Colombini, F., Nishitani, T., Orrù, N. and Pernazza, L.
Osaka J. Math.
48 (2011), 645–673

SOME WELL-POSED CAUCHY PROBLEM FOR SECOND ORDER HYPERBOLIC EQUATIONS WITH TWO INDEPENDENT VARIABLES

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(Received February 19, 2010)

Abstract

In this paper we discuss the C^∞ well-posedness for second order hyperbolic equations $Pu = \partial_t^2 u - a(t,x)\,\partial_x^2 u = f$ with two independent variables (t,x). Assuming that the C^∞ function $a(t,x) \geq 0$ verifies $\partial_t^p a(0,0) \neq 0$ with some p and that the discriminant $\Delta(x)$ of $a(t,x)$ vanishes of finite order at $x = 0$, we prove that the Cauchy problem for P is C^∞ well-posed in a neighbourhood of the origin.

1. Introduction

In this paper we deal with the C^∞ well-posedness of the Cauchy problem for a second order hyperbolic operator with two independent variables $P = \partial_t^2 - a(t,x)\,\partial_x^2$, $(t,x) \in \mathbb{R}^2$:

$$
\begin{cases} Pu = \partial_t^2 u - a(t,x)\,\partial_x^2 u = f, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x) \end{cases} \tag{1.1}
$$

near the origin of $\mathbb{R}^2$, where we always assume that $a(t,x) \geq 0$. In [11] and [12], assuming that $a(t,x)$ is real analytic in (t,x), it is proved that the Cauchy problem for P is C^∞ well-posed. On the other hand, in [4], the authors give a counterexample involving a function $a(t) \in C^\infty([0,T])$, positive for $t > 0$, such that the Cauchy problem for $P = \partial_t^2 - a(t)\,\partial_x^2$ is not C^∞ well-posed. The main feature of this $a(t)$ is that $da(t)/dt$ changes sign infinitely many times when $t \downarrow 0$. There are many works trying to extend the C^∞ well-posedness result in [11] without the analyticity assumptions on $a(t,x)$ (see for example, [1], [2], [3], [5], [8], [10], [13]).

In this paper we assume that $a(t,x)$ is of class C^∞ in (t,x) and essentially a polynomial in t and we discuss the C^∞ well-posedness question under this rather general assumption. If $a(0,0) \neq 0$ then P is strictly hyperbolic and if $a(0,0) = \partial_t a(0,0) = 0$ but $\partial_t^2 a(0,0) \neq 0$ then P is effectively hyperbolic at $(0,0)$ and hence the Cauchy problem is C^∞ well-posed for any lower order term (see [7], [11]). Thus we may assume that

2000 Mathematics Subject Classification. Primary 35L15; Secondary 35L80, 35B30.

$a(0,0) = \partial_t a(0,0) = \partial_t^2 a(0,0) = 0$ without restrictions as far as the C^∞ well-posedness is concerned. We assume that there is a $p \in \mathbb{N}$, $p \geq 3$ such that

$$\partial_t^p a(0,0) \neq 0. \tag{1.2}$$

Then applying the Malgrange preparation theorem we can write

$$a(t,x) = e(t,x)(t^p + a_1(x)t^{p-1} + \cdots + a_p(x)) \tag{1.3}$$

where $e, a_1, \ldots, a_p$ are of class C^∞ in a neighbourhood of the origin and $e(0,0) \neq 0$. Let $\Delta(x)$ be the discriminant of $a(t,x)/e(t,x)$ as a polynomial in t. We call $\Delta(x)$ the discriminant of $a(t,x)$. We now assume that there is $q \in \mathbb{N}$ such that

$$\left(\frac{d}{dx}\right)^q \Delta(0) \neq 0. \tag{1.4}$$

Then we have

Theorem 1.1. *Assume* (1.2) *and* (1.4). *Then the Cauchy problem* (1.1) *is* C^∞ *well-posed in a neighbourhood of the origin.*

One can easily generalize Theorem 1.1 a little bit as follows:

Theorem 1.1′. *Assume that* $b_j(t,x)$, $j = 1, \ldots, r$ *are functions of class* C^∞ *and verify the conditions* (1.2) *and* (1.4) *with some* $p_j, q_j \in \mathbb{N}$ (*the nonnegativity of* $b_j(t,x)$ *is not assumed*) *and that* $a(t,x) = b_1(t,x)^{m_1} \cdots b_r(t,x)^{m_r}$ *where* $m_j \in \mathbb{N}$ *and* $B_j(t,x) = b_j(t,x)^{m_j} \geq 0$ *near the origin. Then the assertion of* Theorem 1.1 *holds.*

In Section 2 we define a weighted energy and in Sections 3 and 4 we derive a priori estimates. In Section 5 we prove Theorem 1.1. Finally in Sections 6, 7 and 8 we construct the weight functions.

2. Energy

Throughout this paper an index x or t will denote respectively a space or time derivative, e.g. $u_x = \partial_x u$ and $k_{n,t} = \partial_t k_n$. As usual, we set $D = \partial_x / i$.

We prove Theorem 1.1 by deriving a priori estimates. Take $\chi(x) \in C_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ in a neighbourhood of the origin; $\chi(x)a(t,x)$ is then defined and of class C^∞ in $[-T, T] \times \mathbb{R}$.

Let us consider an energy

$$\mathcal{E}(t,u) = \sum_{n=0}^{\infty} e^{-ct} A(t)^n \int k_n(t,x)[|u_{n,t}|^2 + \chi(x)a(t,x)|\partial_x u_n|^2 + (n^2+1)|u_n|^2]\, dx$$

where $c > 0$, $A(t) = e^{a-bt}$ with $a, b > 0$ and

$$u_n = \frac{1}{n!} \log^n \langle D \rangle u, \quad \langle \xi \rangle^2 = \xi^2 + 1.$$

Here

$$\langle D \rangle^s u = e^{s \log\langle D \rangle} u = \sum_{n=0}^{\infty} \frac{s^n}{n!} \log^n \langle D \rangle u$$

has the role of a partition of unity. Although $(s^n/n!)\log^n \langle D \rangle$ does not localize the frequencies ξ so much (but see Lemma 3.1 below), it has the advantage that $\partial_\xi^\ell((s^n/n!)\log^n \langle \xi \rangle)$ conserves the same form up to factors $\xi^i \langle \xi \rangle^{-j}$. In order that this energy may work well to derive a priori estimates, the weight functions $k_n(t, x)$ are required to verify some suitable properties. For similar examples of energy see [8], [9] and [13]. Our main task in this paper is then to construct a sequence of weight functions $k_n(t, x)$ for $a(t, x)$ satisfying the properties listed in the next proposition:

Proposition 2.1. *Let $N > 1$ be a given constant and $a(t, x)$ be a nonnegative function of class C^∞ satisfying* (1.2) *and* (1.4). *One can find $T > 0$ and construct a sequence of weight functions $k_n(t, x)$ defined on $[-T, T] \times \mathbb{R}$ verifying the following properties*:

1) *$k_n(t, x)$ is a Lipschitz continuous function and*

$$C_1 2^{-C_2 n} \le k_n(t, x) \le 1.$$

2) $k_{n,t}(t, x) \ge -C_3 e^{C_4 n}$.
3) *We have that*

$$|k_{n,x}(t, x)| \sqrt{\chi(x) a(t, x)} \le C_5 (n + 1) k_n(t, x).$$

4) *We have that*

$$k_{n,t}(t, x) \le -N \frac{|\chi(x) a_t(t, x)|}{\chi(x) a(t, x) + 2^{-2n}} k_n(t, x) + C_6 (n + 1) k_n(t, x).$$

5) $k_{n+1}(t, x) \le C_7 k_n(t, x)$.

The proof of Proposition 2.1 will be given in Sections 6, 7 and 8.

3. Energy estimate

In what follows we write simply $a(t, x)$ instead of $\chi(x)a(t, x)$ and assume that $u \in C^2([-T, T]; \mathcal{S}(\mathbb{R}))$ verifies

$$Pu = \partial_t^2 u - a(t, x)\, \partial_x^2 u = f.$$

Let us define

$$u_{\beta,s,j} = 2^{-n\beta} \frac{D^{\beta+j}}{\langle D\rangle^{s+j}} u \quad \text{and} \quad u_{n,\beta,s,j} = \frac{\log^n \langle D\rangle}{n!} u_{\beta,s,j}. \tag{3.1}$$

With these definitions, $u_{0,0,0} = u$ and $u_n = u_{n,0,0,0}$. We introduce the energy

$$\begin{aligned}\mathcal{E}(t, u) &= \sum_{n=0}^{\infty} \sum_{\beta=0}^{p} \sum_{s=0}^{p+q} \sum_{j=0}^{1} e^{-ct} A^n(t) \int k_n(t, x)[|\partial_t u_{n,\beta,s,j}|^2 + a(t, x)|\partial_x u_{n,\beta,s,j}|^2 \\ &\qquad + (n^2 + 1)|u_{n,\beta,s,j}|^2]\, dx \\ &= \sum_{n=0}^{\infty} \sum_{\beta=0}^{p} \sum_{s=0}^{p+q} \sum_{j=0}^{1} E_n(t, u_{\beta,s,j})\end{aligned}$$

where $k_n(t, x)$ is given by Proposition 2.1 (we will later determine the undefined quantities of this expression, namely a, b in the term $A(t)$, the coefficient c and the number of terms of the sum, that depends on $p, q \in \mathbb{N}$).

Performing the derivative of $E_n(t, u)$ with respect to t we have that

$$\begin{aligned}\frac{d}{dt} E_n(t, u) &= -(c + nb)E_n(t, u) \\ &\quad + e^{-ct} A^n(t) \int k_{n,t}(t, x)[|u_{n,t}|^2 + a(t, x)|\partial_x u_n|^2 + (n^2 + 1)|u_n|^2]\, dx \\ &\quad + e^{-ct} A^n(t) \int k_n(t, x) 2\, \mathrm{Re}(u_{n,tt}\overline{u}_{n,t})\, dx \\ &\quad + e^{-ct} A^n(t) \int k_n(t, x) a_t(t, x)|\partial_x u_n|^2\, dx \\ &\quad + e^{-ct} A^n(t) \int k_n(t, x) a(t, x) 2\, \mathrm{Re}(\partial_x u_n \overline{u}_{n,xt})\, dx \\ &\quad + (n^2 + 1)e^{-ct} A^n(t) \int k_n(t, x) 2\, \mathrm{Re}(u_{n,t}\overline{u}_n)\, dx \\ &= -(c + nb)E_n(t, u) + I_2(u_n) + I_3(u_n) + I_4(u_n) + I_5(u_n) + I_6(u_n).\end{aligned}$$

We then begin studying $I_6(u_n)$: note that

$$I_6(u_n) \le e^{-ct} A^n(t)\Big[\int k_n(n|u_{n,t}|^2 + n^3|u_n|^2)\, dx + \int k_n(|u_{n,t}|^2 + |u_n|^2)\, dx\Big],$$

therefore it is clear that $I_6(u_n)$ can be bounded by $CnE_n(t,u)$. Thus we have that

$$\sum_{n,\beta,s,j} I_6(u_{n,\beta,s,j}) \le C \sum_{n,\beta,s,j} nE_n(t,u_{\beta,s,j}) \tag{3.2}$$

where the sum is taken over $n \in \mathbb{N}$, $0 \le \beta \le p$, $0 \le s \le p+q$ and $j = 0, 1$.

