of $\tilde{B}$. It is a maximal trigonal curve with the combinatorial type of singular fibers $4\tilde{\mathbf{A}}_2$ or $2\tilde{\mathbf{A}}_4 \oplus 2\tilde{\mathbf{A}}_0^*$; its skeleton Sk is shown in Figs. 2 (a) and (b), respectively.

Let $R$ be the region of Sk containing the only improper fiber of $\tilde{B}$. We choose for the reference point $b$ the vertex shown in the figures in grey and take for $\{\alpha_1, \alpha_2, \alpha_3\}$ a canonical basis over $b$ defined by the marking such that $\widehat{12} \subset R$. In both cases, the only nontrivial slope $\varkappa = \alpha_3$ is given by Paragraph 3.5.4 (1), with $(p, q) = (4, 1)$ or $(2, 1)$, respectively. According to Proposition 3.4.8, the fundamental group

$$\pi_1 := \pi_1(\mathbb{P}^2 \setminus C) = \pi_1(\Sigma_1 \setminus (\tilde{B} \cup E))$$

is defined by the relation at infinity and the braid relations resulting from three (out of four) regions $R$, $R_1$, $R_2$ shown in the figures. Using Subsection 3.6, one obtains the following relations: for the set of singularities $\mathbf{A}_6 \oplus 3\mathbf{A}_2$, see Fig. 2 (a):

$$\rho^2 = \alpha_3, \quad \{\alpha_2, \alpha_3\}_3 = \{\alpha_1, \alpha_2\alpha_3\alpha_2^{-1}\}_3 = 1,$$
$$(\alpha_1\alpha_2\alpha_1)\alpha_2(\alpha_1\alpha_2\alpha_1)^{-1} = \alpha_3\alpha_1\alpha_3^{-1}, \quad (\alpha_1\alpha_2)\alpha_1(\alpha_1\alpha_2)^{-1} = \alpha_3\alpha_2\alpha_3^{-1},$$

and for the set of singularities $3\mathbf{A}_4$, see Fig. 2 (b):

$$\rho^2 = \alpha_3, \quad \{\alpha_2, \alpha_3\}_5 = \{\alpha_2, \rho^{-1}\alpha_1\rho\}_5, \quad \alpha_1\alpha_2\alpha_1^{-1} = \alpha_3\alpha_1\alpha_3^{-1}, \quad \alpha_1 = \alpha_3\alpha_2\alpha_3^{-1}.$$

In the former case, the group is known to be infinite, see [7], as it factors to infinite Coxeter's group (2, 3, 7), see [6]. In the latter case, using GAP [17], one can see that

$\pi_1$ is a soluble group of order 320; one has

$$\pi_1/\pi_1' = \mathbb{Z}_5, \quad \pi_1'/\pi_1'' = (\mathbb{Z}_2)^4, \quad \pi_1'' = (\mathbb{Z}_2)^2,$$

where $'$ stands for the commutant.

Certainly, this approach applies as well to other quintics with a double singular point. In particular, one can easily show that the groups of all other irreducible quintics are abelian.

## 4. Proof of Theorem 1.2.1

In Subsection 4.1, we replace a sextic as in the theorem with its *trigonal model*, which is a maximal trigonal curve in an appropriate Hirzebruch surface. Then, in Subsections 4.2 and 4.3, we enumerate the possible skeletons of trigonal models; in view of Theorem 2.6.1, this enumeration suffices to prove Theorem 1.2.1.

**4.1. The trigonal models.** Recall that, due to [11], an irreducible maximizing simple sextic cannot have a singular point of type $\mathbf{D}_{2k}$, $k \geqslant 2$ or more than one singular point from the list $\mathbf{A}_{2k+1}$, $k \geqslant 0$, $\mathbf{D}_{2k+1}$, $k \geqslant 2$, or $\mathbf{E}_7$. Thus, a sextic as in Theorem 1.2.1 has a unique type $\mathbf{D}$ point, which is either $\mathbf{D}_5$ or $\mathbf{D}_p$ with odd $p \geqslant 7$. We will consider the two cases separately.

Both Theorem 1.2.1 and Theorem 1.2.2 are proved by a reduction of sextics to trigonal curves. A key rôle is played by the following two propositions.

**Proposition 4.1.1.** *There is a natural bijection $\phi$, invariant under equisingular deformations, between Zariski open and dense in each equisingular stratum subsets of the following two sets*:
(1) *plane sextics $C$ with a distinguished type $\mathbf{D}_p$, $p \geqslant 7$, singular point $P$ and without linear components through $P$, and*
(2) *trigonal curves $B \subset \Sigma_3$ with a distinguished type $\tilde{\mathbf{A}}_1$ singular fiber $F_{\mathrm{I}}$ and a distinguished type $\tilde{\mathbf{A}}_{p-7}$ ($\tilde{\mathbf{A}}_0^*$ if $p = 7$) singular fiber $F_{\mathrm{II}} \neq F_{\mathrm{I}}$.*
*A sextic $C$ is irreducible if and only if so is $B = \phi(C)$, and $C$ is maximizing if and only if $B$ is maximal and stable.*

**Proposition 4.1.2.** *There is a natural bijection $\phi$, invariant under equisingular deformations, between Zariski open and dense in each equisingular stratum subsets of the following two sets*:
(1) *plane sextics $C$ with a distinguished type $\mathbf{D}_5$ singular point $P$ and without linear components through $P$, and*
(2) *trigonal curves $B \subset \Sigma_4$ with a distinguished type $\tilde{\mathbf{A}}_1$ singular fiber $F_{\mathrm{I}}$ and a distinguished type $\tilde{\mathbf{A}}_3$ singular fiber $F_{\mathrm{II}}$.*
*A sextic $C$ is irreducible if and only if so is $B = \phi(C)$, and $C$ is maximizing if and only if $B$ is maximal and stable.*

The trigonal curve $B = \phi(C)$ corresponding to a sextic $C$ via Propositions 4.1.1 and 4.1.2 is called the *trigonal model* of $C$.

Proof of Propositions 4.1.1 and 4.1.2. Let $C \subset \mathbb{P}^2$ and $P$ be a pair as in the statement. Blow $P$ up and denote by $\tilde{B} \subset \Sigma_1 = \mathbb{P}^2(P)$ the proper transform of $C$; it is a generalized trigonal curve with two points at infinity. We let $B = \phi(C)$ to be the trigonal model of $\tilde{B}$. The inverse transformation consists in the passage from $B$ back to $\tilde{B} \subset \Sigma_1$ and blowing down the exceptional section of $\Sigma_1$.

