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MOLECULAR DECOMPOSITION OF THE MODULATION SPACES

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Abstract

The aim of this paper is to develop a theory of decomposition in the weighted modulation spaces $M^{s,W}_{p,q}$ with $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $W \in A_\infty$, where W belongs to the class of A_∞ defined by Muckenhoupt. For this purpose we shall define molecules for the modulation spaces. As an application we give a simple proof of the boundedness of the pseudo-differential operators with symbols in $M^0_{\infty,\min(1,p,q)}$. We shall deal with dual spaces as well.

1. Introduction

The modulation spaces, introduced by Feichtinger in 1983 (see [6]), are one of the function spaces to investigate growth, decay and regularity of functions or distributions in a way other than the Besov spaces. Several important properties of the modulation spaces such as duality, interpolation theory and atomic decomposition were well investigated by Feichtinger and Gröchenig [6, 7, 8, 9, 10]. Now they are recognized as appropriate function spaces and they are applied to time-frequency analysis and pseudo-differential calculus. For example, by using the theory of the modulation spaces, Sjöstrand and Tachizawa generalized the theory of Calderón–Vaillancourt [4, 25] (see also the work due to Gröchenig–Heil [16]). In recent years, they are also applied to study the global well-posedness of solutions for the Cauchy problem such as KdV and NLS equations [2, 3].

Based on the standard notation of signal analysis, we adopt the following notations.

$$T_a f(x) := f(x-a), \quad M_b f(x) := e^{ib\cdot x} f(x), \quad a, b \in \mathbb{R}^n, \quad f \in \mathcal{S}',$$
$$f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\,dy,$$
$$\mathcal{F} f(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)\exp(-ix\cdot\xi)\,dx,$$
$$\mathcal{F}^{-1} f(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi)\exp(ix\cdot\xi)\,d\xi.$$

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To denote cubes in $\mathbb{R}^n$, we use

$$Q(r) := \{x \in \mathbb{R}^n : \max(|x_1|, \dots, |x_n|) \le r\},$$
$$Q_l := [l_1, l_1 + 1] \times [l_2, l_2 + 1] \times \cdots \times [l_n, l_n + 1]$$

for $r > 0$ and $l \in \mathbb{Z}^n$. It will be helpful to use the notation from [28]. Let $f \in \mathcal{S}'$ and $\tau \in \mathcal{S}$. Then we write

$$\tau(D) f := \mathcal{F}^{-1}(\tau \cdot \mathcal{F} f) = (2\pi)^{-n/2} \mathcal{F}^{-1}\tau * f. \tag{1}$$

As for the Fourier multipliers and the multiplication operators we prefer to avoid superfluous brackets. We shall list some typical examples in this paper: Let $a, b \in \mathbb{R}^n$. Then we write $T_a\phi(D) f := [T_a\phi](D) f$, $M_b\phi(D) f := [M_b\phi](D) f$, $M_b\psi * f := [M_b\psi] * f$. If possible confusion can occur, we bind the function on which the operator acts.

Fix $g \in \mathcal{S} \setminus \{0\}$. Then define

$$\| f : M^s_{p,q} \| := \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\langle f, M_y T_x g \rangle|^p \, dx \right)^{q/p} (1 + |y|)^{sq} \, dy \right)^{1/q}$$

for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Denote by $M^s_{p,q}$ the set of all tempered distributions $f \in \mathcal{S}'$ for which the norm is finite. An important observation is that the function space $M^s_{p,q}$ does not depend on the specific choices of $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. For more details we refer to [12].

In the present paper we consider the weighted modulation spaces. In general by a weighted modulation norm we mean the following norm given by

$$\| f : M^v_{p,q} \| := \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\langle f, M_y T_x g \rangle|^p v(x, y) \, dx \right)^{q/p} dy \right)^{1/q}.$$

Note that $M^s_{p,q}$ is recovered by setting $v(x, y) = (1 + |y|)^{sq}$. There are many important classes of weights.

1. A weight $v : \mathbb{R}^{2n} \to [0, \infty)$ is said to be a submultiplicative, if there exists a constant $C > 0$ such that $v(x + y) \le C v(x) v(y)$ for all $x, y \in \mathbb{R}^{2n}$.
2. Fix a submultiplicative weight v. A weight m is said to be v-moderate, if there exists a constant $C > 0$ such that $m(x + y) \le C v(x) m(y)$ for all $x, y \in \mathbb{R}^{2n}$.
3. A weight is said to be subconvolutive, if $v^{-1} \in L^1(\mathbb{R}^{2n})$ and $v^{-1} * v^{-1} \le c v^{-1}$ for some constant $c > 0$.
4. A weight v is said to satisfy the Gelfand–Raikov–Shilov condition (respectively the Beurling–Domar condition, the logarithmic integral condition), if

$$\lim_{n\to\infty} v(nx)^{1/n} = 1 \quad (\text{resp. } \textstyle\sum_{j=1}^{\infty} \log v(nx)/n < \infty,\ \int_{|x|\ge 1} \log v(x)/|x|^{n+1} \, dx < \infty).$$

It is shown in [15] that the Beurling–Domar condition implies the Gelfand–Raikov–Shilov condition. We refer to [8] for more details of the submultiplicative, moderate and subconvolutive weights not only on $\mathbb{R}^n$ but also on locally compact abelian groups. In the present paper, we consider weights of the form

$$v(x, y) = W(x)(1 + |y|)^s,$$

where $s \in \mathbb{R}$ and W belongs to the class A_∞ of Muckenhoupt. As the example $W(x) = |x|^\alpha$, $\alpha > -n$ shows, it can happen that v fails the submultiplicative condition or the subconvolutive condition. Another similar example shows that v does not necessarily satisfy the Beurling–Domar condition.

