Spreafico, M.
Osaka J. Math.
48 (2011), 41–50

ZETA DETERMINANT AND OPERATOR DETERMINANTS

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(Received May 25, 2009)

Abstract

We apply techniques of zeta functions and regularized products theory to study the zeta determinant of a class of abstract operators with compact resolvent, and in particular the relation with other spectral functions.

1. Introduction

Let ξ be a complex vector bundle over a closed m-dimensional smooth Riemannian manifold M, and H a symmetric non-negative elliptic pseudo differential operator of order $r > 0$ in the space of the L^2 sections of ξ. The zeta function of H is defined by the (absolutely and locally uniformly convergent) series (where the λ_n are the positive eigenvalues of H)

$$\zeta(s, H) = \sum_{n=1}^{\infty} \lambda_n^{-s},$$

when $\mathrm{Re}(s) > m/r$, and by analytic extension elsewhere, and admits a meromorphic extension with $s = 0$ as a regular point (this is equivalent to take the trace $\mathrm{Tr}\, A^{-s}$ of the complex powers of A [8]). In terms of the heat operator

$$\zeta(s, H) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}(\mathrm{Tr}\, \mathrm{e}^{-tH} - \dim \ker H)\, dt,$$

and the asymptotic expansion of the trace of the heat semigroup can be used to obtain the analytic continuation of the zeta function, and information on poles and residues [3]. Following Ray and Singer [6], the zeta regularized determinant of H is defined by

$$\log \det_{\zeta} H = -\frac{d}{ds}\zeta(s, H)\Big|_{s=0},$$

but, the information contained in the heat expansion is not enough for the zeta determinant. Various authors investigated the properties of regularized products and zeta

2000 Mathematics Subject Classification. Primary 58J52; Secondary 58J40, 11M36.

determinants (see for example [7], [12], [1] and [5]). Applying some results in abstract zeta function theory [9] and [11], we prove the following general result concerning the relationship between the zeta determinant and the Fredholm determinant (compare with [2] for the case $r < m$).

Theorem 1.1. *Let H be a self-adjoint non-negative elliptic pseudo differential operator of order r defined in the space of the L^2 sections of some complex vector bundle over a compact Riemannian manifold of dimension m. Then,*

$$\log \det_\zeta H = \frac{1}{[m/r]+1} \operatorname*{Res}_{\lambda=\infty} \log \det'(I - \lambda R(\lambda, H^{[m/r]+1})),$$

where $\operatorname{Res}_{0\lambda=\infty} f(\lambda)$ *denotes the constant term in the asymptotic expansion of* $f(\lambda)$ *for large* λ.

Our approach consists in showing that the spectrum of the operators of this class is a set of complex numbers of a particular type that has been studied in [11], and called of sequence of spectral type (see also [12] for similar approach, and the general formulation in [5]). The main results then follow from general properties of this type of sequences. We present the definition and the main properties of these sequences in the next section, and we state and prove the results for the operators in the last section. Our main reference for operator determinants is [4].

2. Sequences of spectral type

Let $S = \{a_n\}_{n=1}^{\infty}$ be a sequence of non-vanishing complex numbers, ordered as $0 < |a_1| \leq |a_2| \leq \cdots$ (if we need to include the number 0 in the discussion, it will be denoted by $a_0 = 0$) with the unique point of accumulation at infinite. We introduce some spectral functions associated to this type of sequences, and we study their main properties. This section in essentially based on [10] and [11].

The positive real number (possibly infinite)

$$s_0 = \limsup_{n\to\infty} \frac{\log n}{\log|a_n|},$$

is called the exponent of convergence of S, and denoted by $\mathsf{e}(S)$. We are only interested in sequences with $\mathsf{e}(S) = s_0 < \infty$. If this is the case, then there exists a least integer p such that the series $\sum_{n=1}^{\infty} a_n^{-p-1}$ converges absolutely. If s_0 is not an integer, p is the greatest integer less than s_0; if s_0 is an integer, p may be either s_0 or $s_0 - 1$. We assume $s_0 - 1 < p \leq s_0$ (for $s_0 - 1 = p$ if and only if the associated zeta function converges for $s = s_0$ integer, thus up to pathological cases we can take $p = [s_0]$, the integer part). We call the integer p the genus of the sequence S, and we write $p = \mathsf{g}(S)$. The series $\sum_{n=1}^{\infty} a_n^{-s}$ converges uniformly on $\operatorname{Re}(s) \geq s_0 + \epsilon$, for all real $\epsilon > 0$, absolutely