The distinguished fibers $F_{\mathrm{I}}$ and $F_{\mathrm{II}}$ of $B$ correspond to the two points of $\tilde{B}$ at infinity ($F_{\mathrm{I}}$ corresponding to the smooth branch of $C$ at $P$). It is straightforward that, generically, the types of these fibers are as indicated in the statements. If the original sextic $C$ is in a special position with respect to the pencil of lines through $P$, these fibers may degenerate: $F_{\mathrm{I}}$ may degenerate to $\tilde{\mathbf{A}}_1^*$ or $\tilde{\mathbf{A}}_s$, $s > 1$, and $F_{\mathrm{II}}$ may degenerate to $\tilde{\mathbf{A}}_0^{**}$ (in Proposition 4.1.1 with $p = 7$) or to $\tilde{\mathbf{A}}_s$, $s > 3$ (in Proposition 4.1.2). However, using theory of trigonal curves (perturbations of dessins), one can easily see that any such curve $B$ can be perturbed to a generic one, and this perturbation is followed by *equisingular* deformations of $\tilde{B}$ and $C$.

Since $C$ and $B$ are birational transforms of each other, they are either both reducible or both irreducible. The fact that maximizing sextics correspond to stable maximal trigonal curves follows from Theorem 2.4.2: for a generic sextic $C$ as in the statements, one has $\mu(B) = \mu(C) - 6$ in Proposition 4.1.1 and $\mu(B) = \mu(C) - 1$ in Proposition 4.1.2. □

Let $C$ be a sextic as in Theorem 1.2.1. Since $C$ is irreducible, it has a unique type $\mathbf{D}$ point (see above), which we take for the distinguished point $P$. Denote by $B \subset \Sigma_k$, $k = 3$ or 4, the trigonal model of $C$ and let Sk be the skeleton of $B$. Let, further, $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ be the regions of Sk containing the distinguished singular fibers $F_{\mathrm{I}}$ and $F_{\mathrm{II}}$, respectively. Since we assume that $C$ has no type $\mathbf{E}$ singular points, $B$ has no triple points (one has $\mathrm{td}_B \equiv 0$, see Remark 2.6.2) and hence Sk has exactly $2k$ $\bullet$-vertices and has no singular vertices. Thus, due to Theorem 2.6.1, the proof of Theorem 1.2.1 reduces to the enumeration of 3-regular skeletons of irreducible curves with a prescribed number of vertices and with a pair $(R_{\mathrm{I}}, R_{\mathrm{II}})$ of distinguished regions. This is done is Subsections 4.2 and 4.3 below.

**4.2. The case $\mathbf{D}_p$, $p \geqslant 7$.** In this case, Sk has six $\bullet$-vertices, $R_{\mathrm{I}}$ is a bigon, and $R_{\mathrm{II}}$ is a $(p-6)$-gon. Since $p$ is not fixed, one can take for $R_{\mathrm{II}}$ any region of Sk other than $R_{\mathrm{I}}$.

The bigonal region $R_{\mathrm{I}}$ looks as shown in grey in Fig. 3 (a); we will call this region the *insertion*. Removing $R_{\mathrm{I}}$ from Sk and patching the two black edges in the figure to a single edge results in a new 3-regular skeleton Sk$'$ with four $\bullet$-vertices. Conversely, starting from Sk$'$ and placing an insertion at the middle of any of its edges produces

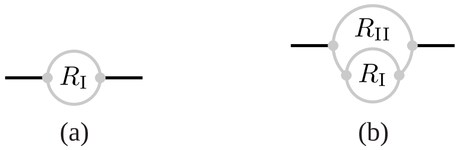


Fig. 3. A bigonal (a) and bibigonal (b) insertions.

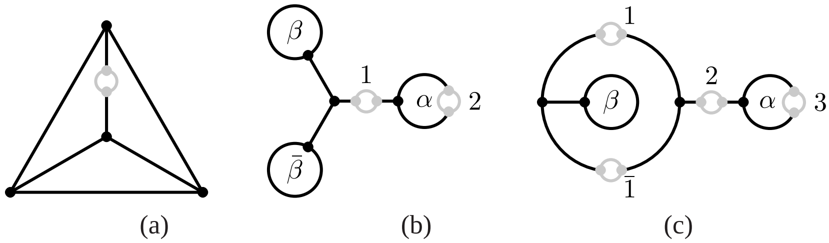


Fig. 4. A type $\mathbf{D}_p$ singular point, $p \geqslant 7$.

a skeleton Sk with six •-vertices and a distinguished bigonal region, which we take for $R_{\rm I}$. Using Theorem 3.6.8, one can see that Sk is the skeleton of an irreducible trigonal curve if and only if so is Sk$'$. There are three such skeletons (see e.g. [9]); they are listed in Fig. 4. Starting from one of these skeletons and varying the position of the insertion (shown in grey and numbered in the figure) and the choice of the second distinguished region $R_{\rm II}$, all up to symmetries of the skeleton, one obtains the 22 deformation families listed in Table 1. (Some rows of the table represent pairs of complex conjugate curves, see comments below.)

**4.2.1. Comments to Tables 1 and 2.** Listed in the tables are combinatorial types of singularities and references to the figures representing the corresponding skeletons. Equal superscripts precede combinatorial types shared by several items in the tables. The 'Count' column lists the numbers $(n_r, n_c)$ of real curves and pairs of complex conjugate curves, so that the total number of curves represented by a row is $n_r + 2n_c$. The last two columns refer to the computation of the fundamental group and indicate the parameters used in this computation. (A parameter list is marked with a $^*$ when the general approach does not work quite well for a particular curve. In this case, more details are found in the subsection referred to in the table.)

REMARK 4.2.2. Items 4 and 5 in Table 1 differ by the choice of the monogonal region $R_{\rm II}$. We assume that nos. 4 and 5 correspond, respectively, to the regions marked with $\alpha$ or $\beta$, $\bar\beta$ in Fig. 4 (b). (In the latter case, the two choices differ by an orientation reversing symmetry, i.e., the two curves are complex conjugate.) Similarly, we assume that nos. 13 and 14 in the table correspond, respectively, to the monogonal regions marked with $\alpha$ and $\beta$ in Fig. 4 (c).