Before we go further, we recall the definition of A_p-weights. In the sequel by a "weight", we mean a non-negative measurable function $W \in L^1_{\mathrm{loc}}$ satisfying $0 < W < \infty$ for a.e. and we define the maximal operator M by

$$Mf(x) := \sup_{\substack{x \in Q \\ Q:\,\text{cube}}} \frac{1}{|Q|} \int_Q |f(y)|\,dy.$$

Let $1 \le p < \infty$. Then we define

$$A_p(W) = \begin{cases} \operatorname*{ess.\,sup}_{x \in \mathbb{R}^n} \dfrac{MW(x)}{W(x)} & \text{if} \quad p = 1, \\ \sup_{Q:\,\text{cube}} \left(\dfrac{1}{|Q|} \displaystyle\int_Q W(x)\,dx \right) \cdot \left(\dfrac{1}{|Q|} \displaystyle\int_Q W(x)^{1/(1-p)}\,dx \right)^{p-1} & \text{if} \quad 1 < p < \infty. \end{cases}$$

The quantity $A_p(W)$ is called the A_p-norm of W, although $A_p(W)$ is not actually a norm (see [20, 21]). Then it is easy to see that

$$A_p(W) \le A_q(W), \quad 1 \le q \le p < \infty.$$

The class A_p of weights is the set of all weights W for which the norm $A_p(W)$ is finite. We also define

$$A_\infty := \bigcup_{1 \le p < \infty} A_p.$$

We remark that $|x|^{-n+\varepsilon} \in A_1$ for all $0 < \varepsilon < n$. If $W \in A_\infty$, then we have

$$\int_{Q(l)} W(x)\,dx \le c\langle l \rangle^M, \quad l \in \mathbb{Z}^n \tag{2}$$

for some $M > 0$ and $c > 0$.

Let W be a weight. Then we define

$$\| f : L^W_p \| := \left(\int_{\mathbb{R}^n} |f(x)|^p W(x)\,dx \right)^{1/p}, \quad 1 \le p < \infty.$$

Let $1 < p < \infty$. Muckenhoupt showed that the maximal operator M is bounded on L_p^W if and only if $W \in A_p$. Muckenhoupt also proved that the weak-(1, 1) estimate, that is,

$$\int_{\{Mf>\lambda\}} W(x)\,dx \le \frac{C}{\lambda}\int |f(x)|W(x)\,dx$$

holds if and only if $W \in A_1$. We refer to [20, 21] for more details.

Having set down the elementary facts on the weights, let us describe the weighted function space $M_{p,q}^{s,W}$. Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. The first author of the present paper noticed that the definition of the unweighted modulation spaces can be described as follows: Pick a function $\phi \in \mathcal{S}$ so that $\mathrm{supp}(\phi) \subset Q(2)$, $\sum_{m\in\mathbb{Z}^n} T_m\phi(x) \equiv 1$ and write $\langle x\rangle := \sqrt{1+|x|^2}$. In [18] we have defined

$$\| f : M_{p,q}^s \| := \left(\sum_{m\in\mathbb{Z}^n} \langle m\rangle^{qs} \|[\mathcal{F}^{-1}T_m\phi] * f : L_p\|^q\right)^{1/q} \tag{3}$$

for $f \in \mathcal{S}'$. It is still possible to establish that different choices of ϕ will give us an equivalent norm.

The main results of this paper can be summarized as follows: Most of the theory of the modulation spaces $M_{p,q}^s$ carries over to the A_∞-weighted cases with $0 < p, q \le \infty$ and $s \in \mathbb{R}$.

Let $W \in A_\infty$ throughout. Then define $\| f_m : l_q(L_p^W)\| := \left(\sum_{m\in\mathbb{Z}^n} \| f_m : L_p^W\|^q\right)^{1/q}$ for a family of measurable functions $\{f_m\}_{m\in\mathbb{Z}^n}$. Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Then the modulation norm is given by

$$\begin{aligned} \| f : M_{p,q}^{s,W}\| &:= \|\langle m\rangle^s T_m\phi(D) f : l_q(L_p^W)\| \\ &= \left(\sum_{m\in\mathbb{Z}^n} \langle m\rangle^{qs} \|[\mathcal{F}^{-1}T_m\phi] * f : L_p^W\|^q\right)^{1/q}. \end{aligned} \tag{4}$$

Here and below we assume that $W \in A_P$ with $1 \le P < \infty$ for the sake of definiteness.

