

## ON WEIGHTED COMPLEX RANDERS METRICS

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### Abstract

In this paper we introduce the weighted complex Randers metric  $F = h + \sum_{i=1}^m |B_i|^{1/i}$  on a complex manifold  $M$ , here  $h$  is a Hermitian metric on  $M$  and  $B_i$ ,  $i = 1, \dots, m$  are holomorphic symmetric forms of weights  $i$  on  $M$ , respectively. These metrics are special case of jet metric studied in Chandler–Wong [6]. Our main theorem is that the holomorphic sectional curvature  $\text{hbsc}_F$  of  $F$  is always less or equal to  $\text{hbsc}_h$ . Using this result we obtain a rigidity result, that is, a compact complex manifold  $M$  of complex dimension  $n$  with a weighted complex Randers metric  $F$  of positive constant holomorphic sectional curvature is isomorphic to  $\mathbb{P}^n$ .

### 1. Introduction

The classical Randers metric in real Finsler geometry is a perturbation of a Riemannian metric by adding a small term  $\epsilon B$  where  $B = \beta_i(x) dx^i$  is a one form. Over a compact manifold this is a Finsler metric for  $\epsilon$  sufficiently small. Alternatively one may add the norm  $|B|$ , in which case the resulting metric is not smooth wherever  $B = 0$ , but the usual theory of Finsler geometry is applicable off this set. In the last few years physicists working in general relativity introduced a term, of the form  $|b_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l|^{1/4}$  to the Lorentz metric. This additional term is the norm of a symmetric form of weight 4. Using this it is possible to construct models of expanding universe without assuming the existence of dark energy (see Chang [7]).

Mathematically it makes sense to introduce terms of any order by adding the norm of symmetric forms of weight  $m$ . These Finsler metrics are said to be of weight  $m$ . Of interest is the curvature of such metrics and equations, such as the Einstein equations, associated to it. The computation turns out to be quite complicated and much work still need to be done.

The purpose of this article is to deal with complex case and assuming that the symmetric forms are holomorphic. In this setting, the weighted holomorphic Randers metrics are special case of jet metrics studied in Chandler–Wong [6] and, as it turns out the curvature is relatively easy to compute. We shall show below how the curvature can be computed using a result in Cao–Wong [5]. The result is then used to obtain a rigidity result (see Section 4). The case where the weight is one was already obtained

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by Chen–Shen [8] along the line of the computation in Aldea–Munteanu [3]. Their technique does not seem to work for general  $m$ .

**2. Cartan connection and Chern–Rund connection**

Let  $M$  be a complex manifold of dimension  $n$  and  $E$  a holomorphic vector bundle of rank  $r$  over  $M$ . Let  $(U, z)$  be a *local holomorphic trivialization* coordinate neighborhood of  $E$  where  $z = (z^1, \dots, z^i, \dots, z^n)$  is a local coordinate on  $U \subset M$ . Let  $e = (e_1, \dots, e_r)$  be a local holomorphic frame field of  $E$  over  $U$ . Every element in  $E|_U$  is of the form  $\zeta^\alpha e_\alpha$  (Einstein summation convention is adopted here). Thus  $\zeta = (\zeta^1, \dots, \zeta^\alpha, \dots, \zeta^r)$  is a fiber coordinate of  $E|_U$ .

- In the sequel, unless otherwise stated explicitly, local computations will always be carried out with respect to coordinate systems  $(z; \zeta)$  defined by local holomorphic trivialization as above.

DEFINITION 2.1. Let  $E$  be a holomorphic vector bundle of rank  $r$  over a complex manifold  $M$ . A real-valued function  $F: E \rightarrow [0, \infty)$  is said to be a Finsler metric along the fibers of  $E$  if

- (i)  $F(z, \zeta) = 0$  if and only if  $\zeta = 0$ ;
- (ii)  $F(z, t\zeta) = |t|F(z, \zeta)$  for all  $t \in \mathbb{C}_* = \mathbb{C} \setminus \{0\}$ .

A Finsler metric is said to be of class  $C^k$  (or simply, continuous if  $k = 0$  and smooth if  $k = \infty$ ) if  $G = F^2$  is of class  $C^k$  on  $E_* = E \setminus \{\text{zero-section}\}$  and that the restriction of  $G$  to each line through the origin of each fiber is also of class  $C^k$  even at the origin.

The projection map  $\pi: E \rightarrow M$  induces a surjective bundle map

$$\pi_*: TE \rightarrow TM$$

and a holomorphic bundle  $\pi^*E$  of rank  $r$  over  $E$  obtained by pulling back  $E$  over  $M$  via the projection  $\pi$ .

$$\pi^*: \pi^*E \rightarrow E.$$

Here  $TM$  and  $TE$  are the holomorphic tangent bundles; some authors use the notations  $T^{1,0}M$  and  $T^{1,0}E$ . The kernel of  $\pi_*$  shall be referred to as the vertical sub-bundle:

$$(2.1) \quad VTE = \ker \pi_* \subset TE.$$

By definition a local holomorphic frame for the vertical field is given by

$$(2.2) \quad \left\{ \frac{\partial}{\partial \zeta^\alpha}, \alpha = 1, 2, \dots, r = \text{rank } E \right\}.$$

The vertical bundle is a holomorphic vector subbundle of rank  $r$  of  $\rho: TE \rightarrow E$ . We shall at times write  $VE$  instead of  $VTE$ , and its restriction to  $E_*$  ( $= E \setminus \{\text{zero-section}\}$ ) as  $VE_*$ , in the sequel. The following lemma is well-known and is easily verified:

**Lemma 2.1.** *The bundles  $\pi^*: \pi^*E \rightarrow E$  and  $\rho|_{VTE}: VTE \rightarrow E$  are canonically holomorphically isomorphic.*

DEFINITION 2.2. A smooth (for the purpose of defining curvature later, class  $C^4$  is sufficient) Finsler metric  $F$  on a holomorphic vector bundle  $E$  is said to be strictly pseudoconvex if the Hermitian form induced by  $G = F^2$ :

$$h_F = ds_F^2 = G_{\alpha\bar{\beta}}(z, \zeta) d\zeta^\alpha \otimes d\bar{\zeta}^{\bar{\beta}}, \quad (z, \zeta) \in E_*$$

is positive definite along the fibers of the vertical bundle  $VE_*$  over  $E_*$ , where

$$G_{\alpha\bar{\beta}}(z, \zeta) = \partial_\alpha \partial_{\bar{\beta}} F^2(z, \zeta) = \frac{\partial^2 F^2}{\partial \zeta^\alpha \partial \bar{\zeta}^{\bar{\beta}}}(z, \zeta), \quad 1 \leq \alpha, \beta \leq r = \text{rank } E.$$

A strictly pseudoconvex Finsler metric  $F$  is only assumed to be smooth on  $E_* = E \setminus \{\text{zero-section}\}$  hence the induced Hermitian metric is only smooth on  $VE_*$ , the vertical subbundle of the tangent bundle  $TE_*$  of  $E_*$ . By Lemma 2.1 we may replace the bundle  $VE$  by  $\pi^*E$ . In the literature some authors use  $VE_*$  other use  $\pi^*E$ . We shall not make any distinction between the two in this article.

It is also well-known and easy to verify that

**Lemma 2.2.** *For a strictly pseudoconvex Finsler metric, we have*

$$G_{\alpha\bar{\beta}}(z, t\zeta) = G_{\alpha\bar{\beta}}(z, \zeta)$$

for all  $t \in \mathbb{C}_* = \mathbb{C} \setminus \{0\}$ . Thus the hermitian form  $h_F$  is well-defined on the projective bundle  $\mathbb{P}(E) = E_*/\mathbb{C}_*$ .

Denote by  $p: \mathbb{P}(E) \rightarrow E$  the canonical projection then the pull-back

$$(2.3) \quad p^*: p^*E \rightarrow \mathbb{P}(E)$$

is a holomorphic vector bundle. Given a strictly pseudoconvex Finsler metric on  $E$  the Hermitian metric  $h_F$  on  $\pi^*E|_{E_*}$  descends to a smooth Hermitian inner product on  $p^*E$  over  $\mathbb{P}(E)$ . The advantage of working on the projective bundle  $\mathbb{P}(E)$  over that of working on  $E$  being that the inner product is everywhere smooth. However it is customary also to work on  $E_*$  using coordinates on  $E$  which is a little less cumbersome than using the projective coordinates. In the complex case the terminologies (such as the Cartan connection, Chern–Rund connection) have not yet been standardized so we shall provide below, precisely, the definitions of these terms and these may differ from what appeared in the literature.

