Zhou, Y.
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THE 3-CANONICAL SYSTEM ON 3-FOLDS OF GENERAL TYPE

YIQUN ZHOU

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Abstract

Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. We classify minimal Gorenstein 3-folds of general type according to the birationality of 3-canonical system on X.

1. Introduction

Let X be a projective variety on $\mathbb{C}$, and K_X is the canonical divisor on X. One may define the m-canonical map ϕ_m corresponding to the complete linear system $|mK_X|$. To study the behavior of ϕ_m, especially the birationality of ϕ_m, has been one of the most important topics of birational geometry. When $\dim X \leq 2$, the behavior of ϕ_m has been thoroughly studied. However, when $\dim X \geq 3$, many problems still remain open.

Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. When $\dim X = 3$, we know that when m is big enough, ϕ_m must be birational, hence it is important to find the universal lower bound N for all 3-folds such that ϕ_m is birational for all $m \geq N$. The known best result is as follows: Chen, Chen and Zhang [5] proved that $m \geq 5$ is enough; Chen and Zhang [6] proved that if X cannot be fibred by a unique family of irreducible curves of geometric genus 2, then $m \geq 4$ is enough; Zhu [12] proved the generic finiteness of ϕ_3. Also in [6], Chen and Zhang proposed an open problem(see [6, 6.4 (2)]): is it possible to characterize the birationality of ϕ_3? We have several examples to illustrate the importance of this problem.

EXAMPLE 1.1 ([6, Example 6.2]). Kobayashi (see [9, Proposition 3.2]) has constructed a family of canonically polarized smooth threefolds Y satisfying the equality

$$K_Y^3 = \frac{4}{3}p_g(Y) - \frac{10}{3}$$

where $p_g(Y) = 7, 10, 13, \ldots$, Chen and Zhang proved that all examplxes above have non-birational 4-canonical maps, thus they have non-birational 3-canonical maps.

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EXAMPLE 1.2. Let S be a surface of type $(K_S^2, p_g(S)) = (2,3)$. Let C be a smooth curve of genus ≥ 2. Take $X = S \times C$. Since ϕ_1 is generically finite, $\dim \phi_1(X) = 3$. When $g(C)$ is big enough, $p_g(X)$ is big enough, hence ϕ_4 is birational by [6, 4.2]. But ϕ_3 is obviously not birational.

This paper makes an effort to answer this open problem, the main theorem is as follows. Notice that the Example 1.1 corresponds to the $d = 2$ situation, and the Example 1.2 corresponds to the $d = 1$ situation.

Theorem 1.1. *Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Denote $d = \dim \phi_1(X)$. If ϕ_3 on X is not birational, then X must be one of the following types*:
(1) $p_g(X) \leq 7$;
(2) $p_g(X) \geq 8$, *while*:
  (1) $d = 3$, *X contains a surface which has a family of curves of genus* 2;
  (2) $d = 2$, *X contains a surface which has a family of curves of genus* ≤ 3;
  (3) $d = 1$, *X contains a surface S such that either S has a family of curves of genus* ≤ 3 *or S satisfies* $p_g(S) = 1$, $K_{S_0}^2 \leq 9$, *where S_0 is the minimal model of S.*

2. The key technical results

2.1. Brief review on curves.

FACT 2.1. Let C be a smooth curve of genus ≥ 2. Assume D is a divisor on C with $\deg(D) \geq 3$. Then the rational map corresponding to $|K_C + D|$ gives a birational morphism onto its image.

2.2. Brief review of relevant results on surfaces. Let S be a smooth minimal surface of general type, according to [2], one has

FACT 2.2. For all $m \geq 5$, ϕ_m is a biratinal morphism onto its image.

FACT 2.3. For $m = 4$, ϕ_m is birational if and only if $(K_S^2, p_g(S)) \neq (1, 2)$.

FACT 2.4. For $m = 3$, ϕ_m is birational if and only if $(K_S^2, p_g(S)) \neq (1,2)$ and $(2,3)$.

FACT 2.5. For $m = 2$, if $K_S^2 \geq 10$, then ϕ_m is birational if and only if S cannot be fibred by a family of irreducible curves of geometric genus 2.