Table 1. Maximal sets of singularities with a type $\mathbf{D}_p$ point, $p \geqslant 7$.

| # | Set of singularities | Figure | Count | $\pi_1$ | Params |
|---|---|---|---|---|---|
| 1 | $\mathbf{D}_{11} \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2$ | 4 (a) | (1, 0) | 5.2 | |
| 2 | $\mathbf{D}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$ | 4 (a) | (1, 0) | 5.3 | (5, 5; 3) |
| 3 | $\mathbf{D}_{19}$ | 4 (b)-1 | (1, 0) | 5.2 | |
| 4 | ${}^1\mathbf{D}_7 \oplus \mathbf{A}_{12}$ | 4 (b)-1 | (1, 0) | 5.3 | (13, –; 1) |
| 5 | ${}^1\mathbf{D}_7 \oplus \mathbf{A}_{12}$ | 4 (b)-1 | (0, 1) | 5.4 | (13, 13; 1) |
| 6 | $\mathbf{D}_{17} \oplus \mathbf{A}_2$ | 4 (b)-2 | (1, 0) | 5.2 | |
| 7 | $\mathbf{D}_9 \oplus \mathbf{A}_{10}$ | 4 (b)-2 | (1, 0) | 5.2 | |
| 8 | $\mathbf{D}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$ | 4 (b)-2 | (0, 1) | 5.5 | (3, 11; 1) |
| 9 | $\mathbf{D}_{13} \oplus \mathbf{A}_6$ | 4 (c)-1, $\bar{1}$ | (0, 1) | 5.2 | |
| 10 | $\mathbf{D}_7 \oplus 2\mathbf{A}_6$ | 4 (c)-1, $\bar{1}$ | (0, 1) | 5.4 | (7, 7; 1) |
| 11 | $\mathbf{D}_{15} \oplus \mathbf{A}_4$ | 4 (c)-2 | (1, 0) | 5.2 | |
| 12 | $\mathbf{D}_{11} \oplus \mathbf{A}_8$ | 4 (c)-2 | (1, 0) | 5.3 | (9, –; 5) |
| 13 | ${}^2\mathbf{D}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$ | 4 (c)-2 | (1, 0) | 5.3 | (9, –; 1) |
| 14 | ${}^2\mathbf{D}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$ | 4 (c)-2 | (1, 0) | 5.5 | ${}^*$(9, –; 1) |
| 15 | $\mathbf{D}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$ | 4 (c)-3 | (1, 0) | 5.2 | |
| 16 | $\mathbf{D}_{11} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$ | 4 (c)-3 | (1, 0) | 5.4 | (3, 7; 5) |
| 17 | $\mathbf{D}_9 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4$ | 4 (c)-3 | (1, 0) | 5.2 | |
| 18 | $\mathbf{D}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$ | 4 (c)-3 | (1, 0) | 5.6 | (3, 7; 1) |

**4.3. The case $\mathbf{D}_5$.** In this case, Sk has eight $\bullet$-vertices and two distinguished regions, a bigon $R_{\mathrm{I}}$ and a quadrilateral $R_{\mathrm{II}}$. If $R_{\mathrm{I}}$ is adjacent to $R_{\mathrm{II}}$, then the two regions form together an insertion shown in grey in Fig. 3 (b); we call this fragment a *bibigon*. As in the previous subsection, removing the insertion and patching together the two black edges, one obtains a 3-regular skeleton $\mathrm{Sk}'$ with four $\bullet$-vertices. The new skeleton $\mathrm{Sk}'$ represents an irreducible curve if and only if so does Sk; hence, $\mathrm{Sk}'$ is one of the three skeletons shown in Fig. 4. Varying the position of the insertion, one obtains items 19–26 in Table 2.

REMARK 4.3.1. Unlike Subsection 4.2, this time the insertion has a certain orientation, which should be taken into account. For this reason, some positions shown in Fig. 4 give rise to two rows in the table. Similarly, most positions shown in Fig. 7 below give rise to two rows in Table 4.

Otherwise (if $R_{\mathrm{I}}$ is not adjacent to $R_{\mathrm{II}}$), removing $R_{\mathrm{I}}$ produces a skeleton $\mathrm{Sk}'$ with six $\bullet$-vertices and a distinguished quadrilateral region $R_{\mathrm{II}}$. Such skeletons can easily be classified; they are shown in Fig. 6 (where $R_{\mathrm{II}}$ is the outer region of the skeleton). Using Theorem 3.6.8, one can see that only one of these skeletons (the last one in Fig. 6, also shown in Fig. 5) represents an irreducible curve. Varying the position of

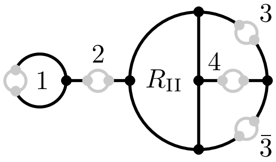


Fig. 5. A type $\mathbf{D}_5$ singular point: irreducible curves.

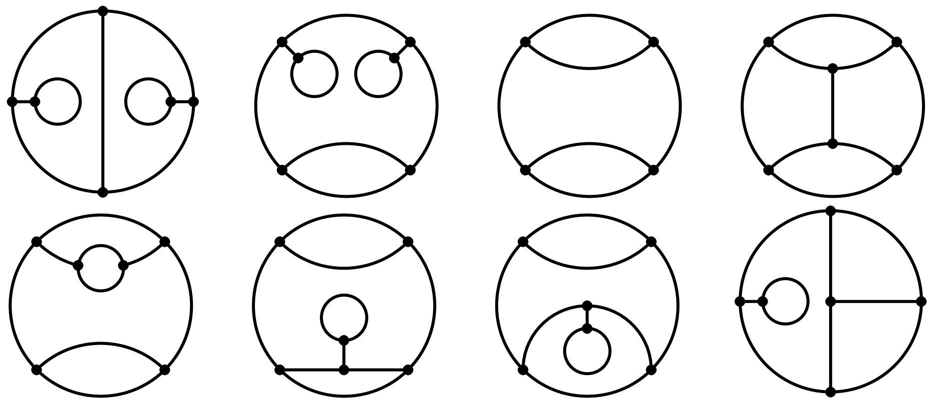

Fig. 6. A type $\mathbf{D}_5$ singular point: all curves.