DEFINITION 2.3. Let  $E$  be a holomorphic vector bundle over a complex manifold  $M$  and  $F$  a strictly pseudoconvex Finsler metric. The Cartan connection and curvature is by definition the Hermitian connection and curvature of  $h_F$ .

Denote by

$$(2.4) \quad \vec{P}^v = \zeta^\alpha \frac{\partial}{\partial \zeta^\alpha}$$

be the vertical position vector field (also known as radial vector field or Liouville vector field). It is clear that  $\vec{P}^v$  is a holomorphic section of  $VE$ . Denote by  $\nabla$  the Cartan connection. Then

$$\nabla \vec{P}^v: TE_* \rightarrow V_{TE_*}, \quad X \mapsto \nabla_X \vec{P}$$

is defined. The horizontal sub-bundle  $HTE$  is by definition

$$(2.5) \quad HTE = \ker \vec{P}^v.$$

A local *smooth* frame for the horizontal bundle is given by:

$$(2.6) \quad \frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - G_{|i}^\alpha \frac{\partial}{\partial \zeta^\alpha}, \quad i = 1, \dots, n = \dim_{\mathbb{C}} M.$$

where

$$G_{|j}^\mu = G^{\bar{\beta}\mu} G_{\bar{\beta}|j}, \quad G_{\bar{\beta}|j} = \frac{\partial^2 G}{\partial \zeta^{\bar{\beta}} \partial z^j}.$$

- A note on the notation is in order: subscripts preceded by  $|$  indicates that it is a base index; those that are not preceded by  $|$  are fiber indices.

The vector fields in (2.2) and (2.6) form a local frame for  $TE$ . By duality,

$$\{dz^1, \dots, dz^n\}$$

is a local frame for horizontal 1-forms and

$$\{\delta \zeta^1, \dots, \delta \zeta^r\}, \quad \delta \zeta^\alpha = d\zeta^\alpha + G_{|i}^\alpha dz^i$$

is a local frame for vertical 1-forms. With these, the Cartan connection forms are decomposed into horizontal and vertical components

$$(2.7) \quad \theta_\alpha^\beta = \Gamma_{\alpha|i}^\beta dz^i + G_{\alpha\mu}^\beta \delta \zeta^\mu.$$

The Cartan curvature is decomposed into four components:

$$(2.8) \quad \Theta_\alpha^\beta = K_\alpha^\beta + P_\alpha^\beta + Q_\alpha^\beta + R_\alpha^\beta,$$

where

$$\begin{aligned} K_\alpha^\beta &= K_{\alpha i \bar{j}}^\beta dz^i \wedge d\bar{z}^{\bar{j}}, & (\mathfrak{h}, \bar{\mathfrak{h}})\text{-component,} \\ \Psi_\alpha^\beta &= P_{\alpha i \bar{v}}^\beta dz^i \wedge \delta\zeta^{\bar{v}}, & (\mathfrak{h}, \bar{\mathfrak{v}})\text{-component,} \\ Q_\alpha^\beta &= Q_{\alpha \mu \bar{j}}^\beta \delta\zeta^\mu \wedge d\bar{\zeta}^{\bar{j}}, & (\mathfrak{v}, \bar{\mathfrak{h}})\text{-component,} \\ R_\alpha^\beta &= R_{\alpha \mu \bar{v}}^\beta \delta\zeta^\mu \wedge \delta\zeta^{\bar{v}}, & (\mathfrak{v}, \bar{\mathfrak{v}})\text{-component.} \end{aligned}$$

The Cartan mixed holomorphic bisectional curvature is, by definition

$$(2.9) \quad k(X, \vec{P}) = \frac{\langle \Theta(X, \bar{X})\vec{P}, \vec{P} \rangle_{h_F}}{\|X\|_{h_F}^2 \|\vec{P}\|_{h_F}^2}$$

for any horizontal vector field  $X$ . It is well-known that

$$\langle P(X, \bar{X})\vec{P}, \vec{P} \rangle_{h_F} = \langle R(X, \bar{X})\vec{P}, \vec{P} \rangle_{h_F} = 0$$

hence

$$(2.10) \quad k(X, \vec{P}) = \frac{\langle K(X, \bar{X})\vec{P}, \vec{P} \rangle_{h_F}}{\|X\|_{h_F}^2 \|\vec{P}\|_{h_F}^2} = \frac{K_{\alpha \bar{\beta} k \bar{l}} X^k X^{\bar{l}} \zeta^\alpha \zeta^{\bar{\beta}}}{\|X\|_{h_F}^2 \|\vec{P}\|_{h_F}^2}.$$

We dropped the superscript and write  $\vec{P}$  for  $\vec{P}^v$ . Observe that

$$(2.11) \quad \|\vec{P}\|_{h_F}^2 = G_{\alpha \bar{\beta}} \zeta^\alpha \zeta^{\bar{\beta}} = G.$$

Consider the case  $E = TM$  is the holomorphic tangent bundle equipped with a strictly pseudoconvex Finsler metric. In this case the vertical tangent bundle and the horizontal tangent bundle are also canonically isomorphic. The isomorphism is determined via the identification

$$\frac{\delta}{\delta z^i} \longleftrightarrow \frac{\partial}{\partial \zeta^i}$$

and extends to the entire bundle by forcing complex linearity. This identification identifies the vertical position vector field  $\vec{P}^v$  with the horizontal vector vector field

$$(2.12) \quad \vec{P}^h = \zeta^i \frac{\delta}{\delta z^i}.$$

The metric  $h_F$  on the vertical bundle is also naturally identified as a metric on the horizontal bundle. We can of course also pull-back the Cartan connection and curvature to the horizontal bundle. The Cartan connection is decomposed as:

$$(2.13) \quad \theta_i^j = \Gamma_{i|k}^j dz^k + G_{i|k}^j \delta \zeta^k$$

where  $\Gamma_{i|k}^j = G_{i|k}^j - G_{|k}^p G_{i|p}^j$  with

$$G_{i|k}^j = G^{\bar{q}j} \frac{\partial^3 G}{\partial \zeta^{\bar{q}} \partial \zeta^i \partial z^k}, \quad G_{i|p}^j = G^{\bar{q}j} \frac{\partial^3 G}{\partial \zeta^{\bar{q}} \partial \zeta^i \partial \zeta^p}, \quad G_{|k}^p = G^{\bar{q}p} \frac{\partial^2 G}{\partial \zeta^{\bar{q}} \partial z^k}.$$

DEFINITION 2.4. Let  $M$  be a complex manifold and  $F$  be a strictly pseudoconvex Finsler metric on  $TM$ . Then  $F$  is said to be Finsler–Kähler if  $\tau^{\text{hor}}$  vanishes, that is the horizontal Christoffel symbols are symmetric, more precisely

$$\Gamma_{i|k}^j = \Gamma_{k|i}^j$$

for all  $i, j$  and  $k$ .

There is another concept known as the weakly Finsler–Kähler condition, which has its origin in the work of Royden [11] on the complex geodesics of the Kobayashi metric:

DEFINITION 2.5. Let  $M$  be a complex manifold and  $F$  be a strictly pseudoconvex Finsler metric on  $TM$ . Then  $F$  is said to be weakly Finsler–Kähler if

$$G^i \Gamma_{i|k}^j G_j = G^i \Gamma_{k|i}^j G_j$$

where  $G^i = G^{\bar{j}i} G_{\bar{j}}$  for all  $i, j$  and  $k$ .