2.3. General method. Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Under the assumption $p_g(X) \geq 2$, we study the canonical map ϕ_1 which is a rational map. Take the modification $\pi\colon X' \to X$, according to Hironaka, such that

(i) X' is smooth.
(ii) The movable part of $|K_{X'}|$ is basepoint free.
(iii) $\pi^*(K_X)$ is linearly equivalent to a divisor with normal crossing support.

Denote by g the composition $\phi_1 \circ \pi$. So $g\colon X' \to W' \subseteq \mathbb{P}^{p_g(X)-1}$ is a morphism. Let $X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of g. We get a commutative diagram as below:

$$\begin{array}{ccc} X' & \xrightarrow{\ f\ } & B \\ {\scriptstyle \pi}\downarrow & \overset{g}{\searrow} & \downarrow{\scriptstyle s} \\ X & \xrightarrow[\ \phi_1\ ]{} & W'. \end{array}$$

We may write $K'_X \sim \pi^*(K_X) + E_\pi \sim M_1 + Z_1$, where M_1 is the movable part of $|K'_X|$, Z_1 is the fixed part, and E_π is a sum of distinct exceptional divisors. We may write $\pi^*(K_X) \sim M_1 + E'_1$, where $E'_1 = Z_1 - E_\pi$ is an effective divisor.

If $\dim \phi_1(X) = 2$, we see that a general fiber S of f is a smooth projective curve C of genus ≥ 2. We say that X is canonically fibred by curves of genus $g = g(C)$.

If $\dim \phi_1(X) = 1$, we see that a general fiber S of f is a smooth projective surface of general type. We say that X is canonically fibred by surfaces with invariants $(c_1^2(S_0), p_g(S))$, where S_0 is the minimal model of S. We may write $M_1 \equiv a_1 S$, where $a_1 \geq p_g(X) - 1$.

A generic irreducible element S of $|M_1|$, means either a general member of $|M_1|$ when $\dim \phi_1(X) \geq 2$ or, otherwise, a general fiber of f.

REMARK 2.1. Throughout the paper $D_1 \sim D_2$ (resp. $=_{\mathbb{Q}}$, or $D_1 \equiv D_2$) means that divisors D_1 and D_2 are linearly equivalent (resp. rD_1 and rD_2 are linearly equivalent for some positive integer r, or D_1 and D_2 are numerically equivalent).

Lemma 2.1 (Chen [6, Theorem 3.6]). *Let X be a minimal projective 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities and assume $p_g(X) \geq 2$. Keep the same notation as above. Pick a generic irreducible element S of $|M_1|$. Suppose, on the smooth surface S, there is a movable linear system $|G|$ and denote by C a generic irreducible element of $|G|$. Set $\xi := (\pi^*(K_X) \cdot C)_{X'}$, and*

$$p := \begin{cases} 1 & \text{If} \quad \dim \phi_1(X) \geq 2, \\ a_1 & \text{otherwise.} \end{cases}$$

Assume there is a rational number $\beta \geq 0$, such that $\pi^*(K_X)|_S - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. Denote $\alpha := (m - 1 - 1/p - 1/\beta)\xi$, set $\alpha_0 = \lceil \alpha \rceil$, then

(1) *Assume there is a positive integer* m *such that the linear system* $|K_S + \lceil (m-2)\pi^*(K_X)|_S \rceil|$ *seperates different generic irreducible elements of* $|G|$, *then* ϕ_m *is birational if one of the following conditions is satisfied*:
   i. $\alpha > 2$;
   ii. $\alpha_0 \geq 2$ *and* C *is non-hyperelliptic*;
   iii. $\alpha > 0$, C *is non-hyperelliptic and* C *is an even divisor of* S.
(2) $\xi \geq (2g(C) - 2 + \alpha_0)/m$ *if one of the following conditions is satisfied*:
   iv. $\alpha > 1$;
   v. $\alpha > 0$, *and* C *is an even divisor.*

3. Proof of the main theorem

We now study the birationality of ϕ_3. Set $d = \dim \phi_1(X)$.

