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# FELLER PROPERTY OF SKEW PRODUCT DIFFUSION PROCESSES

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**Abstract**

Feller property of some diffusion processes and the time changed processes is investigated. Diffusion processes treated here are skew product of one dimensional generalized diffusion processes and the spherical Brownian motion, and the time changed processes are given by additive functional associated with some underlying measure. Concrete expressions of the Dirichlet forms corresponding to time changed processes are also obtained, which may be of non-local type caused by degeneracy of the underlying measures.

## 1. Introduction

Let $s$ be a continuous strictly increasing function on an open interval $I = (l_1, l_2)$, and $m$ be a right continuous nondecreasing function on $I$, where $-\infty \leq l_1 < l_2 \leq \infty$. We denote by $\mathrm{R} = [R_t, P_r^{\mathrm{R}}]$ a one dimensional generalized diffusion process (ODGDP for brief) on $I$ with scale function $s$, speed measure $m$ and no killing measure. We also denote by $\Theta = [\Theta_t, P_\theta^{\Theta}]$ the spherical Brownian motion on $S^{d-1} \subset \mathbb{R}^d$ with generator $(1/2)\Delta$, $\Delta$ being the spherical Laplacian on $S^{d-1}$. In this article we study Feller property of the skew product $\mathrm{X} = [X_t = (R_t, \Theta_{\mathbf{f}(t)}),\ P_{(r,\theta)}^{\mathrm{X}} = P_r^{\mathrm{R}} \otimes P_\theta^{\Theta},\ (r, \theta) \in I \times S^{d-1}]$ with respect to a positive continuous additive functional (PCAF for brief) $\mathbf{f}(t)$ of the ODGDP R. We also study Feller property of time changed processes of the skew product X. In [10] Ogura et al. were concerned with the skew product of a one dimensional diffusion process on $\mathbb{R}^1$ and a $d-1$ dimensional diffusion process on $\mathbb{R}^{d-1}$ with respect to a PCAF, and its time changed process. They showed Feller property of these processes by studying some properties of the corresponding PCAF. We observe behavior of sample paths of R near the end points $l_i$, $i = 1, 2$ to show Feller property of the skew product X. We present Dirichlet forms of the skew product X and time changed processes, which are limit processes appeared in some limit theorem discussed by the first author. Our results ensure that Feller property is preserved in sequences of stochastic processes and their limit processes discussed by her. Dirichlet forms corresponding to time changed processes may be non-local type. Namely, they are expressed

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by diffusion term, jump term and killing term. Our results show that Markov processes corresponding to such non-local type Dirichlet forms satisfy Feller property.

In Section 2 we present Dirichlet forms corresponding to the ODGDP R, the spherical Brownian motion $\Theta$, and the skew product X by employing the results of [4] and [9]. In Section 3 we state Feller property of the skew product X. Section 4 is devoted to time changed processes of the skew product. We show their Feller property. In Section 5 we present Dirichlet form of the time changed process and give some typical examples.

## 2. Preliminaries

**2.1. ODGDP.** Let $s, m, I$, etc. be those given in the preceding section. We denote by $ds$ and $dm$ the measures induced by $s$ and $m$, respectively. We assume that supp$[m]$, the support of $dm$, coincides with $I$. For a function $f$ on $I$, we simply write $f(l_1)$ (resp. $f(l_2)$) in place of $f(l_1+)$ (resp. $f(l_2-)$) provided $f(l_1+)$ (resp. $f(l_2-)$) exists. Let $D(\mathcal{G}_{s,m})$ be the space of all bounded continuous functions $u$ on $I$ satisfying the following two conditions.
(i) There exist a function $f$ on $I$ and two constants $A_1, A_2$ such that

$$u(x) = A_1 + A_2\{s(x) - s(c)\} + \int_{(c,x]} \{s(x) - s(y)\} f(y)\, dm(y), \quad x \in I. \tag{2.1}$$

(ii) For each $i = 1, 2$, $u(l_i) = 0$ if $|m(l_i)| + |s(l_i)| < \infty$.
Throughout this paper we denote by $c$ an arbitrarily fixed point of $I$. The operator $\mathcal{G}_{s,m}$ is defined by the mapping from $u \in D(\mathcal{G}_{s,m})$ to $f$ appeared in (2.1). The operator $\mathcal{G}_{s,m}$ is called the one-dimensional generalized diffusion operator (ODGDO for brief) with $(s, m)$, and $s$ and $m$ are called the scale function and the speed measure, respectively. We set

$$J_{\mu,\nu}(l_i) = \int_{(c,l_i)} d\mu(x) \int_{(c,x]} d\nu(y),$$

for Borel measures $\mu$ and $\nu$ on $I$. Following [3], we call the end point $l_i$ to be

$$\begin{array}{llll} (s, m)\text{-regular} & \text{if} & J_{s,m}(l_i) < \infty \quad \text{and} & J_{m,s}(l_i) < \infty, \\ (s, m)\text{-exit} & \text{if} & J_{s,m}(l_i) < \infty \quad \text{and} & J_{m,s}(l_i) = \infty, \\ (s, m)\text{-entrance} & \text{if} & J_{s,m}(l_i) = \infty \quad \text{and} & J_{m,s}(l_i) < \infty, \\ (s, m)\text{-natural} & \text{if} & J_{s,m}(l_i) = \infty \quad \text{and} & J_{m,s}(l_i) = \infty. \end{array}$$

Recall that

$$\begin{array}{llll} \text{if } l_i \text{ is } (s, m)\text{-regular}, & |m(l_i)| < \infty & \text{and} & |s(l_i)| < \infty, \\ \text{if } l_i \text{ is } (s, m)\text{-exit}, & |m(l_i)| = \infty & \text{and} & |s(l_i)| < \infty, \\ \text{if } l_i \text{ is } (s, m)\text{-entrance}, & |m(l_i)| < \infty & \text{and} & |s(l_i)| = \infty, \\ \text{if } l_i \text{ is } (s, m)\text{-natural}, & |m(l_i)| = \infty & \text{or} & |s(l_i)| = \infty. \end{array}$$

Therefore the above condition (ii) means that the absorbing boundary condition is posed at $l_i$ if it is $(s, m)$-regular. It is known that there exists a strong Markov process $\mathrm{R} = [R_t, P_r^{\mathrm{R}}]$ with the generator $\mathcal{G}_{s,m}$, which is called an ODGDP on $I$ (see [6], [11]).

We denote by $p_t^{\mathrm{R}}$ the semigroup of the ODGDP R, that is,

$$p_t^{\mathrm{R}} f(r) = E^{P_r^{\mathrm{R}}}[f(R_t)] = \int_I p^{\mathrm{R}}(t, r, \xi) f(\xi)\, dm(\xi), \quad t > 0,\ r \in I, \tag{2.2}$$

for $f \in C_b(I)$, where $C_b(A)$ is the set of all bounded continuous functions on a set $A$, $E^P$ stands for the expectation with respect to the probability measure $P$, and $p^{\mathrm{R}}(t,r,\xi)$ denotes the transition probability density of R with respect to $dm$. We note that $p_t^{\mathrm{R}} f \in C_b(I)$ and there exist the following limits for $t > 0$ (see [6], [8]).

$$\lim_{r \to l_i} p_t^{\mathrm{R}} f(r) = 0 \quad \text{if } l_i \text{ is } (s, m)\text{-regular or exit.} \tag{2.3}$$

$$\lim_{r \to l_i} p_t^{\mathrm{R}} f(r) \in \mathbb{R} \quad \text{if } l_i \text{ is } (s, m)\text{-entrance and there exists the limit } f(l_i). \tag{2.4}$$

$$\lim_{r \to l_i} p_t^{\mathrm{R}} f(r) = 0 \quad \text{if } l_i \text{ is } (s, m)\text{-natural and there exists the limit } f(l_i) = 0. \tag{2.5}$$

We consider the following symmetric bilinear form $(\mathcal{E}^{\mathrm{R}}, \mathcal{F}^{\mathrm{R}})$.

$$\begin{aligned} \mathcal{E}^{\mathrm{R}}(u, v) &= \int_I \frac{du}{ds}\frac{dv}{ds}\, ds, \\ \mathcal{F}^{\mathrm{R}} &= \{u \in L^2(I, m) \colon u \text{ is absolutely continuous on } I \\ &\qquad\qquad \text{with respect to } ds \text{ and } \mathcal{E}^{\mathrm{R}}(u, u) < \infty\}. \end{aligned} \tag{2.6}$$

We set $\mathcal{C}^{\mathrm{R}} = \{u \circ s \colon u \in C_0^1(J)\}$, where $J = s(I)$ and $C_0^1(J)$ is the set of all continuously differentiable functions on $J$ with compact support. Then $(\mathcal{E}^{\mathrm{R}}, \mathcal{F}^{\mathrm{R}})$ is a regular, strongly local, irreducible Dirichlet form on $L^2(I, m)$ possessing $\mathcal{C}^{\mathrm{R}}$ as its core and corresponding to the ODGDP $\mathrm{R} = [R_t, P_r^{\mathrm{R}}]$ (see [1], [5]). In the following we write $s^{\mathrm{R}}$ and $m^{\mathrm{R}}$ in place of $s$ and $m$, respectively.

Following [5], we call $\mathcal{E}^{\mathrm{R}}$ to be conservative if $p_t^{\mathrm{R}}1 = 1$, $t > 0$. Since $p_t^{\mathrm{R}}1(r) = P_r^{\mathrm{R}}(t < \sigma_{l_1}^{\mathrm{R}} \wedge \sigma_{l_2}^{\mathrm{R}})$, we see that $p_t^{\mathrm{R}}1 = 1$ if and only if

$$\text{both of} \quad l_i,\ i = 1, 2, \quad \text{are } (s^{\mathrm{R}}, m^{\mathrm{R}})\text{-entrance or natural,} \tag{2.7}$$

where $\sigma_a^{\mathrm{R}}$ stands for the first hitting time to point $a$ for the ODGDP R, that is, $\sigma_a^{\mathrm{R}} = \inf\{t > 0 \colon R_t = a\}$, and $a \wedge b = \min\{a, b\}$. Finally we summarize hitting probability densities. For an open interval $E = (a, b) \subset I$, let $p_E^{\mathrm{R}}(t, \xi, \eta)$ be the transition probability density of the ODGDP on $E$ with the scale function $s^{\mathrm{R}}$ and the speed measure $m^{\mathrm{R}}$. Note that $a$ (resp. $b$) is regular and absorbing if $l_1 < a$ (resp. $b < l_2$). Let denote by $D_{s^{\mathrm{R}}(r)}$ the right derivative with respect to $ds^{\mathrm{R}}(r)$. It is known that there exist the following limits (see [8]).

$$h_E^{\mathrm{R}}(t, r, a) := \lim_{\xi \downarrow a} D_{s^{\mathrm{R}}(\xi)} p_E^{\mathrm{R}}(t, r, \xi) \geq 0,$$
$$h_E^{\mathrm{R}}(t, r, b) := -\lim_{\xi \uparrow b} D_{s^{\mathrm{R}}(\xi)} p_E^{\mathrm{R}}(t, r, \xi) \geq 0,$$

for $t > 0$ and $a < r < b$. Then it holds true that

$$P_r^{\mathrm{R}}(\sigma_a^{\mathrm{R}} < t,\ \sigma_a^{\mathrm{R}} < \sigma_b^{\mathrm{R}}) = \int_0^t h_E^{\mathrm{R}}(u, r, a)\, du, \tag{2.8}$$

$$P_r^{\mathrm{R}}(\sigma_b^{\mathrm{R}} < t,\ \sigma_b^{\mathrm{R}} < \sigma_a^{\mathrm{R}}) = \int_0^t h_E^{\mathrm{R}}(u, r, b)\, du, \tag{2.9}$$

for $t > 0$ and $a < r < b$.