The horizontal component of the Cartan connection:

$$(2.14) \quad \phi_i^j = \Gamma_{i|k}^j dz^k$$

is known as the *Chern–Rund connection* [12] of the horizontal subbundle. The Chern–Rund connection is torsion free if and only if the Finsler metric  $F$  is Finsler–Kähler. Indeed, following Chern [9] we define the (horizontal) torsion of  $\omega$  by:

$$\varpi_D = (D dz^i) \otimes \frac{\delta}{\delta z^i}.$$

It is clear from the definition of the Chern–Rund connection  $D$  that

$$\varpi_D = (-\Gamma_{j|k}^i dz^j \wedge dz^k) \otimes \frac{\delta}{\delta z^i}.$$

We remark in passing that the Cartan connection being torsion free is not equivalent to the metric  $F$  being Finsler–Kähler. The curvature  $\Omega$  of the Chern–Rund connection is given by

$$\Omega = d\omega - \omega \wedge \omega.$$

In contrast to the Cartan curvature, the Chern–Rund curvature:

$$(2.15) \quad \Omega = \bar{\partial}\omega + \partial\omega - \omega \wedge \omega$$

is not of bidegree  $(1, 1)$ . The reason being that the Cartan connection is a metric connection but the Chern–Rund connection is not. Note that in (2.14) the operator  $\partial$  and  $\bar{\partial}$  are operators on  $TTM$ , the tangent bundle of the tangent bundle of  $M$ . Analogous to the case of the Cartan curvature (see (2.8)),  $\Omega$  may be decomposed into four different components:

$$(2.16) \quad \Omega_i^j = (\Omega_i^j)^{h,\bar{h}} + (\Omega_i^j)^{h,\bar{v}} + (\Omega_i^j)^{\bar{h},v} + (\Omega_i^j)^{v,\bar{v}}$$

and that  $(\Omega_i^j)^{v,\bar{v}} = 0$ . It is known that the component  $(\Omega_i^j)^{h,\bar{h}}$  is of bidegree  $(1, 1)$ . In fact, from the definitions (2.13), (2.14) and (2.15) of the Chern–Rund curvature, one gets by a direct computation that  $\Omega$  consists of a component of bidegree  $(2, 0)$  and a component of bidegree  $(1, 1)$ . The  $(2, 0)$ -component is given by

$$\Omega^{2,0} = -\frac{\partial\Gamma_{i|k}^j}{\partial\zeta^l} dz^k \wedge \delta\zeta^l,$$

and the  $(1, 1)$ -component is

$$(2.17) \quad \begin{aligned} (\Omega^{1,1})_i^j &= (\Omega_i^j)^{h,\bar{h}} + (\Omega_i^j)^{h,\bar{v}}, \\ (\Omega_i^j)^{h,\bar{h}} &= -\frac{\delta\Gamma_{i|k}^j}{\delta z^{\bar{l}}} dz^k \wedge dz^{\bar{l}} = K_i^j + G_{i\mu}^j \frac{\delta G_{|k}^\mu}{\delta z^{\bar{l}}} dz^k \wedge dz^{\bar{l}}, \\ (\Omega_i^j)^{h,\bar{v}} &= -\frac{\partial\Gamma_{i|k}^j}{\partial\zeta^{\bar{l}}} dz^k \wedge \delta\zeta^{\bar{l}} = P_i^j + G_{i\mu}^j \frac{\partial G_{|k}^\mu}{\partial\zeta^{\bar{l}}} dz^k \wedge \delta\zeta^{\bar{l}} \end{aligned}$$

where  $K_i^j$  is the  $(h,\bar{h})$ -component and  $P_i^j$  is the  $(h,\bar{v})$ -component of the Cartan curvature.

The Cartan holomorphic bisectonal curvature is defined by

$$(2.18) \quad k_{\Theta}(X, \bar{P}^h) = \frac{2\langle\Theta(X, \bar{X})\bar{P}^h, \bar{P}^h\rangle_{h_F}}{\|X\|_{h_F}^2 \|\bar{P}^h\|_{h_F}^2} = \frac{2\langle K(X, \bar{X})\bar{P}^h, \bar{P}^h\rangle_{h_F}}{\|X\|_{h_F}^2 \|\bar{P}^h\|_{h_F}^2}$$

and the Chern–Rund, holomorphic bisectional curvature is defined by

$$(2.19) \quad k_{\Omega}(X, \vec{P}^h) = \frac{2\langle \Omega(X, \bar{X})\vec{P}^h, \vec{P}^h \rangle_{h_F}}{\|X\|_{h_F}^2 \|\vec{P}^h\|_{h_F}^2} = \frac{2\langle \Omega^{h,\bar{h}}(X, \bar{X})\vec{P}^h, \vec{P}^h \rangle_{h_F}}{\|X\|_{h_F}^2 \|\vec{P}^h\|_{h_F}^2}.$$

Note that (see (2.11))  $\|\vec{P}^h\|_{h_F}^2 = G = F^2$ . The holomorphic sectional curvature is defined by

$$(2.20) \quad k_{\Theta}(\vec{P}^h) = \frac{\langle K(\vec{P}^h, \overline{\vec{P}^h})\vec{P}^h, \vec{P}^h \rangle_{h_F}}{\|\vec{P}^h\|_{h_F}^4},$$

$$(2.21) \quad k_{\Omega}(\vec{P}^h) = \frac{\langle \Omega^{h,\bar{h}}(\vec{P}^h, \overline{\vec{P}^h})\vec{P}^h, \vec{P}^h \rangle_{h_F}}{\|\vec{P}^h\|_{h_F}^4}.$$

Even though the Cartan curvature and the Chern–Rund curvature are different, it turns out that

**Theorem 2.1.** *Let  $F$  be a strictly pseudoconvex Finsler metric on  $TM$  where  $M$  is a complex manifold. Then the Cartan holomorphic bisectional curvature and the Chern–Rund holomorphic bisectional curvature are equal, i.e.,*

$$k_{\Theta}(X, \vec{P}^h) = k_{\Omega}(X, \vec{P}^h).$$

Proof. From here on we shall simply write  $\vec{P}$  for  $\vec{P}^h$ . By (2.17)

$$(\Omega_i^j)^{h,\bar{h}} = K_i^j + G_{i\mu}^j \frac{\delta G_{|k}^{\mu}}{\delta z^{\bar{l}}} dz^k \wedge dz^{\bar{l}}.$$

Since  $\zeta^i G_{ip}^j = 0$ , we have  $\zeta^i (\Omega_i^j)^{h,\bar{h}} = \zeta^i K_i^j$  and the theorem follows from the definitions of the bisectional curvature. □

In fact

$$\langle K(X, \bar{Y})\vec{P}, \vec{P} \rangle_{h_F} = \langle \Omega^{h,\bar{h}}(X, \bar{Y})\vec{P}, \vec{P} \rangle_{h_F} = -G_{j\bar{q}} \frac{\delta \Gamma_{i|k}^j}{\delta z^{\bar{l}}} X^k Y^{\bar{j}} \zeta^i \zeta^{\bar{q}}$$

for any horizontal tangents  $X$  and  $Y$ . The last equality follows from the definition of  $\Omega^{h,\bar{h}}$  (see (2.10) and (2.11)):

$$\Omega^{h,\bar{h}} = -\frac{\delta \Gamma_{i|k}^j}{\delta z^{\bar{l}}} dz^k \wedge dz^{\bar{l}}$$

follow by lowering the index by the metric.



**Corollary 2.1.** *If the Finsler metric  $F$  is Finsler–Kähler then*

$$\langle \Omega^{\text{h},\text{h}}(\vec{P}, \bar{Y})X, \vec{P} \rangle_{h_F} = \langle \Omega^{\text{h},\text{h}}(X, \bar{Y})\vec{P}, \vec{P} \rangle_{h_F}.$$

Proof. From the last identity before the corollary, we get via the Finsler–Kähler condition  $\Gamma_{i|k}^j = \Gamma_{k|i}^j$ ,

$$-G_{j\bar{q}} \frac{\delta \Gamma_{i|k}^j}{\delta z^{\bar{l}}} X^k Y^{\bar{l}} \zeta^i \zeta^{\bar{q}} = -G_{j\bar{q}} \frac{\delta \Gamma_{k|i}^j}{\delta z^{\bar{l}}} X^k Y^{\bar{l}} \zeta^i \zeta^{\bar{q}}.$$

The LHS is

$$\langle \Omega^{\text{h},\text{h}}(X, \bar{Y})\vec{P}, \vec{P} \rangle_{h_F} \quad (= \langle K(X, \bar{Y})\vec{P}, \vec{P} \rangle_{h_F})$$

and the RHS is

$$\langle \Omega^{\text{h},\text{h}}(\vec{P}, \bar{Y})X, \vec{P} \rangle_{h_F}. \quad \square$$

In terms of local coordinates the corollary asserts that

$$\Omega_{i\bar{j}k\bar{l}} \zeta^i \zeta^{\bar{j}} = \Omega_{i\bar{j}k\bar{l}} \zeta^k \zeta^{\bar{j}} = \Omega_{k\bar{j}i\bar{l}} \zeta^i \zeta^{\bar{j}}.$$

REMARK 2.1. In general we do not have

$$\langle K(\vec{P}, \bar{Y})X, \vec{P} \rangle_{h_F} = \langle K(X, \bar{Y})\vec{P}, \vec{P} \rangle_{h_F}.$$

Even if the metric is Finsler–Kähler. The LHS  $\neq \langle \Omega^{\text{h},\text{h}}(\vec{P}, \bar{Y})X, \vec{P} \rangle_{h_F}$ . Thus, in view of the preceding corollary, the Chern–Rund curvature is better behave than the Cartan curvature. In classical (Hermitian–)Kähler geometry, the Cartan curvature is the same as the Chern–Rund curvature hence we do have the symmetry for both.