Table 2. Maximal sets of singularities with a type $\mathbf{D}_5$ point.

| # | Set of singularities | Figure | Count | $\pi_1$ | Params |
|---|---|---|---|---|---|
| 19 | $\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus 2\mathbf{A}_2$ | 4 (a) | (1, 0) | 5.7 | (7, 5, 3) |
| 20 | $\mathbf{D}_5 \oplus \mathbf{A}_{14}$ | 4 (b)-1 | (0, 1) | 5.7 | (15, –, 1) |
| 21 | $\mathbf{D}_5 \oplus \mathbf{A}_{12} \oplus \mathbf{A}_2$ | 4 (b)-2 | (1, 0) | 5.7 | (13, 3, –) |
| 22 | $^{3}\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_4$ | 4 (b)-2 | (1, 0) | 5.7 | $^{*}$(11, 5, –) |
| 23 | $\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_6$ | 4 (c)-1, $\bar{1}$ | (0, 1) | 5.7 | $^{*}$(9, 7, –) |
| 24 | $^{3}\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_4$ | 4 (c)-2 | (0, 1) | 5.7 | (11, –, 1) |
| 25 | $^{4}\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$ | 4 (c)-3 | (1, 0) | 5.7 | (9, 3, –) |
| 26 | $^{5}\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_4$ | 4 (c)-3 | (1, 0) | 5.7 | (5, 7, –) |
| 27 | $\mathbf{D}_5 \oplus (\mathbf{A}_8 \oplus 3\mathbf{A}_2)$ | 5-1 | (1, 0) | 5.9 | |
| 28 | $\mathbf{D}_5 \oplus \mathbf{A}_{10} \oplus 2\mathbf{A}_2$ | 5-2 | (1, 0) | 5.8 | (11, –, 1) |
| 29 | $^{4}\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$ | 5-3, $\bar{3}$ | (0, 1) | 5.8 | (5, 9, 3) |
| 30 | $^{5}\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_4$ | 5-4 | (1, 0) | 5.8 | (5, 5, 7) |

the bigonal insertion $R_{\mathrm{I}}$ (shown in grey and numbered in Fig. 5), one obtains items 27–30 in Table 2.

REMARK 4.3.2. The classification in this subsection could as well be obtained from [3], where all 3-regular skeletons with eight $\bullet$-vertices are listed.

## 5. Proof of Theorem 1.2.2

We fix a sextic $C \subset \mathbb{P}^2$ as in the theorem and consider the fundamental group $\pi_1 := \pi_1(\mathbb{P}^2 \setminus C) = \pi_1(\Sigma_1 \setminus (\tilde{B} \cup E))$, where $\tilde{B} \subset \Sigma_1$ is the generalized trigonal curve obtained by blowing up the type **D** singular point of $C$, cf. Subsection 4.1. The group $\pi_1$ is computed on a case by case basis, using the approach of Section 3 and the skeletons found in Section 4. (We retain the notation $R_{\mathrm{I}}$, $R_{\mathrm{II}}$ for the two distinguished regions containing the improper fibers.) Without further references, finite groups are treated using `GAP` [17]: in most cases, the `Size` function returns `6`, which suffices to conclude that the group is $\mathbb{Z}_6$ (as so is its abelianization).

**5.1. A singular point of type $\mathbf{D}_p$, $p \geqslant 7$.** We take for the reference fiber $F_b$ the fiber over an appropriate vertex $v_{\mathrm{I}}$ in the boundary of $R_{\mathrm{I}}$ and choose a canonical basis $\{\alpha_1, \alpha_2, \alpha_3\}$ in $F$ corresponding to the marking at $v_{\mathrm{I}}$ such that $\widehat{12} \subset R_{\mathrm{I}}$. Next, we choose an appropriate vertex $v_{\mathrm{II}}$ in the boundary of $R_{\mathrm{II}}$ and a canonical basis $\{\beta_1, \beta_2, \beta_3\}$ in the fiber over $v_{\mathrm{II}}$ corresponding to a marking at $v_{\mathrm{II}}$ such that $\widehat{23} \subset R_{\mathrm{II}}$. The translation homomorphisms from $v_{\mathrm{I}}$ to $v_{\mathrm{II}}$ are computed below on a case by case basis.

According to 3.5.3 (1) and 3.5.4 (2), the slopes over $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ are $\varkappa_{\mathrm{I}} = \alpha_1\alpha_2$ and $\varkappa_{\mathrm{II}} = \beta_1$, respectively, and the corresponding braid relations are

$$(5.1.1) \qquad [\alpha_1\alpha_2, \alpha_3] = 1, \quad \sigma_2^{p-6}(\beta_j) = \beta_1\beta_j\beta_1^{-1}, \quad j = 2, 3.$$

Furthermore, in view of the first relation in (5.1.1), the relation at infinity (3.4.6) simplifies to

$$(5.1.2) \qquad (\alpha_1\alpha_2)^2\alpha_3^3 = \beta_1.$$

These four relations are present in any group $\pi_1$.

**5.2. The case of $R_{\mathrm{II}}$ adjacent to $R_{\mathrm{I}}$.** Assume that the region $R_{\mathrm{II}}$ is adjacent to the insertion $R_{\mathrm{I}}$ (nos. 1, 3, 6, 7, 9, 11, 15, and 17 in Table 1). Then $v_{\mathrm{I}}$ and $v_{\mathrm{II}}$ can be chosen to coincide, so that $\beta_j = \alpha_j$, $j = 1, 2, 3$, and the relation at infinity (5.1.2) simplifies further to $\alpha_2\alpha_1\alpha_2\alpha_3^3 = 1$. It follows that $\alpha_3$ commutes with $\alpha_2\alpha_1\alpha_2$; hence, in view of the first relation in (5.1.1), one has $[\alpha_3, \alpha_2] = [\alpha_3, \alpha_1] = 1$. On the other hand, $\alpha_1 = \alpha_2^{-1}\alpha_3^{-3}\alpha_2^{-1}$ belongs to the abelian subgroup generated by $\alpha_2$ and $\alpha_3$. Thus, the group is abelian.

