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THE NUMBER OF MAPPINGS BETWEEN COMPACT RIEMANN SURFACES

FERNANDO CHAMIZO* and YOLANDA FUERTES†

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Abstract

We give bounds for the number of morphisms $f\colon X \to Y$ where X and Y are compact Riemann surfaces. The target surface Y is not necessarily fixed.

1. Introduction

A classical result due to Hurwitz [7] asserts that the number of automorphisms $f\colon X \to X$ of a compact Riemann surface is bounded by $84(g-1)$ where $g \geq 2$ is the genus of X. A natural extension is the study of the number of (non-constant holomorphic) morphisms onto a compact Riemann surface X' of genus g', $1 < g' < g$. Let us denote this number by $\mathcal{N}$,

$$\mathcal{N} = \mathcal{N}(X, X') = \#\{f \text{ morphism } X \to X'\}.$$

De Franchis [2] proved in 1913 that $\mathcal{N}$ is finite and that, in fact,

$$\mathcal{I} = \mathcal{I}(X) = \sum_{X'} \mathcal{N}(X, X')$$

is also finite when X' runs over all possible target surfaces up to isomorphisms (see Remark 3.4 in [5] for a short modern proof).

Since then several authors [6], [8], [9], [11], [12] (see generalizations in [1]) have given effective bounds for $\mathcal{N}$ and $\mathcal{I}$ in terms of g and g'. In contrast with the linear behaviour when $X = X'$, all of these bounds show an exponential growth in g. The best known result for $\mathcal{N}$ is due to Naranjo and Pirola [10] who proved

$$\mathcal{N} \leq 8(g-1)\rho\left(\binom{2g}{1}(2\rho)^{2g-1} + \binom{2g}{3}(2\rho)^{2g-3} + \cdots\right) \tag{1.1}$$

where $\rho = (g-1)/(g'-1)$. The second named author has proved $\mathcal{N} \leq 2(2g'+2)^{2g+2}$ when X is hyperelliptic with a different approach based on Weierstrass points [4].

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On the other hand Tanabe got in [12]

$$\mathcal{I} \leq (2g-3)(g-2)(g-1)2^{2g-3}(2g-1)^{g-1}(2g)^{4g}. \tag{1.2}$$

In this paper we give new bounds for $\mathcal{N}$ and $\mathcal{I}$ simplifying and sharpening [11] and [12]. Our results are easier to write in term of the degree of the morphisms (in fact the same applies for previous results although they are not always written in this way) therefore we define

$$\mathcal{N}_d = \mathcal{N}_d(X, X') = \#\{f \text{ morphism } X \to X' \text{ with } \deg(f) \leq d\}$$

and

$$\mathcal{I}_d = \mathcal{I}_d(X) = \sum_{X'} \mathcal{N}_d(X, X').$$

Theorem 1.1. *We have*

$$\mathcal{N}_d \leq 8(g-1)(2d)^{2g}.$$

Theorem 1.2. *We have*

$$\mathcal{I}_d \leq \binom{2g-2}{d}(2d)^{2g+1}(2g-1)^d.$$

2. Proof of the results

First of all we introduce some notation and recall some basic facts following [3].

For a Riemann surface X of genus g we consider the $2g$-dimensional real vector space $\mathcal{H}(X)$ generated by real harmonic forms on X, and choose a basis of this space, $\{\omega_1, \omega_2, \dots, \omega_{2g}\}$, being dual to a canonical basis for $H_1(X, \mathrm{Z})$. We shall use $\mathcal{Z}_X$ to denote the $2g$-dimensional lattice generated by this basis, $\mathcal{H}(X) = \mathcal{Z}_X \otimes \mathbb{R}$. In $\mathcal{H}(X)$ there is a natural inner product given by $\langle \omega, \eta \rangle = \int \eta \wedge *\omega$, let $\|\cdot\|$ denote the corresponding norm.