if $\mathrm{Re}(s) > s_0$ and diverges if $\mathrm{Re}(s) < s_0$. We define the zeta function associated to S by

$$\zeta(s, S) = \sum_{n=1}^{\infty} a_n^{-s},$$

when $\mathrm{Re}(s) > \mathsf{e}(S)$, and by analytic continuation otherwise. The Weierstrass canonical product

$$\prod_{n=1}^{\infty}\left(1 + \frac{z}{a_n}\right)\mathrm{e}^{\sum_{j=1}^{\mathsf{g}(S)}((-1)^j/j)(z^j/a_n^j)},$$

converges uniformly and absolutely in any bounded closed region of the complex plane, and is an entire function of order $\mathsf{e}(S)$ (this is the first Borel theorem) which vanishes if and only if $z = -a_n$ for some n. We call the open subset $\rho(S) = \mathbb{C} - S$ of the complex plane the resolvent set of S. We define the Gamma function associated to S by the canonical product

$$\frac{1}{\Gamma(z, S)} = \prod_{n=1}^{\infty}\left(1 + \frac{z}{a_n}\right)\mathrm{e}^{\sum_{j=1}^{\mathsf{g}(S)}((-1)^j/j)(z^j/a_n^j)}.$$

For further convenience, we use the variable $\lambda = -z$ for the Gamma function. When necessary in order to define the meromorphic branch of an analytic function, the domain for λ will be the open subset $\mathbb{C} - [0, \infty)$ of the complex plane. For all $\lambda \in \rho(S)$, we define

$$\frac{1}{\Gamma(-\lambda, S)} = \prod_{n=1}^{\infty}\left(1 + \frac{-\lambda}{a_n}\right)\mathrm{e}^{\sum_{j=1}^{\mathsf{g}(S)}((-1)^j/j)((-\lambda)^j/a_n^j)}. \tag{2.1}$$

For each non-negative integer k, we define the following functions

$$r_k(\lambda, S) = k!\sum_{n=1}^{\infty}\left(\frac{(-1)^k}{(\lambda - a_n)^{k+1}} + \sum_{j=0}^{\mathsf{g}(S)-k-1}\binom{j+k}{j}\frac{\lambda^j}{a_n^{j+k+1}}\right),$$

where in particular,

$$r_0(\lambda, S) = \sum_{n=1}^{\infty}\left(\frac{1}{\lambda - a_n} + \sum_{j=0}^{\mathsf{g}(S)-1}\frac{\lambda^j}{a_n^{j+1}}\right).$$

Lemma 2.1. *For all k,*

$$r_k(\lambda, S) = -\frac{d^{k+1}}{d\lambda^{k+1}}\log\Gamma(-\lambda, S)$$

and

$$r_0(\lambda, S) = r(\lambda, S) = -\frac{d}{d\lambda}\log\Gamma(-\lambda, S).$$

Lemma 2.2. *If* $k = \mathsf{g} = \mathsf{g}(S)$, *then*

$$r_{\mathsf{g}}(\lambda, S) = -\mathsf{g}!\, \zeta(\mathsf{g}+1, S-\lambda) = -\mathsf{g}! \sum_{n=1}^{\infty} (a_n - \lambda)^{-\mathsf{g}-1}.$$

This is a uniformly convergent series and we can take the limit for $\lambda \to \infty$ *that is* 0. *This implies that* $r(\lambda, S)$ *can not have a term like* $(-\lambda)^m$ *and consequently* $\log \Gamma(-\lambda, S)$ *can not have a term like* $(-\lambda)^{m+1}$, *with* $m \geq \mathsf{g}$.