The bisectional curvature is a form of bidegree (1, 1)

$$B_{\Omega} = 2B_{k\bar{l}} dz^k \wedge dz^{\bar{l}}, \quad B_{k\bar{l}} = \Omega_{i\bar{q}k\bar{l}} \zeta^i \zeta^{\bar{q}} = -G_{j\bar{q}} \frac{\delta \Gamma_{i|k}^j}{\delta z^{\bar{l}}} \zeta^i \zeta^{\bar{q}}.$$

**Theorem 2.2.** *The condition that the holomorphic sectional curvature  $k_{\Omega}(\vec{P}) = c$  is a constant is equivalent to the condition that the holomorphic bisectional curvature form  $B_{\Omega} = 2c(GG_{a\bar{b}} + G_a G_{\bar{b}}) dz^a \wedge dz^{\bar{b}}$ .*

Proof. The equation for constant ( $= c$ ) holomorphic sectional curvature is given by

$$B_{i\bar{j}} \zeta^i \zeta^{\bar{j}} = G_p \frac{\delta G_{i|j}^p}{\delta z^{\bar{j}}} \zeta^i \zeta^{\bar{j}} = cG^2.$$

Since  $\Omega_{i\bar{q}k\bar{l}}$  is homogeneous of bidegree  $(0, 0)$  we have

$$\zeta^a \frac{\partial B_{k\bar{l}}}{\partial \zeta^a} = \zeta^{\bar{b}} \frac{\partial B_{k\bar{l}}}{\partial \zeta^{\bar{b}}} = 0$$

and this implies that

$$B_{a\bar{b}} = \frac{\partial^2 (B_{k\bar{l}} \zeta^k \zeta^{\bar{l}})}{\partial \zeta^a \partial \zeta^{\bar{b}}} = c(GG_{a\bar{b}} + G_a G_{\bar{b}}). \quad \square$$

### 3. Ampleness and bisectonal curvature

Let  $E$  be a holomorphic vector bundle of rank  $r \geq 2$  over a complex manifold  $M$  and let  $\mathcal{L}^{-1} = \mathcal{L}_{\mathbb{P}(E)}^{-1}$  over  $\mathbb{P}(E)$  be the “tautological” line bundle. This bundle is determined by the condition that its restriction to every fiber  $\mathcal{L}^{-1}|_{\mathbb{P}(E)_z}$  is the tautological line bundle over the projective space  $\mathbb{P}(E)_z$ . The dual is denoted by  $\mathcal{L} = \mathcal{L}_{\mathbb{P}(E)}$  and is known as the *Serre line bundle*. By definition the bundle space of  $\mathcal{L}^{-1}$  is the blowing up of the bundle space of  $E$  along the zero section. Thus there is a canonical isomorphism  $\mathcal{L}^{-1} \setminus \{\text{zero-section}\} \cong E \setminus \{\text{zero-section}\}$  compatible with the respective  $\mathbb{C}^*$  structure associated to the respective bundle structures. Let  $H$  be a Hermitian metric along the fibers of  $\mathcal{L}^{-1}$  which, via the preceding isomorphism, determines uniquely a Finsler metric on  $E$ :

$$(3.1) \quad F: E \rightarrow \mathbb{R}_{\geq 0}, \quad F(z, \zeta) = \|\beta^{-1}(z, \zeta)\|_H$$

where  $\beta: \mathcal{L}^{-1} \rightarrow E$  is the blowing up map along the zero section. It was shown earlier that a Finsler metric  $F$  on  $E$  is identified with a Hermitian metric  $h_F$  along the fibers of the “tautological” line bundle  $\mathcal{L}^{-1}$  over  $\mathbb{P}(E)$ . The  $(1, 1)$ -form

$$(3.2) \quad \frac{\sqrt{-1}}{4\pi} \partial\bar{\partial} \log G, \quad G = F^2: E \rightarrow \mathbb{R}_{\geq 0}$$

descends to the Chern form  $c_1(\mathcal{L}, h_F^*)$  of  $\mathcal{L}$  with the dual metric  $h_F^*$ , that is

$$(3.3) \quad \frac{\sqrt{-1}}{4\pi} \partial\bar{\partial} \log G = [\ ]^*(c_1(\mathcal{L}, h_F^*)) = -[\ ]^*(c_1(\mathcal{L}^{-1}, h_F)) = -\frac{\sqrt{-1}}{4\pi} \bar{\partial}\partial \log G$$

where  $[\ ]: E_* \rightarrow \mathbb{P}(E)$  is the Hopf fibration.

Moreover, the condition that  $F$  is strictly pseudoconvex is equivalent to the conditions that

- (1)  $c_1(\mathcal{L}|_{\mathbb{P}(E)_z}, h_F^*|_{\mathbb{P}(E)_z})$  is positive definite on  $\mathbb{P}(E)_z \cong \mathbb{P}^{r-1}$  (and that  $\mathcal{L}|_{\mathbb{P}(E)_z} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(1)$  is the hyperplane bundle on  $\mathbb{P}^{r-1}$ ) and
- (2)  $z \mapsto c_1(\mathcal{L}|_{\mathbb{P}(E)_z}, h_F^*|_{\mathbb{P}(E)_z})$  is smooth.

The following theorem is due to Cao–Wong ([5]):

**Theorem 3.1.** *Let  $(E, F)$  be a strictly pseudoconvex Finsler holomorphic vector bundle,  $\text{rank } E \geq 2$ , over a complex Finsler manifold  $M$ . Let  $h_F^*$  be the Hermitian metric along the fibers of the hyperplane line bundle  $\mathcal{L}$  over  $\mathbb{P}(E)$  associated to the Finsler metric  $F$  via (3.1) and  $K$  be the  $(h, \bar{h})$ -component of the curvature of the Cartan connection associated to the Finsler metric  $F$ . Then the Chern form  $c_1(\mathcal{L}, h_F^*)$  is positive definite on  $\mathbb{P}(E)$  if and only if the mixed holomorphic bisectional curvature is strictly negative in the direction of  $X$  and the radial vector field  $\vec{P}$ , on  $E_*$ :*

$$\langle K(X, \bar{X})\vec{P}, \vec{P} \rangle_{G_V} = K_{\alpha\bar{\beta}k\bar{l}}\zeta^\alpha\bar{\zeta}^\beta X^k\bar{X}^l < 0.$$

for all nonzero horizontal tangent vector  $X$ .

In fact the following formula is obtained in [5]:

$$\begin{aligned} (3.4) \quad & [ \ ]^* c_1(\mathcal{L}, h_F^*) \\ &= \frac{\sqrt{-1}}{4\pi} \partial\bar{\partial} \log G \\ &= [ \ ]^*(c_1(\mathcal{L}|_{\mathbb{P}(E)_z}, h_F^*|_{\mathbb{P}(E)_z})) - \frac{\sqrt{-1}}{4\pi} \frac{\langle K(\cdot, \cdot)\vec{P}, \vec{P} \rangle_{h_F}}{G}. \end{aligned}$$

Since  $[ \ ]^*(c_1(\mathcal{L}|_{\mathbb{P}(E)_z}, h_F^*|_{\mathbb{P}(E)_z}))$  is positive definite (see the remark before the theorem) in the fiber directions and is trivial in the horizontal directions while the second term is trivial in the fiber directions and is positive definite in the horizontal directions if and only if the mixed bisectional curvature:

$$k(X, P) = \frac{\langle K(X, \bar{X})\vec{P}, \vec{P} \rangle_{h_F}}{\|X\|_{h_F}^2 G}$$

is strictly negative for non-zero horizontal tangent vector  $X$ , we conclude that  $c_1(\mathcal{L}, h_F)$  is positive-definite (resp. positive semi-definite) if and only if  $k(X, P) < 0$  (resp.  $\leq 0$ ).