The argument above applies to a reducible maximizing sextic $C$ as well, provided that $C$ is covered by Proposition 4.1.1 (i.e., $C$ has a type $\mathbf{D}_p$, $p \geqslant 7$, singular point $P$ and

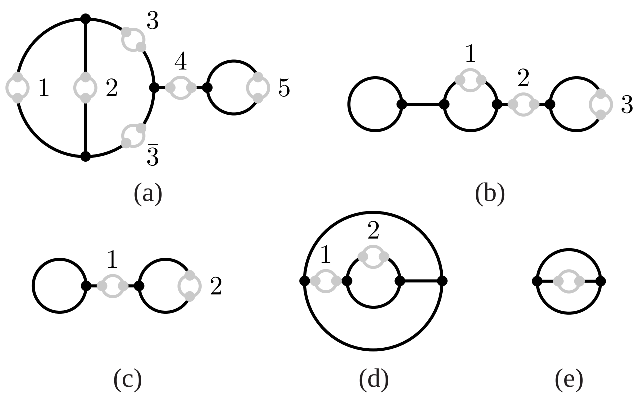


Fig. 7. A type $\mathbf{D}_p$ singular point, $p \geqslant 7$: reducible curves.

has no linear components through $P$) and the distinguished regions $R_{\mathrm{I}}$, $R_{\mathrm{II}}$ of the skeleton Sk of the trigonal model $B$ of $C$ are adjacent to each other. Such curves can easily be enumerated similar to Subsection 4.2, by reducing Sk to a skeleton Sk$'$ with at most four •-vertices (see e.g. [9] and Fig. 7; note that this time we do not require that Sk$'$ should have exactly four •-vertices as we accept curves $B$ with $\tilde{\mathbf{D}}$ type singular fibers). The resulting sets of singularities are listed in Table 3.

REMARK 5.2.1. In Figs. 7 (c) and (e), in addition to $R_{\mathrm{II}}$ one should also choose one of the remaining regions to contain the $\tilde{\mathbf{D}}$ type singular fiber (as one should have

Table 3. Some reducible sextics with abelian fundamental groups.

| Set of singularities | Figure |
|---|---|
| The splitting $C_3 + C_3$ | |
| $\mathbf{D}_{14} \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$ | 7 (a)-1 |
| $\mathbf{D}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$ | 7 (a)-1 |
| $\mathbf{D}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ | 7 (a)-4 |
| $\mathbf{D}_{18} \oplus \mathbf{A}_1$ | 7 (b)-2 |
| $\mathbf{D}_{14} \oplus \mathbf{D}_5$ | 7 (c)-1 |
| The splitting $C_4 + C_2$ | |
| $\mathbf{D}_{11} \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$ | 7 (a)-2 |
| $\mathbf{D}_{11} \oplus \mathbf{A}_7 \oplus \mathbf{A}_1$ | 7 (a)-3 |
| $\mathbf{D}_9 \oplus \mathbf{A}_7 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ | 7 (a)-5 |
| $\mathbf{D}_{10} \oplus \mathbf{A}_9$ | 7 (b)-1 |
| $\mathbf{D}_9 \oplus \mathbf{A}_9 \oplus \mathbf{A}_1$ | 7 (b)-3 |
| $\mathbf{D}_{10} \oplus \mathbf{D}_9$ | 7 (c)-2 |
| $\mathbf{D}_9 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5$ | 7 (c)-2 |

| Set of singularities | Figure |
|---|---|
| The splitting $C_5 + C_1$ | |
| $\mathbf{D}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$ | 7 (a)-2 |
| $\mathbf{D}_{14} \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$ | 7 (a)-3 |
| $\mathbf{D}_{14} \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ | 7 (a)-5 |
| $\mathbf{D}_{16} \oplus \mathbf{A}_3$ | 7 (b)-1 |
| $\mathbf{D}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ | 7 (b)-3 |
| $\mathbf{D}_{12} \oplus \mathbf{D}_7$ | 7 (c)-2 |
| $\mathbf{D}_{12} \oplus \mathbf{D}_5 \oplus \mathbf{A}_2$ | 7 (c)-2 |
| The splitting $C_3 + C_2 + C_1$ | |
| $\mathbf{D}_{12} \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_1$ | 7 (d)-1 |
| $\mathbf{D}_{12} \oplus 2\mathbf{A}_3 \oplus \mathbf{A}_1$ | 7 (d)-2 |
| $\mathbf{D}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_3 \oplus \mathbf{A}_1$ | 7 (d)-2 |
| $\mathbf{D}_{10} \oplus \mathbf{D}_8 \oplus \mathbf{A}_1$ | 7 (e) |
| $\mathbf{D}_{10} \oplus \mathbf{D}_6 \oplus \mathbf{A}_3$ | 7 (e) |

$\sum \mathrm{td}_B = 1$, see Remark 2.6.2). The degrees of the components of $C$ can be determined using Theorem 3.6.8, which describes the components in terms of generators (see Remark 3.6.9), and the abelianization of relation (5.1.2).

**5.3. The case of $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ connected by a single edge.** Assume that $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ are connected by a single edge of the skeleton (nos. 2, 4, 12, and 13 in Table 1) and choose vertices $v_{\mathrm{I}}$ and $v_{\mathrm{II}}$ so that this connecting edge is $[v_{\mathrm{I}}, v_{\mathrm{II}}]$. Then the translation homomorphism $\mathfrak{m}_{[3,1]}$, see 3.6.1, is given by

$$\beta_1 = \alpha_2\alpha_3\alpha_2^{-1}, \quad \beta_2 = \alpha_2, \quad \beta_3 = (\alpha_2\alpha_3)^{-1}\alpha_1(\alpha_2\alpha_3). \tag{5.3.1}$$

Furthermore, in addition to (5.1.1) and (5.1.2), one has the braid relations

$$\{\alpha_1, \alpha_3\}_l = \{\alpha_2, \alpha_3\}_m = 1 \tag{5.3.2}$$

from the two regions adjacent to $R_{\mathrm{I}}$, see Corollary 3.6.5. (In most cases, these two regions coincide. Since $[\alpha_1\alpha_2, \alpha_3] = 1$, see (5.1.1), the relation $\{\alpha_1, \alpha_2\alpha_3\alpha_2^{-1}\}_l = 1$ given by the corollary is equivalent to $\{\alpha_1, \alpha_3\}_l = 1$.) Trying the possible values of $(l, m; p)$, see Table 1, with `GAP` [17], one concludes that all four groups are abelian.

**5.4. The case of $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ connected by two edges.** Assume that $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ are connected by a chain $\zeta$ of two edges (nos. 5, 10, and 16 in Table 1) and choose reference vertices $v_{\mathrm{I}}$ and $v_{\mathrm{II}}$ at the two ends of $\zeta$. The translation homomorphism $\mathfrak{m}_{[3,1]} \circ \mathfrak{m}_{[3,1]}$ along $\zeta$ is given by

$$\beta_1 = \delta^{-1}\alpha_1\delta, \quad \beta_2 = \alpha_2, \quad \beta_3 = \rho^{-1}\delta\rho.$$

where $\delta = \alpha_2\alpha_3\alpha_2^{-1}$, and relations (5.1.1), (5.1.2), and (5.3.2) with the values of $(l,m;p)$ given in Table 1 suffice to show that all three groups are abelian.