We use the notation $\Sigma_{\theta,a} = \{z \in \mathbb{C} \mid |\arg(z-a)| \leq \theta/2\}$, with $a \geq \delta > 0$, $0 < \theta < \pi$. We use $D_{\theta,a} = \mathbb{C} - \Sigma_{\theta,a}$, for the complementary (open) domain and $\Lambda_{\theta,a} = \partial\Sigma_{\theta,a} = \{z \in \mathbb{C} \mid |\arg(z-a)| = \theta/2\}$, oriented counter clockwise, for the boundary. With this notation, we define now a particular subclass of sequences.

DEFINITION 2.3. Let $S = \{a_n\}_{n=1}^{\infty}$ be a sequence of non-vanishing complex number with unique accumulation point at infinity and $\mathsf{e}(S) < \infty$. We say that S is a sequence of spectral type if the following conditions hold:
(1) there exist $a > 0$ and $0 < \theta < \pi$, such that S is contained in the interior of the sector $\Sigma_{\theta,a}$;
(2) the logarithm of the associated Gamma function as a uniform asymptotic expansion for large $\lambda \in D_{\theta,a}(S) = \mathbb{C} - \Sigma_{\theta,a}$ of the following form

$$\log \Gamma(-\lambda, S) = \sum_{\alpha} \sum_{k=0}^{K_\alpha} a_{\alpha,k} (-\lambda)^{\alpha} \log^k(-\lambda) + o((-\lambda)^{\alpha_N}),$$

where $\{\alpha\}$ is a decreasing sequence of real numbers $\alpha_0 \neq 0 > \alpha > \cdots > \alpha_N \geq -\infty$, and $k = 0, 1, \ldots, K_\alpha \in \mathbb{N}$, for each α.

We call the number α_N the order of the sequence S, and we use the notation $\mathsf{o}(S)$. We call the open set $D_{\theta,a}(S)$ the asymptotic domain of S.

Note that the number N appearing in the above definition gives the number of elements in the sequence $\{\alpha\}$, this number will appear in the following as related to the number of poles of the associated zeta function. We say that S is a sequence of spectral type of infinite order if this number is not finite, and we write $\mathsf{o}(S) = \infty$.

REMARK 2.4. The point $\lambda = 0$ belongs by definition to the domain of analicity of $\Gamma^{-1}(-\lambda, S)$ for a sequence of spectral type, and $\log \Gamma(-\lambda, S)$ has a zero of order $\mathsf{g}(S) + 1$ at $\lambda = 0$.

DEFINITION 2.5. A sequence of spectral type is called regular if the coefficients $a_{\alpha,k}$ in the expansion of the logarithm of the associated Gamma function vanish for all $k \neq 0, 1$.

REMARK 2.6. Let S be a regular sequence of spectral type with $\mathsf{o}(S) = \alpha_N$. Then, we have

$$\log \Gamma(-\lambda, S) = \sum_{j=0}^{N} a_{\alpha_j,0}(-\lambda)^{\alpha_j} + \sum_{j=0}^{N} a_{\alpha_j,1}(-\lambda)^{\alpha_j} \log(-\lambda) + O((-\lambda)^{\alpha_N}),$$

for large λ in $D_{\theta,a}(S)$, where $\alpha_0 \neq 0 > \alpha_1 > \cdots > \alpha_N = \mathsf{o}(S)$.

REMARK 2.7. If S is a regular sequence of spectral type, then $\alpha_0 \leq \mathsf{e}(S)$.

We introduce a further spectral function for a sequence of spectral type, namely we define the heat function associated to S by

$$f(t, S) = 1 + \sum_{n=1}^{\infty} \mathrm{e}^{-ta_n}. \tag{2.2}$$

Note that only the first condition of Definition 2.3 is actually necessary for the convergence of the series appearing in equation (2.2). Eventually, we state the main results about regular sequences of spectral type in the present context.