Next, let us consider $I_2(u_n)$ and $I_4(u_n)$ (the terms $I_3(u_n)$ and $I_5(u_n)$ will be estimated together in the next section). Note that

$$k_n a_t |\partial_x u_n|^2 \le k_n \frac{|a_t|}{a+2^{-2n}} a|\partial_x u_n|^2 + k_n \frac{|a_t|}{a+2^{-2n}} 2^{-2n}|\partial_x u_n|^2. \tag{3.3}$$

With a slight abuse of notation we will set $A = A(0)$ in what follows.

Lemma 3.1. *For every $t \in [-T,T]$ (for a suitably small T) and every fixed s, j, if p and A are large enough we have that*

$$\sum_n A^n(t) \sum_{\beta=0}^{p} \int k_n \frac{|a_t|}{a+2^{-2n}} 2^{-2n} |\partial_x u_{n,\beta,s,j}|^2 \, dx$$
$$\le \sum_n A^n(t) \sum_{\beta=1}^{p} \int k_n \frac{|a_t|}{a+2^{-2n}} |u_{n,\beta,s,j}|^2 \, dx + C \sum_n A^n(t) \int k_n |u_{n,0,s,j}|^2 \, dx.$$

Proof. Let us denote by $\|u\|$ the $L^2(\mathbb{R})$ norm of $u(t,\cdot)$. Obviously

$$k_n \frac{|a_t|}{a+2^{-2n}} 2^{-2n} |\partial_x u_{n,\beta,s,j}|^2 = k_n \frac{|a_t|}{a+2^{-2n}} |u_{n,\beta+1,s,j}|^2$$

if $0 \le \beta < p$. If $\beta = p$, noting that $|a_t| \le C$ and $k_n \le 1$ by Proposition 2.1 (and fixing s, j and setting $w = u_{0,s,j}$, $w_n = u_{n,0,s,j}$) we have that

$$\begin{aligned}
&\sum_n A^n(t) 2^{-2n(p+1)} \int k_n \frac{|a_t|}{a+2^{-2n}} |D^{p+1} w_n|^2 \, dx \\
&\le C_1 \sum_n A^n(t) 2^{-2np} \|\langle D\rangle^{p+1} w_n\|^2 \\
&\le C_1 \sum_n A^n(t) 2^{-2np} \left\| \sum_m (p+1)^m \frac{\log^{m+n}\langle D\rangle}{m!\,n!} w \right\|^2 \\
&\le C_2 \sum_{m,n} A^n(t) 2^{-2np} (m+1)^2 (p+1)^{2m} \left\| \frac{\log^{m+n}\langle D\rangle}{m!\,n!} w \right\|^2 \\
&\le C_2 \sum_{m,n} A(t)^{m+n} 2^{-2(m+n)p} A(t)^{-m} (m+1)^2 \\
&\qquad \times 2^{2mp} 2^{2m(p+1)} 2^{2(m+n)} \left\| \frac{\log^{m+n}\langle D\rangle}{(m+n)!} w \right\|^2.
\end{aligned} \tag{3.4}$$

Set $\mu = m+n$; choosing p large enough, by Proposition 2.1 we can have that $k_\mu 2^{2\mu(p-1)} \geq C_3 > 0$. Observe that whatever the choice of b may be, we can suppose that $A(t) \geq A/2$ for $t \in [-T, T]$ simply decreasing T; on the other hand, we also choose A large with respect to $2^2 \cdot 2^{4p+2} \cdot 2$, so that (taking into account that $\sum_{m=0}^{\infty} 1/2^m = 2$), the last line in (3.4) can be bounded by

$$2C_2 \sum_\mu A^\mu 2^{-2\mu(p-1)} \|w_\mu\|^2 \leq C_4 \sum_\mu A^\mu \int k_\mu |w_\mu|^2 \, dx.$$

This ends the proof of Lemma 3.1. □

Recall now that by 4) of Proposition 2.1

$$k_n \frac{|a_t|}{a + 2^{-2n}} \leq -\frac{1}{N} k_{n,t} + \frac{C}{N}(n+1)k_n. \tag{3.5}$$

By Lemma 3.1 and (3.3), (3.5) we see that (for every fixed s and j)

$$\sum_{n,\beta} I_4(u_{n,\beta,s,j}) \leq -\frac{1}{N} \sum_{n,\beta} e^{-ct} A^n(t) \int k_{n,t}(a|\partial_x u_{n,\beta,s,j}|^2 + |u_{n,\beta,s,j}|^2) \, dx$$
$$+ C \sum_{n,\beta} n E_n(u_{\beta,s,j}).$$

From 4) of Proposition 2.1 we have that $k_{n,t} \leq C(n+1)k_n$, thus, since $1 - 1/N > 0$, we obtain that

$$\sum_{n,\beta} I_4(u_{n,\beta,s,j}) + \sum_{n,\beta} I_2(u_{n,\beta,s,j}) \leq C \sum_{n,\beta} n E_n(u_{\beta,s,j}). \tag{3.6}$$

4. Energy estimate (continued)

We turn to $I_5(u_n)$. Note that

$$\begin{aligned} I_5(u_n) &= 2e^{-ct} A^n(t) \int k_n a(t,x) \operatorname{Re}(u_{n,x} \overline{u}_{n,xt}) \, dx \\ &= -2e^{-ct} A^n(t) \int k_{n,x} a(t,x) \operatorname{Re}(u_{n,x} \overline{u}_{n,t}) \, dx \\ &\quad - 2e^{-ct} A^n(t) \int k_n a_x(t,x) \operatorname{Re}(u_{n,x} \overline{u}_{n,t}) \, dx \\ &\quad - 2e^{-ct} A^n(t) \int k_n a(t,x) \operatorname{Re}(u_{n,xx} \overline{u}_{n,t}) \, dx \\ &= J_1(u_n) + J_2(u_n) + J_3(u_n). \end{aligned}$$

By 3) of Proposition 2.1 we have

$$|J_1(u_n)| \leq Ce^{-ct} A^n(t) \int n k_n (|u_{n,t}|^2 + a(t,x)|u_{n,x}|^2) \, dx \leq CnE_n(u) \tag{4.1}$$

and from the Glaeser inequality, applied to $a \geq 0$, it follows that

$$|J_2(u_n)| \leq Ce^{-ct}A^n(t)\int k_n(|u_{n,t}|^2 + a(t,x)|u_{n,x}|^2)\,dx \leq CE_n(u). \tag{4.2}$$

We still have to estimate

$$J_3(u_{n,\beta,s,j}) = -2e^{-ct}A^n(t)\int k_n(t,x)a(t,x)\,\mathrm{Re}(\partial_x^2 u_{n,\beta,s,j}\ \partial_t\overline{u}_{n,\beta,s,j})\,dx;$$

but note that

$$\begin{aligned} &I_3(u_{n,\beta,s,j}) + J_3(u_{n,\beta,s,j}) \\ &= 2e^{-ct}A^n(t)\int k_n\,\mathrm{Re}\left(\left[\frac{\log^n\langle D\rangle}{n!}\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}, a\right]\partial_x^2 u\cdot c_{n,\beta}\ \partial_t\overline{u}_{n,\beta,s,j}\right)dx \\ &\quad + 2e^{-ct}A^n(t)\int k_n(t,x)\,\mathrm{Re}(f_{n,\beta,s,j}\ \partial_t\bar{u}_{n,\beta,s,j})\,dx \end{aligned} \tag{4.3}$$

where $c_{n,\beta} = 2^{-n\beta}$ and $\beta = 0, 1, \ldots, p$, $s = 0, 1, \ldots, p+q$, $j = 0, 1$ and $f_{n,\beta,s,j}$ is defined as in (3.1).

We rewrite the commutator as

$$\begin{aligned} &\left[\frac{\log^n\langle D\rangle}{n!}\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}, a(t,x)\right]\partial_x^2 u_{n,\beta,s,j}\cdot c_{n,\beta} \\ &= \sum_{1\leq l<p+q+2-s}\frac{(-i)^l}{l!}\ \partial_x^l a\Phi^{(l)}_{\beta,s,j}(D)\,\partial_x^2 u\cdot c_{n,\beta} + R(u_{n,\beta,s,j}) \end{aligned} \tag{4.4}$$

where

$$\Phi_{\beta,s,j}(\xi) = \frac{\log^n\langle\xi\rangle}{n!}\frac{\xi^{\beta+j}}{\langle\xi\rangle^{s+j}}$$

and

$$\begin{aligned} R(u_{n,\beta,s,j}) = \frac{-1}{(m-1)!}\iiint_0^1 & e^{ix\xi}\,\Phi^{(m)}_{\beta,s,j}(\eta+\theta(\xi-\eta)) \\ & \times (1-\theta)^{m-1}(\xi-\eta)^m\hat{a}(t,\xi-\eta)\eta^2\hat{u}(t,\eta)c_{n,\beta}\,d\theta\,d\eta\,d\xi \end{aligned}$$

with $m = p+q+2-s$. Here $\hat{a}(t,\xi)$ denotes the Fourier transform of $a(t,x)$ with respect to x.

As a consequence, writing $r = p + q$, we see that

$$
\begin{aligned}
&I_3(u_{n,\beta,s,j}) + J_3(u_{n,\beta,s,j}) \\
&\le e^{-ct}\frac{1}{n+1}A^n(t)\int k_n\Big|\sum_{1\le l<m}\frac{(-i)^l}{l!}\,\partial_x^l a\Phi^{(l)}_{\beta,s,j}(D)\,\partial_x^2 uc_{n,\beta}\Big|^2\,dx \\
&\quad + e^{-ct}(n+1)A^n(t)\int k_n|\partial_t u_{n,\beta,s,j}|^2\,dx \\
&\quad + e^{-ct}\frac{1}{n+1}A^n(t)\int k_n|R(u_{n,\beta,s,j})|^2\,dx \\
&\quad + e^{-ct}(n+1)A^n(t)\int k_n|\partial_t u_{n,\beta,s,j}|^2\,dx \\
&\quad + e^{-ct}A^n(t)\int k_n(t,x)|f_{n,\beta,s,j}|^2\,dx + e^{-ct}A^n(t)\int k_n|\partial_t u_{n,\beta,s,j}|^2\,dx.
\end{aligned}
\tag{4.5}
$$

The second, fourth and sixth term are smaller than $CnE_n(u_{\beta,s,j})$ for some $C > 0$. We keep the fifth one as it is and study the other two in the following two lemmas; we start with the first term.