**2.2. Spherical Brownian motion.** Next we consider the spherical Brownian motion $\mathrm{BM}(S^d)$ on $S^d \subset \mathbb{R}^{d+1}$ with generator $(1/2)\Delta$, where $\Delta$ is the spherical Laplacian on $S^d$. Itô and McKean [6] showed that the spherical Brownian motion is described as the skew product of the Legendre process $\mathrm{LEG}(d) = \{\varphi_t\}$ with the generator

$$\frac{1}{2}(\sin\varphi)^{1-d} \frac{\partial}{\partial\varphi}(\sin\varphi)^{d-1} \frac{\partial}{\partial\varphi}, \quad 0 < \varphi < \pi, \tag{2.10}$$

and an independent spherical Brownian motion $\mathrm{BM}(S^{d-1})$ with respect to the PCAF $\int_0^t (\sin\varphi_s)^{-2}\, ds$. Fukushima and Oshima [4] determined the Dirichlet form corresponding to the skew product $(X_t^{(1)}, X_{A_t}^{(2)})$, where $\{X_t^{(i)}\}$, $i = 1, 2$, are independent conservative Markov processes on state space $X^{(i)}$, and $A_t$ is a PCAF of $\{X_t^{(1)}\}$. They presented the Dirichlet form corresponding to the spherical Brownian motion $\mathrm{BM}(S^d)$ as an application of their results. More precisely, let $X^{(1)} = (0, \pi)$, $X_1^{(2)} = \mathbb{T}$ $(= \mathbb{R}^1/[0, 2\pi])$ the torus, and $X_d^{(2)} = X^{(1)} \times X_{d-1}^{(2)}$ $(d \geq 2)$. In the following $X_d^{(2)}$ is identified with $S^d$ $(\subset \mathbb{R}^{d+1})$. Then $dm_d^{(1)}(\varphi) = (\sin\varphi)^d\, d\varphi$ $(d \geq 1)$ are the measures on $X^{(1)}$, $dm_1^{(2)}(\theta) = d\theta$ is the measure on $X_1^{(2)}$, and $m_d^{(2)} = m_{d-1}^{(1)} \otimes m_{d-1}^{(2)}$ $(d \geq 2)$ are measures on $X_d^{(2)}$. We consider the following symmetric bilinear forms.

$$\mathcal{E}^1(u, v) = \frac{1}{2} \int_{X_1^{(2)}} \frac{du}{d\theta} \frac{dv}{d\theta}\, d\theta, \quad u, v \in C^\infty(X_1^{(2)}), \tag{2.11}$$

$$\begin{aligned} &\mathcal{E}^d(f, g) \\ &= \int_{X_{d-1}^{(2)}} \mathcal{E}^{d-1,(1)}(f(\cdot, \theta), g(\cdot, \theta))\, dm_{d-1}^{(2)}(\theta) \\ &\quad + \int_{X^{(1)}} \mathcal{E}^{d-1}(f(\varphi, \cdot), g(\varphi, \cdot))\, d\mu_{d-1}(\varphi), \quad f, g \in C_0^\infty(X_d^{(2)}),\ d \geq 2, \end{aligned} \tag{2.12}$$

where $C^\infty(A)$ (resp. $C_0^\infty(A)$) stands for the set of all infinitely continuously differentiable functions on a set $A$ (resp. with compact support), $d\mu_{d-1}(\varphi) = (\sin\varphi)^{-2}\, dm_{d-1}^{(1)}(\varphi) = (\sin\varphi)^{d-3}\, d\varphi$ and

$$\mathcal{E}^{d-1,(1)}(u, v) = \frac{1}{2}\int_{X^{(1)}} \frac{du}{d\varphi}\frac{dv}{d\varphi}(\sin\varphi)^{d-1}\, d\varphi, \quad u, v \in C_0^\infty(X^{(1)}). \tag{2.13}$$

We note that $(\mathcal{E}^1, C^\infty(X_1^{(2)}))$ and $(\mathcal{E}^{d-1,(1)}, C_0^\infty(X^{(1)}))$ are closable on $L^2(X_1^{(2)}, m_1^{(2)})$ and $L^2(X^{(1)}, m_{d-1}^{(1)})$, respectively. Their closures are regular Dirichlet forms, which are denoted by $(\mathcal{E}^1, \mathcal{F}^1)$ and $(\mathcal{E}^{d-1,(1)}, \mathcal{F}^{d-1,(1)})$, respectively. The former is corresponding to the circular Brownian motion BM$(S^1)$ and the latter is corresponding to LEG$(d)$ with generator (2.10). By virtue of [4] and [6], $(\mathcal{E}^d, C_0^\infty(X_d^{(2)}))$ is closable on $L^2(X_d^{(2)}, m_d^{(2)})$ and the closure $(\mathcal{E}^d, \mathcal{F}^d)$ is a regular Dirichlet form corresponding to BM$(S^d)$. In the following we denote by $\Theta = [\Theta_t, P_\theta^\Theta]$ and $(\mathcal{E}^\Theta, \mathcal{F}^\Theta)$ the spherical Brownian motion BM$(S^{d-1})$ and the corresponding Dirichlet form $(\mathcal{E}^{d-1}, \mathcal{F}^{d-1})$, respectively.

We denote by $p_t^\Theta$ the semigroup of the spherical Brownian motion $\Theta$, that is,

$$\begin{aligned} p_t^\Theta f(\theta) &= E^{P_\theta^\Theta}[f(\Theta_t)] \\ &= \int_{S^{d-1}} p^\Theta(t, \theta, \varphi) f(\varphi)\, dm_{d-1}^{(2)}(\varphi), \quad t > 0,\ \theta \in S^{d-1}, \end{aligned} \tag{2.14}$$

for $f \in C_b(S^{d-1})$, where $p^\Theta(t, \theta, \varphi)$ stands for the transition probability density of $\Theta$. It is known that $p^\Theta(t, \theta, \varphi)$ is represented by spherical harmonics $S_n^l$, that is,

$$p^\Theta(t, \theta, \varphi) = \sum_{n=0}^\infty e^{-\gamma_n t} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi), \tag{2.15}$$

where $\gamma_n = (1/2)n(n+d-2)$, $\kappa(n) = (2n+d-2)\cdot(n+d-3)!/n!\,(d-2)!$ which is the number of spherical harmonics of weight $n$, $(1/2)\Delta S_n^l = -\gamma_n S_n^l$, and

$$\int_{S^{d-1}} S_n^l S_n^k\, dm_{d-1}^{(2)} = \begin{cases} 1, & l = k, \\ 0, & l \neq k, \end{cases}$$

(see [2], [6]). We set $A_{d-1} = \int_{S^{d-1}} dm_{d-1}^{(2)}$ (the total area of the spherical surface $S^{d-1}$), so that $S_0^1 = A_{d-1}^{-1/2}$. Note that $\kappa(0) = 1$. When $d = 2$, (2.15) is reduced to

$$\begin{aligned} &p^\Theta(t, \theta, \varphi) \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^\infty e^{-n^2 t/2}\{\cos n\theta \cos n\varphi + \sin n\theta \sin n\varphi\} \\ &= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^\infty e^{-n^2 t/2} \cos n(\theta - \varphi). \end{aligned} \tag{2.16}$$

**2.3. Skew product.** Now we turn to a skew product of $\mathrm{R} = [R_t, P_r^{\mathrm{R}}]$ and $\Theta = [\Theta_t, P_\theta^\Theta]$. It is known that the ODGDP R has the local time $l^{\mathrm{R}}(t,r)$ which is continuous with respect to $(t,r) \in [0,\infty) \times I$ and satisfies $\int_0^t 1_A(R_u)\,du = \int_A l^{\mathrm{R}}(t,r)\,dm^{\mathrm{R}}(r)$, $t > 0$, for every measurable set $A \subset I$ (see [6]), where $1_A$ is the indicator for a set $A$. Let $\nu$ be a Radon measure on $I$ and assume that $\mathrm{supp}[\nu]$, the support of $\nu$, coincides with $I$. We set

$$\mathbf{f}(t) = \int_I l^{\mathrm{R}}(t,r)\,d\nu(r). \tag{2.17}$$

Since $\mathrm{supp}[\nu] = I$, we see that

$$P_r^{\mathrm{R}}(\mathbf{f}(t) > 0,\ t > 0) = 1, \quad r \in I. \tag{2.18}$$

We assume (2.7). Let $\mathrm{X} = [X_t = (R_t, \Theta_{\mathbf{f}(t)}),\ P^{\mathrm{X}}_{(r,\theta)} = P_r^{\mathrm{R}} \otimes P_\theta^\Theta,\ (r,\theta) \in I \times S^{d-1}]$ be the skew product of the ODGDP R and the spherical Brownian motion $\Theta$ with respect to the PCAF $\mathbf{f}(t)$, and set

$$\begin{aligned}\mathcal{E}^{\mathrm{X}}(f,g) = &\int_{S^{d-1}} \mathcal{E}^{\mathrm{R}}(f(\,\cdot\,,\theta), g(\,\cdot\,,\theta))\,dm^{(2)}_{d-1}(\theta)\\ &+ \int_I \mathcal{E}^\Theta(f(r,\,\cdot\,), g(r,\,\cdot\,))\,d\nu(r),\end{aligned} \tag{2.19}$$

for $f, g \in \mathcal{C}^{\mathrm{X}}$, where $\mathcal{C}^{\mathrm{X}} = \{f(s^{\mathrm{R}}(r),\theta) \colon f \in C_0^\infty(J \times S^{d-1})\}$ and $J = s^{\mathrm{R}}(I)$. Then by means of Theorem 1.1 of [4] and (2.18), we immediately obtain the following result. So we omit the proof.