REMARK 3.1. The preceding theorem is applicable to the case  $E = TM$ . In this case, by Theorem 2.1 the Chern–Rund holomorphic bisectional curvature and the Cartan holomorphic bisectional curvature coincide, hence ampleness of the tangent bundle is equivalent to the positivity of the Chern–Rund holomorphic bisectional curvature.

DEFINITION 3.1. (1) A holomorphic line bundle  $L$  over a compact complex manifold is said to be very ample if the map  $\Phi = [s_0, \dots, s_N]: M \rightarrow \mathbb{P}^N$  where  $s_0, \dots, s_N$  is a basis of  $H^0(M, L^m)$  is a holomorphic embedding. It is said to be ample if there exists a positive integer  $m$  such that  $L^m (= \mathcal{L}^{\otimes m})$  is very ample.

(2) A holomorphic vector bundle  $E$  of rank  $\geq 2$  is said to be ample if the Serre line bundle  $\mathcal{L}$  over  $\mathbb{P}(E^*)$  is an ample line bundle over  $\mathbb{P}(E^*)$  where  $E^*$  is the dual of  $E$ .

REMARK 3.2. Note that the ampleness of a vector bundle  $E$  is formulated in terms of the Serre line bundle over the dual projective bundle  $\mathbb{P}(E^*)$  and not  $\mathbb{P}(E)$ . On the other hand a vector bundle  $E^*$  is ample if and only if the Serre line bundle over  $\mathbb{P}(E)$  is ample.

The following result is a consequence of the preceding theorem:

**Corollary 3.1.** *Let  $E$  be a rank  $r \geq 2$  holomorphic vector bundle over a compact complex manifold  $M$  and for any positive integer  $k$  let  $\mathcal{L}_{\mathbb{P}(\odot^k E)}$  be the “hyperplane bundle” over  $\mathbb{P}(\odot^k E)$ . Then the following statements are equivalent:*

- (1)  $E^*$  is ample.
- (2)  $\mathcal{L}_{\mathbb{P}(E)}$  is ample.
- (3)  $\odot^k E^*$  is ample for some positive integer  $k$ .
- (4)  $\mathcal{L}_{\mathbb{P}(\odot^k E)}$  is ample for some positive integer  $k$ .
- (5)  $\odot^k E^*$  is ample for all positive integer  $k$ .
- (6)  $\mathcal{L}_{\mathbb{P}(\odot^k E)}$  is ample for all positive integer  $k$ .
- (7) There exists a strictly pseudoconvex Finsler metric along the fibers of  $E$  with negative mixed holomorphic bisectional curvature.
- (8) There exists a positive integer  $k$  and a strictly pseudoconvex Finsler metric along the fibers of  $\odot^k E$  with negative mixed holomorphic bisectional curvature.
- (9) For any positive integer  $k$  there exists a strictly pseudoconvex Finsler metric along the fibers of  $\odot^k E$  with negative mixed holomorphic bisectional curvature.
- (10) There exists a positive integer  $m$  and a Hermitian metric along the fibers of  $\odot^m E$  with negative mixed holomorphic bisectional curvature.

The result of the preceding corollary applies also to the dual, in which case we have:

**Corollary 3.2.** *Let  $E$  be a rank  $r \geq 2$  holomorphic vector bundle over a compact complex manifold  $M$  and for any positive integer  $k$  let  $\mathcal{L}_{\mathbb{P}(\odot^k E)}$  be the “hyperplane bundle” over  $\mathbb{P}(\odot^k E)$ . Then the following statements are equivalent:*

- (1)  $E$  is ample.
- (2)  $\mathcal{L}_{\mathbb{P}(E^*)}$  is ample.
- (3)  $\odot^k E$  is ample for some positive integer  $k$ .
- (4)  $\mathcal{L}_{\mathbb{P}(\odot^k E^*)}$  is ample for some positive integer  $k$ .
- (5)  $\odot^k E$  is ample for all positive integer  $k$ .
- (6)  $\mathcal{L}_{\mathbb{P}(\odot^k E^*)}$  is ample for all positive integer  $k$ .

- (7) *There exists a strictly pseudoconvex Finsler metric along the fibers of  $E$  with positive mixed holomorphic bisectional curvature.*
- (8) *For some positive integer  $k$  there exists a strictly pseudoconvex Finsler metric along the fibers of  $\odot^k E$  with positive mixed holomorphic bisectional curvature.*
- (9) *For all positive integer  $k$  there exists a strictly pseudoconvex Finsler metric along the fibers of  $\odot^k E$  with positive mixed holomorphic bisectional curvature.*
- (10) *There exists a positive integer  $m$  and a Hermitian metric along the fibers of  $\odot^m E$  with positive mixed holomorphic bisectional curvature.*

REMARK 3.3. In the precedings, the mixed holomorphic bisectional curvature is the the mixed holomorphic bisectional curvature of the Cartan connection. If  $E = TM$  then the Chern–Rund connection is also defined and by Theorem 2.1 the Cartan bisectional curvature and the Chern–Rund bisectional curvature are identical.

A famous theorem of Mori ([10] valid over any algebraically closed field of characteristic zero) asserts that

**Theorem (Mori).** *A compact complex manifold with ample tangent bundle is biholomorphic to  $\mathbb{P}^n$ .*

It is well-known that the converse is true, namely, the tangent bundle of a projective space is ample. This together with Theorem 3.1 yields

**Theorem 3.2.** *Let  $M$  be a compact complex manifold. Then the following statements are equivalent.*

- (1)  $M \cong \mathbb{P}^n$ .
- (2)  $TM$  is ample.
- (3) *There exists a strictly pseudoconvex Finsler metric on  $TM$  with positive bisectional curvature.*

**Corollary 3.3.** *Let  $M$  be compact complex manifold. Assume that there is a strictly pseudoconvex Finsler metric such that the holomorphic sectional curvature of the Chern–Rund is a constant. Then  $M \cong \mathbb{P}^n$ .*

Proof. By Theorem 2.2, the Chern–Rund holomorphic sectional curvature is a constant  $c$  if and only if

$$B_{k\bar{l}} dz^k \wedge dz^{\bar{l}} = 2\Omega_{i\bar{q}k\bar{l}} dz^k \wedge dz^{\bar{l}} = c(GG_{k\bar{l}} + G_k G_{\bar{l}}) dz^k \wedge dz^{\bar{l}}.$$

Thus the bisectional curvature is given by

$$\frac{2B_{k\bar{l}} X^k X^{\bar{l}}}{\|X\|_{h_F}^2 G} = \frac{2\Omega_{i\bar{q}k\bar{l}} \zeta^i \zeta^{\bar{q}}}{\|X\|_{h_F}^2 G} = c \frac{(GG_{k\bar{l}} X^k X^{\bar{l}} + G_k G_{\bar{l}} X^k X^{\bar{l}})}{\|X\|_{h_F}^2 G} = c \left( 1 + \frac{G_k G_{\bar{l}} X^k X^{\bar{l}}}{\|X\|_{h_F}^2 G} \right)$$

as we have, by definition,  $\|X\|_{h_F}^2 = G_{k\bar{l}}X^kX^{\bar{l}}$ . Since  $G_kG_{\bar{l}}X^kX^{\bar{l}}$  is non-negative, we conclude that the bisectional curvature is bounded below by  $c > 0$ . By Theorem 3.2 this is equivalent to the condition that the tangent bundle is ample. Thus  $M \cong \mathbb{P}^n$  by Mori's theorem.  $\square$

REMARK 3.4. (1) The holomorphic bisectional form was first introduced in Cao–Wong [5]. Later this is also studied by Aldea [1] and [2]. They considered (in our notations)  $B_\Omega$  satisfying the condition

$$B_\Omega = (\phi G G_{k\bar{l}} + \psi G_k G_{\bar{l}}) dz^k \wedge d\bar{z}^{\bar{l}}$$

where  $\phi$  and  $\psi$  are positive functions. We would like to point out that by the theorem of Cao–Wong, the space is  $\mathbb{P}^n$ .

(2) The proof in the preceding theorem shows that, if the holomorphic sectional curvature is a constant  $c$  then the holomorphic bisectional curvature is pinched between  $1/2$  and  $1$ .