**5.5. The case of $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ connected by three edges.** Assume that $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ are connected by a chain $\zeta$ of three edges (nos. 8 and 14 in Table 1) and choose $v_{\mathrm{I}}$ and $v_{\mathrm{II}}$ at the ends of $\zeta$. Under an appropriate choice of $\zeta$, the translation homomorphism $\mathfrak{m}_{[2,1]} \circ \mathfrak{m}_{[3,1]} \circ \mathfrak{m}_{[3,1]}$ is

$$\beta_1 = \delta^{-1}\rho\alpha_2\rho^{-1}\delta, \quad \beta_2 = \delta^{-1}\alpha_1\delta, \quad \beta_3 = \rho^{-1}\delta\rho,$$

where $\delta = \alpha_2\alpha_3\alpha_2^{-1}$. (To make this homomorphism uniform, for no. 8 we take for $R_{\mathrm{II}}$ the monogon marked with $\beta$ in Fig. 4 (b). Since the two curves in no. 8 are complex conjugate, their groups are isomorphic.)

For no. 8, relations (5.1.1), (5.1.2), and (5.3.2) with $(l, m; p) = (3, 11; 1)$ suffice to show that the group is abelian. For no. 14, one should also take into account the relation $\{\beta_1, \beta_2\}_5 = 1$ resulting from the pentagon adjacent to $R_{\mathrm{II}}$.

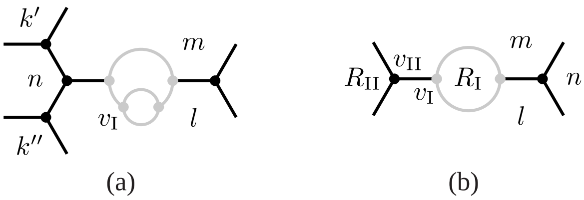


Fig. 8. Regions used in the computation.

**5.6. The remaining case: no. 18 in Table 1.** In this case, the regions $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ are connected by a chain $\zeta$ of four edges. Choosing $v_{\mathrm{I}}$ and $v_{\mathrm{II}}$ at the ends of $\zeta$, one obtains the translation homomorphism $\mathfrak{m}_{[2,1]} \circ \mathfrak{m}_{[3,1]} \circ \mathfrak{m}_{[3,1]} \circ \mathfrak{m}_{[3,1]}$ given by

$$\beta_1 = \delta^{-1}\alpha_1^{-1}\delta\rho\alpha_2\rho^{-1}\delta^{-1}\alpha_1\delta, \quad \beta_2 = \delta^{-1}\alpha_1^{-1}\delta\alpha_1\delta, \quad \beta_3 = \rho^{-1}\delta^{-1}\alpha_1\delta\rho, \tag{5.6.1}$$

where $\delta = \alpha_2\alpha_3\alpha_2^{-1}$. Relations (5.1.1), (5.1.2), and (5.3.2) with $(l, m; p) = (3, 7; 1)$ suffice to show that the group is abelian.

**5.7. A singular point of type $\mathbf{D}_5$, a bibigonal insertion.** In the case of a type $\mathbf{D}_5$ singular point, choose a pair of reference vertices $v_{\mathrm{I}}$ and $v_{\mathrm{II}}$ and canonical bases $\{\alpha_1, \alpha_2, \alpha_3\}$ over $v_{\mathrm{I}}$ and $\{\beta_1, \beta_2, \beta_3\}$ over $v_{\mathrm{II}}$ similar to Subsection 5.1.

If the skeleton Sk of $B$ has a bibigonal insertion, $v_{\mathrm{I}}$ and $v_{\mathrm{II}}$ can be chosen to coincide, see Fig. 8 (a), so that one has $\beta_j = \alpha_j$, $j = 1, 2, 3$. According to 3.5.3, the slopes over $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ are $\varkappa_{\mathrm{I}} = \alpha_1\alpha_2$ and $\varkappa_{\mathrm{II}} = (\alpha_2\alpha_3)^2$, respectively, the braid relations become

$$[\alpha_1\alpha_2, \alpha_3] = [\alpha_1, (\alpha_2\alpha_3)^2] = 1, \tag{5.7.1}$$

and the relation at infinity (3.4.6) simplifies to

$$\alpha_3\rho^3 = (\alpha_2\alpha_3)^2. \tag{5.7.2}$$

Besides, $\pi_1$ has extra relations

$$\{\alpha_1, \alpha_3\}_l = \{\alpha_2, \rho^{-1}\alpha_1\rho\}_m = \{\alpha_2, \rho^{-1}\alpha_2\alpha_3\alpha_2^{-1}\rho\}_n = 1 \tag{5.7.3}$$

resulting from the $l$-, $m$-, and $n$-gonal regions marked in the figure.

From the second relation in (5.7.1) it follows that $(\alpha_2\alpha_3)^2$ commutes with $\rho$; then (5.7.2) implies that $(\alpha_2\alpha_3)^2$ also commutes with $\alpha_3$ and hence with $\alpha_2$. Thus, $(\alpha_2\alpha_3)^2$ is a central element and we replace $\pi_1$ with its quotient $G := \pi_1/(\alpha_2\alpha_3)^2$. (Otherwise, the coset enumeration may fail in `GAP`.) Using `GAP` [17], we show that $G = \mathbb{Z}_2$; then $\pi_1$ is a central extension of a cyclic group, hence abelian.

In most cases, for the conclusion that $G = \mathbb{Z}_2$ it suffices to use relations (5.7.1)–(5.7.3) with the values of parameters $(l, m, n)$ listed in Table 2. For nos. 22

and 23 in Table 2, one should also take into account the relations

$$\alpha_2 = \rho^{-1}\alpha_2\rho^{-1}\alpha_2\alpha_3\alpha_2^{-1}\rho\alpha_2^{-1}\rho \quad \text{(for no. 22)},$$
$$\alpha_2 = \rho^{-1}\alpha_2\rho^{-1}\alpha_1\rho\alpha_2^{-1}\rho \quad \text{(for no. 23)}$$

resulting from appropriate monogonal regions of Sk.

This computation applies as well to a reducible maximizing sextic $C$, provided that it is covered by Proposition 4.1.2, the skeleton Sk of the trigonal model $B$ of $C$ has a bibigonal insertion, and $C$ splits into two components (Figs. 7 (a)–(c) and Table 4; the latter condition assures that the abelianization $G/[G,G]$ is finite). This time, the central element $(\alpha_2\alpha_3)^2$ has infinite order in the abelianization $\pi_1/[\pi_1,\pi_1]$ and hence one has $[\pi_1,\pi_1] = [G,G]$.