**Proposition 2.1.** *We assume* (2.7). *Then the form* $(\mathcal{E}^{\mathrm{X}}, \mathcal{C}^{\mathrm{X}})$ *is closable on* $L^2(I \times S^{d-1}, m^{\mathrm{R}} \otimes m^{(2)}_{d-1})$. *The closure* $(\mathcal{E}^{\mathrm{X}}, \mathcal{F}^{\mathrm{X}})$ *is a regular Dirichlet form and it is corresponding to the skew product* X.

Let denote by $p_t^{\mathrm{X}}$ the semigroup of the skew product X, that is,

$$p_t^{\mathrm{X}} f(r,\theta) = E^{P_r^{\mathrm{R}} \otimes P_\theta^\Theta}[f(R_t, \Theta_{\mathbf{f}(t)})], \quad t > 0,\ (r,\theta) \in I \times S^{d-1}, \tag{2.20}$$

for $f \in C_b(I \times S^{d-1})$. By virtue of (2.15) we obtain the following

$$\begin{aligned}p_t^{\mathrm{X}} f(r,\theta) &= \int_{S^{d-1}} E^{P_r^{\mathrm{R}}}[f(R_t,\varphi)p^\Theta(\mathbf{f}(t),\theta,\varphi)]\,dm^{(2)}_{d-1}(\varphi)\\ &= \sum_{n=0}^\infty \sum_{l=1}^{\kappa(n)} S_n^l(\theta) \int_{S^{d-1}} S_n^l(\varphi) E^{P_r^{\mathrm{R}}}[f(R_t,\varphi)e^{-\gamma_n \mathbf{f}(t)}]\,dm^{(2)}_{d-1}(\varphi).\end{aligned} \tag{2.21}$$

## 3. Feller property of the skew product

Let $\mathrm{X} = [X_t = (R_t, \Theta_{\mathbf{f}(t)}),\ P^{\mathrm{X}}_{(r,\theta)} = P^{\mathrm{R}}_r \otimes P^{\Theta}_{\theta},\ (r,\theta) \in I \times S^{d-1}]$ be the skew product of the ODGDP R and the spherical Brownian motion $\Theta$ with respect to the PCAF $\mathbf{f}(t)$ defined in the preceding section. We go forward with our argument under the assumption (2.7). We show Feller property of the skew product X. Since $E^{P^{\mathrm{R}}_r}[f(R_t,\eta)e^{-\gamma_n \mathbf{f}(t)}]$ is continuous in $r \in I$ (see [6]), we immediately obtain the following result by means of (2.21), so we omit the proof.

**Proposition 3.1.** *Let $f \in C_b(I \times S^{d-1})$ and $t > 0$. Then $p^{\mathrm{X}}_t f \in C_b(I \times S^{d-1})$.*

Next we observe the behavior of $p^{\mathrm{X}}_t f(r,\theta)$ as $r \to l_i$.

**Theorem 3.2.** *Let $i = 1, 2$, $t > 0$ and $f \in C_b(I \times S^{d-1})$.*
(i) *Assume that the end point $l_i$ is $(s^{\mathrm{R}}, m^{\mathrm{R}})$-entrance, and the measure $\nu$ satisfies*

$$\left| \int_{(c,l_i)} s^{\mathrm{R}}(r)\, d\nu(r) \right| = \infty. \tag{3.1}$$

*Further assume that there exists the limit $\lim_{r\to l_i} f(r,\theta)$ for any $\theta \in S^{d-1}$. Then there exist the following limits.*

$$E^{P^{\mathrm{R}}_{l_i}}[f(R_t,\theta)] := \lim_{r\to l_i} E^{P^{\mathrm{R}}_r}[f(R_t,\theta)], \quad \theta \in S^{d-1}. \tag{3.2}$$

$$\lim_{r\to l_i} p^{\mathrm{X}}_t f(r,\theta) = \frac{1}{A_{d-1}} \int_{S^{d-1}} E^{P^{\mathrm{R}}_{l_i}}[f(R_t,\varphi)]\, dm^{(2)}_{d-1}(\varphi), \quad \theta \in S^{d-1}. \tag{3.3}$$

*Note that the limit* (3.3) *is independent of $\theta$.*
(ii) *Assume that the end point $l_i$ is $(s^{\mathrm{R}}, m^{\mathrm{R}})$-natural and $f$ satisfies*

$$\lim_{r\to l_i} \sup_{\theta\in S^{d-1}} |f(r,\theta)| = 0.$$

*Then*

$$\lim_{r\to l_i} p^{\mathrm{X}}_t f(r,\theta) = 0, \quad \theta \in S^{d-1}. \tag{3.4}$$

Proof. (i) We only show the statement for $i = 1$. Assume that the end point $l_1$ is $(s^{\mathrm{R}}, m^{\mathrm{R}})$-entrance, and there exists the limit $\lim_{r\to l_1} f(r,\theta)$ for any $\theta \in S^{d-1}$. Then, by means of (2.4), there exists the limit

$$E^{P^{\mathrm{R}}_{l_1}}[f(R_t,\theta)] := \lim_{r\to l_1} E^{P^{\mathrm{R}}_r}[f(R_t,\theta)], \quad \theta \in S^{d-1}.$$

We claim that, if $\nu$ satisfies (3.1),

$$\lim_{r\to l_1} E^{P^{\mathrm{R}}_r}[f(R_t,\theta)e^{-C\mathbf{f}(t)}] = 0, \quad \theta \in S^{d-1}, \tag{3.5}$$

for any positive constant $C$. This fact is obtained by Itô and McKean [6]. Their idea is as follows. Since the support of $m^{\mathrm{R}}$ coincides with $I$, we can employ the argument in [6] and find that the time changed process $\mathrm{Q} = [R_{\mathbf{f}^{-1}(t)}, P_r^{\mathrm{R}}]$ is an ODGDP with the scale function $s^{\mathrm{R}}$ and the speed measure $\nu$, where $\mathbf{f}^{-1}$ is the inverse of $\mathbf{f}$. Since the end point $l_1$ is $(s^{\mathrm{R}}, m^{\mathrm{R}})$-entrance, we see $s^{\mathrm{R}}(l_1) = -\infty$. Combining this with (3.1), we find that the end point $l_1$ is $(s^{\mathrm{R}}, \nu)$-natural. Since $l_1$ is $(s^{\mathrm{R}}, m^{\mathrm{R}})$-entrance, we have

$$\limsup_{a\to l_1}\limsup_{r\to l_1} P_r^{\mathrm{R}}(\mathbf{f}(t) = \infty,\ t < \sigma_a^{\mathrm{R}}) \le \limsup_{a\to l_1}\limsup_{r\to l_1} P_r^{\mathrm{R}}(t < \sigma_a^{\mathrm{R}}) = 0. \tag{3.6}$$

Since $l_1$ is $(s^{\mathrm{R}}, \nu)$-natural and $\mathbf{f}(\sigma_a^{\mathrm{R}})$ is the first hitting time to the point $a$ for the ODGDP Q (see [6]), we obtain

$$\lim_{r\to l_1} E^{P_r^{\mathrm{R}}}[e^{-\mathbf{f}(\sigma_a^{\mathrm{R}})}] = 0, \quad a \in I.$$

Therefore

$$\begin{aligned}
&\liminf_{a\to l_1}\liminf_{r\to l_1} P_r^{\mathrm{R}}(\mathbf{f}(t) = \infty,\ t > \sigma_a^{\mathrm{R}})\\
&\ge \liminf_{a\to l_1}\liminf_{r\to l_1} P_r^{\mathrm{R}}(\mathbf{f}(\sigma_a^{\mathrm{R}}) = \infty,\ t > \sigma_a^{\mathrm{R}})\\
&= \liminf_{a\to l_1}\liminf_{r\to l_1} P_r^{\mathrm{R}}(t > \sigma_a^{\mathrm{R}}) = 1,
\end{aligned}$$

where we used the fact that $l_1$ is $(s^{\mathrm{R}}, m^{\mathrm{R}})$-entrance. Thus we obtain that

$$\lim_{r\to l_1} P_r^{\mathrm{R}}(\mathbf{f}(t) = \infty) = 1, \quad t > 0,$$

which implies (3.5). By using (2.21) and (3.5), we arrive at

$$\begin{aligned}
\lim_{r\to l_1} p_t^{\mathrm{X}} f(r,\theta) &= S_0^1(\theta)\int_{S^{d-1}} S_0^1(\varphi) E^{P_{l_1}^{\mathrm{R}}}[f(R_t,\varphi)]\, dm_{d-1}^{(2)}(\varphi)\\
&= \frac{1}{A_{d-1}}\int_{S^{d-1}} E^{P_{l_1}^{\mathrm{R}}}[f(R_t,\varphi)]\, dm_{d-1}^{(2)}(\varphi).
\end{aligned}$$

(ii) Assume that the end point $l_i$ is $(s^{\mathrm{R}}, m^{\mathrm{R}})$-natural and $\lim_{r\to l_i}\sup_{\theta\in S^{d-1}}|f(r,\theta)| = 0$. We set $h(r) = \sup_{\theta\in S^{d-1}}|f(r,\theta)|$. Then by means of (2.5) and (2.20),

$$\limsup_{r\to l_i}\sup_{\theta\in S^{d-1}}|p_t^{\mathrm{X}} f(r,\theta)| \le \limsup_{r\to l_i} E^{P_r^{\mathrm{R}}}[h(R_t)] = \lim_{r\to l_i} p_t^{\mathrm{R}} h(r) = 0.$$

Thus we obtain (3.4). □

## 4. Feller property of time changed processes

Let $\mathrm{X} = [X_t = (R_t, \Theta_{\mathbf{f}(t)}),\ P_{(r,\theta)}^{\mathrm{X}} = P_r^{\mathrm{R}} \otimes P_\theta^{\Theta},\ (r,\theta) \in I \times S^{d-1}]$ be the skew product of the ODGDP R and the spherical Brownian motion $\Theta$ with respect to the PCAF $\mathbf{f}(t)$

defined in Section 2. In this section we consider a time changed process of X and show its Feller property under the assumption (2.7).