#### 4. Complex Randers metrics

A continuous complex Finsler metric is said to be a generalized complex Finsler metric if it is strictly pseudoconvex outside the set where it is not smooth.

DEFINITION 4.1. A generalized complex Finsler metric  $F$  on  $TM$ , where  $M$  is a complex manifold, is said to be a (classical) complex Randers metric if it is of the form:

$$F = h + |\beta|, \quad h = (h_{i\bar{j}}(z)\zeta^i\bar{\zeta}^{\bar{j}})^{1/2}, \quad \beta = b_i(z)\zeta^i$$

where  $h_{i\bar{j}}(z)\zeta^i\bar{\zeta}^{\bar{j}}$  is a Hermitian metric,  $\beta = b_i(z)\zeta^i$  is a global form of bidegree  $(1, 0)$  and

$$|\beta|^2 = (b_i(z)\zeta^i)\overline{(b_i(z)\zeta^i)}.$$

It is said to be a holomorphic complex Randers metric if the 1-form  $\beta$  is globally holomorphic.

REMARK 4.1. (1) On a general complex manifold there may not exist any non-trivial global holomorphic forms. Thus a non-trivial holomorphic complex Randers metric might not even exist.

(2) A holomorphic line bundle might not have any non-trivial global holomorphic section but there may exist non-trivial global holomorphic sections of  $L^m$  for integer  $m$ . For instance an ample line bundle might not have any non-trivial global holomorphic section but for  $m \gg 0$ ,  $L^m$  is very ample and there exist enough global holomorphic sections to embed  $M$  as a holomorphic submanifold of a projective space.

(3) By the theorem of Grothendieck–Serre, there is an isomorphism between  $H^0(\mathbb{P}(E), \mathcal{L}^m)$  and  $H^0(M, \odot^m E)$ . Thus

$$H^0(\mathbb{P}(E), \mathcal{L}^m) \neq \{0\} \quad \text{if and only if} \quad H^0\left(M, \odot^m E\right) \neq \{0\}.$$

Taking  $E = T^*M$  we see immediately that it is possible that  $H^0(M, \odot E) = \{0\}$  but  $H^0(M, \odot^m E) \neq \{0\}$ . Namely, there may exist non-trivial global holomorphic symmetric forms of weight  $m$  even though there is no non-trivial global holomorphic 1-forms.

(4) A Randers metric is not smooth at those points  $(z, \zeta)$  such that  $b_i(z)\zeta^i = 0$ . These occur, besides the zero section which is of complex dimension  $n = \dim_{\mathbb{C}} M$ , also along the subvariety  $Z = \{(z, \zeta) \mid b_i(z)\zeta^i = 0\}$  which is of real dimension  $4n - 2$  (complex dimension  $2n - 1$  if  $\beta$  is holomorphic) in  $TM$ . Thus, strictly speaking the metric is not smooth in the sense of Definition 2.1. We shall however allow this mild singularity in this section. We shall, of course, assume that  $F$  is strictly pseudoconvex wherever it is smooth. Observe also that, in the case of compact manifolds, we may control  $\beta$  by a parameter, namely

$$F_\epsilon = A + \epsilon|\beta|.$$

For  $\epsilon > 0$  small, this is a small deformation of a Hermitian metric and is certainly strictly pseudoconvex wherever it is smooth.

**Theorem 4.1** (Chen–Shen). *Let  $F = h + |\beta|$  be a complex Randers metric with*

$$h = (h_{i\bar{j}}(z)\zeta^i\bar{\zeta}^j)^{1/2}, \quad \beta = b_i(z)\zeta^i$$

*with  $\beta$  holomorphic. Then the horizontal holomorphic sectional curvature of the Cartan–Chern–Rund connection is given by*

$$k_F(z, \zeta) = \frac{h^3}{F^3}k_h(\zeta) - \frac{h}{F^3C}(b_{i;j}\zeta^i\zeta^j)(\overline{b_{i;j}\zeta^i\zeta^j})$$

*where  $k_h(\zeta)$  is the holomorphic sectional curvature of the Hermitian metric  $h$  and  $b_{i;j}$  is the covariant derivative of  $b_i$  in the variable  $\zeta^j$  with respect to the Hermitian metric  $h$ .*

**Corollary 4.1** (Chen–Shen). *Let  $F = h + |\beta|$  be a complex Randers metric with*

$$h = (h_{i\bar{j}}(z)\zeta^i\bar{\zeta}^j)^{1/2}, \quad \beta = b_i(z)\zeta^i$$

*with  $h$  Kähler and  $\beta$  holomorphic. If  $F$  is of positive constant holomorphic sectional curvature outside of the singular set then  $M \cong \mathbb{P}^n$ .*

The preceding results can be found in [8].

Denote by  $\odot^m T^*M$  the  $m$ -fold symmetric product of the cotangent bundle  $T^*M$ . A general smooth section of  $\odot^m T^*M$ , henceforth referred to as a smooth (resp. holomorphic) symmetric form of weight  $m$ , is of the form

$$\beta = \beta_{i_1, \dots, i_m}(z) dz^{i_1} \odot \dots \odot dz^{i_m}$$

where  $\beta_{i_1, \dots, i_m}$  are smooth (resp. holomorphic) functions. Each of these may be identified with a smooth (resp. holomorphic) function on  $TM$ , homogeneous of degree (or *weight*)  $m$  in the fiber variables:

$$\beta(z; \zeta) = \beta_{i_1, \dots, i_m}(z) \zeta^{i_1} \dots \zeta^{i_m}, \quad (z; \zeta) \in TM.$$

It is immediately clear that

$$\beta(z; \lambda \zeta) = \beta_{i_1, \dots, i_m}(z) (\lambda \zeta)^{i_1} \odot \dots \odot (\lambda \zeta)^{i_m} = \lambda^m \beta(z; \zeta),$$

hence we have

$$|\beta(z; \lambda \zeta)|^{1/m} = |\lambda| |\beta(z; \zeta)|,$$

For example, if  $\beta_1, \dots, \beta_m$  are holomorphic 1-forms then their symmetric product  $\beta_1 \odot \dots \odot \beta_m$  is a symmetric form of weight  $m$ . In fact each

$$\odot^{i_1} \beta_1 \odot^{i_2} \beta_2 \dots \odot^{i_m} \beta_m, \quad i_1 + \dots + i_m = m, \quad i_j \geq 0$$

is a symmetric form of weight  $m$ . This also shows that  $(H^0(M, \odot^p T^*M))$  is the space of global holomorphic symmetric forms of weight  $p$

$$\dim H^0\left(M, \odot^p T^*M\right) \leq \dim H^0\left(M, \odot^q T^*M\right)$$

for  $q \geq p$ .

DEFINITION 4.2. A generalized complex Finsler metric  $F$  on  $TM$ , where  $M$  is a complex manifold, is said to be a complex Randers metric of weight  $m$  if it is of the form:

$$F = h + |B_1| + |B_2|^{1/2} + \dots + |B_m|^{1/m}, \quad h = (h_{i\bar{j}}(z) \zeta^i \bar{\zeta}^{\bar{j}})^{1/2}$$

with  $h_{i\bar{j}}(z) \zeta^i \bar{\zeta}^{\bar{j}}$  is a Hermitian metric and

$$|B_i|^{1/i} = \sum_{j=1}^{n_i} |\beta_{ij}|^{1/i}$$



where each  $\beta_{i,j}$  is a global smooth symmetric form of weight  $i$  and for each  $i$ , the symmetric forms  $\beta_{i1}, \dots, \beta_{in_i}$  are linearly independent. It is said to be a holomorphic complex Randers metric of weight  $m$  if the symmetric forms  $\beta_{i,j}$  are globally holomorphic.

We shall only deal with the holomorphic case. Holomorphic Randers metrics are special cases of jet metrics (see Chandler–Wong [5]).