A fundamental technique in harmonic analysis is to represent a function or distribution as a linear combination of functions of an elementary form. We shall investigate the structure of weighted modulation spaces and discuss several applications of this technique. For example, the "Gabor expansion" for the modulation spaces is discussed in Gröchenig [12] and Galperin–Samarah [11]. The heart of the matter of this expansion is to decompose a function into a linear combination of elements of the family $\{T_l M_m g\}_{m;\, l\in\mathbb{Z}^n}$ which is created by just one "atomic" function g. However, such atomic decomposition has some disadvantages in analyzing the pseudo-differential operators. In general, it is not the case that the pseudo-differential operators map the family $\{T_l M_m g\}_{m;\, l\in\mathbb{Z}^n}$ to another one created by an atomic function again. To overcome this disadvantage, we introduce the "molecular" decomposition. Molecules are mapped to molecules again by pseudo-differential operators (Lemma 3.2). We refer to [1, 17] for the definition of the molecules for different modulation spaces.

DEFINITION 1.1 (Molecule). Let $s \in \mathbb{R}$. Suppose that $K, N \in \mathbb{N}$ are large enough and fixed. A C^K-function $\tau : \mathbb{R}^n \to \mathbb{C}$ is said to be an $(s; m, l)$-molecule, if it satisfies

$$|\partial^\alpha (e^{-im\cdot x}\tau(x))| \le \langle m\rangle^{-s}\langle x-l\rangle^{-N}, \quad x \in \mathbb{R}^n$$

for $|\alpha| \le K$. Also set

$$\mathcal{M}^s := \left\{ M = \{mol^s_{ml}\}_{m,l\in\mathbb{Z}^n} \subset C^K : \begin{array}{l} \text{there exists } c>0 \text{ such that} \\ c\cdot mol^s_{ml} \text{ is an } (s;m,l)\text{-molecule for every } m,l\in\mathbb{Z}^n \end{array} \right\}.$$

The integers K and N are taken sufficiently large, say, $K, N \ge 10[n/\min(1, p/P, q)] + M + 10$, where $[a]$ denotes the integer part of $a \in \mathbb{R}$ and M is a positive number appearing in (2).

Next, we introduce a sequence space $m^W_{p,q}$ to describe the condition of the coefficients of the molecular decomposition.

DEFINITION 1.2 (Sequence space $m^W_{p,q}$). Let $0 < p, q \le \infty$. Given $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n}$, define

$$\|\lambda : m^W_{p,q}\| := \left\| \left\{ \sum_{l\in\mathbb{Z}^n} \lambda_{ml}\chi_{Q_l} \right\}_m : l_q(L^W_p) \right\|.$$

Here a natural modification is made when p and/or q is infinite. The sequence $m^W_{p,q}$ is the set of doubly indexed sequences $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n}$ for which the quasi-norm $\|\lambda : m^W_{p,q}\|$ is finite.

With these definitions in mind, we shall present our main theorem in this paper.

Theorem 1.3. *Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Let $\kappa \in \mathcal{S}$ be taken so that $\chi_{Q(3)} \le \kappa \le \chi_{Q(3+1/100)}$.*
1. *Set $mol^s_{ml} := \langle m\rangle^{-s} T_l M_m[\mathcal{F}^{-1}\kappa]$. The decomposition, called Gabor decomposition, holds for $M^{s,W}_{p,q}$. More precisely, we have $\{mol^s_{ml}\}_{m,l\in\mathbb{Z}^n} \in \mathcal{M}^s$ and the mapping*

$$f \in M^{s,W}_{p,q} \mapsto \lambda = \{\langle m\rangle^s T_m \phi(D) f(l)\}_{m,l\in\mathbb{Z}^n} \in m^W_{p,q}$$

is bounded. Furthermore, any $f \in M^{s,W}_{p,q}$ admits the following Gabor decomposition

$$f = \sum_{m,l\in\mathbb{Z}^n} \lambda_{ml}\cdot mol^s_{ml}, \quad \lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n} = \{\langle m\rangle^s T_m\phi(D) f(l)\}_{m,l\in\mathbb{Z}^n} \in m^W_{p,q}. \tag{5}$$

2. *Suppose we are given* $M = \{mol^s_{ml}\}_{m,l \in \mathbb{Z}^n} \in \mathcal{M}^s$ *and* $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n} \in m^W_{p,q}$. *Then*

$$f := \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot mol^s_{ml} \tag{6}$$

converges unconditionally in the topology of $\mathcal{S}'$. *Furthermore* f *belongs to* $M^{s,W}_{p,q}$ *and satisfies the quasi-norm estimate* $\| f : M^{s,W}_{p,q} \| \le C \|\lambda : m^W_{p,q}\|$. *In particular if* $0 < p, q < \infty$, *then the convergence of* (6) *takes place in* $M^{s,W}_{p,q}$.

In [11] Galperin and Samarah obtained the following result.