Let $\mu$ be a non-trivial Radon measure on $I$ and set

$$\mathbf{g}(t) = \int_I l^{\mathrm{R}}(t, r)\, d\mu(r), \quad t > 0. \tag{4.1}$$

We denote by $\tau(t)$ the right continuous inverse of $\mathbf{g}(t)$. We consider the time changed process $\mathrm{Y} = [Y_t = (R_{\tau(t)}, \theta_{\mathbf{f}(\tau(t))}),\ P^{\mathrm{Y}}_{(r,\theta)} = P^{\mathrm{R}}_r \otimes P^{\Theta}_{\theta},\ (r, \theta) \in I \times S^{d-1}]$. Let denote by $p^{\mathrm{Y}}_t$ the semigroup of Y, that is,

$$p^{\mathrm{Y}}_t f(r, \theta) = E^{P^{\mathrm{R}}_r \otimes P^{\Theta}_{\theta}}[f(R_{\tau(t)}, \Theta_{\mathbf{f}(\tau(t))})], \quad t > 0,\ (r, \theta) \in I \times S^{d-1}, \tag{4.2}$$

for $f \in C_b(I \times S^{d-1})$. By virtue of (2.15) we obtain the following

$$\begin{aligned} &p^{\mathrm{Y}}_t f(r, \theta) \\ &= \int_{S^{d-1}} E^{P^{\mathrm{R}}_r}[f(R_{\tau(t)}, \varphi) p^{\Theta}(\mathbf{f}(\tau(t)), \theta, \varphi)]\, dm^{(2)}_{d-1}(\varphi) \\ &= \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S^l_n(\theta) \int_{S^{d-1}} S^l_n(\varphi) E^{P^{\mathrm{R}}_r}[f(R_{\tau(t)}, \varphi) e^{-\gamma_n \mathbf{f}(\tau(t))}]\, dm^{(2)}_{d-1}(\varphi). \end{aligned} \tag{4.3}$$

Note that the time changed process $\mathrm{U} = [R_{\tau(t)}, P^{\mathrm{R}}_r]$ is an ODGDP with the scale function $s^{\mathrm{R}}$ and the speed measure $\mu$. We set $\Lambda = \mathrm{supp}[\mu]$ and $\Gamma = \Lambda \times S^{d-1}$. Also note that the time changed process Y is essentially defined on $\Gamma$. Since $E^{P^{\mathrm{R}}_r}[f(R_{\tau(t)}, \varphi) e^{-\gamma_n \mathbf{f}(\tau(t))}]$ is continuous in $r \in \Lambda$ (see [6]), the following result is obvious by means of (4.3). So we omit the proof.

**Proposition 4.1.** *Let $f \in C_b(\Gamma)$ and $t > 0$. Then $p^{\mathrm{Y}}_t f \in C_b(\Gamma)$.*

We observe the behavior of $p^{\mathrm{Y}}_t f(r, \theta)$ as $r\ (\in \Lambda) \to l_1$ (resp. $l_2$) when $l_1 = \inf \Lambda$ (resp. $l_2 = \sup \Lambda$).

**Theorem 4.2.** *Let $f \in C_b(\Gamma)$ and $t > 0$. The following properties hold true for the end point $l_i$ satisfying $l_1 = \inf \Lambda$ or $l_2 = \sup \Lambda$.*
(i) *If the end point $l_i$ is $(s^{\mathrm{R}}, \mu)$-regular or exit, then*

$$\lim_{r\ (\in \Lambda) \to l_i} p^{\mathrm{Y}}_t f(r, \theta) = 0, \quad \theta \in S^{d-1}. \tag{4.4}$$

(ii) *Assume that the end point $l_i$ is $(s^{\mathrm{R}}, \mu)$-entrance, and the measure $\nu$ satisfies* (3.1). *Further assume that there exists the limit $\lim_{r\ (\in \Lambda) \to l_i} f(r, \theta)$ for any $\theta \in S^{d-1}$. Then* *there exist the following limits.*

$$E^{P^{\mathrm{R}}_{l_i}}[f(R_{\tau(t)},\theta)] := \lim_{r\,(\in\Lambda)\to l_i} E^{P^{\mathrm{R}}_r}[f(R_{\tau(t)},\theta)], \quad \theta\in S^{d-1}. \tag{4.5}$$

$$\lim_{r\,(\in\Lambda)\to l_i} p^{\mathrm{Y}}_t f(r,\theta) = \frac{1}{A_{d-1}}\int_{S^{d-1}} E^{P^{\mathrm{R}}_{l_i}}[f(R_{\tau(t)},\varphi)]\,dm^{(2)}_{d-1}(\varphi), \quad \theta\in S^{d-1}. \tag{4.6}$$

*Note that the limit* (4.6) *is independent of* $\theta$.
(iii) *Assume that the end point* $l_i$ *is* $(s^{\mathrm{R}},\mu)$*-natural and* $f$ *satisfies*

$$\lim_{r\,(\in\Lambda)\to l_i}\ \sup_{\theta\in S^{d-1}} |f(r,\theta)| = 0.$$

*Then* (4.4) *holds true.*

Proof. We may assume that $l_1 = \inf\Lambda$. We show the statements for $l_1$.

(i) Assume that the end point $l_1$ is $(s^{\mathrm{R}},\mu)$-regular or exit. By virtue of (2.3) for U we get

$$\limsup_{r\,(\in\Lambda)\to l_1} |E^{P^{\mathrm{R}}_r}[f(R_{\tau(t)},\theta)e^{-\mathbf{f}(\tau(t))}]| \le \limsup_{r\,(\in\Lambda)\to l_1} E^{P^{\mathrm{R}}_r}[|f(R_{\tau(t)},\theta)|] = 0, \quad \theta\in S^{d-1}.$$

Combining this with the dominated convergence theorem and (4.3), we obtain the statement (i).

(ii) Assume that the end point $l_1$ is $(s^{\mathrm{R}},\mu)$-entrance, and there exists the limit $\lim_{r\,(\in\Lambda)\to l_1} f(r,\theta)$ for any $\theta\in S^{d-1}$. Then, by means of (2.4) for the ODGDP U, there exists the limit

$$E^{P^{\mathrm{R}}_{l_1}}[f(R_{\tau(t)},\theta)] := \lim_{r\,(\in\Lambda)\to l_1} E^{P^{\mathrm{R}}_r}[f(R_{\tau(t)},\theta)], \quad \theta\in S^{d-1}.$$

Note that $\lim_{r\,(\in\Lambda)\to l_1} P^{\mathrm{R}}_r(\tau(t)>0)=1$. Therefore, by the same argument as for (3.5), we obtain

$$\lim_{r\,(\in\Lambda)\to l_1} E^{P^{\mathrm{R}}_r}[f(R_{\tau(t)},\theta)e^{-C\mathbf{f}(\tau(t))}] = 0, \quad \theta\in S^{d-1}, \tag{4.7}$$

for any positive constant $C$. Combining this with (4.3), we find

$$\begin{aligned}\lim_{r\,(\in\Lambda)\to l_1} p^{\mathrm{Y}}_t f(r,\theta) &= S^1_0(\theta)\int_{S^{d-1}} S^1_0(\varphi)E^{P^{\mathrm{R}}_{l_1}}[f(R_{\tau(t)},\varphi)]\,dm^{(2)}_{d-1}(\varphi)\\ &= \frac{1}{A_{d-1}}\int_{S^{d-1}} E^{P^{\mathrm{R}}_{l_1}}[f(R_{\tau(t)},\varphi)]\,dm^{(2)}_{d-1}(\varphi).\end{aligned}$$

(iii) Assume that the end point $l_1$ is $(s^{\mathrm{R}},\mu)$-natural and

$$\lim_{r\,(\in\Lambda)\to l_1}\ \sup_{\theta\in S^{d-1}} |f(r,\theta)| = 0.$$

We set $h(r)=\sup_{\theta\in S^{d-1}}|f(r,\theta)|$. Then by means of (2.5) for the ODGDP U and (4.2),

$$\limsup_{r\,(\in\Lambda)\to l_1}\sup_{\theta\in S^{d-1}}|p_t^{\mathrm{Y}}f(r,\theta)|\le\limsup_{r\,(\in\Lambda)\to l_1}E^{P_r^{\mathrm{R}}}[h(R_{\tau(t)})]=0.$$

Thus we obtain (4.4). □

## 5. Dirichlet form of the time changed process

In this section, we derive the Dirichlet form $(\mathcal{E}^{\mathrm{Y}},\mathcal{F}^{\mathrm{Y}})$ of the time changed process Y defined in the preceding section. Y is a time changed process of X. X is the skew product of R and $\Theta$ with respect to $\mathbf{f}$ defined by (2.17), and the Dirichlet form $(\mathcal{E}^{\mathrm{X}},\mathcal{F}^{\mathrm{X}})$ corresponding to X is given in Proposition 2.1. In the following we assume (2.7) and that

(5.1) for any compact set $B\subset I$, there exists a positive constant $M_B$ satisfying $1_B(r)\,ds^{\mathrm{R}}(r)\le M_B1_B(r)\,dm^{\mathrm{R}}(r)$.

We note that the measure $\mu\otimes m_{d-1}^{(2)}$ charges no set of zero $\mathcal{E}^{\mathrm{X}}$-capacity. For this, it is enough to show that, for every compact set $B\subset I$, there is a positive constant $C$ such that

$$\int_{B\times S^{d-1}}|u(r,\theta)|\,d\mu(r)\,dm_{d-1}^{(2)}(\theta)\le C\mathcal{E}_1^{\mathrm{X}}(u,u)^{1/2},\quad u\in\mathcal{C}^{\mathrm{X}},\tag{5.2}$$

that is, $1_B(r)\,d\mu(r)\,dm_{d-1}^{(2)}(\theta)$ is of finite energy integral, where $\mathcal{E}_1^{\mathrm{X}}(u,u)=\mathcal{E}^{\mathrm{X}}(u,u)+(u,u)_{L^2(m^{\mathrm{R}}\otimes m_{d-1}^{(2)};I\times S^{d-1})}$. Let $\Phi$ be an element of $C_0^\infty(J)$ such that $\Phi(s^{\mathrm{R}}(r))=1$ for $r\in B$. We set $D=\operatorname{supp}[\Phi\circ s^{\mathrm{R}}]$. Then we find that

$$\begin{aligned}&\int_{B\times S^{d-1}}|u(r,\theta)|\,d\mu(r)\,dm_{d-1}^{(2)}(\theta)\\&\le\mu(B)A_{d-1}^{1/2}\biggl\{\sqrt{2}\mathcal{E}^{\mathrm{X}}(u,u)^{1/2}\biggl(\int_J\Phi(\xi)^2\,d\xi\biggr)^{1/2}\\&\qquad+M_D^{1/2}\biggl(\int_{I\times S^{d-1}}u(r,\theta)^2\,dm^{\mathrm{R}}(r)\,dm_{d-1}^{(2)}(\theta)\biggr)^{1/2}\biggl(\int_J\Phi'(\xi)^2\,d\xi\biggr)^{1/2}\biggr\},\end{aligned}$$

which implies (5.2). We note that $\mathbf{g}(t)$ defined by (4.1) is a PCAF of X and $P_{(r,\theta)}^{\mathrm{X}}(\mathbf{g}(t)>0,\ t>0)=1$ for $(r,\theta)\in\Gamma$. Employing Theorem 6.2.1 in [5], we see that the Dirichlet form $(\mathcal{E}^{\mathrm{Y}},\mathcal{F}^{\mathrm{Y}})$ is regular on $L^2(\Gamma,\mu\otimes m_{d-1}^{(2)})$ and has $\mathcal{C}^{\mathrm{X}}|_\Gamma$ as a core, where $\mathcal{C}^{\mathrm{X}}|_\Gamma=\{u|_\Gamma : u\in\mathcal{C}^X\}$.