Lemma 4.1. *We have that*

$$
\begin{aligned}
&e^{-ct}\sum_{n,\beta,s,j}\frac{1}{n+1}A^n(t)\int k_n\Big|\sum_{1\le l<m}\frac{(-i)^l}{l!}\,\partial_x^l a\Phi^{(l)}_{\beta,s,j}(D)\,\partial_x^2 uc_{n,\beta}\Big|^2\,dx \\
&\le C\sum_{n,\beta,s,j}(n+1)E_n(u_{\beta,s,j}).
\end{aligned}
$$

Proof. We write $r = p + q$ and let n stay fixed for the moment. The left-hand side can then be estimated by

$$
C(p,q)\sum_{\beta\le p, s\le r, j}\frac{1}{n+1}A^n(t)\int k_n\sum_{1\le l<m}\frac{1}{(l!)^2}\big|\partial_x^l a\Phi^{(l)}_{\beta,s,j}(D)\,\partial_x^2 uc_{n,\beta}\big|^2\,dx. \tag{4.6}
$$

We first consider the term with $l = 1$ of this expression:

$$
\begin{aligned}
&|\partial_x a\Phi^{(1)}_{\beta,s,j}(D)\,\partial_x^2 uc_{n,\beta}| \\
&= \Big|\partial_x a\Big[\frac{\log^{n-1}\langle D\rangle}{(n-1)!}\frac{D^{\beta+j+1}}{\langle D\rangle^{s+j+2}} \\
&\quad + \frac{\log^n\langle D\rangle}{n!}\Big(\frac{(\beta+j)D^{\beta+j-1}}{\langle D\rangle^{s+j}} - (s+j)\frac{D^{\beta+j+1}}{\langle D\rangle^{s+j+2}}\Big)\Big]\partial_x^2 uc_{n,\beta}\Big|
\end{aligned}
$$

$$\leq C\sqrt{a}\Big(\Big|\frac{D^{\beta+j+2}}{\langle D\rangle^{s+j+2}}\,\partial_x u_{n-1}\Big| + (p+1)\Big|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_n\Big| + (s+1)\Big|\frac{D^{\beta+j+2}}{\langle D\rangle^{s+j+2}}\,\partial_x u_n\Big|\Big)c_{n,\beta}$$
$$\leq C_1\sqrt{a}\Big(\Big|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_{n-1}\Big|c_{n-1,\beta} + \Big|\frac{D^{\beta+j}}{\langle D\rangle^{s+j+2}}\,\partial_x u_{n-1}\Big|c_{n-1,\beta} + \Big|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_n\Big|c_{n,\beta} + \Big|\frac{D^{\beta+j}}{\langle D\rangle^{s+j+2}}\,\partial_x u_n\Big|c_{n,\beta}\Big).$$

Here we have used $D^2 = \langle D\rangle^2 - 1$ and

$$\frac{c_{n,\beta}}{c_{n',\beta'}} \leq 1, \quad n' \leq n,\ \beta' \leq \beta. \tag{4.7}$$

Thus (4.6) with $l = 1$ can be estimated by

$$C\sum_{\beta\leq p, s\leq r, j}\frac{1}{n+1}A^n(t)\int k_n\Big[a\Big|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_{n-1}c_{n-1,\beta}\Big|^2 + a\Big|\frac{D^{\beta+j}}{\langle D\rangle^{s+j+2}}\,\partial_x u_{n-1}c_{n-1,\beta}\Big|^2 + a\Big|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}\,\partial_x u_n c_{n,\beta}\Big|^2 + a\Big|\frac{D^{\beta+j}}{\langle D\rangle^{s+j+2}}\,\partial_x u_n c_{n,\beta}\Big|^2\Big]\,dx$$
$$\leq C\frac{1}{n+1}\sum_{\beta\leq p, s\leq r, j}(E_{n-1}(u_{\beta,s,j}) + E_n(u_{\beta,s,j})) + C\frac{1}{n+1}\sum_{\beta\leq p, r+1\leq s\leq r+2, j}(E_{n-1}(u_{\beta,s,j}) + E_n(u_{\beta,s,j}))$$

because $k_n \leq Ck_{n-1}$ by 5) of Proposition 2.1 and $A^n(t) \leq CA(t)^{n-1}$.

We next consider the terms with $l \geq 2$. Note that one can write

$$\Big[\frac{\log^n\langle\xi\rangle}{n!}\,\frac{\xi^{\beta+j}}{\langle\xi\rangle^{s+j}}\Big]^{(l)}\xi^2 = \sum_{h=0}^{\min\{l,n\}}\ \sum_{\substack{l_1\geq h, l_1+l_2=l\\ l_2\leq\beta+2+j+l_1}} C_{h,l_1,l_2}\,\frac{\log^{n-h}\langle\xi\rangle}{(n-h)!}\,\frac{\xi^{\beta+2+j+l_1-l_2}}{\langle\xi\rangle^{s+j+2l_1}} \tag{4.8}$$

for some constants C_{h,l_1,l_2} whose absolute values are bounded by a constant depending on p and q, but not on n. If $2 + j + l_1 - l_2$ is even and nonnegative, then using $\xi^2 = \langle\xi\rangle^2 - 1$ the right-hand side can be written as

$$\sum_{h=0}^{\min\{l,n\}}\ \sum_{s\leq s'\leq s+2r+3}\ \sum_{\beta'\leq\beta}\ \sum_{j=0}^{1} C_{h,\beta',s',j}\,\frac{\log^{n-h}\langle\xi\rangle}{(n-h)!}\,\frac{\xi^{\beta'+j}}{\langle\xi\rangle^{s'+j}} \tag{4.9}$$

(because $2 + j + l_1 - l_2 \le j + 2l_1$ for $l \ge 2$) where $|C_{h,\beta',s',j}|$ is bounded by a constant independent of n. The same argument applied to the case in which $2+j+l_1-l_2$ is odd and nonnegative shows that the right-hand side can be written in the same form (4.9). Then (4.6) with $l \ge 2$ can be bounded by

$$C(p,q)\sum_{\substack{\beta\le p,j\\ s\le 3r+3}}\sum_{h=0}^{\min\{r+1-s,n\}}\frac{1}{n+1}A^n(t)\int k_n(t,x)\sum_{j=0}^{1}\left|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}u_{n-h}\right|^2 c_{n-h,\beta}^2\,dx$$

because of (4.7). This is bounded by

$$C(p,q,A)\sum_{\beta\le p,s\le 3r+3,j}\ \sum_{h=n-r-1}^{n}\frac{1}{h+1}E_h(u_{\beta,s,j})$$

because we can suppose $A(t) \le 2A$. We now need to deal with the terms with $s > r$:

$$\sum_{\beta\le p,r<s\le 3r+3,j}\frac{1}{n+1}A^n(t)\int k_n(t,x)\sum_{j=0}^{1}\left|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}u_n\right|^2 c_{n,\beta}^2\,dx.$$

But since $k_n \le 1$ by 1) of Proposition 2.1 and $\beta \le p$, $s \ge r = p + q$, we have

$$\begin{aligned}
&\sum_n\frac{1}{n+1}A^n(t)\int k_n(t,x)\sum_{j=0}^{1}\left|\frac{D^{\beta+j}}{\langle D\rangle^{s+j}}u_n\right|^2 c_{n,\beta}^2\,dx\\
&\le C\sum_n A^n(t)\int|\langle D\rangle^{-q}u_n|^2\,dx \le C\int\left(\sum_n A^{n/2}(t)\langle\xi\rangle^{-q}\frac{\log^n\langle\xi\rangle}{n!}\right)^2|\hat{u}|^2\,d\xi\\
&\le C\int(\langle\xi\rangle^{-q+\sqrt{A(t)}})^2|\hat{u}|^2\,d\xi \le C\int|u|^2\,dx \le C_2\int k_0(t,x)|u_0|^2\,dx
\end{aligned}$$

provided $q > \sqrt{2A} > \sqrt{A(t)}$. □

It remains to estimate the third term of (4.5), the one containing $|R(u_{n,\beta,s,j})|^2$.

Lemma 4.2. *We have that*

$$\sum_{n,\beta,s,j}\frac{1}{n+1}A^n\int k_n|R(u_{n,\beta,s,j})|^2\,dx \le C(p,q,A)\int k_0(t,x)|u_0|^2\,dx \tag{4.10}$$

for large q.

Proof. Recall that the left-hand side of (4.10) is by definition

$$\sum_{n,\beta,s,j} A^n(t)\int k_n\left|\int e^{ix\xi}\left(\iint_0^1 \Phi^{(m)}_{\beta,s,j}(\eta+\theta(\xi-\eta))\frac{1}{(m-1)!}(1-\theta)^{m-1}\right.\right.$$
$$\left.\left.\times(\xi-\eta)^m\hat{a}(t,\xi-\eta)\eta^2\hat{u}(t,\eta)\,d\theta\,d\eta\right)d\xi\right|^2 c^2_{n,\beta}\,dx$$

which by Parseval's formula is bounded by

$$\sum_{n,\beta,s,j} A^n(t)\int\left|\iint_0^1 \Phi^{(m)}_{\beta,s,j}(\eta+\theta(\xi-\eta))\frac{1}{(m-1)!}(1-\theta)^{m-1}\right.$$
$$\left.\times(\xi-\eta)^m\hat{a}(t,\xi-\eta)\eta^2\hat{u}(t,\eta)\,d\theta\,d\eta\right|^2 d\xi$$

because $k_n \le 1$ and $c_{n,\beta}\le 1$. From (4.9) it is enough to estimate terms of the form

$$C(A,p,q)\sum_n A^n(t)\int\left|\iint_0^1 \frac{\log^n\langle\eta+\theta(\xi-\eta)\rangle}{n!}\frac{(\eta+\theta(\xi-\eta))^{\beta_1+j}}{\langle\eta+\theta(\xi-\eta)\rangle^{s_1+j}}\right.$$
$$\left.\times(\xi-\eta)^m\hat{a}(t,\xi-\eta)\eta^2\hat{u}(t,\eta)\,d\theta\,d\eta\right|^2 d\xi$$

with

$$s_1-\beta_1\ge s+m-p=q+2.$$

Applying the inequality $\langle\eta+\xi\rangle^s\le 2^{|s|}\langle\eta\rangle^s\langle\xi\rangle^{|s|}$ we see that this is bounded by (writing $\hat{u}(\eta)$ for $\hat{u}(t,\eta)$ and $\hat{a}(\eta)$ for $\hat{a}(t,\eta)$)

$$C(A,p,q)\sum_n A^n\int\left|\iint_0^1 \frac{\log^n\langle\eta+\theta(\xi-\eta)\rangle}{n!}\frac{1}{\langle\eta+\theta(\xi-\eta)\rangle^{q+2}}\,d\theta\right.$$
$$\left.\times|(\xi-\eta)^m\hat{a}(\xi-\eta)|\,|\eta^2\hat{u}(\eta)|\,d\theta\,d\eta\right|^2 d\xi$$
$$\le C\sum_n(3^2A)^n\int\left(\int\langle\xi-\eta\rangle^{m+q+2}|\hat{a}(\xi-\eta)|\frac{\log^n\langle\eta\rangle}{n!}\frac{1}{\langle\eta\rangle^q}|\hat{u}(\eta)|\,d\eta\right)^2 d\xi$$
$$+C\sum_n(3^2A)^n\int\left(\int\langle\xi-\eta\rangle^{m+q+2}\frac{\log^n\langle\xi-\eta\rangle}{n!}|\hat{a}(\xi-\eta)|\frac{1}{\langle\eta\rangle^q}|\hat{u}(\eta)|\,d\eta\right)^2 d\xi$$
$$+C\sum_n(3^2A)^n\int\left(\frac{\log^n 2}{n!}\int\langle\xi-\eta\rangle^{m+q+2}\,|\hat{a}(\xi-\eta)|\frac{1}{\langle\eta\rangle^q}|\hat{u}(\eta)|\,d\eta\right)^2 d\xi$$