Theorem 1.4. *Let* $0 < p, q \le \infty$ *and* $f \in \mathcal{S}'$. *Assume that* $\nu \colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ *is submultiplicative. Assume in addition that* $m \colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ *is* ν*-moderate. Fix* $g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ *and let* $\alpha, \beta > 0$ *be sufficiently small.*

Then f *satisfies*

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\langle f, M_x T_\omega g \rangle|^p m(x, \omega)^p \, dx \right)^{q/p} d\omega \right)^{1/q} < \infty$$

if and only if f *satisfies*

$$\left(\sum_{m \in \mathbb{Z}^n} \left(\sum_{k \in \mathbb{Z}^n} |\langle f, M_{m\alpha} T_{k\beta} g \rangle|^p m(\alpha m, \beta k)^p \right)^{q/p} \right)^{1/q} < \infty.$$

If this is the case, f *admits the decomposition* (5) *in* Theorem 1.3.

We remark that the part 1 of Theorem 1.3 is contained in Theorem 1.4 in the framework of the weighted setting if $w \equiv 1$ and $m \equiv 1$. Note that the result in Theorem 1.4 does not cover our result when $w(x) = |x|^{-n+\varepsilon}$ for $0 < \varepsilon < n$. But the main contribution of this paper is the part 2 of Theorem 1.3, which has never explicitly appeared in any literature at least for our class of weight functions. This result is important because pseudo-differential operators do not map $mol^s_{ml} = \langle m \rangle^{-s} T_l M_m \mathcal{F}\kappa$ to a function of the same form. All we can say is that the mapped one belongs to $\mathcal{M}^s$ (Lemma 3.2). In other words, pseudo-differential operators map the function f with the decomposition (5) to another one with the decomposition (6). We can, however, recover the norm of the mapped function by virtue of Theorem 1.3 2. Actually we take this advantage to show some boundedness result of pseudodifferential operators (Theorem 3.4).

Finally we describe the organization of this paper. In the next section, which is the heart of this paper, we investigate the molecular decomposition of the modulation spaces. In Section 2, we prove our main result Theorem 1.3. Although the proof of the decomposition result part 1 is just a suitable modification of the argument in [11], we include it for reader's convenience. Our main concern is, however, the proof of the synthesis result part 2. In Section 3 we investigate the pseudo-differential operators whose symbol belongs to $S^0_{0,0}$. Recall that a symbol class $S^m_{\rho,\delta}$ with $m \in \mathbb{R}$ and $0 \le \rho, \delta \le 1$ is the set of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$-functions a satisfying $|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}$. We remark that $M^0_{p,q}$-boundedness of the pseudo-differential operators with symbols in the Hörmander class $S^m_{\rho,\delta}$ was obtained in [11, 16, 25] with $1 \le p, q \le \infty$. As an application of $M^{s,W}_{p,q}$-boundedness of this result and the decomposition result in Section 2 we shall prove that the pseudo-differential operator with symbols in $M^0_{\infty,\min(1,p,q)}(\mathbb{R}^n \times \mathbb{R}^n)$ is bounded on $M^{0,W}_{p,q}$. We remark that in [12, 16] Gröchenig and Heil proved this result in the case when $1 \le p, q \le \infty$. Recently there are many literatures proving the boundedness on the modulation spaces of the pseudo-differential operators with symbols in the Sjöstrand class (see [12, 17, 22]). In particular Gröchenig established this type of boundedness by using the almost-diagonization. Here we shall use our decomposition results directly. What is new about this result is the fact that we have proved the counterpart for general parameters $0 < p, q \le \infty$ and the A_∞-weighted setting, and the point that we do not have to rely on the dual argument. We refer to [23, 24] for non-negative results on the boundedness of the pseudo-differential operators. In Section 4 we exhibit another application of the results in Section 2. In [19] the first author investigated the dual space of $M^0_{p,q}$ with $0 < p, q < \infty$. However, the definitive result when $0 < p \le 1 \le q < \infty$ was missing. We exploit the molecular decomposition along with the method used in [5]. In the present paper we shall supplement this missing part. The proof is again based on the molecular decomposition obtained in Section 2.

2. Molecular decomposition in $M^{s,W}_{p,q}$

In this section we deal with the molecular decomposition, in particular, the synthesis property. We assume that $\phi \in \mathcal{S}$ is a positive function satisfying

$$\operatorname{supp}(\phi) \subset Q(2), \qquad \sum_{m \in \mathbb{Z}^n} T_m \phi(x) \equiv 1. \tag{7}$$

As preliminaries we collect two important results on the band-limited distributions.