The following lemma is easily obtained, so the proof is omitted.

**Lemma 5.1.** *Assume that* $\int_\Lambda ds^{\mathrm{R}} > 0$. *Let* $u \in \mathcal{C}^{\mathrm{X}}$ *and put* $f = u|_\Gamma$. *Then there exists the limit*

$$\partial^*_{s^{\mathrm{R}}} f(r,\theta) := \lim_{r'(\in \Lambda)\to r} \frac{f(r',\theta) - f(r,\theta)}{s^{\mathrm{R}}(r') - s^{\mathrm{R}}(r)} = \lim_{r'\to r} \frac{u(r',\theta) - u(r,\theta)}{s^{\mathrm{R}}(r') - s^{\mathrm{R}}(r)},$$

*for* $ds^{\mathrm{R}}$*-a.e.* $r \in \Lambda$ *and every* $\theta \in S^{d-1}$.

If $\Lambda = I$, then $\mathcal{E}^{\mathrm{Y}}(u, v) = \mathcal{E}^{\mathrm{X}}(u, v)$ for $u, v \in \mathcal{C}^{\mathrm{X}}$. Therefore we are restricted to the case that $I \setminus \Lambda \neq \emptyset$. For a set $E \subset I$ we put

$$\mathcal{E}^{\mathrm{R}}_E(u, v) = \int_E \frac{du}{ds^{\mathrm{R}}} \frac{dv}{ds^{\mathrm{R}}}\, ds^{\mathrm{R}},$$

$$\mathcal{E}^{\mathrm{X}}_E(f, g) = \int_{S^{d-1}} \mathcal{E}^{\mathrm{R}}_E(f(\,\cdot\,,\theta), g(\,\cdot\,,\theta))\, dm^{(2)}_{d-1}(\theta) + \int_E \mathcal{E}^{\Theta}(f(r,\,\cdot\,), g(r,\,\cdot\,))\, d\nu(r).$$

We note that $I \setminus \Lambda = \bigcup_{k\in K} I_k$, a finite or a countable disjoint union of open intervals $I_k = (a_k, b_k)$ with the end points belonging to $\Lambda \cup \{l_1, l_2\}$. Since $\mathcal{C}^{\mathrm{X}}|_\Gamma$ is a core of $(\mathcal{E}^{\mathrm{Y}}, \mathcal{F}^{\mathrm{Y}})$, we fix a $u \in \mathcal{C}^{\mathrm{X}}$ and set $f = u|_\Gamma$. Then $f \in \mathcal{F}^{\mathrm{Y}}$ and

$$\mathcal{E}^{\mathrm{Y}}(f, f) = \mathcal{E}^{\mathrm{X}}(H_\Gamma u, H_\Gamma u), \tag{5.3}$$

where $H_\Gamma u(r,\theta) = E^{P^{\mathrm{X}}_{(r,\theta)}}[u(X_{\sigma^{\mathrm{X}}_\Gamma}); \sigma^{\mathrm{X}}_\Gamma < \infty]$, and $\sigma^{\mathrm{X}}_\Gamma = \inf\{t > 0\colon X_t \in \Gamma\}$. By means of (2.19) and (5.3) we see that

$$\mathcal{E}^{\mathrm{Y}}(f, f) = \mathcal{E}^{\mathrm{X}}_\Lambda(H_\Gamma u, H_\Gamma u) + \sum_{k\in K} \mathcal{E}^{\mathrm{X}}_{I_k}(H_\Gamma u, H_\Gamma u). \tag{5.4}$$

**Lemma 5.2.** *It holds true that*

$$\begin{aligned}&\mathcal{E}^{\mathrm{X}}_\Lambda(H_\Gamma u, H_\Gamma u)\\ &= \int_\Gamma \partial^*_{s^{\mathrm{R}}} f(r,\theta)^2\, ds^{\mathrm{R}}(r)\, dm^{(2)}_{d-1}(\theta) + \int_\Lambda \mathcal{E}^{\Theta}(f(r,\,\cdot\,), f(r,\,\cdot\,))\, d\nu(r).\end{aligned} \tag{5.5}$$

*If* $\int_\Lambda ds^{\mathrm{R}}(r) = 0$, *then the first term of the right hand side vanishes.*

Proof. Since $P^{\mathrm{X}}_{(r,\theta)}(\sigma^{\mathrm{X}}_\Gamma = 0) = 1$ for $(r,\theta) \in \Gamma$, $H_\Gamma u = u = f$ on $\Gamma$. Combining this with Lemma 5.1, we obtain (5.5). □

We are going to derive an explicit form of $\mathcal{E}^{\mathrm{X}}_{I_k}(H_\Gamma u, H_\Gamma u)$. For $r \in I_k = (a_k, b_k)$ and $\theta, \varphi \in S^{d-1}$, we set

$$G_{k,1}(r;\theta,\varphi) = E^{P^{\mathrm{R}}_r}[p^{\Theta}(\mathbf{f}(\sigma^{\mathrm{R}}_{b_k}), \theta, \varphi); \sigma^{\mathrm{R}}_{b_k} < \sigma^{\mathrm{R}}_{a_k}], \tag{5.6}$$

$$G_{k,2}(r;\theta,\varphi) = E^{P^{\mathrm{R}}_r}[p^{\Theta}(\mathbf{f}(\sigma^{\mathrm{R}}_{a_k}), \theta, \varphi); \sigma^{\mathrm{R}}_{a_k} < \sigma^{\mathrm{R}}_{b_k}]. \tag{5.7}$$

By means of (2.15) we see that

$$G_{k,1}(r;\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{l=1}^{\kappa(n)}S_n^l(\theta)S_n^l(\varphi)E^{P_r^{\mathrm{R}}}[e^{-\gamma_n\mathbf{f}(\sigma_{b_k}^{\mathrm{R}})};\sigma_{b_k}^{\mathrm{R}}<\sigma_{a_k}^{\mathrm{R}}], \tag{5.8}$$

$$G_{k,2}(r;\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{l=1}^{\kappa(n)}S_n^l(\theta)S_n^l(\varphi)E^{P_r^{\mathrm{R}}}[e^{-\gamma_n\mathbf{f}(\sigma_{a_k}^{\mathrm{R}})};\sigma_{a_k}^{\mathrm{R}}<\sigma_{b_k}^{\mathrm{R}}], \tag{5.9}$$

for $r\in I_k=(a_k,b_k)$ and $\theta,\varphi\in S^{d-1}$.

**Lemma 5.3.** *Let $r\in I_k$ and $\theta\in S^{d-1}$. If $l_1=a_k<b_k<l_2$, then*

$$H_\Gamma u(r,\theta)=\int_{S^{d-1}}f(b_k,\varphi)G_{k,1}(r;\theta,\varphi)\,dm_{d-1}^{(2)}(\varphi). \tag{5.10}$$

*If $l_1<a_k<b_k=l_2$, then*

$$H_\Gamma u(r,\theta)=\int_{S^{d-1}}f(a_k,\varphi)G_{k,2}(r;\theta,\varphi)\,dm_{d-1}^{(2)}(\varphi). \tag{5.11}$$

*If $l_1<a_k<b_k<l_2$, then*

$$H_\Gamma u(r,\theta)=\int_{S^{d-1}}\{f(a_k,\varphi)G_{k,2}(r;\theta,\varphi)+f(b_k,\varphi)G_{k,1}(r;\theta,\varphi)\}\,dm_{d-1}^{(2)}(\varphi). \tag{5.12}$$

Proof. Let $l_1<a_k<b_k<l_2$, $r\in I_k$ and $\theta\in S^{d-1}$. Note that $P_{(r,\theta)}^{\mathrm{X}}(\sigma_\Gamma^{\mathrm{X}}=\sigma_{a_k}^{\mathrm{R}}\wedge\sigma_{b_k}^{\mathrm{R}}<\infty)=1$. Therefore, by virtue of (2.15), we find that

$$\begin{aligned}
H_\Gamma u(r,\theta)&=E^{P_{(r,\theta)}^{\mathrm{X}}}[u(R_{\sigma_\Gamma^{\mathrm{X}}},\Theta_{\mathbf{f}(\sigma_\Gamma^{\mathrm{X}})});\sigma_\Gamma^{\mathrm{X}}<\infty]\\
&=\sum_{n=0}^{\infty}\sum_{l=1}^{\kappa(n)}S_n^l(\theta)\int_{S^{d-1}}S_n^l(\varphi)E^{P_r^{\mathrm{R}}}\big[u\big(R_{\sigma_{a_k}^{\mathrm{R}}\wedge\sigma_{b_k}^{\mathrm{R}}},\varphi\big)e^{-\gamma_n\mathbf{f}(\sigma_{a_k}^{\mathrm{R}}\wedge\sigma_{b_k}^{\mathrm{R}})}\big]\,dm_{d-1}^{(2)}(\varphi)\\
&=\sum_{n=0}^{\infty}\sum_{l=1}^{\kappa(n)}S_n^l(\theta)\int_{S^{d-1}}S_n^l(\varphi)\big\{f(a_k,\varphi)E^{P_r^{\mathrm{R}}}\big[e^{-\gamma_n\mathbf{f}(\sigma_{a_k}^{\mathrm{R}})};\sigma_{a_k}^{\mathrm{R}}<\sigma_{b_k}^{\mathrm{R}}\big]\\
&\qquad\qquad+f(b_k,\varphi)E^{P_r^{\mathrm{R}}}\big[e^{-\gamma_n\mathbf{f}(\sigma_{b_k}^{\mathrm{R}})};\sigma_{b_k}^{\mathrm{R}}<\sigma_{a_k}^{\mathrm{R}}\big]\big\}\,dm_{d-1}^{(2)}(\varphi).
\end{aligned}$$

Combining this with (5.8) and (5.9), we obtain (5.12).

Let $l_1=a_k<b_k<l_2$, $r\in I_k$ and $\theta\in S^{d-1}$. Then $P_{(r,\theta)}^{\mathrm{X}}(\sigma_\Gamma^{\mathrm{X}}=\sigma_{b_k}^{\mathrm{R}}<\infty)=P_{(r,\theta)}^{\mathrm{X}}(\sigma_\Gamma^{\mathrm{X}}=\sigma_{b_k}^{\mathrm{R}}<\sigma_{a_k}^{\mathrm{R}})$. Therefore we obtain (5.10) in the same way as above.