with $C = 3C(A, p, q)$. By the Schwarz inequality the first integral is estimated by

$$
\begin{aligned}
&C_1(A, p, q) \sum_n A^n 3^{2n} \int \Big(\int \langle \xi - \eta_1 \rangle^{m+q+2} |\hat{a}(t, \xi - \eta_1)| \, d\eta_1 \\
&\qquad\qquad \times \int \langle \xi - \eta \rangle^{m+q+2} |\hat{a}(t, \xi - \eta)| \frac{|\hat{u}_n(\eta)|^2}{\langle \eta \rangle^{2q}} \, d\eta \Big) \, d\xi \\
&\le C_1(A, p, q) \Big(\int \langle \eta_1 \rangle^{m+q+2} |\hat{a}(t, \eta_1)| \, d\eta_1 \Big)^2 \sum_n A^n 3^{2n} \int \frac{|\hat{u}_n(\eta)|^2}{\langle \eta \rangle^{2q}} \, d\eta \\
&\le C_2(A, p, q) \int \Big(\sum_n A^{n/2} 3^n \frac{|\hat{u}_n(\eta)|}{\langle \eta \rangle^q} \Big)^2 \, d\eta \\
&\le C_2(A, p, q) \int \big| \langle \eta \rangle^{3\sqrt{A} - q} |\hat{u}(\eta)| \big|^2 d\eta \\
&\le C_2(A, p, q) \int |\hat{u}(\eta)|^2 \, d\eta \le C_3(A, p, q) \int k_0(t, x) |u_0|^2 \, dx.
\end{aligned}
$$

Here we choose first A large and then q so that $q > 3\sqrt{A}$.

The second term is bounded by

$$
\begin{aligned}
&C_4(A, p, q) \sum_n A^n 3^{2n} \Big(\int \langle \eta_1 \rangle^{m+q+2} \frac{\log^n \langle \eta_1 \rangle}{n!} |\hat{a}(t, \eta_1)| \, d\eta_1 \Big)^2 \int \frac{|\hat{u}(\eta)|^2}{\langle \eta \rangle^{2q}} \, d\eta \\
&\le C_5(A, p, q) \Big(\sum_n A^{n/2} 3^n \int \langle \eta_1 \rangle^{m+q+2} \frac{\log^n \langle \eta_1 \rangle}{n!} |\hat{a}(t, \eta_1)| \, d\eta_1 \Big)^2 \int |\hat{u}(\eta)|^2 d\eta \\
&\le C_6(A, p, q) \Big(\int \langle \eta_1 \rangle^{m+q+2+3\sqrt{A}} |\hat{a}(t, \eta_1)| \, d\eta_1 \Big)^2 \int |\hat{u}(\eta)|^2 \, d\eta \\
&\le C_7(A, p, q) \int |\hat{u}(\eta)|^2 \, d\eta.
\end{aligned}
$$

The last term can be estimated similarly and so we end the proof of Lemma 4.2. □

From (4.1), (4.2), (4.5), Lemma 4.1 and Lemma 4.2 it follows that

$$
\sum_{n,\beta,s,j} \{ I_3(u_{n,\beta,s,j}) + I_5(u_{n,\beta,s,j}) \} \le C \sum_{n,\beta,s,j} n E_n(u_{\beta,s,j}) + [f(t)]^2 \tag{4.11}
$$

where

$$
[f(t)]^2 = e^{-ct} \sum_{n,\beta,s,j} A^n(t) \int k_n(t, x) \Big| \frac{\log^n \langle D \rangle}{n!} \frac{D^{\beta+j}}{\langle D \rangle^{s+j}} f(t, x) 2^{-n\beta} \Big|^2 \, dx.
$$

5. Proof of Theorem 1.1

Summing up the estimates (3.2), (3.6) and (4.11) we have that

$$\frac{d}{dt}\mathcal{E}(t,u) \le [f(t)]^2$$

and hence

$$\mathcal{E}(t,u) \le \mathcal{E}(t_0,u) + \int_{t_0}^{t} [f(s)]^2\,ds \tag{5.1}$$

for $-T \le t_0 \le t \le T$. Let us denote by $\|u\|_r$ the standard norm in the Sobolev space $H^r(\mathbb{R})$. Then we have

Proposition 5.1. *There is $r_1 \in \mathbb{N}$ such that for any $r_2 \in \mathbb{R}$ we can find C such that*

$$\|u_t(t)\|_{r_2}^2 + \|u(t)\|_{r_2}^2 \le C\left(\|u_t(t_0)\|_{r_1+r_2}^2 + \|u(t_0)\|_{r_1+r_2+1}^2 + \int_{t_0}^{t} \|f(s,\,\cdot\,)\|_{r_1+r_2}^2\,ds\right)$$

for any $-T \le t_0 \le t \le T$ and for $u \in C^2([-T,T];\mathcal{S}(\mathbb{R}))$ verifying $Pu = f$.

Proof. It is clear that

$$[u(t)]^2 \ge e^{-ct}c_0 \int |u(t,x)|^2\,dx = c_0 e^{-ct}\|u\|^2$$

because $k_0(t,x) \ge c_0 > 0$ by 1) of Proposition 2.1 (the notation $[\,\cdot\,]$ is defined at the end of last section). This together with (5.1) shows that

$$\|u_t(t)\|^2 + \|u(t)\|^2 \le C\left(\mathcal{E}(t_0,u) + \int_{t_0}^{t} [f(s)]^2\,ds\right). \tag{5.2}$$

On the other hand we see that

$$\begin{aligned}
[u(t)]^2 &\le 2e^{-ct}\sum_{n,\beta,s} A^n(t)\|u_n\|_{\beta-s}^2 \le C_1 e^{-ct}\sum_{n} A^n(t)\|u_n\|_p^2\\
&\le C_1 e^{-ct}\int \langle\xi\rangle^{2p}|\hat{u}|^2\left(\sum_n A(t)^{n/2}\frac{\log^n\langle\xi\rangle}{n!}\right)^2 d\xi\\
&\le C_1 e^{-ct}\int \langle\xi\rangle^{2p+2\sqrt{A(t)}}|\hat{u}|^2\,d\xi \le e^{-ct}\|u\|_{r_1}^2
\end{aligned}$$

with $r_1 = p + \sqrt{2A(0)}$ because we can suppose $A(t) \le 2A(0)$ for $-T \le t \le T$. Similarly, we have that

$$\begin{aligned} &e^{-ct} \sum_{n,\beta,s,j} A^n(t) \int k_n(t,x)a(t,x) \left| \frac{D^{\beta+j}}{\langle D\rangle^{s+j}} \, \partial_x u_n(t,x) 2^{-n\beta} \right|^2 dx \\ &\le 2e^{-ct} \sum_{n,\beta,s} A^n(t) \|u_n\|^2_{\beta-s+1} \le C_2 e^{-ct} \sum_n A^n(t) \|u_n\|^2_{p+1} \\ &\le C_2 e^{-ct} \|u\|^2_{r_1+1}. \end{aligned}$$

Taking (5.1) and (5.2) into account we get that

$$\|u_t(t)\|^2 + \|u(t)\|^2 \le C_3 \left(\|u_t(t_0)\|^2_{r_1} + \|u(t_0)\|^2_{r_1+1} + \int_{t_0}^t \|f(s)\|^2_{r_1} \, ds \right). \tag{5.3}$$

Repeating the same arguments as in Sections 3 and 4 for

$$u_{n,\beta,\gamma,s,j} = 2^{-n\beta} \frac{\log^n \langle D\rangle}{n!} \frac{D^{\beta+\gamma+j}}{\langle D\rangle^{s+j}} u$$

with $\gamma = 0, 1, \ldots, r_2$, we obtain the desired result. □

Proposition 5.2. *There is $r_1 \in \mathbb{N}$ such that for any $r_2 \in \mathbb{R}$ one can find C such that*

$$\|u_t(t)\|^2_{r_2} + \|u(t)\|^2_{r_2} \le C \left(\|u_t(t_0)\|^2_{r_1+r_2} + \|u(t_0)\|^2_{r_1+r_2+1} + \int_{t_0}^t \|f(s,\cdot)\|^2_{r_1+r_2} \, ds \right)$$

for any $-T \le t_0 \le t \le T$ and for any $u \in C^2([-T,T]; \mathcal{S}(\mathbb{R}))$ satisfying

$$P^* u = \partial_t^2 u - a(t,x)\, \partial_x^2 u - 2a_x(t,x)\, \partial_x u - a_{xx}(t,x) u = f.$$

Proof. To check the proposition it suffices to estimate

$$F(u_n) = 2e^{-ct} A^n(t) \int k_n(t,x) \, \mathrm{Re} \left[\frac{\log^n \langle D\rangle}{n!} (2a_x \, \partial_x u + a_{xx} u) \cdot \bar{u}_{n,t} \right] dx. \tag{5.4}$$

Since

$$\begin{aligned} &\frac{\log^n \langle D\rangle}{n!} (2a_x \, \partial_x u + a_{xx} u) \\ &= 2a_x \partial_x u_n + a_{xx} u_n + 2 \left[\frac{\log^n \langle D\rangle}{n!}, a_x \right] \partial_x u + \left[\frac{\log^n \langle D\rangle}{n!}, a_{xx} \right] u \end{aligned}$$

repeating the same arguments as in Section 4 we get that

$$\sum_{n,\beta,s,j} F(u_{n,\beta,s,j}) \le C \sum_{n,\beta,s,j} E_n(u_{\beta,s,j}):$$

this proves the desired assertion. □

By Propositions 5.1 and 5.2, we can apply standard arguments of functional analysis to conclude Theorem 1.1 (see, for example, Section 23.2 in [6]).

To check Theorem 1.1′ we first note that if $k_{jn}(t, x)$, $n \in \mathbb{N}$ are weight functions for $B_j(t, x) \geq 0$ verifying Proposition 2.1 then

$$k_n(t, x) = \prod_{j=1}^{r} k_{jn}(t, x), \quad n \in \mathbb{N}$$

are weight functions for $\prod_{j=1}^{r} B_j(t, x)$ verifying Proposition 2.1. Thus to show Theorem 1.1′ we can assume that $r = 1$. Write $m = m_1$ and $B_1(t, x) = b(t, x)^m$. Note that if m is odd and hence $b(t, x) \geq 0$ near the origin then the proof is obvious because the weight functions for $b(t, x)$ given in Proposition 2.1 are also weight functions for $b(t, x)^m$. Let m be even and hence $b(t, x)^m = [b(t, x)^2]^{m/2}$. Repeating the same arguments as in Sections 6 and 7 with minor changes such as

$$k_{m,t_0(x_0)}(t, x) = \exp\left[N \int_{I_m(x)\cap[t_0(x_0),t]} \frac{|b_t(s, x)|}{|b(s, x)|}\, ds\right]$$

for $t > t_0(x_0)$ and $k_{m,t_0(x_0)}(t, x) = 1$ if $t \leq t_0(x_0)$ with $I_m(x) = \{s \mid 2^{-m} \leq |b(t, x)| \leq 2^{-m+2}\}$ we obtain the required weight functions for $b(t, x)^2$ which is also the required weight functions for $[b(t, x)^2]^{m/2}$.