Lemma 2.1 ([27, Chapter 1]). *Let $0 < \eta < \infty$. Then there exists $c > 0$ such that*

$$\sup_{y \in \mathbb{R}^n} \langle y \rangle^{-n/\eta} |f(x-y)| \le c M^{(\eta)} f(x)$$

for all $f \in \mathcal{S}'$ with $\operatorname{diam}(\operatorname{supp}(\mathcal{F} f)) \le 10$, *where $M^{(\eta)}$ is a powered maximal operator*:

$$M^{(\eta)} f(x) := \sup_{\substack{x \in Q \\ Q:\ cube}} \left(\frac{1}{|Q|} \int_Q |f(y)|^\eta \, dy \right)^{1/\eta}. \tag{8}$$

We note that under our notation the well-known maximal inequality reads

$$\|M^{(\eta)} f : L_p^W\| \le c\|f : L_p^W\|, \quad 0 < \eta < p \le \infty. \tag{9}$$

Let $M \in \mathbb{N}$. Denote by W_2^M the Sobolev space consisting of $f \in L_2$ satisfying

$$\|f : W_2^M\| := \|\langle * \rangle^M \cdot \mathcal{F} f : L_2\| < \infty.$$

The following is a slight modification of the result in [27, Chapter 1].

Lemma 2.2 ([27, Chapter 1]). *Let $W \in A_P$ with $1 \le P < \infty$. Let $0 < p \le \infty$ and $M \in \mathbb{N}$ with $M > n/\min(1, p/P) - n/2$. Set*

$$H(D)f(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} H(\xi)\mathcal{F} f(\xi)e^{ix\cdot\xi}\, d\xi$$

for $H \in \mathcal{S}$ and $f \in \mathcal{S}'$. Then there exists a constant $c > 0$ independent of $R > 0$ so that

$$\|H(D)f : L_p^W\| \le c\|H(R\,\cdot\,) : W_2^M\| \cdot \|f : L_p^W\|,$$

whenever $H \in W_2^M$ and $f \in L_p^W \cap \mathcal{S}'$ with $\operatorname{diam}(\operatorname{supp}(\mathcal{F} f)) \le R$.

From this lemma we can easily deduce that the definition of the function space $M_{p,q}^{s,W}$ does not depend on the choice of $\phi \in \mathcal{S}$ satisfying (7).

The following well-known lemma is used to prove the decomposition results. For example, we refer for the proof to the paper [5] due to M. Frazier and B. Jawerth, who took originally a full advantage of this equality.

Lemma 2.3 ([5]). *Let $f \in \mathcal{S}'$ with frequency support contained in $Q(2)$, where we have defined*

$$Q(2) = \{x \in \mathbb{R}^n : \max(|x_1|, |x_2|, \dots, |x_n|) \le 2\}.$$

Assume in addition that $\kappa \in \mathcal{S}$ is supported on $Q(2)$ and that

$$\sum_{l \in \mathcal{S}} T_l \kappa = 1.$$

Then we have

$$f = (2\pi)^{-n/2} \sum_{l \in \mathbb{Z}^n} f(l) \cdot T_l[\mathcal{F}^{-1}\kappa]. \tag{10}$$

This result is well-known. However for the sake of convenience for readers we supply the proof.

Proof. First we take a test function $\tau \in \mathcal{S}$ arbitrarily. Then the support condition on f gives us

$$\langle \mathcal{F} f, \tau \rangle = \langle \mathcal{F} f, \kappa \cdot \kappa \cdot \tau \rangle. \tag{11}$$

We consider

$$\tau^*(x) := \sum_{l \in \mathbf{Z}^n} \kappa(x - 2\pi l)\tau(x - 2\pi l),$$

which is $2\pi\mathbb{Z}$-periodic. Expand τ^* to the Fourier series. Then we obtain

$$\tau^*(x) = \sum_{m \in \mathbf{Z}^n} a_m \exp(* \cdot mi), \tag{12}$$

where the coefficient is given by

$$\begin{aligned} a_m &= \frac{1}{(2\pi)^n} \int_{Q(\pi)} \tau^*(x) \exp(x \cdot mi)\, dx \\ &= \frac{1}{(2\pi)^n} \int_{Q(\pi)} \left(\sum_{l \in \mathbf{Z}^n} \kappa(x - 2\pi l)\tau(x - 2\pi l) \right) \exp(x \cdot mi)\, dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \kappa(x)\tau(x) \exp(x \cdot mi)\, dx. \end{aligned}$$

Here, $Q(\pi) = \{x \in \mathbb{R}^n \colon \max(|x_1|, |x_2|, \ldots, |x_n|) \le \pi\}$. Taking into account the support condition of the functions, we obtain

$$\kappa(x)\tau(x) = \kappa(x)\tau^*(x) = \sum_{m \in \mathbf{Z}^n} a_m \kappa(x) \exp(* \cdot mi). \tag{13}$$

We write out (11) in full by using (12) and (13).

$$\begin{aligned} \langle \mathcal{F} f, \tau \rangle &= \sum_{m \in \mathbf{Z}^n} a_m \langle \mathcal{F} f, \kappa \exp(* \cdot mi) \rangle \\ &= \sum_{m \in \mathbf{Z}^n} \frac{1}{(2\pi)^n} \langle \kappa \exp(-* \cdot mi), \tau \rangle \cdot \langle \mathcal{F} f, \kappa \exp(* \cdot mi) \rangle \\ &= \left\langle \left\{ \sum_{m \in \mathbf{Z}^n} \frac{1}{(2\pi)^n} \langle \mathcal{F} f, \kappa \exp(* \cdot mi) \rangle \cdot \kappa \exp(-* \cdot mi) \right\}, \tau \right\rangle. \end{aligned}$$