Proposition 2.8. *Let S be a regular sequence of spectral type with order $\mathsf{o}(S) = \alpha_N$. Then, the associated heat function has the following asymptotic expansion for $t \to 0^+$*

$$f(t, S) - 1 = \sum_{j=0}^{N} \sum_{k=0}^{1} c_{\alpha_j,k} t^{-\alpha_j} \log^k t + o(t^{-\alpha_N}),$$

where

$$c_{\alpha_j,0} = \frac{1}{\Gamma(-\alpha_j)}(a_{\alpha_j,0} + \psi(-\alpha_j)a_{\alpha_j,1}),$$
$$c_{\alpha_j,1} = -\frac{a_{\alpha_j,1}}{\Gamma(-\alpha_j)}.$$

Proposition 2.9. *Let S be a regular sequence of spectral type of order $\mathsf{o}(S) = \alpha_N \leq 0$. Then, the associated zeta function is holomorphic in the complex half-plane $\mathrm{Re}(s) > \alpha_N - \epsilon$ (positive small ϵ), up to a finite set of poles. The poles in the half plane $\mathrm{Re}(s) > \alpha_N - \epsilon$ are at most $N + 1$, are located at $s = \alpha_N, \alpha_{N-1}, \ldots, \alpha_0 \leq \mathsf{e}(S)$,*

and are of order at most 2 *with residues*

$$\operatorname*{Res}_{s=\alpha_j}{}_2\, \zeta(s, S) = -\frac{c_{\alpha_j,1}}{\Gamma(\alpha_j)} \quad (= 0, \text{ if } \alpha_j \in \mathbb{Z}),$$

$$\operatorname*{Res}_{s=\alpha_j}{}_1\, \zeta(s, S) = \begin{cases} \dfrac{c_{\alpha_j,0}}{\Gamma(\alpha_j)} + c_{\alpha_j,1}\dfrac{\psi(\alpha_j)}{\Gamma(\alpha_j)}, & \alpha_j \neq 0, -1, -2, \ldots, -[|\alpha_N|], \\ (-1)^{-\alpha_j+1}(-\alpha_j)!\, c_{\alpha_j,1}, & \alpha_j = -1, -2, \ldots, -[|\alpha_N|], \end{cases}$$

$$\operatorname*{Res}_{s=\alpha_j}{}_0\, \zeta(s, S) = (-1)^{\alpha_j}(-\alpha_j)!\, c_{\alpha_j,0}, \quad \alpha_j = -1, -2, \ldots, -[|\alpha_N|];$$

in particular, $s = 0$ *is a regular point with* $\zeta(0, S) = c_{0,0}$.

REMARK 2.10. It is important to observe that all the formulas given Proposition 2.9 can be written using exclusively the coefficients $a_{\alpha_j,k}$ appearing in the asymptotic expansion of the Gamma function. This follows from Proposition 2.8 or by direct computations.

Proposition 2.11. *Let* S *be a regular sequence of spectral type with* $\mathsf{o}(S) \leq 0$. *Then, the zeta function associated to* S *is holomorphic at* $s = 0$ *and satisfies the following expansion near* $s = 0$

$$\zeta(s, S) = -a_{0,1} - a_{0,0}s + O(s^2),$$

where the $a_{\alpha,k}$ *are the coefficients in the expansion of the associated logarithmic Gamma function* $\log \Gamma(-\lambda, S)$.

3. Spectral functions for operators with compact resolvent

Let $\mathcal{H}$ be an (infinite dimensional) complete separable Hilbert space, and A a closed operator in $\mathcal{H}$. We use the notation $R(\lambda, A) = (\lambda I - A)^{-1}$ for the resolvent of A, $\rho(A)$ for the resolvent set, $\sigma(A)$ for the spectrum, and $\sigma_0(A)$ for $\sigma(A) - \{0\}$. For a compact operator T, we denote by $s_n(T)$, $n \in \mathbb{N}$, the singular values of T (with the convention that $\|s_n(T)\| > 0$ for $n > 0$). We denote by $\mathcal{B}_p(\mathcal{H})$ the set of the compact operators T such that $\sum_{n=1}^{\infty} s_n(T)^p < \infty$. Thus, $\mathcal{B}_1(\mathcal{H})$ is the trace class, and for $T \in \mathcal{B}_1(\mathcal{H})$ the trace of T is defined by

$$\operatorname{Tr} T = \sum_{n=1}^{\infty} \lambda_n(T),$$

where $\lambda_n(T)$ are the positive eigenvalues of T.