6. Construction of the weight functions

To prove Proposition 2.1 it turns out that the notation is simpler if we construct the reciprocal functions $1/k_n(t, x)$; we will denote them again by k_n and list in the proposition below the analogous properties that they should enjoy.

Proposition 6.1. *Let $N > 0$ be a given constant. Then there is $T > 0$, a sequence of weight functions $k_n(t,x) \in W^{1,\infty}((-T,T)\times\mathbb{R})$ and some positive constants $C_1,\dots,C_8$ (all depending on N except C_6) such that*

1) $1 \leq k_n(t, x) \leq C_1 e^{C_2 n}$,
2) $0 \leq \partial_t k_n(t, x) \leq C_3 e^{C_4 n}$,
3) *in a neighbourhood of the origin we have*

$$|\partial_x k_n(t, x)| \sqrt{a(t, x)} \leq C_5 n k_n(t, x),$$

4) *in a neighbourhood of the origin we have*

$$\frac{\partial_t k_n(t, x)}{k_n(t, x)} \geq \frac{N}{C_6} \frac{|a_t(t, x)|}{a(t, x) + 2^{-2n}} - C_7 n,$$

5) $k_{n-1} \leq C_8 k_n$.

Proof. The proof is fairly long: we need several steps and we will finish it in the last section. Recall that one can write

$$a(t, x) = e(t, x)(t^p + a_1(x)t^{p-1} + \cdots + a_p(x))$$

in a neighbourhood U of the origin and that, changing the scale of the t coordinate if necessary and using Glaeser's inequality, we may assume that, in U, $0 \leq a(t,x) \leq 1$ and

$$|\partial_x \sqrt{a(t, x)}| \leq L = \frac{1}{320(p+1)}.$$

Let ϵ be a positive number. Since the functions

$$a(t, x) - \epsilon, \quad a(t, x) - 16\epsilon$$

are regular in t, we can write also them as a non-zero function multiplied by a Weierstrass polynomial in a neighbourhood of (0, 0). Let $\Delta_1(x, \epsilon)$ be the discriminant of $a(t, x) - \epsilon$ and $\Delta_2(x, \epsilon)$ the discriminant of $a(t, x) - 16\epsilon$. We observe that up to maybe changing T the equations $a(t, x) - \epsilon = 0$, $a(t, x) - 16\epsilon = 0$, $t + T = 0$ and $t - T = 0$ have mutually distinct solutions in t for small x and $\epsilon > 0$.

Let $\Delta(x, \epsilon) = \Delta_1(x, \epsilon)\Delta_2(x, \epsilon)$; since $\Delta(x, 0)$ vanishes of order $2q$ at $x = 0$ by hypothesis (1.4) we can write, for d sufficiently small,

$$\Delta(x, \epsilon) = c(x, \epsilon)(x^{2q} + c_1(\epsilon)x^{2q-1} + \cdots + c_{2q}(\epsilon))$$

for $|x| < d$ and $|\epsilon| < \epsilon_0$. For $\epsilon > 0$ fixed ($\epsilon < \epsilon_0$), $\Delta(\cdot, \epsilon)$ has at most $2q$ real zeros for $|x| < d$:

$$x_1(\epsilon) \leq x_2(\epsilon) \leq \cdots \leq x_{q_1-1}(\epsilon)$$

where $q_1 - 1$ is the number of real zeros, in x, of $\Delta(x, \epsilon)$ and depends on ϵ. Taking $\epsilon_0 > 0$ and $\delta > 0$ ($\delta \ll d$) small we may assume that $-d + \delta < x_1(\epsilon)$ and $x_{q_1-1}(\epsilon) < d - \delta$ for $|\epsilon| < \epsilon_0$.

Let us call J_δ the interval $(-d + \delta, d - \delta)$; we can assume that $U = [-T, T] \times J_\delta$.

We now divide the interval J_δ into q_1 subintervals $A_j(\epsilon) = (x_{j-1}(\epsilon), x_j(\epsilon))$, $j = 1, \ldots, q_1$, where $x_0(\epsilon) = -d + \delta$, $x_{q_1}(\epsilon) = d - \delta$. For $x \in A_j(\epsilon)$ we can define p_j real functions

$$-T = t_{j1}(x, \epsilon) < \cdots < t_{jp_j}(x, \epsilon) = T$$

which are the roots in t of

$$(a(t, x) - \epsilon)(a(t, x) - 16\epsilon)(t + T)(t - T)$$

contained in the interval $[-T, T]$ and are continuous in $x \in A_j(\epsilon)$. In general p_j depends on j and ϵ; nevertheless, we always have $2 \le p_j \le 2p + 2$. We will at times make the dependence on ϵ implicit to simplify the notation.

Let us fix an integer m and put $\epsilon = 2^{-2m}$. We suppose that $2^{-2m} < \epsilon_0$, that is $m > m_0$; later we will deal with the case $m \le m_0$. We choose one $A_j(2^{-2m})$ and one of the functions $t_{jl}(x, 2^{-2m})$ defined on it and denote it by $t_0(x, 2^{-2m})$ (or $t_0(x)$) for the time being, to avoid clumsiness (we will need to revert to the usual notation from Lemma 6.2 on). Note that either $t_0(x, 2^{-2m}) = \pm T$, or $a(t_0(x, 2^{-2m}), x) = 2^{-2m}$ or $a(t_0(x, 2^{-2m}), x) = 2^{-2m+4}$ in $A_j(2^{-2m})$. Define $b_{t_0}(t, x)$ by

$$b_{t_0}(t, x) = \sqrt{a(t_0(x), x)}$$

if $t \le t_0(x)$ and

$$b_{t_0}(t, x) = \sqrt{a(t_0(x), x)} + \int_{t_0(x)}^{t} |\partial_s \sqrt{a(s, x)}|\, ds$$

if $t > t_0(x)$. Note that $b_{t_0}(t, x)$ is nondecreasing in t and $b_{t_0}(t, x) \ge \sqrt{a(t, x)}$ for $t > t_0(x)$. Define

$$Q_h = (h2^{-m} - 2^{-m-1}, h2^{-m} + 2^{-m-1})$$

for $h \in \mathbb{Z}$. We choose $x_h \in Q_h \cap A_j(2^{-2m})$ (if this set is not empty) and set $x'_h = x_h + 2^{-m}$. For m large, $2^{-m} < \delta$ and $x_h \in A_j(2^{-2m})$ implies $x'_h \in (-d, d)$ (here x_h and x'_h depend on j).

Let us put

$$\phi_{h,t_0}(t, x) = \left(\left(4 - \frac{|x - x_h|}{b_{t_0}(t, x_h)}\right) \vee 0\right) \wedge 1$$

and define

$$k_{m,t_0(x_0)}(t, x) = \exp\left[N \int_{I_m(x)\cap[t_0(x_0),t]} \frac{|a_t(s, x)|}{a(s, x)}\, ds\right] \tag{6.1}$$

if $t > t_0(x_0)$ and $k_{m,t_0(x_0)}(t, x) = 1$ if $t \le t_0(x_0)$. Here N is a positive number, $x_0 \in A_j(2^{-2m})$ and

$$I_m(x) = \{s \mid 2^{-2m} \le a(s, x) \le 2^{-2m+4}\}.$$

We now set

$$\tilde{k}_{m,t_0}(t, x) = \sup_h [k_{m,t_0(x_h)}(t, x_h) k_{m,t_0(x_h)}(t, x'_h) \phi_{h,t_0}(t, x)] \vee 1$$

where the supremum is taken over all h such that $Q_h \cap A_j(2^{-2m}) \ne \emptyset$ (therefore it is indeed a maximum over a finite set). Products of functions $\tilde{k}_{m,t_0}(t, x)$ as t_0 varies among all the possible choices will be factors in the desired weight function $k_n(t, x)$.

Lemma 6.1. *We have*
1) $1 \le \tilde{k}_{m,t_0}(t,x) \le \exp[2N(p+1)\log 2^4]$,
2) $\partial_t \tilde{k}_{m,t_0}(t,x) \ge 0$,
3) $\partial_t \tilde{k}_{m,t_0}(t,x) \le C_9 2^m \tilde{k}_{m,t_0}(t,x)$,
4) $|\partial_x \tilde{k}_{m,t_0}(t,x)|\sqrt{a(t,x)} \le 2\exp[2N(p+1)\log 2^4]\tilde{k}_{m,t_0}(t,x)$.

Proof. Since $a(t,x)$ is a polynomial in t of degree p, 1) is easily checked. From

$$\partial_t k_{m,t_0(x_h)}(t,x_h) \ge 0,\ \partial_t k_{m,t_0(x_h)}(t,x_h') \ge 0,\ \partial_t \phi_{h,t_0}(t,x) \ge 0 \tag{6.2}$$

it follows that $\partial_t \tilde{k}_{m,t_0}(t,x) \ge 0$.

To prove 3) note that

$$\partial_t k_{m,t_0(x_h)}(t,x_h) \le N\frac{|a_t|}{a} k_{m,t_0(x_h)}(t,x_h) \le NC2^m k_{m,t_0(x_h)}(t,x_h),$$
$$\partial_t k_{m,t_0(x_h)}(t,x_h') \le N\frac{|a_t|}{a} k_{m,t_0(x_h)}(t,x_h') \le NC2^m k_{m,t_0(x_h)}(t,x_h'),$$
$$\partial_t \phi_{h,t_0} \le \frac{|x-x_h|}{b_{t_0}(t,x_h)}\frac{|\partial_t b_{t_0}(t,x_h)|}{b_{t_0}(t,x_h)} \le 4\frac{C}{2^{-m}} = 4C2^m.$$