3.1. The case $d = 3$.

Proposition 3.1. *Let* X *be a projective minimal Gorenstein* 3*-fold of general type with* $\mathbb{Q}$*-factorial terminal singularities. Assume* $p_g(X) \geq 8$, *and* $d = 3$. *Then* ϕ_3 *is birational unless* X *contains a surface which has a family of curves of genus* ≤ 2.

Proof. According to our general method, a generic irreducible element S of $|M_1|$ is a smooth projective surface of general type. It is sufficient to verify the birationality for $\phi_3|_S$ by virtue of the Matsuki–Tankeev principle [3, 2.1]. We consider the subsystem $|K_{X'} + \pi^*(K_X) + S| \subset |3K'_X|$. By the Kawamata–Viehweg vanishing theorem [8], we have the surjective map

$$H^0(X', K_{X'} + \pi^*(K_X) + S) \to H^0(S, K_S + L)$$

where $L = \pi^*(K_X)|_S$ is a nef and big divisor on S. If $|L|$ gives a birational map, then so does $|K_S + L|$. Otherwise, $|L|$ gives a generically finite map of degree ≥ 2. Noting that $h^0(S, L) \geq p_g(X) - 1 \geq 7$, we have $L^2 \geq 2(h^0(S, L) - 2) \geq 10$. If $|K_S + L|$ doesn't give a birational map, then according to Reider's result [10], there is a free pencil of curves on S with a generic irreducible element C, such that $L \cdot C = 1$ or 2. On the other hand, $L \cdot C \geq 2$ since $|L|$ gives a generically finite map on C and C is a curve of genus ≥ 2. Therefore we have $L \cdot C = 2$. Moreover, by the Riemann–Roch fomular and Clifford's theorem, we have $h^0(C, L|_C) = 2$. By

$$H^0(S, L - C) \to H^0(S, L) \to H^0(C, L|_C), \tag{3.1}$$

we have

$$h^0(S, L - C) + h^0(C, L|_C) \geq h^0(S, L),$$
$$h^0(S, L) \leq h^0(S, L - C) + 2.$$

We can replace L in (3.1) by $L-nC$ and have n sequences relatively. By $C^2=0$ we have $(L-C)|_C = L|_C$. Since $h^0(S,L)\geq p_g(X)-1$, we have $h^0(S,L)\leq h^0(S,L-nC)+2n$ when $p_g(X)>2n+1$. So there exists an effective divisor E such that $L=_{\mathbb{Q}} nC+E$.

Set $F=C+(1/n)E$, so $L=_{\mathbb{Q}} nF$. Since $p_g(X)\geq 8$, we can choose $n=3$, so $\deg F|_C=(1-1/n)L\cdot C>1$.

By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0\left(K_S+\left\lceil L-\frac{1}{n}E\right\rceil\right)\to H^0(C,K_C+\lceil (n-1)F\rceil|_C).$$

Let M_3 be the movable part of $|K_{X'}+\pi^*(K_X)+S|$, N be the movable part of $|K_S+\lceil L-(1/n)E-C\rceil|$, then $M_3|_S\geq N$ by [3, Lemma 2.7].

So we have

$$3L\cdot C=3\pi^*(K_X)|_S\cdot C\geq \deg(K_C+2F|_C)\geq 2g(C)-2+\frac{8}{3}=2g(C)+\frac{2}{3},$$

then we can derive that $g(C)\leq 2$.

Therefore, ϕ_3 is birational if X does not contain a surface which has a family of curves of genus 2. □

3.2. The case $d=2$.

Proposition 3.2. *Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Assume $p_g(X)\geq 6$, and $d=2$. Then ϕ_3 is birational unless X contains a surface which has a family of curves of genus ≤ 3.*

Proof. Let S be a generic element of $|M_1|$. So $S|_S\equiv aC$, where $a\geq p_g(X)-2$, and C is a general fiber of f and is a smooth curve. Denote $L=\pi^*(K_X)|_S>S|_S$. By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0(X',K_{X'}+\pi^*(K_X)+S)\to H^0(S,K_S+\lceil L\rceil).$$