The reason for taking sum in the definition above is to reduce the dimension of the singular set. For example if  $T^*M$  is spanned (this means that global holomorphic forms span the fibers of  $T^*M$ , i.e., at each point  $z \in M$  there exists a global holomorphic 1-form  $\beta$  such that  $\beta(x) \neq 0$ ). Thus, in this case if we take  $|B_1| = \sum |\beta_{1j}|$  where  $\beta_{11}, \dots, \beta_{1n_1}$  is a basis of the space of all global holomorphic 1-forms (this is finite dimensional if  $M$  is compact). The ‘spanned’ condition means that the zeros of  $|B_1|$  is precisely the zero section of  $TM$ . In other words, the generalized metric  $F = h + |B_1|$  is positive definite and smooth outside the zero section of  $TM$ , that is, it is a bona fide smooth Finsler metric. As remarked earlier, the number of linearly independent symmetric forms of weight  $m$  is non-decreasing. thus the singular set is also non-increasing as we increase the weight.

Another motivation of studying Randers metric of higher weights comes from the physicists in their attempt in understanding dark matters and dark energy (see [7]). For this they add a correction term of the type, in their notation,

$$(dx \otimes dx \otimes dx \otimes dx)^{1/4};$$

i.e., a symmetric form of weight 4 to the usual Lorentz metric. In their case the computation of the curvature is very complicated. The situation in the complex case is much easier.

Suppose that we are given a holomorphic complex Randers metric of weight  $m$ :

$$F = h + |B_1| + |B_2|^{1/2} + \dots + |B_m|^{1/m}$$

constructed from holomorphic symmetric forms. The computation of the holomorphic section curvature is based on the lemma below. Recall that, for a locally integrable function, the derivatives exist in the sense of currents (distributions) and that a subharmonic function is locally integrable.

**Lemma 4.1.** *Let  $A$  and  $B$  be non-negative smooth functions,  $AB \neq 0$ . Then*

$$\begin{aligned} & dd^c \log(A + B) \\ &= \frac{1}{A + B} (A dd^c \log A + B dd^c \log B) + \frac{AB}{(A + B)^2} \left( d \log \frac{A}{B} \wedge d^c \log \frac{A}{B} \right) \end{aligned}$$

on the set where  $AB \neq 0$ .

Proof. We get, by direct computation,

$$\begin{aligned}
 & dd^c \log(A + B) \\
 &= \frac{(A + B)(dd^c A + dd^c B) - (dA + dB) \wedge (d^c A + d^c B)}{(A + B)^2} \\
 &= \frac{A dd^c A - dA \wedge d^c A + B dd^c B - dB \wedge d^c B}{(A + B)^2} + T \\
 &= \frac{A^2}{(A + B)^2} dd^c \log A + \frac{B^2}{(A + B)^2} dd^c \log B + T
 \end{aligned}$$

where

$$T = \frac{B dd^c A + A dd^c B - (dA \wedge d^c B + dB \wedge d^c A)}{(A + B)^2}$$

and that

$$dd^c \log A = \frac{A dd^c A - dA \wedge d^c A}{A^2}, \quad dd^c \log B = \frac{B dd^c B - dB \wedge d^c B}{B^2}.$$

The term  $T$  can be rewritten in a more manageable form by expressing

$$\begin{aligned}
 B dd^c A &= \frac{B}{A}(A^2 dd^c \log A + dA \wedge d^c A), \\
 A dd^c B &= \frac{A}{B}(B^2 dd^c \log B + dB \wedge d^c B),
 \end{aligned}$$

and  $dA = A d \log A$ ,  $d^c A = A d^c \log A$ ,  $dB = B d \log B$ ,  $d^c B = B d^c \log B$ . Then

$$\begin{aligned}
 & B dd^c A + A dd^c B - (dA \wedge d^c B + dB \wedge d^c A) \\
 &= AB(dd^c \log A + dd^c \log B + d \log A \wedge d^c \log A + d \log B \wedge d^c \log B \\
 &\quad - d \log A \wedge d^c \log B - d \log B \wedge d^c \log A) \\
 &= AB[dd^c \log A + dd^c \log B + (d \log A - d \log B) \wedge (d^c \log A - d^c \log B)] \\
 &= AB \left( dd^c \log A + dd^c \log B + d \log \frac{A}{B} \wedge d^c \log \frac{A}{B} \right).
 \end{aligned}$$

Thus we get

$$T = \frac{AB}{(A + B)^2} \left( dd^c \log A + dd^c \log B + d \log \frac{A}{B} \wedge d^c \log \frac{A}{B} \right)$$

and

$$dd^c \log(A + B) = \frac{A dd^c \log A + B dd^c \log B}{A + B} + \frac{AB}{(A + B)^2} d \log \frac{A}{B} \wedge d^c \log \frac{A}{B}$$

as claimed.  $\square$

**Corollary 4.2.** *With the same assumptions as in Lemma 4.1, we have*

$$dd^c \log(A + B) \geq \frac{1}{A + B} (A dd^c \log A + B dd^c \log B)$$

and equality holds if and only if  $A/B$  is a constant on the set  $AB \neq 0$ .

Proof. Since  $A$  and  $B$  are positive and

$$d \log \frac{A}{B} \wedge d^c \log \frac{A}{B} \geq 0$$

we deduce readily that

$$dd^c \log(A + B) \geq \frac{A dd^c \log A + B dd^c \log B}{A + B}$$

and equality holds if and only if

$$d \log \frac{A}{B} \wedge d^c \log \frac{A}{B} = 0.$$

This occurs if and only if

$$\partial \log \frac{A}{B} \wedge \bar{\partial} \log \frac{A}{B} = 0.$$

The form  $\partial \log(A/B)$  is of bidgree  $(1, 0)$  and  $\bar{\partial} \log(A/B)$  is of bidgree  $(0, 1)$ . Thus the preceding identity holds if and only if  $A/B$  is holomorphic and also conjugate holomorphic, i.e.,  $A/B$  is a constant.  $\square$

The preceding lemma and corollary can be extended:

**Lemma 4.2.** *Let  $A_i, i = 1, \dots, q$  be non-negative continuous functions. Assume that  $\prod A_i \neq 0$ . Then on the set  $\prod_{i=1}^q A_i \neq 0$ ,*

$$dd^c \log \sum_{i=1}^q A_i = \frac{1}{\sum_{k=1}^q A_k} \sum_{i=1}^q A_i dd^c \log A_i + T$$

where  $T$  is a non-negative term

$$\sum_{1 \leq i < j \leq q} \frac{A_i A_j}{(\sum_{k=1}^q A_k)^2} \left( d \log \frac{A_i}{A_j} \wedge d^c \log \frac{A_i}{A_j} \right).$$

Consequently, we have

$$dd^c \log \sum_{i=1}^q A_i \geq \frac{1}{\sum_{k=1}^q A_k} \sum_{i=1}^q A_i dd^c \log A_i$$

and equality holds if and only if  $A_i/A_j = c_{ij} = \text{constant}$  for all  $1 \leq i, j \leq q$ .

Proof. The case  $q = 2$  was established above. The proof can be completed via induction. By the case  $q = 2$ , Then

$$\begin{aligned} dd^c \log \sum_{i=1}^q A_i &= \frac{\sum_{j=1}^{q-1} A_j}{\sum_{k=1}^q A_k} dd^c \log \sum_{i=1}^{q-1} A_i + \frac{A_q}{\sum_{k=1}^q A_k} dd^c \log A_q \\ &\quad + \frac{\sum_{j=1}^{q-1} A_j A_q}{(\sum_{k=1}^q A_k)^2} \left( d \log \frac{\sum_{i=1}^{q-1} A_i}{A_q} \wedge d^c \log \frac{\sum_{i=1}^{q-1} A_i}{A_q} \right). \end{aligned}$$

By induction,

$$\begin{aligned} dd^c \log \sum_{i=1}^{q-1} A_i &= \frac{1}{\sum_{k=1}^{q-1} A_k} \sum_{i=1}^{q-1} A_i dd^c \log A_i + \sum_{1 \leq i < j \leq q-1} \frac{A_i A_j}{(\sum_{k=1}^{q-1} A_k)^2} \left( d \log \frac{A_i}{A_j} \wedge d^c \log \frac{A_i}{A_j} \right). \end{aligned}$$

Substitute this into the preceding identity and using the identities

$$d \log \sum_{i=1}^{q-1} A_i = \sum_{i=1}^{q-1} \frac{A_i}{\sum_{k=1}^{q-1} A_k} d \log A_i, \quad d^c \log \sum_{i=1}^{q-1} A_i = \sum_{i=1}^{q-1} \frac{A_i}{\sum_{k=1}^{q-1} A_k} d^c \log A_i$$

we get the desired identity.  $\square$

**Corollary 4.3.** *If, in the preceding lemma,  $\log A_i$  is plurisubharmonic for all  $i$  then*

$$dd^c \log \sum_{i=1}^q A_i = \sum_{1 \leq i < j \leq q} \frac{A_i A_j}{(\sum_{k=1}^q A_k)^2} \left( d \log \frac{A_i}{A_j} \wedge d^c \log \frac{A_i}{A_j} \right)$$