Lemma 3.1. *Let* A *be a closed operator in the Hilbert space* $\mathcal{H}$. *Assume there exist non-negative integers* k *and* p, *such that* $(R(\lambda, A))^k \in \mathcal{B}_p(\mathcal{H})$ *for some* $\lambda \in \rho(A)$. *Then,* $\sigma_0(A) = \{\lambda_n\}_{n=1}^{\infty}$ *is a sequence of non-vanishing complex numbers with unique*

accumulation point at infinity, finite exponent of convergence and genus $\mathrm{g}(A) = kp - 1$. *If we further assume that* $\sigma_0(A)$ *is contained in some sector* $\Sigma_{a,\theta}$, *then* $\sigma_0(A)$ *is a regular sequence of spectral type.*

Proof. By definition, $(R(\lambda, A))^{kp}$ is compact. Thus, $\sigma_0(R(\lambda, A))$ consists of an at most countable number of eigenvalues with finite multiplicity and has no point of accumulation except possibly 0, and is contained in some closed disk centered at the origin of the complex plane. Let $\sigma_0(R(\lambda, A)) = \{\lambda_n(R(\lambda, A))\}_{n=1}^{\infty}$. By the spectral mapping theorem for bounded operator $\sigma_0((R(\lambda, A))^k) = \{\lambda_n^k(R(\lambda, A))\}_{n=1}^{\infty}$. By definition

$$\sum_{n=1}^{\infty} |\lambda_n((R(\lambda, A))^k)|^p < \infty.$$

By the spectral mapping theorem for unbounded operator there is a bijection of the set $\sigma(-R(\lambda, A))$ onto the set $\sigma(A) \cup \{\infty\}$. This implies that $\sigma_0(A) = \{\lambda_n = 1/\lambda_n(R(\lambda, A)) + \lambda\}_{n=1}^{\infty}$. Therefore the series $\sum_{n=1}^{\infty} |\lambda_n|^{-kp}$, converges. This proves the first implication. For the second one, we can use Lemma 2.2. □

Lemma 3.1 allows to introduce for an operator A of that type, all the spectral functions and the other invariants defined in Section 2 for sequences, just using the spectrum of A as underlying sequence. For example we can introduce the zeta function, $\zeta(s, \sigma_0(A))$. For simplicity, we use the notation $\zeta(s, A)$ instead, namely we simply put the operator in the place of the non-vanishing part of its spectrum. In particular, we have the following general abstract result.

Theorem 3.2. *Let* A *be a closed operator in the Hilbert space* $\mathcal{H}$. *Assume there exist non-negative integers* k *and* p, *such that* $(R(\lambda, A))^k \in \mathcal{B}_p(\mathcal{H})$ *for some* $\lambda \in \rho(A)$. *Assume that* $\sigma_0(A) \subset \Sigma_{a,\theta}$, *for some* $a > 0$ *and* $0 < \theta < \pi$. *Then,*

$$\log \det_\zeta A = \operatorname*{Res}_{\lambda=\infty} \log \Gamma(-\lambda, A).$$

If the resolvent $R(\lambda, A)$ itself is of trace class, then $k = p = 1$ and hence the genus is trivial, namely $\mathrm{g}(A) = 0$. The trace is related to the function $r_0(\lambda, A)$ introduced in Section 2

$$\operatorname{Tr} R(\lambda, A) - \frac{\dim \ker A}{\lambda} = r_0(\lambda, A) = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n}.$$

Let us first proceed with the assumption that $\ker A = \{0\}$. Thus, for all $\lambda \in \rho(A)$, $\lambda R(\lambda, A)$ is of trace class, and for $|\lambda| \, \|R(\lambda, A)\| < 1$, the operator (this is well posed, since $\|R(\lambda, A)\| \leq \|R(\lambda, A)\|_1 < \infty$)

$$\log(I - \lambda R(\lambda, A)) = -\lambda R(\lambda, A) - \frac{\lambda^2}{2} R(\lambda, A)^2 + \cdots$$

is well defined and belongs to $\mathcal{B}_1(\mathcal{H})$. Thus, we can define its determinant as

$$\det(I - \lambda R(\lambda, A)) = e^{\operatorname{Tr}\log(I - \lambda R(\lambda, A))}.$$