Finally observe that $\langle \mathcal{F} f, \kappa \cdot \exp(* \cdot mi) \rangle = (2\pi)^{n/2} f(m)$ from the definition of $f(x)$. Since τ is arbitrary, we finally obtain

$$\mathcal{F} f(x) = (2\pi)^{n/2} \sum_{m \in \mathbf{Z}^n} \frac{1}{(2\pi)^n} f(m) \cdot \kappa \exp(-x \cdot mi).$$

By taking the inverse Fourier transform to both sides, we have the desired result. □

It is convenient to transform (10) to the form in which we use in the present paper:

$$f = \sum_{m \in \mathbb{Z}^n} T_m \phi(D) f = (2\pi)^{-n/2} \sum_{m \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} T_m \phi(D) f(l) \cdot T_l M_m [\mathcal{F}^{-1} \kappa] \right). \tag{14}$$

Finally we need a lemma, which is of use for analysis of the modulation spaces.

Lemma 2.4. *Let $0 < p, q \le \infty$. Let $\{F_m\}_{m \in \mathbb{Z}^n}$ be a sequence of positive measurable functions. Set*

$$G_m := \sum_{l \in \mathbb{Z}^n} \langle l - m \rangle^{-N} F_m$$

for $m \in \mathbb{Z}^n$. Then we have

$$\| G_m : l_q(L_p^W) \| \le c \| F_m : l_q(L_p^W) \|$$

for some constant $c > 0$ as long as $N > 2n \max(1, 1/p) \max(1/q, (q-1)/q)$.

Before the proof, we remark that the following fundamental inequality holds.

$$(a+b)^v \le a^v + b^v, \quad 0 < v \le 1, \quad a, b > 0. \tag{15}$$

Proof. Let us set $\eta = \min(1, p)$. Then we have

$$\| f + g : L_p^W \|^\eta \le \| f : L_p^W \|^\eta + \| g : L_p^W \|^\eta$$

for all functions f and g. Using this inequality, we have

$$\begin{aligned} \| G_m : l_q(L_p^W) \|^\eta &= \left(\sum_{m \in \mathbb{Z}^n} (\| G_m : L_p^W \|^\eta)^{q/\eta} \right)^{\eta/q} \\ &\le \left(\sum_{m \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} \langle l - m \rangle^{-N\eta} \| F_l : L_p^W \|^\eta \right)^{q/\eta} \right)^{\eta/q}. \end{aligned}$$

If $q < \eta$, then we have

$$\left(\sum_{l \in \mathbb{Z}^n} \langle l - m \rangle^{-N\eta} \| F_l : L_p^W \|^\eta \right)^{q/\eta} \le \sum_{l \in \mathbb{Z}^n} \langle l - m \rangle^{-Nq} \| F_l : L_p^W \|^q$$

by virtue of (15).

If $q \geq \eta$, then we instead use the Hölder inequality to obtain

$$\begin{aligned}
&\left(\sum_{l\in\mathbb{Z}^n}\langle l-m\rangle^{-N\eta}\|F_l : L_p^W\|^\eta\right)^{q/\eta}\\
&\leq \left(\sum_{l\in\mathbb{Z}^n}\langle l-m\rangle^{-N\eta q'/2}\right)^{1/q'}\cdot\sum_{l\in\mathbb{Z}^n}\langle l-m\rangle^{-Nq/2}\|F_l : L_p^W\|^q\\
&\leq c\sum_{l\in\mathbb{Z}^n}\langle l-m\rangle^{-Nq/2}\|F_l : L_p^W\|^q.
\end{aligned}$$

As a result, we obtain

$$\left(\sum_{l\in\mathbb{Z}^n}\langle l-m\rangle^{-N\eta}\|F_l : L_p^W\|^\eta\right)^{q/\eta} \leq c\sum_{l\in\mathbb{Z}^n}\langle l-m\rangle^{-Nq/2}\|F_l : L_p^W\|^q$$

for all $0 < q < \infty$. Inserting this estimate, we obtain

$$\|G_m : l_q(L_p^W)\|^\eta \leq c\left(\sum_{m\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}\langle l-m\rangle^{-Nq/2}\|F_l : L_p^W\|^q\right)^{\eta/q} = c\|F_m : l_q(L_p^W)\|^\eta.$$

This is the desired result. □

2.1. Proof of (5). The proof will be based on the boundedness of the Hardy–Littlewood maximal operator, which is natural in our framework using the band-limited distributions, while the proof given by Galperin and Samarah relies on the precise estimate for the convolution.

As for the first assertion of Theorem 1.3, $\{mol_{ml}^s\}_{m,l\in\mathbb{Z}^n} \in \mathcal{M}^s$ is clear, once we fix K sufficiently large in the definition of molecules (Definition 1.1).