We also obtain (5.11) by the same argument as that for (5.10). □

By virtue of a general theory of ODGDP's, there exist the following limits (see [8]).

$$J_k^{1,1}(\theta, \varphi) := \lim_{r \downarrow a_k} D_{s^{\mathrm{R}}(r)} G_{k,2}(r; \theta, \varphi). \tag{5.13}$$

$$J_k^{1,2}(\theta, \varphi) := \lim_{r \downarrow a_k} D_{s^{\mathrm{R}}(r)} G_{k,1}(r; \theta, \varphi). \tag{5.14}$$

$$J_k^{2,1}(\theta, \varphi) := -\lim_{r \uparrow b_k} D_{s^{\mathrm{R}}(r)} G_{k,2}(r; \theta, \varphi). \tag{5.15}$$

$$J_k^{2,2}(\theta, \varphi) := -\lim_{r \uparrow b_k} D_{s^{\mathrm{R}}(r)} G_{k,1}(r; \theta, \varphi). \tag{5.16}$$

We denote by $\mathcal{M}$ the product measure $m_{d-1}^{(2)} \otimes m_{d-1}^{(2)}$.

**Lemma 5.4.** (i) *Let $l_1 = a_k < b_k < l_2$. Then*

$$\begin{aligned} &\mathcal{E}_{I_k}^{\mathrm{X}}(H_\Gamma u, H_\Gamma u) \\ &= \frac{1}{2} \int_{S^{d-1} \times S^{d-1}} \{ f(b_k, \theta) - f(b_k, \varphi)\}^2 J_k^{2,2}(\theta, \varphi)\, d\mathcal{M}(\theta, \varphi) \\ &\quad + \frac{1}{s^{\mathrm{R}}(b_k) - s^{\mathrm{R}}(l_1)} \int_{S^{d-1}} f(b_k, \theta)^2\, dm_{d-1}^{(2)}(\theta). \end{aligned} \tag{5.17}$$

*The second term of the right hand side vanishes if $s^{\mathrm{R}}(l_1) = -\infty$.*
(ii) *Let $l_1 < a_k < b_k = l_2$. Then*

$$\begin{aligned} &\mathcal{E}_{I_k}^{\mathrm{X}}(H_\Gamma u, H_\Gamma u) \\ &= \frac{1}{2} \int_{S^{d-1} \times S^{d-1}} \{ f(a_k, \theta) - f(a_k, \varphi)\}^2 J_k^{1,1}(\theta, \varphi)\, d\mathcal{M}(\theta, \varphi) \\ &\quad + \frac{1}{s^{\mathrm{R}}(l_2) - s^{\mathrm{R}}(a_k)} \int_{S^{d-1}} f(a_k, \theta)^2\, dm_{d-1}^{(2)}(\theta). \end{aligned} \tag{5.18}$$

*The second term of the right hand side vanishes if $s^{\mathrm{R}}(l_2) = \infty$.*

Proof. We assume $l_1 = a_k < b_k < l_2$, and write $a$ and $b$ in place of $a_k$ and $b_k$, respectively. By means of Green's formula, (5.10) and (5.16),

$$\begin{aligned} &\mathcal{E}_{I_k}^{\mathrm{X}}(H_\Gamma u, H_\Gamma u) \\ &= \int_{S^{d-1}} H_\Gamma u(b, \theta) \lim_{r \uparrow b} D_{s^{\mathrm{R}}(r)} H_\Gamma u(r, \theta)\, dm_{d-1}^{(2)}(\theta) \\ &= -\int_{S^{d-1} \times S^{d-1}} f(b, \theta) f(b, \varphi) J_k^{2,2}(\theta, \varphi)\, d\mathcal{M}(\theta, \varphi) \\ &= \frac{1}{2} \int_{S^{d-1} \times S^{d-1}} \{ f(b, \theta) - f(b, \varphi)\}^2 J_k^{2,2}(\theta, \varphi) d\mathcal{M}(\theta, \varphi) \\ &\quad - \frac{1}{2} \int_{S^{d-1} \times S^{d-1}} \{ f(b, \theta)^2 + f(b, \varphi)^2\} J_k^{2,2}(\theta, \varphi)\, d\mathcal{M}(\theta, \varphi). \end{aligned}$$

Noting $J_k^{2,2}(\theta, \varphi) = J_k^{2,2}(\varphi, \theta)$, we get

$$\begin{aligned}
&-\frac{1}{2}\int_{S^{d-1}\times S^{d-1}}\{f(b,\theta)^2 + f(b,\varphi)^2\}\, J_k^{2,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi)\\
&= -\int_{S^{d-1}\times S^{d-1}} f(b,\theta)^2 J_k^{2,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi)\\
&= \int_{S^{d-1}} f(b,\theta)^2 \lim_{r\uparrow b} D_{s^{\mathrm{R}}(r)} H_\Gamma 1(r,\theta)\, dm_{d-1}^{(2)}(\theta)\\
&= \frac{1}{s^{\mathrm{R}}(b_k) - s^{\mathrm{R}}(l_1)} \int_{S^{d-1}} f(b,\theta)^2\, dm_{d-1}^{(2)}(\theta).
\end{aligned}$$

Here we used the following fact for the last equality.

$$H_\Gamma 1(r,\theta) = P^{\mathrm{X}}_{(r,\theta)}(\sigma^{\mathrm{X}}_\Gamma < \infty) = P^{\mathrm{R}}_r(\sigma^{\mathrm{R}}_b < \sigma^{\mathrm{R}}_a) = \frac{s^{\mathrm{R}}(r) - s^{\mathrm{R}}(a)}{s^{\mathrm{R}}(b) - s^{\mathrm{R}}(a)},$$

(see [6]). Thus we arrive at the first assertion. In the same way as above we obtain the second assertion. □

**Lemma 5.5.** *Let $l_1 < a_k < b_k < l_2$. Then*

$$\begin{aligned}
&\mathcal{E}^{\mathrm{X}}_{I_k}(H_\Gamma u, H_\Gamma u)\\
&= \frac{1}{2}\int_{S^{d-1}\times S^{d-1}}\{f(a_k,\theta) - f(a_k,\varphi)\}^2 J_k^{1,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi)\\
&\quad + \frac{1}{2}\int_{S^{d-1}\times S^{d-1}}\{f(a_k,\theta) - f(b_k,\varphi)\}^2 J_k^{1,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi)\\
&\quad + \frac{1}{2}\int_{S^{d-1}\times S^{d-1}}\{f(b_k,\theta) - f(a_k,\varphi)\}^2 J_k^{2,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi)\\
&\quad + \frac{1}{2}\int_{S^{d-1}\times S^{d-1}}\{f(b_k,\theta) - f(b_k,\varphi)\}^2 J_k^{2,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi).
\end{aligned} \tag{5.19}$$

Proof. We set $a = a_k$ and $b = b_k$. By means of Green's formula, (5.12), (5.13), (5.14), (5.15) and (5.16),

$$\begin{aligned}
&\mathcal{E}^{\mathrm{X}}_{I_k}(H_\Gamma u, H_\Gamma u)\\
&= \int_{S^{d-1}} H_\Gamma u(b,\theta) \lim_{r\uparrow b} D_{s^{\mathrm{R}}(r)} H_\Gamma u(r,\theta)\, dm_{d-1}^{(2)}(\theta)\\
&\quad - \int_{S^{d-1}} H_\Gamma u(a,\theta) \lim_{r\downarrow a} D_{s^{\mathrm{R}}(r)} H_\Gamma u(r,\theta)\, dm_{d-1}^{(2)}(\theta)
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{S^{d-1}\times S^{d-1}} f(b,\theta) f(a,\varphi) J_k^{2,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad - \int_{S^{d-1}\times S^{d-1}} f(b,\theta) f(b,\varphi) J_k^{2,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad - \int_{S^{d-1}\times S^{d-1}} f(a,\theta) f(a,\varphi) J_k^{1,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad - \int_{S^{d-1}\times S^{d-1}} f(a,\theta) f(b,\varphi) J_k^{1,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&= \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{ f(b,\theta) - f(a,\varphi)\}^2 J_k^{2,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad - \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{ f(b,\theta)^2 + f(a,\varphi)^2\} J_k^{2,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad + \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{ f(b,\theta) - f(b,\varphi)\}^2 J_k^{2,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad - \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{ f(b,\theta)^2 + f(b,\varphi)^2\} J_k^{2,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad + \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{ f(a,\theta) - f(a,\varphi)\}^2 J_k^{1,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad - \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{ f(a,\theta)^2 + f(a,\varphi)^2\} J_k^{1,1}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad + \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{ f(a,\theta) - f(b,\varphi)\}^2 J_k^{1,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\quad - \frac{1}{2}\int_{S^{d-1}\times S^{d-1}} \{ f(a,\theta)^2 + f(b,\varphi)^2\} J_k^{1,2}(\theta,\varphi)\, d\mathcal{M}(\theta,\varphi) \\
&\equiv V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8.
\end{aligned}
$$

In the same way as above, we also find

$$
\begin{aligned}
&\mathcal{E}_{I_k}^{\mathrm{X}}(H_\Gamma(u^2), H_\Gamma 1) \\
&= -\int_{S^{d-1}\times S^{d-1}} f(b,\theta)^2 \{ J_k^{2,1}(\theta,\varphi) + J_k^{2,2}(\theta,\varphi)\}\, d\mathcal{M}(\theta,\varphi) \\
&\quad - \int_{S^{d-1}\times S^{d-1}} f(a,\theta)^2 \{ J_k^{1,1}(\theta,\varphi) + J_k^{1,2}(\theta,\varphi)\}\, d\mathcal{M}(\theta,\varphi) \\
&= -\int_{S^{d-1}\times S^{d-1}} f(a,\varphi)^2 \{ J_k^{2,1}(\theta,\varphi) + J_k^{1,1}(\theta,\varphi)\}\, d\mathcal{M}(\theta,\varphi) \\
&\quad - \int_{S^{d-1}\times S^{d-1}} f(b,\varphi)^2 \{ J_k^{2,2}(\theta,\varphi) + J_k^{1,2}(\theta,\varphi)\}\, d\mathcal{M}(\theta,\varphi).
\end{aligned}
$$

Combining this with $H_\Gamma 1(r,\theta) = P_r^{\mathrm{R}}(\sigma_a^{\mathrm{R}} \wedge \sigma_b^{\mathrm{R}} < \infty) = 1$, we have

$$
V_2 + V_4 + V_6 + V_8 = \mathcal{E}_{I_k}^{\mathrm{X}}(H_\Gamma(u^2), H_\Gamma 1) = 0.
$$

Therefore we obtain the conclusion of the lemma. $\square$

By virtue of Lemmas 5.2, 5.4, and 5.5, we arrive at the following theorem.