Thus we see that

$$\begin{aligned}
&\partial_t[k_{m,t_0(x_h)}(t,x_h)k_{m,t_0(x_h)}(t,x_h')\phi_{h,t_0}(t,x)]\\
&\le 2NC2^m[k_{m,t_0(x_h)}(t,x_h)k_{m,t_0(x_h)}(t,x_h')\phi_{h,t_0}(t,x)]\\
&\quad + 4C2^m \exp[2N(p+1)\log 2^4]\\
&\le \{2NC2^m + 4C2^m \exp[2N(p+1)\log 2^4]\}\tilde{k}_{m,t_0}(t,x)
\end{aligned}$$

which shows that

$$\partial_t \tilde{k}_{m,t_0}(t,x) \le C_9 2^m \tilde{k}_{m,t_0}(t,x).$$

We turn to assertion 4). If $\tilde{k}_{m,t_0}(t,x) = 1$ then $\partial_x \tilde{k}_{m,t_0} = 0$ and hence the assertion clearly holds. If $\tilde{k}_{m,t_0}(t,x) > 1$, let the supremum in the definition of $\tilde{k}_{m,t_0}$ be attained for a certain index $\bar{h}$. Then it is clear that we have $t > t_0(x_{\bar{h}})$ and $\phi_{\bar{h},t_0}(t,x) > 0$. Thus $|x - x_{\bar{h}}| \le 4b_{t_0}(t,x_{\bar{h}})$, so that

$$|\sqrt{a(t,x)} - \sqrt{a(t,x_{\bar{h}})}| \le \frac{1}{4}|x - x_{\bar{h}}| \le b_{t_0}(t,x_{\bar{h}})$$

and hence

$$\sqrt{a(t,x)} \le \sqrt{a(t,x_{\bar{h}})} + b_{t_0}(t,x_{\bar{h}}) \le 2b_{t_0}(t,x_{\bar{h}})$$

because $b_{t_0}(t, x) \geq \sqrt{a(t, x)}$ for $t > t_0(x)$. Now we have that

$$|\partial_x \phi_{\bar{h},t_0}(t, x)| \sqrt{a(t, x)} \leq \frac{\sqrt{a(t, x)}}{b_{t_0}(t, x_{\bar{h}})} \leq 2$$

so that

$$|\partial_x \tilde{k}_{m,t_0}(t, x)| \sqrt{a(t, x)} \leq 2 \exp[2N(p+1) \log 2^4] \leq 2 \exp[2N(p+1) \log 2^4] \tilde{k}_{m,t_0}(t, x)$$

and hence 4). □

Lemma 6.2. *Let $(t, x) \in U$ be a point such that $x \in A_j(2^{-2m})$, $t_{jl}(x, 2^{-2m}) < t < t_{jl+1}(x, 2^{-2m})$ and $2^{-2m+1} \leq a(t, x) \leq 2^{-2m+3}$. If*

$$\tilde{k}_{m,t_{jl}}(t, x) = \left[k_{m,t_{jl}(x_{\bar{h}})}(t, x_{\bar{h}}) \cdot k_{m,t_{jl}(x_{\bar{h}})}\left(t, x'_{\bar{h}}\right) \cdot \phi_{\bar{h},t_{jl}}(t, x)\right]$$

(that is, the supremum in the definition of $\tilde{k}_{m,t_{jl}}$ is attained at index $\bar{h}$), then $|x - x_{\bar{h}}| \leq 160(p+1)/9 \cdot 2^{-m}$.

Proof. We consider the interval Q_i that contains x. Let $x_i \in Q_i \cap A_j(2^{-2m})$: $|x - x_i| \leq 2^{-m}$ and $x'_i = x_i + 2^{-m}$ (it may happen that $x'_i \notin A_j(2^{-2m})$). For y between x and x_i we have $|\sqrt{a(t, y)} - \sqrt{a(t, x)}| \leq 2^{-m-2}$ so that

$$2^{-2m} < a(t, y) < 2^{-2m+4}$$

and $t_{jl}(y, 2^{-2m}) < t < t_{jl+1}(y, 2^{-2m})$. So we see that

$$2^{-2m} < a(t, x_i) < 2^{-2m+4}. \tag{6.3}$$

Suppose $k_{m,t_{jl}(x_i)}(t, x_i) = 1$: it follows that $a_t(s, x_i) = 0$ for all s such that $t_{jl}(x_i, 2^{-2m}) < s < t$, so that

$$a(t, x_i) = a(t_{jl}(x_i), x_i) = 2^{-2m} \quad \text{or} \quad 2^{-2m+4}$$

which contradicts (6.3). Thus we have $k_{m,t_{jl}(x_i)}(t, x_i) > 1$ and hence also

$$k_{m,t_{jl}(x_i)}(t, x_i) k_{m,t_{jl}(x_i)}(t, x'_i) > 1.$$

Since

$$\phi_{i,t_{jl}}(t, x) \geq \left(\left(4 - \frac{2^{-m}}{b_{t_{jl}}(t, x_i)}\right) \vee 0\right) \wedge 1 = 1$$

because $b_{t_{jl}}(t, x_i) \geq \sqrt{a(t_{jl}(x_i), x_i)} \geq 2^{-m}$, we see that

$$\tilde{k}_{m,t_{jl}}(t, x) = \sup_h [k_{m,t_{jl}(x_h)}(t, x_h) k_{m,t_{jl}(x_h)}(t, x'_h) \phi_{h,t_{jl}}(t, x)] > 1.$$

Assume now that when the index is $\bar{h}$ the supremum is attained. Then

$$|x - x_{\bar{h}}| \le 4b_{t_{jl}}(t, x_{\bar{h}})$$

and $t > t_{jl}(x_{\bar{h}})$ (since $k_{m,t_{jl}(x_{\bar{h}})}(t, x_{\bar{h}})k_{m,t_{jl}(x_{\bar{h}})}\big(t, x'_{\bar{h}}\big) > 1$). Consider the smallest value $\bar{t}$ such that

$$\sqrt{a(\bar{t}, x_{\bar{h}})} = \sup_{t_{jl}(x_{\bar{h}}) \le r \le t} \sqrt{a(r, x_{\bar{h}})};$$

noting that $b_{t_{jl}}(t, x_{\bar{h}})$ is nondecreasing in t, it is easy to see that

$$\sqrt{a(\bar{t}, x_{\bar{h}})} \le b_{t_{jl}}(t, x_{\bar{h}}) \le (p+1)\sqrt{a(\bar{t}, x_{\bar{h}})}.$$

We first consider the case in which $t_{jl}(x) < \bar{t}$ $(\le t < t_{jl+1}(x))$. We observe that

$$\sqrt{a(\bar{t}, x)} = \alpha 2^{-m}$$

with α between 1 and 4; then

$$\begin{aligned}\left|\sqrt{a(\bar{t}, x_{\bar{h}})} - \alpha 2^{-m}\right| &\le L|x - x_{\bar{h}}| \le 4Lb_{t_{jl}}(t, x_{\bar{h}}) \\ &\le 4L(p+1)\sqrt{a(\bar{t}, x_{\bar{h}})} \le \frac{1}{10}\sqrt{a(\bar{t}, x_{\bar{h}})}.\end{aligned}$$

We obtain that $(10/11)\alpha 2^{-m} \le \sqrt{a(\bar{t}, x_{\bar{h}})} \le (10/9)\alpha 2^{-m}$ and hence that

$$|x - x_{\bar{h}}| \le 4(p+1)\frac{10}{9}\alpha 2^{-m}.$$

We consider now the other case, i.e. when $t_{jl}(x) \ge \bar{t}$. Since $t_{jl}(x_{\bar{h}}) \le \bar{t}$ and $t_{jl}(x) \ge \bar{t}$, there exists some ξ between x and $x_{\bar{h}}$ such that $t_{jl}(\xi) = \bar{t}$ and hence

$$\sqrt{a(\bar{t}, \xi)} = 2^{-m} \quad \text{or} \quad \sqrt{a(\bar{t}, \xi)} = 2^{-m+2}.$$

Noting that

$$\begin{aligned}\left|\sqrt{a(\bar{t}, x_{\bar{h}})} - \sqrt{a(\bar{t}, \xi)}\right| &\le L|\xi - x_{\bar{h}}| \le 4Lb_{t_{jl}}(t, x_{\bar{h}}) \\ &\le 4L(p+1)\sqrt{a(\bar{t}, x_{\bar{h}})} \le \frac{1}{10}\sqrt{a(\bar{t}, x_{\bar{h}})}\end{aligned}$$

we conclude as before that

$$\frac{10}{11}\alpha 2^{-m} \le \sqrt{a(\bar{t}, x_{\bar{h}})} \le \frac{10}{9}\alpha 2^{-m}, \quad |x - x_{\bar{h}}| \le 4(p+1)\frac{10}{9}\alpha 2^{-m}$$

where $\alpha = 1$ or 4. Thus we have $|x - x_{\bar{h}}| \le (160/9)\cdot(p+1)2^{-m}$ which ends the proof. $\square$

Lemma 6.3. *Let $(t,x)\in U$ be a point such that*

$$2^{-2m+1}\le a(t,x)\le 2^{-2m+3}:$$

there exist j and l such that

$$\partial_t\tilde{k}_{m,t_{jl}}\ge \frac{N}{C_{11}}\frac{|a_t(t,x)|}{a(t,x)}\tilde{k}_{m,t_{jl}}-C_{12}\tilde{k}_{m,t_{jl}}.$$

Proof. We choose j, l such that

$$x\in A_j(2^{-2m}),\quad t_{jl}(x,2^{-2m})<t<t_{jl+1}(x,2^{-2m}).$$

Applying Lemma 6.2 and keeping the same notations, we have that

$$|\sqrt{a(t,x_{\bar{h}})}-\sqrt{a(t,x)}|\le L|x_{\bar{h}}-x|\le \frac{1}{18}\cdot 2^{-m}$$

so that $2^{-2m}<a(t,x_{\bar{h}})<2^{-2m+4}$. The same inequality holds for $a\left(t,x'_{\bar{h}}\right)$. This shows that

$$t\in I_m(x_{\bar{h}})\cap I_m\left(x'_{\bar{h}}\right).$$

Then we have that

$$\partial_t\left[k_{m,t_{jl(x_{\bar{h}})}}(t,x_{\bar{h}})k_{m,t_{jl(x_{\bar{h}})}}\left(t,x'_{\bar{h}}\right)\right]\phi_{\bar{h},t_{jl}}(t,x)\ge N\left[\frac{|a_t(t,x_{\bar{h}})|}{a(t,x_{\bar{h}})}+\frac{\left|a_t\left(t,x'_{\bar{h}}\right)\right|}{a\left(t,x'_{\bar{h}}\right)}\right]\tilde{k}_{m,t_{jl}}(t,x).$$

Note that by Taylor's formula

$$\begin{aligned}a_t(t,x)&=a_t(t,x_{\bar{h}})+a_{tx}(t,x_{\bar{h}})(x-x_{\bar{h}})+R_2(x-x_{\bar{h}}),\\ a_t\left(t,x'_{\bar{h}}\right)&=a_t(t,x_{\bar{h}})+a_{tx}(t,x_{\bar{h}})2^{-m}+R_2(2^{-m})\end{aligned}$$

where R_2 is the remainder of second order, which proves that

$$\begin{aligned}|a_t(t,x)|&\le|a_t(t,x_{\bar{h}})|+\frac{160}{9}\cdot(p+1)\left(|a_t(t,x_{\bar{h}})|+\left|a_t\left(t,x'_{\bar{h}}\right)\right|\right)+C_{10}2^{-2m}\\ &\le\left(\frac{160}{9}\cdot(p+1)+1\right)|a_t(t,x_{\bar{h}})|+\frac{160}{9}\cdot(p+1)\left|a_t\left(t,x'_{\bar{h}}\right)\right|+C_{10}2^{-2m}.\end{aligned}$$

Thus one has that

$$\begin{aligned}\frac{|a_t(t,x)|}{a(t,x)}&\le\left(\frac{160}{9}\cdot(p+1)+1\right)\left(\frac{|a_t(t,x_{\bar{h}})|}{a(t,x)}+\frac{\left|a_t\left(t,x'_{\bar{h}}\right)\right|}{a(t,x)}\right)+C_{10}\\ &\le C_{11}\left(\frac{|a_t(t,x_{\bar{h}})|}{a(t,x_{\bar{h}})}+\frac{\left|a_t\left(t,x'_{\bar{h}}\right)\right|}{a\left(t,x'_{\bar{h}}\right)}\right)+C_{10}\end{aligned}$$

where $C_{11}=16((160/9)\cdot(p+1)+1)$. These prove that

$$\partial_t\tilde{k}_{m,t_{jl}}(t,x)\ge\frac{N}{C_{11}}\frac{|a_t(t,x)|}{a(t,x)}\tilde{k}_{m,t_{jl}}(t,x)-\frac{C_{10}}{C_{11}}N\tilde{k}_{m,t_{jl}}(t,x)$$

which is the desired assertion. □

7. Construction of the weight functions (continued)

We now construct the second kind of factor $\tilde{k}'_{n,t_0}(t,x)$ which appears in the weight functions $k_n(t,x)$. The construction is largely analogous to what was done above for factors of the first kind.