We may write $L\equiv aC+E''$, so $L-C-(1/a)E''$ is nef and big. By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0\left(S,K_S+\left\lceil L-\frac{1}{a}E''\right\rceil\right)\to H^0(C,K_C+D)$$

where $D:=\lceil L-C-(1/a)E''\rceil|_C$. If $\deg\lfloor D\rfloor=(1-(1/a))L\cdot C\geq 3$, then $|K_C+\lceil L\rceil|_C-C|$ gives a birational map by Fact 2.1. By the Matsuki–Tankeev principle, we can derive that $|K_{X'}+\pi^*(K_X)+S|$ gives a birational map, and the birationality of ϕ_3 follows.

We now prove that $(1-1/a)L\cdot C>2$.

Let $G=M_1|_S$, G is composed of a pencil and $G\equiv aC$. In Lemma 2, we may take $p=1$, $\beta=2$, and $\xi=L\cdot C$, $\alpha=(m-2.5)\xi$. By [3], ϕ_4 is generically finite, this means hat $|\lfloor 4\pi^*(K_X)\rfloor|$ maps a general C onto a curve. Thus $4\pi^*(K_X)|_S\cdot C\geq 2$, and $\xi\geq 1/2$.

Let $m=7$, we have $(m-4)\xi\geq 1$. By Lemma 2, $\xi\geq(2g(C)-2+\alpha_0)/m$, follows $\xi\geq(2g(C)-2)/2.5$. When $g(C)\geq 4$, we have $\xi\geq 3$. Hence $(1-1/a)\xi>2$, since $a\geq p_g(X)-2\geq 4$.

Therefore, if X does not contain a surface which has a family of curves of genus ≤ 3, then ϕ_3 is birational. □

3.3. The case $d=1$.

Lemma 3.1 ([6, Lemma 3.7]). *Keep the same notation as in* Subsection 2.3 *and with p as in* Lemma 2.1. *Assume $d=1$, and $g(B)=0$. Let S be a general fiber of $g\colon X'\to W'$. Let $\sigma: S\to S_0$ be the contraction onto the minimal model. Then*

$$\pi^*(K_X)|_S-\frac{p}{p+1}\sigma^*(K_{S_0})$$

is pseudo-effective.

Proposition 3.3. *Let X be a projective minimal Gorenstein* 3*-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Assume $p_g(X)\geq 6$, and $d=1$. Then ϕ_3 is birational unless X contains a surface S which has a family of curves of genus ≤ 3 or which satisfies $p_g(S)=1$, $K_{S_0}^2\leq 9$, where S_0 is the minimal model of S.*

Proof. Let S be a general fiber of f. Then S is a smooth projective surface of general type. We have $M_1\equiv aS$, with $a\geq p_g(X)-1$. Denote $L=\pi^*(K_X)|_S$. Let $\sigma: S\to S_0$ be the contraction onto the minimal model. Denote $F'=\pi_*(S)$. Notice that $K_X\cdot F'^2$ is an even integer [5, 2.1] and $K_X\cdot F'^2\geq 0$.

CASE 1. The case $K_X\cdot F'^2>0$.

Notice that $K_X\equiv aF'+E'$, where E' is an effective divisor. We have $L^2=K_X^2F'\geq aK_XF'^2\geq 2(p_g(X)-1)\geq 10$. By Reider's result, if $|K_X+L|$ doesn't give a birational map, there exists a free pencil of curves on S with a generic irreducible element C, such that $L\cdot C\leq 2$. By Lemma 3.1 and [5, Claim 3.3], we always have

$$\left(\pi^*(K_X)|_S-\frac{p}{p+1}\sigma^*(K_{S_0})\right)\cdot C\geq 0,$$

then we can derive

$$\frac{p}{p+1}\sigma^*(K_{S_0})\cdot C\leq 2,$$

so

$$\sigma^*(K_{S_0})\cdot C\leq 2,\ \text{since}\ p\geq p_g(X)-1\geq 5.$$

Let $\bar{C}$ be the image of C under σ, then we have $K_{S_0} \cdot \bar{C} \leq 2$.