Table 4. Some reducible sextics with a type $\mathbf{D}_5$ point.

| Set of singularities | Figure | Params | $[\pi_1,\pi_1]$ |
|---|---|---|---|
| The splitting $C_3 + C_3$ $(G/[G,G] = \mathbb{Z}_4 \oplus \mathbb{Z}_3)$ | | | |
| $\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2$ | 7 (a)-1 | (6, 8, 3; –, –) | $SL(2,\mathbb{F}_7)$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_9 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$ | 7 (a)-1 | (10, 4, 3; –, –) | $\{1\}$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ | 7 (a)-4 | (12, 12, 1; –, –) | $\{1\}$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_{13} \oplus \mathbf{A}_1$ | 7 (b)-2 | (14, 14, 1; –, –) | $\{1\}$ |
| $\mathbf{D}_{14} \oplus \mathbf{D}_5$ | 7 (c)-1 | *(–, –, 1; –, –) | $\{1\}$ |
| $2\mathbf{D}_5 \oplus \mathbf{A}_9$ | 7 (c)-1 | (10, 10, 1; –, –) | $\mathbb{Z}_5$ |
| The splitting $C_4 + C_2$ $(G/[G,G] = \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ | | | |
| $\mathbf{D}_5 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_4$ | 7 (a)-2 | (6, 5, 6; –, –) | see 5.7.6 |
| $\mathbf{D}_5 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$ | 7 (a)-3, $\bar{3}$ | (10, 5, 2; –, –) | $\{1\}$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_9 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$ | 7 (a)-5 | (10, 3, 10; –, 2) | $\{1\}$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_3$ | 7 (b)-1 | (12, 4, 12; –, 1) | $\mathbb{Z}_4$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ | 7 (b)-3 | (12, 3, 12; –, 2) | see 5.7.7 |
| $\mathbf{D}_{12} \oplus \mathbf{D}_5 \oplus \mathbf{A}_2$ | 7 (c)-2 | (–, 3, –; –, 1) | $\{1\}$ |
| $\mathbf{D}_7 \oplus \mathbf{D}_5 \oplus \mathbf{A}_7$ | 7 (c)-2 | (8, –, 8; –, 1) | $\mathbb{Z}_4$ |
| $2\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$ | 7 (c)-2 | (8, 3, 8; –, –) | see 5.7.8 |
| The splitting $C_5 + C_1$ $(G/[G,G] = \mathbb{Z}_8)$ | | | |
| $\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_3$ | 7 (a)-2 | (7, 4, 6; –, –) | $\{1\}$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$ | 7 (a)-3, $\bar{3}$ | (7, 8, 2; –, –) | $\{1\}$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$ | 7 (a)-5 | (5, 8, 8; 2, –) | $\{1\}$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_9 \oplus \mathbf{A}_5$ | 7 (b)-1 | (6, 10, 10; 1, –) | $\{1\}$ |
| $\mathbf{D}_5 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$ | 7 (b)-3 | (5, 10, 10; 2, –) | $\{1\}$ |
| $\mathbf{D}_{10} \oplus \mathbf{D}_5 \oplus \mathbf{A}_4$ | 7 (c)-2 | (5, –, –; 1, –) | $\{1\}$ |
| $\mathbf{D}_9 \oplus \mathbf{D}_5 \oplus \mathbf{A}_5$ | 7 (c)-2 | (–, 6, 6; 1, –) | $\{1\}$ |
| $2\mathbf{D}_5 \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$ | 7 (c)-2 | (5, 6, 6; –, –) | $\{1\}$ |

For some curves, we also take into account the additional relations

$$\{\rho^{-1}\delta\rho, \delta^{-1}\alpha_1\delta\}_{k'} = 1 \quad \text{or} \quad \{\rho\alpha_2\rho^{-1}, \delta^{-1}\alpha_1\delta\}_{k''} = 1, \tag{5.7.4}$$

$\delta = \alpha_2\alpha_3\alpha_2^{-1}$, resulting from the $k'$- and $k''$-gonal regions marked in Fig. 8 (a). As usual, we always skip the relations corresponding to a region of Sk containing a type $\tilde{\mathbf{D}}$ singular fiber of $B$ (as these relations differ from those indicated above).

For most curves, relations (5.7.1)–(5.7.4) with the values of $(l, m, n; k', k'')$ given in Table 4 suffice to identify $\pi_1$, either because the group is already abelian or due to Proposition 3.4.8. The few special cases are discussed below.

**5.7.5. The set of singularities $\mathbf{D}_{14} \oplus \mathbf{D}_5$.** To show that $\pi_1$ is abelian, one needs to take into account the additional relation

$$(\alpha_2\alpha_3)\alpha_2(\alpha_2\alpha_3)^{-1} = \rho\delta\rho^{-1}$$

resulting from the other monogonal region of Sk.

**5.7.6. The set of singularities $\mathbf{D}_5 \oplus 2\mathbf{A}_5 \oplus \mathbf{A}_4$.** In this case, `GAP` [17] shows that the commutant $[\pi_1, \pi_1]$ is one of the five perfect groups of order 7680. I do not know which of the five groups it is.

**5.7.7. The set of singularities $\mathbf{D}_5 \oplus \mathbf{A}_{11} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$.** One has $[\pi_1, \pi_1] = \mathbb{Z}$. Although $[\pi_1, \pi_1]$ is infinite, it can be simplified using the `GAP` commands

```
P := PresentationNormalClosure(g, Subgroup(g, [g.1/g.2]));
SimplifyPresentation(P);
```

which return a presentation with a single generator and no relations.