Let $f \in M_{p,q}^{s,W}$. Then we expand f according to (14):

$$f = (2\pi)^{-n/2}\sum_{m\in\mathbb{Z}^n}\left(\sum_{l\in\mathbb{Z}^n}T_m\phi(D)f(l)\cdot T_l M_m[\mathcal{F}^{-1}\kappa]\right).$$

Thus, if we set $\lambda_{ml} := (2\pi)^{-n/2}\langle m\rangle^s T_m\phi(D)f(l)$, $mol_{ml}^s := \langle m\rangle^{-s}T_l M_m[\mathcal{F}^{-1}\kappa]$ then we obtain a decomposition of f

$$f = \sum_{m,l\in\mathbb{Z}^n}\lambda_{ml}\cdot mol_{ml}^s\,. \tag{16}$$

Let us check that this decomposition fulfills the desired property in Theorem 1.3. Because we are going to utilize the maximal inequality (9), the expression in the right-hand side is agreeable.

Lemma 2.1 gives us $\left|\sum_{l\in\mathbb{Z}^n}\lambda_{ml}\chi_{Q_l}(x)\right| \le cM^{(\eta)}[\langle m\rangle^s T_m\phi(D)f](x)$ with η slightly less than $\min(1, p/P)$. Now that η is less than $\min(1, p/P)$, we can remove the maximal operator to obtain

$$\|\lambda : m^W_{p,q}\| \le c\|M^{(\eta)}[\langle m\rangle^s T_m\phi(D)f] : l_q(L^W_p)\| \le c\|f : M^{s,W}_{p,q}\|. \tag{17}$$

(17) together with (16) concludes the proof of the decomposition part of Theorem 1.3.

2.2. An equivalent quasi-norm. Having obtained a decomposition result, we are now going to be oriented to the synthesis part. To do this we need an equivalent quasi-norm. Feichtinger [6] defined the modulation spaces in the way described in the following theorem when $1 \le p, q \le \infty$. In [18], the first author extended the definition to the case $0 < p < 1$ or $0 < q < 1$ under the unweighted situation $W \equiv 1$ although we have to restrict the class for ψ. Such generalization was carried out by a simple modification of the argument in [12]. But the following theorem is a non-trivial extension of the result in [18] to the weighted case.

Theorem 2.5. *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $\psi \in \mathcal{S}$ be a positive function satisfying a non-degenerate condition: $\mathcal{F}\psi \neq 0$ on $Q(2)$. Then there exists a constant $c > 0$ such that, for all $f \in M^{s,W}_{p,q}$,*

$$c^{-1}\|f : M^{s,W}_{p,q}\| \le \|\langle k\rangle^s M_k\psi * f : l_q(L^W_p)\| \le c\|f : M^{s,W}_{p,q}\|.$$

To prove the theorem we need one more calculation.

Lemma 2.6. *Let $\tau, \theta \in \mathcal{S}$. Suppose that θ is compactly supported. Then for all $M \in \mathbb{N}$ there exists $c_{M,\alpha}$ depending only on τ, θ, α and M such that*

$$|\partial^\alpha(T_l\theta\cdot T_m\tau)(x)| \le c_{M,\alpha}\langle l-m\rangle^{-M} \quad \textit{for all} \quad x, l, m \in \mathbb{R}^n. \tag{18}$$

Proof. By the Leibnitz rule and the Peetre inequality $\langle a+b\rangle \le \sqrt{2}\langle a\rangle\cdot\langle b\rangle$, we have

$$|\partial^\alpha(T_l\theta\cdot T_m\tau)(x)| \le c_{M,\alpha}\langle x-l\rangle^{-M}\cdot\langle x-m\rangle^{-M} \le c_{M,\alpha}\langle l-m\rangle^{-M},$$

proving (18). □

With Lemmas 2.1, 2.2 and 2.6 in mind, let us complete the proof of Theorem 2.5.

Proof of Theorem 2.5. We shall first prove

$$\|\langle k\rangle^s M_k\psi * f : l_q(L^W_p)\| \le c\|f : M^{s,W}_{p,q}\| \tag{19}$$

and then

$$\|f : M^{s,W}_{p,q}\| \le c\|\langle k\rangle^s M_k\psi * f : l_q(L^W_p)\|. \tag{20}$$

We can assume by replacing ϕ, if necessary, even that

$$\sum_{l\in\mathbb{Z}^n} T_l\phi \equiv 1. \tag{21}$$

For the proof of (19) we decompose $M_k\psi * f$ by using (21)

$$M_k\psi * f = \sum_{l\in\mathbb{Z}^n} M_k\psi * [T_l\phi(D)f]. \tag{22}$$

$M_k\psi * f$ having been decomposed in (22), we are to estimate each summand. To do this, we rewrite the summand as