If $\bar{C}^2 = 0$, then $K_{S_0} \cdot \bar{C} = 2g(\bar{C}) - 2 \leq 2$, so $g(\bar{C}) \leq 2$, which means S has a family of curves of genus 2.

If $\bar{C}^2 > 0$, by the Hodge index theorem, we can derive that $K_{S_0}^2 \cdot \bar{C}^2 \leq (K_{S_0} \cdot \bar{C})^2 \leq 4$. But $K_{S_0}^2 \geq L^2 > 10$, which is a contradiction.

CASE 2. The case $K_X \cdot F'^2 = 0$.

One always has $\pi^*(K_X)|_S \sim \sigma^*(K_{S_0})$ by [5, Claim 3.3]. Notice that

$$\pi^*(K_X) - S - \frac{1}{a}E_1' =_{\mathbb{Q}} \left(1 - \frac{1}{a}\right)\pi^*(K_X).$$

Applying the vanishing theorem, one has the surjective map:

$$H^0\left(X', K_{X'} + \left\lceil 2\pi^*(K_X) - \frac{1}{a}E_1' \right\rceil\right) \to H^0\left(S, K_S + \left\lceil \left(2 - \frac{1}{a}\right)\pi^*(K_X) \right\rceil\Big|_S\right).$$

Since

$$K_S + \left\lceil \left(2 - \frac{1}{a}\right)\pi^*(K_X) \right\rceil\Big|_S \geq K_S + \sigma^*(K_{S_0}) + \left\lceil \left(1 - \frac{1}{a}\right)\pi^*(K_X) \right\rceil\Big|_S,$$

by Fact 2.5, $|K_S + \sigma^*(K_{S_0}) + \lceil (1 - 1/a)\pi^*(K_X) \rceil|_S|$ doesn't give a birational map only if $K_{S_0}^2 \leq 9$ or S has a family of curves of genus 2.

When $p_g(S) \geq 2$, there exists a family of curves with its general member $C' \subset |L|$. By the Kawamata–Viehweg vanishing theorem, we have

$$\left| K_S + C' + \left\lceil \left(1 - \frac{1}{a}\right)L \right\rceil \right|\Big|_{C'} = |K_{C'} + D|,$$

where $\deg D \geq (1 - 1/a)L \cdot C'$. If $|K_{C'} + D|$ doesn't give a birational map, we have $(1 - 1/a)L \cdot C' \leq 2$, then $L \cdot C' = \sigma^*(K_{S_0}) \cdot C' = K_{S_0} \cdot \sigma_*(C') \leq 2$. By $p_g(S) \geq 2$, we have $K_{S_0}^2 \geq 2p_g(S_0) - 4 \geq 0$.

If $K_{S_0}^2 \leq 1$, we have $(\sigma_*(C'))^2 \leq K_{S_0}^2 \leq 1$. Therefore, by the Riemann–Roch fomular, we have $2(p_a(\sigma_*(C')) - 2) = (K_{S_0} + \sigma_*(C')) \cdot \sigma_*(C') \leq 2$, by which we conclude $g(C') \leq p_a(\sigma_*(C')) \leq 2$.

If $K_{S_0}^2 \geq 2$, we have two cases. When $\sigma_*(C')^2 \leq 1$, it is easy to derive $g(C') \leq 2$. When $\sigma_*(C')^2 \geq 2$, by the Hodge index theorem, we have $4 \leq K_{S_0}^2 \cdot \sigma_*(C')^2 \leq (K_{S_0} \cdot \sigma_*(C'))^2 \leq 4$. The equality holds, so $K_{S_0} \equiv \sigma(C')$ and $K_{S_0}^2 = 2$. Therefore by the Riemann–Roch formula, we can conclude that $g(C') \leq 3$. □

By Proposition 3.1, Proposition 3.2 and Proposition 3.3, we can summarize that the main theorem is proved.

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School of Mathematical Sciences
Fudan University
Shanghai 200433
P.R. China
e-mail: 062018004@fudan.edu.cn