Let ϵ be a positive number. Since the function

$$a(t,x) - 16\epsilon$$

is regular in t, then we can write it as a non-zero function multiplied by a Weierstrass polynomial in a neighbourhood of $(0,0)$. Let $\Delta(x,\epsilon)$ be the discriminant. Since $\Delta(x,0)$ vanishes of order q at $x = 0$, from the assumption (1.4) we can write

$$\Delta(x,\epsilon) = c(x,\epsilon)(x^q + c_1(\epsilon)x^{q-1} + \cdots + c_q(\epsilon))$$

for $|x| < d$ and $|\epsilon| < \epsilon_0$. For $\epsilon > 0$ fixed ($\epsilon < \epsilon_0$), $\Delta(\,\cdot\,,\epsilon)$ has at most q real zeros for $|x| < d$;

$$x_1(\epsilon) \le x_2(\epsilon) \le \cdots \le x_{q_1-1}(\epsilon).$$

As in Section 6, we may assume that $-d+\delta < x_1(\epsilon)$, $x_{q_1-1}(\epsilon) < d-\delta$ for $|\epsilon| < \epsilon_0$. We divide the interval $J'_\delta = (-d+\delta, d-\delta)$ into q_1 subintervals $A'_j(\epsilon) = (x_{j-1}(\epsilon), x_j(\epsilon))$, where $x_0(\epsilon) = -d+\delta$, $x_{q_1}(\epsilon) = d-\delta$. For $x \in A'_j(\epsilon)$ we can define p_j real functions $(0 \le p_j \le p+2)$

$$-T = t_{j1}(x,\epsilon) < \cdots < t_{jp_j}(x,\epsilon) = T$$

which are the roots of

$$(a(t,x) - 16\epsilon)(t+T)(t-T) = 0$$

contained in the interval $[-T, T]$ and are continuous in $x \in A'_j(\epsilon)$.

Let us fix an integer n and put $\epsilon = 2^{-2n}$. Take $A'_j(2^{-2n})$ and call $t_0(x, 2^{-2n})$ one of the functions defined on it. Note that either $t_0 = \pm T$ or $a(t_0(x,2^{-2n}),x) = 2^{-2n+4}$ in $A'_j(2^{-2n})$. Define $b'_{t_0}(t,x)$ by

$$b'_{t_0}(t,x) = \sqrt{a(t_0(x),x)} + 2^{-n}$$

if $t > t_0(x)$ and

$$b'_{t_0}(t,x) = \sqrt{a(t_0(x),x)} + \int_{t_0(x)}^{t} |\partial_s \sqrt{a(s,x)}|\, ds + 2^{-n}$$

if $t > t_0(x)$. Note that $b'_{t_0}(t,x)$ is nondecreasing in t and $b'_{t_0}(t,x) \ge \sqrt{a(t,x)} + 2^{-n}$ for $t > t_0(x)$. We then define

$$Q_h = (h2^{-n} - 2^{-n-1}, h2^{-n} + 2^{-n-1})$$

for $h \in \mathbb{Z}$; we choose $x_h \in Q_h \cap A'_j(2^{-2n})$ (if this set is not empty) and set $x'_h = x_h + 2^{-n}$. For n large, $x_h \in A'_j(2^{-2n})$ implies $x'_h \in (-d, d)$. Put

$$\phi'_{h,t_0}(t, x) = \left(\left(4 - \frac{|x - x_h|}{b'_{t_0}(t, x_h)}\right) \vee 0\right) \wedge 1$$

and define (since $x_0 \in A'_j(2^{-2n})$) $k'_{n,t_0(x_0)}(t, x) = 1$ if $t \le t_0(x_0)$ and

$$k'_{n,t_0(x_0)}(t, x) = \exp\left[N \int_{I'_n(x)\cap[t_0(x_0),t]} \frac{|a_t(s, x)|}{2^{-2n}}\, ds\right]$$

if $t > t_0(x_0)$. Here N is the positive constant given in the definition (6.1) of $k_{m,t_0(x_0)}(t,x)$ and

$$I'_n(x) = \{s \mid a(s, x) \le 2^{-2n+4}\}.$$

We now define $\tilde{k}'_{n,t_0}(t, x)$ by

$$\tilde{k}'_{n,t_0}(t, x) = \sup_h [k'_{n,t_0(x_h)}(t, x_h) k'_{n,t_0(x_h)}(t, x'_h) \phi'_{h,t_0}(t, x)] \vee 1$$

where the supremum is taken over all h such that $Q_h \cap A'_j(2^{-2n}) \ne \emptyset$.

This $\tilde{k}'_{n,t_0}(t, x)$ enjoys analogous properties as $\tilde{k}_{m,t_0}(t, x)$ listed in Lemma 6.1.

Lemma 7.1. *We have*

1) $1 \le \tilde{k}'_{n,t_0}(t, x) \le \exp[2N(p + 1)2^4]$,
2) $\partial_t \tilde{k}'_{n,t_0}(t, x) \ge 0$,
3) $\partial_t \tilde{k}'_{n,t_0}(t, x) \le C_1 2^n \tilde{k}'_{n,t_0}(t, x)$,
4) $|\partial_x \tilde{k}'_{n,t_0}(t, x)| \sqrt{a(t, x)} \le 2 \exp[2N(p + 1)2^4] \tilde{k}'_{n,t_0}(t, x)$.

Proof. To check 2) it is enough to observe that

$$\partial_t k'_{n,t_0(x_h)}(t, x_h) \ge 0, \quad \partial_t k'_{n,t_0(x_h)}(t, x'_h) \ge 0, \quad \partial_t \phi'_{h,t_0}(t, x) \ge 0. \tag{7.1}$$

To see 3) note that

$$\partial_t k'_{n,t_0(x_h)}(t, x_h) \le N \frac{|a_t|}{2^{-2n}} k'_{n,t_0(x_h)}(t, x_h) \le N C_2 2^n k'_{n,t_0(x_h)}(t, x_h),$$
$$\partial_t k'_{n,t_0(x_h)}(t, x'_h) \le N \frac{|a_t|}{2^{-2n}} k'_{n,t_0(x_h)}(t, x'_h) \le N C_2 2^n k'_{n,t_0(x_h)}(t, x'_h).$$

On the other hand we have that

$$\partial_t \phi'_{h,t_0} \leq \frac{|x - x_h|}{b'_{t_0}(t, x_h)} \frac{|\partial_t b'_{t_0}(t, x_h)|}{b'_{t_0}(t, x_h)} \leq 4 \frac{C_3}{2^{-n}} = 4C_3 2^n$$

and hence that

$$\begin{aligned}
&\partial_t [k'_{n,t_0(x_h)}(t, x_h) k'_{n,t_0(x_h)}(t, x'_h) \phi'_{h,t_0}(t, x)] \\
&\leq 2NC_2 2^n [k'_{n,t_0(x_h)}(t, x_h) k'_{n,t_0(x_h)}(t, x'_h) \phi'_{h,t_0}(t, x)] \\
&\quad + 4C_3 2^n \exp[2N(p+1)2^4] \\
&\leq \{2NC_2 2^n + 4C_3 2^n \exp[2N(p+1)2^4]\} \tilde{k}'_{n,t_0}(t, x)
\end{aligned}$$

which implies that

$$\partial_t \tilde{k}'_{n,t_0}(t, x) \leq C_4 2^n \tilde{k}'_{n,t_0}(t, x).$$

We turn to the proof of 4). If $\tilde{k}'_{n,t_0}(t, x) = 1$ then $\partial_x \tilde{k}'_{n,t_0} = 0$ and nothing is to be proved. Assume that this is not the case. Let $\bar{h}$ be an index such that the supremum in the definition of $\tilde{k}'_{n,t_0}$ is attained for that index. We have $k'_{n,t_0(x_{\bar{h}})}(t, x_{\bar{h}}) k'_{n,t_0(x_{\bar{h}})}\big(t, x'_{\bar{h}}\big) \phi'_{\bar{h},t_0}(t, x) > 1$, $t > t_0(x_{\bar{h}})$ and $\phi'_{\bar{h},t_0}(t, x) > 0$. We have thus $|x - x_{\bar{h}}| \leq 4b'_{t_0}(t, x_{\bar{h}})$, so that

$$|\sqrt{a(t, x)} - \sqrt{a(t, x_{\bar{h}})}| \leq \frac{1}{4} |x - x_{\bar{h}}| \leq b'_{t_0}(t, x_{\bar{h}})$$

and hence

$$\sqrt{a(t, x)} \leq \sqrt{a(t, x_{\bar{h}})} + b'_{t_0}(t, x_{\bar{h}}) \leq 2b'_{t_0}(t, x_{\bar{h}}).$$

From this it follows that

$$|\partial_x \phi'_{\bar{h},t_0}(t, x)| \sqrt{a(t, x)} \leq \frac{\sqrt{a(t, x)}}{b'_{t_0}(t, x_{\bar{h}})} \leq 2$$

so that

$$|\partial_x \tilde{k}'_{n,t_0}(t, x)| \sqrt{a(t, x)} \leq 2 \exp[2N(p+1)2^4] \leq 2 \exp[2N(p+1)2^4] \tilde{k}'_{n,t_0}(t, x)$$

which shows 4). □

Lemma 7.2. *Let (t, x) be in $[-T, T] \times J'_\delta$ be a point such that $a(t, x) \leq 2^{-2n+3}$, $x \in A'_j(2^{-2n})$ and $t_{jl}(x, 2^{-2n}) < t < t_{jl+1}(x, 2^{-2n})$. If the supremum of*

$$k'_{n,t_{jl}(x_h)}(t, x_h) \cdot k'_{n,t_{jl}(x_h)}(t, x'_h) \cdot \phi_{h,t_{jl}}(t, x)$$

on the set of indices h such that $Q_h \cap A'_j(2^{-2n}) \neq \emptyset$ is attained for index $\bar{h}$, then $|x - x_{\bar{h}}| \leq (200(p+1)/9) \cdot 2^{-n}$.