**Theorem 5.6.** *Assume* $\Lambda \neq I$, (2.7) *and* (5.1). *Then the Dirichlet form* $(\mathcal{E}^{\mathrm{Y}}, \mathcal{F}^{\mathrm{Y}})$ *of* Y *is regular on* $L^2(\Gamma, \mu \otimes m^{(2)}_{d-1})$ *and has* $\mathcal{C}^{\mathrm{X}}|_{\Gamma}$ *as a core. For* $f \in \mathcal{C}^{\mathrm{X}}|_{\Gamma}$, *the Dirichlet form* $(\mathcal{E}^{\mathrm{Y}}, \mathcal{F}^{\mathrm{Y}})$ *is given by the following.*

(5.20)

$$\begin{aligned}
\mathcal{E}^{\mathrm{Y}}(f, f) = &\int_{\Gamma} \partial^*_{s^{\mathrm{R}}} f(r,\theta)^2 \, ds^{\mathrm{R}}(r) \, dm^{(2)}_{d-1}(\theta) + \int_{\Lambda} \mathcal{E}^{\Theta}(f(r,\cdot), f(r,\cdot)) \, d\nu(r) \\
&+ \frac{1}{2} \sum_{k \in K : l_1 < a_k < b_k \leq l_2} \int_{S^{d-1} \times S^{d-1}} \{ f(a_k,\theta) - f(a_k,\varphi)\}^2 J^{1,1}_k(\theta,\varphi) \, d\mathcal{M}(\theta,\varphi) \\
&+ \frac{1}{2} \sum_{k \in K : l_1 \leq a_k < b_k < l_2} \int_{S^{d-1} \times S^{d-1}} \{ f(b_k,\theta) - f(b_k,\varphi)\}^2 J^{2,2}_k(\theta,\varphi) \, d\mathcal{M}(\theta,\varphi) \\
&+ \frac{1}{2} \sum_{k \in K : l_1 < a_k < b_k < l_2} \int_{S^{d-1} \times S^{d-1}} \{ f(a_k,\theta) - f(b_k,\varphi)\}^2 J^{1,2}_k(\theta,\varphi) \, d\mathcal{M}(\theta,\varphi) \\
&+ \frac{1}{2} \sum_{k \in K : l_1 < a_k < b_k < l_2} \int_{S^{d-1} \times S^{d-1}} \{ f(b_k,\theta) - f(a_k,\varphi)\}^2 J^{2,1}_k(\theta,\varphi) \, d\mathcal{M}(\theta,\varphi) \\
&+ I_1(f) + I_2(f).
\end{aligned}$$

*Here the first term of the right hand side vanishes in case that* $\int_{\Lambda} ds^{\mathrm{R}}(r) = 0$. *The last two terms* $I_i(f)$, $i = 1, 2$ *should be read as*

$$I_1(f) = \begin{cases} \dfrac{1}{s^{\mathrm{R}}(b_k) - s^{\mathrm{R}}(l_1)} \displaystyle\int_{S^{d-1}} f(b_k,\theta)^2 dm^{(2)}_{d-1}(\theta) \\ \qquad \text{if} \quad l_1 = a_k < b_k < l_2 \quad \text{and} \quad s^{\mathrm{R}}(l_1) > -\infty, \\ 0 \quad \text{otherwise}, \end{cases}$$

$$I_2(f) = \begin{cases} \dfrac{1}{s^{\mathrm{R}}(l_2) - s^{\mathrm{R}}(a_k)} \displaystyle\int_{S^{d-1}} f(a_k,\theta)^2 dm^{(2)}_{d-1}(\theta) \\ \qquad \text{if} \quad l_1 < a_k < b_k = l_2 \quad \text{and} \quad s^{\mathrm{R}}(l_2) < \infty, \\ 0 \quad \text{otherwise}. \end{cases}$$

EXAMPLE 5.7. Let $d \geq 2$ and R be the Bessel process on $I = (0, \infty)$ with the generator $\mathcal{G}^{\mathrm{R}} = (1/2)(d^2/dr^2 + ((d-1)/r)(d/dr))$. We may set $ds^{\mathrm{R}}(r) = 2r^{1-d}\, dr$ and $dm^{\mathrm{R}}(r) = r^{d-1}\, dr$. Note that the assumption (5.1) is satisfied. The end point 0 is $(s^{\mathrm{R}}, m^{\mathrm{R}})$-entrance and the end point $\infty$ is $(s^{\mathrm{R}}, m^{\mathrm{R}})$-natural. We set

$$\mathbf{h}(t) = \int_I l^{\mathrm{R}}(t, r) r^{d-3}\, dr = \int_0^t R_s^{-2}\, ds, \quad t > 0.$$

In the same way as in [6], we obtain the following.

$$E^{P_r^{\mathrm{R}}}[e^{-\gamma_n \mathbf{h}(\sigma_b^{\mathrm{R}})}] = \left(\frac{r}{b}\right)^n, \quad 0 < r < b. \tag{5.21}$$

$$E^{P_r^{\mathrm{R}}}[e^{-\gamma_n \mathbf{h}(\sigma_a^{\mathrm{R}})}] = \left(\frac{a}{r}\right)^{d-2+n}, \quad a < r < \infty. \tag{5.22}$$

$$E^{P_r^{\mathrm{R}}}[e^{-\gamma_n \mathbf{h}(\sigma_a^{\mathrm{R}})}; \sigma_a^{\mathrm{R}} < \sigma_b^{\mathrm{R}}] = \frac{(b/r)^{d-2+n} - (r/b)^n}{(b/a)^{d-2+n} - (a/b)^n}, \quad a < r < b. \tag{5.23}$$

$$E^{P_r^{\mathrm{R}}}[e^{-\gamma_n \mathbf{h}(\sigma_b^{\mathrm{R}})}; \sigma_b^{\mathrm{R}} < \sigma_a^{\mathrm{R}}] = \frac{(r/a)^n - (a/r)^{d-2+n}}{(b/a)^n - (a/b)^{d-2+n}}, \quad a < r < b. \tag{5.24}$$

Here $0 < a < b < \infty$ and $n \geq 0$, where, if $d = 2$ and $n = 0$, (5.23) and (5.24) are reduced to (5.25) and (5.26), respectively.

$$P_r^{\mathrm{R}}(\sigma_a^{\mathrm{R}} < \sigma_b^{\mathrm{R}}) = \frac{\log b/r}{\log b/a}, \quad a < r < b, \tag{5.25}$$

$$P_r^{\mathrm{R}}(\sigma_b^{\mathrm{R}} < \sigma_a^{\mathrm{R}}) = \frac{\log r/a}{\log b/a}, \quad a < r < b. \tag{5.26}$$

We note that the functions given by (5.21)–(5.24) satisfy the equation $\mathcal{G}^{\mathrm{R}} g(r) = \gamma_n r^{-2} g(r)$.

(i) We first consider the case that $d\nu(r) = r^{-2}\, dm^{\mathrm{R}}(r) = r^{d-3}\, dr$. Then

$$\mathbf{f}(t) = \int_I l^{\mathrm{R}}(t, r) r^{-2}\, dm^{\mathrm{R}}(r) = \int_0^t R_s^{-2}\, ds,$$

hence the skew product $\mathrm{X} = [(R_t, \Theta_{\mathbf{f}(t)}), P_r^{\mathrm{R}} \otimes P_\theta^\Theta, (r, \theta) \in I \times S^{d-1}]$ is reduced to $d$-dimensional Brownian motion BM($d$). The assumption (3.1) is also satisfied for the end points 0 and $\infty$. It is well known that the statements (i) and (ii) of Theorem 3.2 are valid for BM($d$).

(ii) Let $d\mu(r) = 1_{(0,a)}(r)\, dm^{\mathrm{R}}(r)$ and $d\nu(r) = 1_{(0,a)}(r)\, d\omega(r) + 1_{(a,\infty)}(r) r^{-2}\, dm^{\mathrm{R}}(r)$, where $0 < a < \infty$ and $\omega$ is a Radon measure on $I$ such that $\mathrm{supp}[\omega] = I$ and $\left|\int_0^a s^{\mathrm{R}}(r)\, d\omega(r)\right| = \infty$. By virtue of Theorem 5.6, we get the following. For $f \in \mathcal{C}^{\mathrm{X}}|_{(0,a)\times S^{d-1}}$,

$$\begin{aligned} \mathcal{E}^{\mathrm{Y}}(f, f) = {} & \frac{1}{2} \int_{(0,a)\times S^{d-1}} \frac{\partial f}{\partial r}(r, \theta)^2 r^{d-1}\, dr\, dm_{d-1}^{(2)}(\theta) \\ & + \int_{(0,a)} \mathcal{E}^\Theta(f(r, \cdot), f(r, \cdot))\, d\omega(r) \\ & + \frac{1}{2} \int_{S^{d-1}\times S^{d-1}} \{f(a, \theta) - f(a, \varphi)\}^2 J(\theta, \varphi)\, d\mathcal{M}(\theta, \varphi) + I(f), \end{aligned} \tag{5.27}$$

where

$$I(f) = \begin{cases} \dfrac{d-2}{2} a^{d-2} \displaystyle\int_{S^{d-1}} f(a, \theta)^2\, dm_{d-1}^{(2)}(\theta), & \text{if} \quad d \geq 3, \\ 0, & \text{if} \quad d = 2. \end{cases} \tag{5.28}$$

Since $P_r^{\mathrm{R}}(\mathbf{f}(\sigma_a^{\mathrm{R}}) = \mathbf{h}(\sigma_a^{\mathrm{R}})) = 1$ for $a < r < \infty$, $J(\theta, \varphi)$ is given as follows.

$$\begin{aligned} J(\theta, \varphi) &= \lim_{r\downarrow a} D_{s^{\mathrm{R}}(r)} \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi) E^{P_r^{\mathrm{R}}}[e^{-\gamma_n \mathbf{h}(\sigma_a^{\mathrm{R}})}] \\ &= \lim_{r\downarrow a} D_{s^{\mathrm{R}}(r)} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^{d-2+n} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi). \end{aligned} \tag{5.29}$$

Especially, if $d = 2$, then

$$\begin{aligned} J(\theta, \varphi) &= \lim_{r\downarrow a} D_{s^{\mathrm{R}}(r)} \left\{ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos n(\theta - \varphi) \right\} \\ &= \frac{1}{\pi} \lim_{r\downarrow a} D_{s^{\mathrm{R}}(r)} \frac{(a/r)\cos(\theta-\varphi) - (a/r)^2}{1 - 2(a/r)\cos(\theta - \varphi) + (a/r)^2} \\ &= \frac{1}{4\pi} \frac{1}{1 - \cos(\theta - \varphi)} = \left(8\pi \sin^2 \frac{\theta - \varphi}{2}\right)^{-1}. \end{aligned} \tag{5.30}$$

Therefore $\mathcal{E}^{\mathrm{Y}}$ corresponding to the case $d = 2$ is given as follows.

$$\begin{aligned} \mathcal{E}^{\mathrm{Y}}(f, f) &= \frac{1}{2} \int_{(0,a)\times S^1} \frac{\partial f}{\partial r}(r, \theta)^2 r\, dr\, d\theta + \frac{1}{2} \int_{(0,a)\times S^1} \frac{\partial f}{\partial \theta}(r, \theta)^2\, d\omega(r)\, d\theta \\ &\quad + \frac{1}{16\pi} \int_{S^1\times S^1} \{f(a, \theta) - f(a, \varphi)\}^2 \frac{1}{\sin^2((\theta - \varphi)/2)}\, d\theta\, d\varphi. \end{aligned}$$