**5.7.8. The set of singularities $2\mathbf{D}_5 \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$.** One has

$$\pi_1'/\pi_1'' = \mathbb{Z} \oplus \mathbb{Z}_3$$

and

$$\pi_1'' = Q_8 := \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H},$$

where we abbreviate $\pi_1' = [\pi_1, \pi_1]$ and $\pi_1'' = [\pi_1', \pi_1']$. The first statement is straightforward. For the second one, consider the normal closure $H$ of $\alpha_1\alpha_2^{-1}$ in $G$. It is an order 3 subgroup of $G'$ and one has $H/[H, H] = \mathbb{Z}$. Hence $[H, H] = [G', G'] = G''$. Simplifying the presentation of $H$ given by `GAP` in the same way as in Paragraph 5.7.7, one obtains two generators $\kappa_2$, $\kappa_3$ and three relations

$$\kappa_3^4 = 1, \quad \kappa_3^2\kappa_2^{-1}\kappa_3^2\kappa_2 = 1, \quad \kappa_2^{-2}\kappa_3^{-1}\kappa_2\kappa_3^{-1}\kappa_2\kappa_3 = 1.$$

From the first two relations it follows that $\kappa_3^2$ is a central element. Then the third one, rewritten in the form $\kappa_2^3 = (\kappa_2\kappa_3^{-1})^3\kappa_3^2$, implies that $\kappa_2^3$ is also central. Since the image of $\kappa_2^3$ in $H/[H,H]$ has infinite order, $[H,H]$ is equal to the commutant of the quotient $H/\kappa_2^3$. The latter group is finite and its commutant is $Q_8$.

**Proposition 5.7.9.** *Let $C$ be one of the sextics listed in* Tables 3 *and* 4, *and let $C'$ be an irreducible perturbation of $C$ preserving the distinguished type $\mathbf{D}_5$ singular point. Then $\pi_1(\mathbb{P}^2 \setminus C') = \mathbb{Z}_6$.*

Proof. It suffices to consider one of the seven sextics $C$ in Table 4 that have non-abelian groups. Since the distinguished type $\mathbf{D}_5$ point is preserved, the perturbation of $C$ is followed by a perturbation $B \to B'$ of its trigonal model, or a perturbation $\tilde{B} \to \tilde{B}'$ of the generalized trigonal curve in $\Sigma_1 = \mathbb{P}^2(P)$ which is actually used in the computation of $\pi_1$: one can assume that a proper type $\mathbf{A}_{2r-1}$ or $\mathbf{D}_{2r-1}$ singular point $Q$ of $\tilde{B}$ is perturbed so that the intersection $\tilde{B}' \cap M_Q$ is connected, where $M_Q$ is a Milnor ball about $Q$.

If the point $Q$ that is perturbed is of type $\mathbf{D}_{2r-1}$, the inclusion homomorphism $\pi_1(M_Q \setminus \tilde{B}) \to \pi_1(\Sigma_1 \setminus \tilde{B})$ is onto (as $M_Q$ contains all three generators in a fiber sufficiently close to $Q$). On the other hand, for any perturbation $\tilde{B} \to \tilde{B}'$ with $\tilde{B}' \cap M_Q$ connected, the group $\pi_1(M_Q \setminus \tilde{B}')$ is abelian (see [16]; the maximal perturbation with the connectedness property is $\mathbf{D}_{2r-1} \to \mathbf{A}_{2r-2}$.)

If $Q$ is of type $\mathbf{A}_{2r-1}$, the group of $\tilde{B}'$ is found similar to that of $\tilde{B}$: it suffices to replace the corresponding (necessarily even) parameter(s) in $(l,m,n;k',k'')$, see Table 4, with its maximal odd divisor. Considering curves and parameters one by one and using `GAP` [17], one concludes that all groups are abelian. $\square$

**5.8. Other curves not of torus type.** In all three cases (nos. 28–30 in Table 2), the distinguished regions $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ are connected by a single edge, see Fig. 5. Choose vertices $v_{\mathrm{I}}$, $v_{\mathrm{II}}$ introduced in Subsection 5.7 as shown in Fig. 8 (b). Then the translation homomorphism from $v_{\mathrm{I}}$ to $v_{\mathrm{II}}$ is given by (5.3.1). Hence, the braid relations from $R_{\mathrm{I}}$ and $R_{\mathrm{II}}$ and the relation at infinity become

$$[\alpha_1\alpha_2,\, \alpha_3] = [(\rho^{-1}\alpha_1\rho\alpha_2)^2,\, \alpha_3] = 1$$

and

$$\alpha_3\rho^3 = (\alpha_2\rho^{-1}\alpha_1\rho)^2,$$

respectively. Consider also the relations

$$\{\alpha_1,\, \alpha_3\}_l = \{\alpha_2,\, \alpha_3\}_m = \{\alpha_2^{-1}\alpha_1\alpha_2,\, \rho^{-1}\alpha_2\rho\}_n = 1$$

resulting from the $l$-, $m$-, and $n$-gonal regions marked in Fig. 8 (b). Using the values of $(l,\, m,\, n)$ given in Table 2, one concludes that all three groups are abelian.

**5.9. The curve of torus type (no. 27 in Table 2).** The skeleton Sk is the one shown in Fig. 5, with the bigonal insertion labelled with 1. Take for $v_{\mathrm{I}}$ and $v_{\mathrm{II}}$, respectively, the upper vertex of the insertion and the center of the large circle in the figure. Then a complete set of relations for $\pi_1$ is

$$[\alpha_1\alpha_2, \alpha_3] = \{\alpha_2, \alpha_3\}_3 = \{\alpha_1, \alpha_3\}_9 = 1,$$
$$[(\beta_1\beta_2)^2, \beta_3] = \{\beta_2, \beta_3\} = \{\beta_1, \beta_2\beta_3\beta_2^{-1}\}_3 = 1,$$
$$\alpha_3\rho^3 = (\beta_1\beta_2)^2,$$

where $\beta_1$, $\beta_2$, $\beta_3$ are as in (5.6.1). (We use the marking at $v_{\mathrm{II}}$ such that $\widehat{12} \subset R_{\mathrm{II}}$. Along an appropriate path of length 4, the translation homomorphism from $v_{\mathrm{I}}$ to $v_{\mathrm{II}}$ is $\mathfrak{m}_{[2,1]} \circ \mathfrak{m}_{[3,1]} \circ \mathfrak{m}_{[3,1]} \circ \mathfrak{m}_{[3,1]}$.) Using the GAP commands

```
P := PresentationNormalClosure(g, Subgroup(g, [g.1/g.3]));
SimplifyPresentation(P);
```

one finds that $[\pi_1, \pi_1]$ is a free group on two generators. Since there is a canonical (perturbation) epimorphism $\pi_1 \twoheadrightarrow \bar{\mathbb{B}}_3$ and all groups involved are residually finite, hence Hopfian, the above epimorphism is an isomorphism. (This approach to using GAP [17] to treat a group 'suspected' to be isomorphic to $\bar{\mathbb{B}}_3$ was suggested to me by E. Artal Bartolo.)

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