Proof. We follow the proof of Lemma 6.2. We consider the interval Q_i that contains x. Let $x_i \in Q_i \cap A'_j(2^{-2n})$: $|x - x_i| \le 2^{-n}$ and $x'_i = x_i + 2^{-n}$ (x'_i may not belong to $A'_j(2^{-2n})$). For y between x and x_i we have $|\sqrt{a(t,y)} - \sqrt{a(t,x)}| \le 2^{-n-2}$ so that

$$a(t,y) < 2^{-2n+4}$$

and $t_{jl}(y, 2^{-2n}) < t < t_{jl+1}(y, 2^{-2n})$. So we see that

$$a(t,x_i) < 2^{-2n+4}.$$

If $k'_{n,t_{jl}(x_i)}(t,x_i) = 1$ it follows that $a_t(s,x_i) = 0$ for $t_{jl}(x_i, 2^{-2n}) < s < t$ so that

$$a(t,x_i) = a(t_{jl}(x_i), x_i) = 2^{-2n+4}$$

which is a contradiction. Thus we have that $k'_{n,t_{jl}(x_i)}(t,x_i) > 1$ and hence

$$k'_{n,t_{jl}(x_i)}(t,x_i) \cdot k'_{n,t_{jl}(x_i)}(t,x'_i) > 1.$$

Note that

$$\phi'_{i,t_{jl}}(t,x) \ge \left(\left(4 - \frac{2^{-n}}{b'_{t_{jl}}(t,x_i)}\right) \vee 0\right) \wedge 1 = 1$$

since $b'_{t_{jl}}(t,x_i) \ge 2^{-n}$. So we see that

$$\sup_h [k'_{n,t_{jl}(x_h)}(t,x_h) k'_{n,t_{jl}(x_h)}(t,x'_h) \phi'_{h,t_{jl}}(t,x)] > 1.$$

Suppose that the supremum is attained for a certain index $\bar{h}$. Then

$$|x - x_{\bar{h}}| \le 4 b'_{t_{jl}}(t, x_{\bar{h}})$$

and $t > t_{jl}(x_{\bar{h}})$ (since $k'_{n,t_{jl}(x_{\bar{h}})}(t,x_{\bar{h}}) k'_{n,t_{jl}(x_{\bar{h}})}\big(t,x'_{\bar{h}}\big) > 1$). Consider the first value $\bar{t}$ at which

$$\sqrt{a(\bar{t}, x_{\bar{h}})} = \sup_{t_{jl}(x_{\bar{h}}) \le r \le t} \sqrt{a(r, x_{\bar{h}})}$$

then we see as before that

$$\sqrt{a(\bar{t}, x_{\bar{h}})} + 2^{-n} \le b'_{t_{jl}}(t, x_{\bar{h}}) \le (p+1)\left(\sqrt{a(\bar{t}, x_{\bar{h}})} + 2^{-n}\right).$$

We first treat the case in which $t_{jl}(x) < \bar{t}$ ($\le t < t_{jl+1}(x)$). Note that

$$\sqrt{a(\bar{t}, x)} + 2^{-n} = \alpha 2^{-n}$$

with α between 1 and 5. Thus one has

$$\left|\sqrt{a(\bar t, x_{\bar h})} + 2^{-n} - \alpha 2^{-n}\right| \le L|x - x_{\bar h}| \le 4Lb'_{t_{jl}}(t, x_{\bar h})$$
$$\le 4L(p+1)\Big(\sqrt{a(\bar t, x_{\bar h})} + 2^{-n}\Big) \le \frac{1}{10}\Big(\sqrt{a(\bar t, x_{\bar h})} + 2^{-n}\Big).$$

Then $(10/11)\alpha 2^{-n} \le \sqrt{a(\bar t, x_{\bar h})} + 2^{-n} \le (10/9)\alpha 2^{-n}$ and hence

$$|x - x_{\bar h}| \le 4(p+1)\frac{10}{9}\alpha 2^{-n}.$$

We turn to the other case, i.e., if $t_{jl}(x) \ge \bar t$. Since $t_{jl}(x_{\bar h}) \le \bar t$ and $t_{jl}(x) \ge \bar t$ there exists ξ between x and $x_{\bar h}$ such that $t_{jl}(\xi) = \bar t$. That is

$$\sqrt{a(\bar t, \xi)} = 2^{-n+2}$$

and then

$$\left|\sqrt{a(\bar t, x_{\bar h})} + 2^{-n} - \sqrt{a(\bar t, \xi)} - 2^{-n}\right| \le L|\xi - x_{\bar h}| \le 4Lb'_{t_{jl}}(t, x_{\bar h})$$
$$\le 4L(p+1)\Big(\sqrt{a(\bar t, x_{\bar h})} + 2^{-n}\Big)$$
$$\le \frac{1}{10}\Big(\sqrt{a(\bar t, x_{\bar h})} + 2^{-n}\Big).$$

We conclude as before that

$$\frac{10}{11}\alpha 2^{-n} \le \sqrt{a(\bar t, x_{\bar h})} + 2^{-n} \le \frac{10}{9}\alpha 2^{-n}, \quad |x - x_{\bar h}| \le 4(p+1)\frac{10}{9}\alpha 2^{-n}$$

where $\alpha = 5$. This gives $|x - x_{\bar h}| \le (200/9)\cdot(p+1)2^{-n}$ and hence the assertion. □

Lemma 7.3. *Let* $(t, x) \in [-T, T] \times J'_\delta$ *with*

$$a(t, x) \le 2^{-2n+3}:$$

there exists j, l *such that*

$$\partial_t \tilde k'_{n,t_{jl}}(t, x) \ge \frac{N}{C_6}\frac{|a_t(t, x)|}{a(t, x) + 2^{-2n}}\tilde k'_{n,t_{jl}}(t, x) - C_7 \tilde k'_{n,t_{jl}}(t, x).$$

Proof. We choose j and l so that $x \in A'_j(2^{-2n})$ and $t_{jl}(x, 2^{-2n}) < t < t_{jl+1}(x, 2^{-2n})$. By Lemma 7.2 (using again $\bar h$ for a maximal index) we have that

$$|\sqrt{a(t, x_{\bar h})} - \sqrt{a(t, x)}| \le L|x_{\bar h} - x| \le \frac{5}{72}\cdot 2^{-n}$$

so that $a(t, x_{\bar{h}}) < 2^{-2n+4}$. We have the same inequality for $a\left(t, x'_{\bar{h}}\right)$ and hence

$$t \in I'_n(x_{\bar{h}}) \cap I'_n\left(x'_{\bar{h}}\right).$$

Therefore we have

$$\begin{aligned}
&\partial_t[k'_{n,t_{jl}(x_{\bar{h}})}(t, x_{\bar{h}})k'_{n,t_{jl}(x_{\bar{h}})}\left(t, x'_{\bar{h}}\right)]\phi'_{\bar{h},t_{jl}}(t, x) \\
&\geq N\left[\frac{|a_t(t, x_{\bar{h}})|}{2^{-2n}} + \frac{\left|a_t\left(t, x'_{\bar{h}}\right)\right|}{2^{-2n}}\right]\tilde{k}'_{m,t_{jl}}(t, x).
\end{aligned}$$

Note that again by Taylor's formula

$$\begin{aligned}
a_t(t, x) &= a_t(t, x_{\bar{h}}) + a_{tx}(t, x_{\bar{h}})(x - x_{\bar{h}}) + R_2(x - x_{\bar{h}}), \\
a_t\left(t, x'_{\bar{h}}\right) &= a_t(t, x_{\bar{h}}) + a_{tx}(t, x_{\bar{h}})2^{-n} + R_2(2^{-n}).
\end{aligned}$$

From this we get

$$\begin{aligned}
|a_t(t, x)| &\leq |a_t(t, x_{\bar{h}})| + \frac{200}{9} \cdot (p+1)\left(|a_t(t, x_{\bar{h}})| + \left|a_t\left(t, x'_{\bar{h}}\right)\right|\right) + C_5 2^{-2n} \\
&\leq \left(\frac{200}{9} \cdot (p+1) + 1\right)|a_t(t, x_{\bar{h}})| + \frac{200}{9} \cdot (p+1)\left|a_t\left(t, x'_{\bar{h}}\right)\right| + C_5 2^{-2n}
\end{aligned}$$

so that

$$\begin{aligned}
\frac{|a_t(t, x)|}{a(t, x) + 2^{-2n}} &\leq \left(\frac{200}{9} \cdot (p+1) + 1\right)\left(\frac{|a_t(t, x_{\bar{h}})|}{a(t, x) + 2^{-2n}} + \frac{\left|a_t\left(t, x'_{\bar{h}}\right)\right|}{a(t, x) + 2^{-2n}}\right) + C_5 \\
&\leq C_6\left(\frac{|a_t(t, x_{\bar{h}})|}{2^{-2n}} + \frac{\left|a_t\left(t, x'_{\bar{h}}\right)\right|}{2^{-2n}}\right) + C_5
\end{aligned}$$

where $C_6 = ((200/9) \cdot (p+1) + 1)$. Thus we conclude

$$\partial_t \tilde{k}'_{n,t_{jl}}(t, x) \geq \frac{N}{C_6} \frac{|a_t(t, x)|}{a(t, x) + 2^{-2n}} \tilde{k}'_{n,t_{jl}}(t, x) - \frac{C_5}{C_6} N \tilde{k}'_{n,t_l}(t, x)$$

and so Lemma 7.3 is proved. □

8. Proof of Proposition 6.1

Let $n \in \mathbb{N}$ be such that $n \geq m_0 + 1$. We set

$$\tilde{k}_m = \prod_{j,l} \tilde{k}_{m,t_{jl}}, \quad m = m_0, m_0 + 1, \ldots, n-1$$

and

$$\tilde{k}'_n = \prod_{j,l} \tilde{k}'_{n,t_{jl}}$$

where the product is taken over $j = 1, \dots, q_1$, $l = 0, 1, \dots, p_j$. For $0 \le m \le m_0 - 1$ we choose $\tilde{k}_m = 1$ and for $0 \le n \le m_0$ we also choose $\tilde{k}'_n = 1$. We finally define

$$k_n(t, x) = \tilde{k}_1 \cdot \tilde{k}_2 \cdot \dots \cdot \tilde{k}_{n-1} \cdot \tilde{k}'_n.$$

Then properties 1)–4) follow from Lemmas 6.1, 6.3, 7.1, 7.3. We now check 5). Since

$$\begin{aligned} k_{n-1} &= \tilde{k}_1 \tilde{k}_2 \cdots \tilde{k}_{n-2} \tilde{k}'_{n-1}, \\ k_n &= \tilde{k}_1 \tilde{k}_2 \cdots \tilde{k}_{n-1} \tilde{k}'_n \end{aligned}$$

hence

$$\frac{k_{n-1}}{k_n} = \frac{\tilde{k}'_{n-1}}{\tilde{k}_{n-1} \tilde{k}'_n}.$$

Here note that $\tilde{k}_{n-1} \ge 1$ since $\tilde{k}_{n-1} = \prod_{j,l} \tilde{k}_{m,t_{jl}}$ and $\tilde{k}_{m,t_{jl}}(t, x) \ge 1$ for any possible value of j and l. Similarly we have $\tilde{k}'_n \ge 1$. On the other hand we have that

$$\tilde{k}'_{n-1} = \prod_{j,l} \tilde{k}'_{m,t_{jl}} \le \exp[2N(2p+2)2^4(p+2)(q+1)]:$$

in fact there are at most $(p+2)(q+1)$ functions in the product. This indeed proves

$$\frac{k_{n-1}}{k_n} \le C. \qquad \square$$

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