*if and only if  $\log A_i$  is pluriharmonic for all  $i$ .*

Proof. The function  $\log A_i$  is plurisubharmonic if and only if  $dd^c \log A_i \geq 0$ , hence

$$dd^c \log \sum_{i=1}^q A_i = \sum_{1 \leq i < j \leq q} \frac{A_i A_j}{(\sum_{k=1}^q A_k)^2} \left( d \log \frac{A_i}{A_j} \wedge d^c \log \frac{A_i}{A_j} \right)$$

if and only if

$$\sum_{i=1}^q A_i dd^c \log A_i = 0$$

if and only if  $dd^c \log A_i = 0$  ( $\log A_i$  is pluriharmonic) for all  $i$ .  $\square$

REMARK 4.2. Suppose that  $A_i = |\beta_i|^{1/m}$  where  $b_i$  is a holomorphic symmetric form of weight  $m$ . Then  $\log A_i$  is plurisubharmonic and pluriharmonic outside of the set  $\beta_i = 0$ . In particular, we have  $dd^c \log \sum_i |\beta_i|^{1/m} \geq 0$ .

For a Finsler metric  $F$ , the holomorphic bisectional curvature of  $F$  will be denoted by  $\text{hbsc}_F$ . The next theorem is the main result of this article:

**Theorem 4.2.** *Let  $F = h + \sum_{i=1}^m |B_i|^{1/i}$  be a holomorphic weighted complex Randers metric on a complex manifold  $M$ . Then*

$$\text{hbsc}_F \leq \frac{h}{F} \text{hbsc}_h \leq \text{hbsc}_h$$

outside the singular set of  $F$ .

Proof. Let  $F = h + \sum_{i=1}^m |B_i|^{1/i}$  be a holomorphic weighted complex Randers metric and  $G = F^2 = h^2 + 2h|B| + |B|^2$  where  $|B| = \sum_{i=1}^m |B_i|^{1/i}$ . Then

$$dd^c \log G - \phi = \frac{h^2}{G} dd^c \log h^2 + \frac{2h|B|}{G} dd^c \log h|B| + \frac{|B|^2}{G} dd^c \log |B|^2$$

with

$$dd^c \log |B|^2 - \psi = \sum_{i=1}^m \frac{|B_i|^{1/i}}{|B|} dd^c \log |B_i|^{2/i}$$

where

$$\begin{aligned} \phi &= \frac{2h^3|B|}{G^2} \left( d \log \frac{h}{2|B|} \wedge d^c \log \frac{h}{2|B|} \right) + \frac{h^2|B|^2}{G^2} \left( d \log \frac{h^2}{|B|^2} \wedge d^c \log \frac{h^2}{|B|^2} \right) \\ &+ \frac{2h|B|^3}{G^2} \left( d \log \frac{2h}{|B|} \wedge d^c \log \frac{2h}{|B|} \right) > 0 \end{aligned}$$

and

$$\psi = \frac{1}{2} \sum_{1 \leq i < j \leq m} \frac{|B_i|^{1/i} |B_j|^{1/j}}{|B|^2} \left( d \log \frac{|B_i|^{2/i}}{|B_j|^{2/j}} \wedge d^c \log \frac{|B_i|^{2/i}}{|B_j|^{2/j}} \right) > 0.$$

Thus

$$dd^c \log G - \phi = \frac{h^2}{G} dd^c \log h^2 + \frac{2h|B|}{G} dd^c \log h + \frac{|B|}{F} dd^c \log |B|^2$$

which we rewrite as (using  $F = h + |B|$ )

$$dd^c \log G - \phi - \frac{|B|}{F} \psi = \frac{h}{F} dd^c \log h^2 + \frac{1}{F} \sum_{i=1}^m |B_i|^{1/i} dd^c \log |B_i|^{2/i}.$$

Now  $dd^c \log|B_i|^{2/i} = 2 dd^c \log \sum_{j=1}^{n_i} |\beta_{ij}|^{1/i}$  and we get

$$dd^c \log|B_i|^{2/i} - \gamma_i = \sum_{j=1}^{n_i} \frac{|\beta_{ij}|^{2/i}}{|B_i|^{2/i}} dd^c \log|\beta_{ij}|^{2/i}$$

where

$$\gamma_i = \frac{1}{2} \sum_{1 \leq j < k \leq n_i} \frac{|\beta_{ij}|^{1/i} |\beta_{ik}|^{1/i}}{|B_i|^{2/i}} \left( d \log \frac{|\beta_{ij}|^{2/i}}{|\beta_{ik}|^{2/i}} \wedge d^c \log \frac{|\beta_{ij}|^{2/i}}{|\beta_{ik}|^{2/i}} \right) > 0.$$

Since  $\beta_{ij}$  is holomorphic,  $\log|\beta_{ij}|$  is pluriharmonic hence  $dd^c \log|\beta_{ij}| = 0$ . Thus the preceding identity reduces to

$$dd^c \log|B_i|^{2/i} - \gamma_i = 0$$

and we arrive at the identity

$$\frac{h}{F} dd^c \log h^2 = dd^c \log G - \phi - \frac{|B|}{F} \psi - \frac{1}{F} \sum_{i=1}^m |B_i|^{1/i} \gamma_i.$$

This implies that

$$dd^c \log G \geq \frac{h}{F} dd^c \log h^2.$$

Denote by  $h_F$  the Hermitian metric on the tautological line bundle  $\mathcal{L}^{-1}$  over  $\mathbb{P}(TM)$  associate to the Hermitian metric  $F$  on  $E$ . There is also the Hermitian metric  $h_0$  on the  $\mathcal{L}^{-1}$  notation associate to the Hermitian metric  $h$  on  $TM$ . The preceding simply means that (by (3.4) in Section 3)

$$c_1(\mathcal{L}^{-1}, h_F) \geq \frac{h}{F} c_1(\mathcal{L}^{-1}, h_0).$$

By the result of Cao–Wong (see (3.4)) this is equivalent to the condition that

$$\text{hbsc}_F \leq \frac{h}{F} \text{hbsc}_h.$$

Since  $h \leq F$ , we certainly have  $\text{hbsc}_F \leq \text{hbsc}_h$ . □

REMARK 4.3. We get as an immediate consequence that the holomorphic sectional curvature of  $h$  is also greater than or equal to that of  $F$ . This extends the result of Chen and Shen which was established under the condition that  $F = h + |b_i(z)\zeta^i|$  and that  $h$  is Kähler.

**Corollary 4.4.** *Let  $F = h + |B|$  be a holomorphic complex Randers metric of weight  $m$  on a compact complex manifold  $M$ . If the holomorphic sectional curvature of  $F$  is a positive constant then  $M \cong \mathbb{P}^n$ .*

*Proof.* By Remark 3.4, the holomorphic sectional curvature of  $F$  is a constant  $c$  implies that the holomorphic bisectional curvature of  $F$  is  $\geq c/2 > 0$  outside the singular set of  $F$ . By the preceding theorem the holomorphic bisectional curvature of  $h$  is also  $\geq c/2 > 0$  outside the singular set of  $F$ . Since the singular set is of codimensional at least one and  $h$  is non-singular ( $h$  is Hermitian) the holomorphic bisectional curvature of  $h$  is  $\geq c/2 > 0$  everywhere. This implies that  $M \cong \mathbb{P}^n$  (by Cao–Wong’s theorem).  $\square$

REMARK 4.4. (1) This extends the result of Chen and Shen which was established under the condition that  $F = h + |b_i(z)\zeta^i|$  and that  $h$  is Kähler. Note that Chen and Shen’s proof does not work for the more general situation of the preceding theorem.

(2) The assumption that  $M$  is compact (by compactness we always mean without boundary) is important in the last corollary. Even though it is true that the formula for bisectional curvature is local, to get to ampleness in Cao–Wong [5] we need compactness; otherwise there is no Kodaira vanishing theorem nor embedding theorem, even though the Chern form is positive definite. In the case of compact with boundary there will be boundary term when you integrate.

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