Since

$$I - \lambda R(\lambda, A) = (I - \lambda A^{-1})^{-1},$$

we can define the determinant

$$\det((I - \lambda A^{-1})^{-1}) = \det(I - \lambda R(\lambda, A)),$$

and using the spectral representation, if $\sigma(A) = \{\lambda_n\}_{n=1}^{\infty}$,

$$\log\det((I - \lambda A^{-1})^{-1}) = -\log\prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right),$$

and therefore

$$\begin{aligned}\det(I - \lambda R(\lambda, A)) &= \det((I - \lambda A^{-1})^{-1}) \\ &= \prod_{n=1}^{\infty}\frac{1}{1 - \lambda/\lambda_n}.\end{aligned}$$

This gives the gamma function introduced in Section 2, namely

$$\det(I - \lambda R(\lambda, A)) = \det((I - \lambda A^{-1})^{-1}) = \Gamma(-\lambda, A).$$

If A has non trivial kernel, let $\sigma(A) = \{\lambda_n\}_{n=0}^{\infty}$. Then, we define the determinant by

$$\det'(I - \lambda R(\lambda, A)) = \prod_{n=1}^{\infty}\frac{1}{1 - \lambda/\lambda_n},$$

where the zero eigenvalue is omitted.

We have proved the following result.

Proposition 3.3. *Let A be a closed operator in the Hilbert space $\mathcal{H}$ with resolvent of trace class. Assume that $\sigma_0(A) \subset \Sigma_{a,\theta}$, for some $a > 0$ and $0 < \theta < \pi$. Then,*

$$\log\det_{\zeta} A = \operatorname*{Res}_{\lambda=\infty}\log\det'(I - \lambda R(\lambda, A)).$$

This result extends to the more interesting class of operators considered in Lemma 3.1 as follows.

Proposition 3.4. *Let A be a closed operator in the Hilbert space $\mathcal{H}$. Assume there exist non-negative integers k and p, such that $(R(\lambda, A))^k \in \mathcal{B}_p(\mathcal{H})$ for some $\lambda \in \rho(A)$. Assume that $\sigma_0(A) \subset \Sigma_{a,\theta}$, for some $a > 0$ and $0 < \theta < \pi$. Then,*

$$\log \det{}_\zeta A = \operatorname*{Res}_{\lambda=\infty} \log(\det{}'(I - \lambda R(\lambda, A^{kp})))^{1/kp}.$$

Proof. Let $\sigma_0 = \{\lambda_n\}_{n=1}^{\infty}$. By Lemma 3.1, $\mathsf{g}(A) = kp - 1$. It is easy to see that this implies that $\mathsf{g}(A^{kp}) = 0$, and therefore $R(\lambda, A^{kp})$ is of trace class. Thus, the operator A^{kp} satisfies all the hypothesis of Proposition 3.3. The statement follows since $\zeta(as, S) = \zeta(s, S^a)$, and $\zeta'(0, S^a) = a\zeta'(0, S)$. □

Eventually, we turn to elliptic operators, and to the proof of Theorem 1.1. The advantage of working with this class of operators is due to the fact that some of the hypothesis of Proposition 3.4 are automatically satisfied. For, if H is a self-adjoint non-negative elliptic pseudo differential operator of order r defined in the space of the L^2 sections of some complex vector bundle over a compact Riemannian manifold of dimension m, then, $\sigma_0(H)$ is a sequence with exponent of convergence m/r and genus $\mathsf{g} = [m/r]$. This means that $(R(\lambda, H))^{\mathsf{g}+1}$ is of trace class, and therefore all the hypothesis of Proposition 3.4 are satisfied.

ACKNOWLEDGEMENTS. We thank the referee for his/her comments and remarks that improved the presentation.

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