Any morphism $f\colon X \to X'$ of degree d induces a linear map, its pullback, $f^*\colon \mathcal{H}(X') \to \mathcal{H}(X)$ that also maps $\mathcal{Z}_{X'}$ into $\mathcal{Z}_X$. In fact by the definition of the degree

$$\|f^*\omega\| = \sqrt{d}\|\omega\| \quad \text{for any} \quad \omega \in \mathcal{H}(X')$$

(note that norms indicated with the same symbol are defined in different spaces). By duality there is another linear map, the pushforward, $f_*\colon \mathcal{H}(X) \to \mathcal{H}(X')$, still preserving the lattices, satisfying $(f_* \circ f^*)(\omega) = d\omega$ for $\omega \in \mathcal{H}(X')$ and with the same norm operator, in particular

$$\|f_*\eta\| \leq \sqrt{d}\|\eta\| \quad \text{for any} \quad \eta \in \mathcal{H}(X).$$

Proof of Theorem 1.1. Fix a non-zero form $e' \in \mathcal{Z}_{X'}$ of minimal norm and consider the equivalence relation in the set of morphisms $X \to X'$ of degree $\le d$ given by $f \sim h$ if $f^*e' = \pm h^*e'$. By Corollary 3.2 of [10] the cardinal of each class is at most $8(g-1)$. It remains to prove that the number of equivalence classes is $\le (2d)^{2g}$.

Recall that $f^*e' \in \mathcal{Z}_X$ and $\dim \mathcal{Z}_X = 2g$ then the coordinates of f^*e' form a vector in $\mathbb{Z}^{2g}$ and there are at most $(2d)^{2g}$ possibilities for its reduction (mod $2d$). Assume that there were more than $(2d)^{2g}$ equivalence classes, then pigeonhole principle assures the existence of two non-equivalent morphisms f and h such that $f^*e' - h^*e' = 2d\eta$ with $\eta \in \mathcal{Z}_X - \{0\}$. By the positivity of $(f_* - h_*)(f^* - h^*)$ its kernel coincides with that of $f^* - h^*$ and we deduce $\eta \notin (\operatorname{Ker} f_*) \cap (\operatorname{Ker} h_*)$, in particular

$$\|e'\| \le \max(\|f_*\eta\|, \|h_*\eta\|) \le \sqrt{d}\|\eta\|.$$

Connecting this with the triangle inequality

$$2\sqrt{d}\|e'\| \le \|f^*e' - h^*e'\| \le \|f^*e'\| + \|h^*e'\| \le 2\sqrt{d}\|e'\|.$$

Then inequalities become equalities. The central one implies $f^*e' = \lambda h^*e'$, $\lambda \in \mathbb{R}$, and the last one $\|f^*e'\| = \|h^*e'\|$ (because $\|f^*e'\|, \|h^*e'\| \le \sqrt{d}\|e'\|$). Consequently $\lambda = \pm 1$ and f and h are in the same class against our assumption. □

Proof of Theorem 1.2. Let N_d be the number of equivalence classes of morphisms of degree exactly d modulo isomorphisms counted by $\mathcal{I}_d$. We are going to prove

$$N_d \le 2\binom{2g-2}{d}(2d)^{2g}(2g-1)^d, \tag{2.1}$$

that gives

$$\mathcal{I}_d \le 2\sum_{j=1}^{d}\binom{2g-2}{j}(2j)^{2g}(2g-1)^j. \tag{2.2}$$

The bound in the statement follows using the monotonicity of binomial coefficients in the range $j \le g-1$ (which is assured by the Riemann–Hurwitz formula).

To prove (2.1), given $f\colon X \to X'_i$ and $h\colon X \to X'_j$ in the set of morphisms of degree d, we define $f \sim h$ if $f^*e'_i = \pm h^*e'_j$ where $e'_i \in \mathcal{Z}_{X'_i}$ and $e'_j \in \mathcal{Z}_{X'_j}$ are non-zero fixed elements of minimal norm. By Lemma 3 of [12] we have

$$N_d \le 2\binom{2g-2}{d}(2g-1)^d E$$

where E is the number of equivalence classes of $\sim$, and (2.1) follows if $E \le (2d)^{2g}$.