Since the assumption of Theorem 4.2 (ii) is satisfied, the time changed process corresponding to (5.27) has Feller property in the sense of Proposition 4.1 and Theorem 4.2 (ii).
(iii) Let $d\mu(r) = 1_{(a,\infty)}(r)\, dm^{\mathrm{R}}(r)$ and $d\nu(r) = 1_{(0,a)}(r) r^{-2}\, dm^{\mathrm{R}}(r) + 1_{(a,\infty)}(r)\, d\omega(r)$, where $0 < a < \infty$ and $\omega$ is a Radon measure on $I$ such that $\mathrm{supp}[\omega] = I$. By virtue of Theorem 5.6, we get the following. For $f \in \mathcal{C}^{\mathrm{X}}|_{(a,\infty)\times S^{d-1}}$,

$$\begin{aligned} \mathcal{E}^{\mathrm{Y}}(f, f) &= \frac{1}{2} \int_{(a,\infty)\times S^{d-1}} \frac{\partial f}{\partial r}(r, \theta)^2 r^{d-1}\, dr\, dm_{d-1}^{(2)}(\theta) \\ &\quad + \int_{(a,\infty)} \mathcal{E}^{\Theta}(f(r, \cdot), f(r, \cdot))\, d\omega(r) \\ &\quad + \frac{1}{2} \int_{S^{d-1}\times S^{d-1}} \{f(a, \theta) - f(a, \varphi)\}^2 J(\theta, \varphi)\, d\mathcal{M}(\theta, \varphi). \end{aligned} \tag{5.31}$$

Here $J(\theta, \varphi)$ is given as follows. Since $P_r^{\mathrm{R}}(\mathbf{f}(\sigma_a^{\mathrm{R}}) = \mathbf{h}(\sigma_a^{\mathrm{R}})) = 1$ for $0 < r < a$,

$$\begin{aligned} J(\theta, \varphi) &= -\lim_{r\uparrow a} D_{s^{\mathrm{R}}(r)} \sum_{n=0}^{\infty} \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi) E^{P_r^{\mathrm{R}}}[e^{-\gamma_n \mathbf{h}(\sigma_a^{\mathrm{R}})}] \\ &= -\lim_{r\uparrow a} D_{s^{\mathrm{R}}(r)} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \sum_{l=1}^{\kappa(n)} S_n^l(\theta) S_n^l(\varphi). \end{aligned} \tag{5.32}$$

When $d = 2$,

$$
\begin{aligned}
J(\theta, \varphi) &= -\lim_{r\uparrow a} D_{s^{\mathrm{R}}(r)}\left\{\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\varphi)\right\} \\
&= -\frac{1}{\pi}\lim_{r\uparrow a} D_{s^{\mathrm{R}}(r)}\frac{(r/a)\cos(\theta-\varphi)-(r/a)^2}{1-2(r/a)\cos(\theta-\varphi)+(r/a)^2} \qquad (5.33)\\
&= \frac{1}{4\pi}\frac{1}{1-\cos(\theta-\varphi)} = \left(8\pi\sin^2\frac{\theta-\varphi}{2}\right)^{-1}.
\end{aligned}
$$

Therefore $\mathcal{E}^{\mathrm{Y}}$ corresponding to the case $d = 2$ is given as follows.

$$
\begin{aligned}
\mathcal{E}^{\mathrm{Y}}(f, f) &= \frac{1}{2}\int_{(a,\infty)\times S^1}\frac{\partial f}{\partial r}(r,\theta)^2 r\,dr\,d\theta + \frac{1}{2}\int_{(a,\infty)\times S^1}\frac{\partial f}{\partial \theta}(r,\theta)^2\,d\omega(r)\,d\theta \\
&\quad + \frac{1}{16\pi}\int_{S^1\times S^1}\{f(a,\theta)-f(a,\varphi)\}^2\frac{1}{\sin^2((\theta-\varphi)/2)}\,d\theta\,d\varphi.
\end{aligned}
$$

Since the assumption of Theorem 4.2 (iii) is satisfied, the time changed process corresponding to (5.31) has Feller property in the sense of Proposition 4.1 and Theorem 4.2 (iii).
(iv) Let $d\mu(r) = 1_{(a,b)}(r)\,dm^{\mathrm{R}}(r)$ and $d\nu(r) = 1_{(0,a)\cup(b,\infty)}(r)r^{-2}\,dm^{\mathrm{R}}(r) + 1_{(a,b)}(r)\,d\omega(r)$, where $0 < a < b < \infty$ and $\omega$ is a Radon measure on $I$ such that $\mathrm{supp}[\omega] = I$. By virtue of Theorem 5.6, we get the following. For $f \in \mathcal{C}^{\mathrm{X}}|_{(a,b)\times S^{d-1}}$,

$$
\begin{aligned}
\mathcal{E}^{\mathrm{Y}}(f, f) &= \frac{1}{2}\int_{(a,b)\times S^{d-1}}\frac{\partial f}{\partial r}(r,\theta)^2 r^{d-1}\,dr\,dm^{(2)}_{d-1}(\theta) \\
&\quad + \int_{(a,b)}\mathcal{E}^{\Theta}(f(r,\cdot), f(r,\cdot))\,d\omega(r) \\
&\quad + \frac{1}{2}\int_{S^{d-1}\times S^{d-1}}\{f(a,\theta)-f(a,\varphi)\}^2 J_1(\theta,\varphi)\,d\mathcal{M}(\theta,\varphi) \qquad (5.34)\\
&\quad + \frac{1}{2}\int_{S^{d-1}\times S^{d-1}}\{f(b,\theta)-f(b,\varphi)\}^2 J_2(\theta,\varphi)\,d\mathcal{M}(\theta,\varphi) \\
&\quad + I(f),
\end{aligned}
$$

where $I(f)$ is given by (5.28) with $b$ in place of $a$, $J_1(\theta, \varphi)$ is given by (5.32), and $J_2(\theta, \varphi)$ is given by (5.29) with $b$ in place of $a$. Therefore, if $d = 2$, then

$$
J_1(\theta,\varphi) = J_2(\theta,\varphi) = \left(8\pi\sin^2\frac{\theta-\varphi}{2}\right)^{-1}.
$$

Further $\mathcal{E}^{\mathrm{Y}}$ corresponding to the case $d=2$ is given as follows.

$$\begin{aligned}\mathcal{E}^{\mathrm{Y}}(f,f)&=\frac{1}{2}\int_{(a,b)\times S^1}\frac{\partial f}{\partial r}(r,\theta)^2 r\,dr\,d\theta+\frac{1}{2}\int_{(a,b)\times S^1}\frac{\partial f}{\partial\theta}(r,\theta)^2\,d\omega(r)\,d\theta\\&\quad+\frac{1}{16\pi}\int_{S^1\times S^1}\{f(a,\theta)-f(a,\varphi)\}^2\frac{1}{\sin^2((\theta-\varphi)/2)}\,d\theta\,d\varphi\\&\quad+\frac{1}{16\pi}\int_{S^1\times S^1}\{f(b,\theta)-f(b,\varphi)\}^2\frac{1}{\sin^2((\theta-\varphi)/2)}\,d\theta\,d\varphi.\end{aligned}$$

Since the assumption of Theorem 4.2 (ii) is satisfied, the time changed process corresponding to (5.34) has Feller property in the sense of Proposition 4.1

(v) Let $d\mu(r)=\delta_a(dr)$ and $d\nu(r)=r^{-2}\,dm^{\mathrm{R}}(r)+C\delta_a(dr)$, where $0<a<\infty$, $\delta_a$ stands for the unit measure concentrated at a point $a$ and $C$ is a positive number. By virtue of Theorem 5.6, we get the following. For $f\in\mathcal{C}^{\mathrm{X}}|_{\{a\}\times S^{d-1}}$,

$$\begin{aligned}\mathcal{E}^{\mathrm{Y}}(f,f)&=C\mathcal{E}^{\Theta}(f(a,\cdot),f(a,\cdot))\\&\quad+\frac{1}{2}\int_{S^{d-1}\times S^{d-1}}\{f(a,\theta)-f(a,\varphi)\}^2J(\theta,\varphi)\,d\mathcal{M}(\theta,\varphi)+I(f),\end{aligned}\tag{5.35}$$

where $I(f)$ is given by (5.28) and $J(\theta,\varphi)$ is given as follows. Since $P_r^{\mathrm{R}}(\mathbf{f}(\sigma_a^{\mathrm{R}})=\mathbf{h}(\sigma_a^{\mathrm{R}}))=1$ for $r\neq a$,

$$\begin{aligned}J(\theta,\varphi)&=-\lim_{r\uparrow a}D_{s^{\mathrm{R}}(r)}\sum_{n=0}^{\infty}\sum_{l=1}^{\kappa(n)}S_n^l(\theta)S_n^l(\varphi)E^{P_r^{\mathrm{R}}}[e^{-\gamma_n\mathbf{h}(\sigma_a^{\mathrm{R}})}]\\&\quad+\lim_{r\downarrow a}D_{s^{\mathrm{R}}(r)}\sum_{n=0}^{\infty}\sum_{l=1}^{\kappa(n)}S_n^l(\theta)S_n^l(\varphi)E^{P_r^{\mathrm{R}}}[e^{-\gamma_n\mathbf{h}(\sigma_a^{\mathrm{R}})}]\\&=-\lim_{r\uparrow a}D_{s^{\mathrm{R}}(r)}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\sum_{l=1}^{\kappa(n)}S_n^l(\theta)S_n^l(\varphi)\\&\quad+\lim_{r\downarrow a}D_{s^{\mathrm{R}}(r)}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{d-2+n}\sum_{l=1}^{\kappa(n)}S_n^l(\theta)S_n^l(\varphi).\end{aligned}$$

When $d=2$, by means of (5.30) and (5.33),

$$J(\theta,\varphi)=\left(4\pi\sin^2\frac{\theta-\varphi}{2}\right)^{-1}.$$

Therefore $\mathcal{E}^{\mathrm{Y}}$ corresponding to the case $d=2$ is given as follows.

$$\begin{aligned}\mathcal{E}^{\mathrm{Y}}(f,f)&=\frac{C}{2}\int_{S^1}\frac{\partial f}{\partial\theta}(a,\theta)^2\,d\theta\\&\quad+\frac{1}{8\pi}\int_{S^1\times S^1}\{f(a,\theta)-f(a,\varphi)\}^2\frac{1}{\sin^2((\theta-\varphi)/2)}\,d\theta\,d\varphi.\end{aligned}$$

Since the assumption of Theorem 4.2 (ii) is satisfied, the time changed process corresponding to (5.35) has Feller property in the sense of Proposition 4.1.

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