Suppose on the contrary that $E > (2d)^{2g}$, then by pigeonhole principle, there exist f and h with $f^*e'_i \neq \pm h^*e'_j$, such that

$$f^*e'_i - h^*e'_j = 2d\eta \quad \text{for} \quad \eta \in \mathcal{Z}_X - \{0\}. \tag{2.3}$$

If $\eta \in \operatorname{Ker} f_*$ then (2.3) implies $de'_i - f_*h^*e'_j = 0$ hence $\|e'_i\| \leq \|e'_j\|$. The same applies for $\eta \in \operatorname{Ker} h_*$ and we have

$$\max(\|e'_i\|, \|e'_j\|) \leq \max(\|f_*\eta\|, \|h_*\eta\|) \leq \sqrt{d}\|\eta\|$$

whenever $\eta \notin (\operatorname{Ker} f_*) \cap (\operatorname{Ker} h_*)$, that we can assume because $\eta \in (\operatorname{Ker} f_*) \cap (\operatorname{Ker} h_*) = (\operatorname{Im} f^*)^\perp \cap (\operatorname{Im} h^*)^\perp = (\operatorname{Im} f^* + \operatorname{Im} h^*)^\perp$ is incompatible with (2.3). Using this inequality and (2.3) we obtain

$$2\sqrt{d}\max(\|e'_i\|, \|e'_j\|) \leq \|f^*e'_i - h^*e'_j\| \leq \|f^*e'_i\| + \|h^*e'_j\| = \sqrt{d}(\|e'_i\| + \|e'_j\|)$$

and this leads to a contradiction for $f^*e'_i \neq \pm h^*e'_j$. □

3. Comparison with previous results

The Riemann–Hurwitz formula implies $d \leq \rho$ with $\rho = (g-1)/(g'-1)$ as in (1.1) and one recovers bounds for $\mathcal{N}$ and $\mathcal{I}$ just substituting d by ρ (or better by $\lfloor \rho \rfloor$) in our results.

Note that the bound (1.1) can be written as

$$4(g-1)\rho[(2\rho+1)^{2g} - (2\rho-1)^{2g}].$$

In [10] it is claimed that the leading term of the function between brackets is $4g(2\rho)^{2g-1}$ but this is a little misleading because the exponential behaviour in g avoids any noticeable cancellation in the subtraction. The ratio with respect to the bound in Theorem 1.1 is

$$\frac{\rho}{2}\left[\left(1+\frac{1}{2\rho}\right)^{2g} - \left(1-\frac{1}{2\rho}\right)^{2g}\right] \sim \rho \sinh(g'-1)$$

where $\sim$ indicates the same asymptotics for large values of ρ. Note that then (1.1) shows an exponential growth in g' in comparison with Theorem 1.1.

On the other hand (1.2) divided by our bound in Theorem 1.2 in the worst case scenario $d = g-1$ gives

$$\frac{(2g-3)(g-2)(g-1)2^{2g-3}(2g)^{4g}}{\binom{2g-2}{g-1}(2g-2)^{2g+1}} \sim \frac{e^2\sqrt{\pi g}}{2} g^2(2g)^{2g}$$

where we have employed Stirling asymptotic formula $N! \sim N^N e^{-N}\sqrt{2\pi N}$ giving $\binom{2N}{N} \sim 2^{2N}/\sqrt{\pi N}$ and $(g/(g-1))^{2g} \to e^2$ as $g \to \infty$.

In fact in the proof of (1.2) Tanabe gets a bound depending on ρ (see p. 3063 in [12]) to be compared with (2.1). In this case the quotient is

$$\frac{(2g-2g'+1)(4d^2+1)^{2g}}{2(2d)^{2g}} \geq \frac{1}{2}(2d)^{2g}$$

that still grows exponentially.

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Fernando Chamizo
Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid
28049 Madrid
Spain
e-mail: fernando.chamizo@uam.es

Yolanda Fuertes
Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid
28049 Madrid
Spain
e-mail: yolanda.fuertes@uam.es