$$\begin{aligned} M_k\psi * [T_l\phi(D)f](x) &= c_n\mathcal{F}^{-1}(T_k[\mathcal{F}\psi]\cdot\mathcal{F}(T_l\phi(D)f))(x) \\ &= c_n T_k[\mathcal{F}\psi](D)T_l\phi(D)f(x) \\ &= c_n[T_k[\mathcal{F}\psi]\cdot T_l\tilde{\kappa}](D)T_l\phi(D)f(x) \\ &= c_n\int_{\mathbb{R}^n}\mathcal{F}^{-1}[T_k[\mathcal{F}\psi]\cdot T_l\tilde{\kappa}](y)T_l\phi(D)f(x-y)\,dy, \end{aligned}$$

where $\tilde{\kappa}\in\mathcal{S}$ is an auxiliary compactly supported function that equals 1 on $\mathrm{supp}(\phi)$. By virtue of Lemma 2.6 we have

$$|\mathcal{F}^{-1}[T_k[\mathcal{F}\psi]\cdot T_l\tilde{\kappa}](y)| \le c_N\langle l-k\rangle^{-N}\cdot\langle y\rangle^{-N}, \tag{23}$$

where N is taken arbitrarily large. Let $\eta := \min(1,p)/2$. From Lemma 2.1 we have

$$|T_l\phi(D)f(x-y)| \le cM^{(\eta)}[T_l\phi(D)f](x)\cdot\langle y\rangle^{n/\eta}. \tag{24}$$

Recall that N is still at our disposal. Thus, if we take N large enough and combine (23) and (24), we obtain

$$|M_k\psi * T_l\phi(D)f(x)| \le c\langle l-k\rangle^{-2N}\cdot M^{(\eta)}[T_l\phi(D)f](x).$$

Therefore, inserting this estimate and using the boundedness of $M^{(\eta)}$, we have

$$\begin{aligned} &\|\langle k\rangle^s M_k\psi * f : L_p^W\|^{\min(1,p/P)} \\ &\le \sum_{l\in\mathbb{Z}^n}\|\langle k\rangle^s M_k\psi * T_l\phi(D)f : L_p^W\|^{\min(1,p/P)} \\ &\le c\sum_{l\in\mathbb{Z}^n}\langle l-k\rangle^{-(2N-s)\min(1,p/P)}\cdot\|\langle l\rangle^s M^{(\eta)}[T_l\phi(D)f] : L_p^W\|^{\min(1,p/P)} \\ &\le c\sum_{l\in\mathbb{Z}^n}\langle l-k\rangle^{-(2N-s)\min(1,p/P)}\cdot\|\langle l\rangle^s T_l\phi(D)f : L_p^W\|^{\min(1,p/P)}. \end{aligned}$$

Here we have used $\langle a+b\rangle \le \sqrt{2}\langle a\rangle\cdot\langle b\rangle$ again.

By Lemma 2.4 and the fact that N is sufficiently large we obtain

$$\|\langle k\rangle^s M_k\psi * f : L^W_p\|^{\min(1,p)} \le c\left(\sum_{l\in\mathbb{Z}^n}\langle l-k\rangle^{-Nq}\cdot\|\langle l\rangle^s T_l\phi(D)f : L^W_p\|^q\right)^{1/u}.$$

Therefore, if we arrange this inequality, we are led to

$$\|\langle k\rangle^s M_k\psi * f : L^W_p\|^q \le c\sum_{l\in\mathbb{Z}^n}\langle l-k\rangle^{-Nq}\cdot\|\langle l\rangle^s T_l\phi(D)f : L^W_p\|^q. \tag{25}$$

If we add (25) over $k\in\mathbb{Z}^n$, then we obtain (19).

Now we prove (20). For this purpose we pick a smooth bump function $\eta_0\colon \mathbb{R}\to\mathbb{R}$ so that $\chi_{(-1,1)}\le\eta_0\le\chi_{(-2,2)}$. Set $\eta(x):=\eta_K(x):=\eta_0(K^{-1}x_1)\eta_0(K^{-1}x_2)\cdots\eta_0(K^{-1}x_n)$ with K large. We let $\eta^\sharp:=\eta(2^{-1}*)$ and $M:=[n/\min(1,p)-n/2]+1$. Then we have, taking into account the size of the supports of functions, that

$$\|f : M^{s,W}_{p,q}\| = \left(\sum_{k\in\mathbb{Z}^n}\|\langle k\rangle^s T_k\eta^\sharp(D)T_k\eta(D)T_k\phi(D)f : L^W_p\|^q\right)^{1/q}.$$

Since $\mathcal{F}\psi$ never vanishes on $\mathrm{supp}(\phi)$, the function $\Phi:=\phi/\mathcal{F}\psi$ is well-defined. Note that

$$T_k\phi(D)f = T_k\Phi(D)[M_k\psi * f].$$

Thus, using this decomposition and the translation invariance of W^M_2, we obtain

$$\begin{aligned}\|f : M^{s,W}_{p,q}\| &= \left(\sum_{k\in\mathbb{Z}^n}\|\langle k\rangle^s T_k\eta^\sharp(D)T_k\Phi(D)T_k\eta(D)[M_k\psi * f] : L^W_p\|^q\right)^{1/q}\\ &\le c\left(\sum_{k\in\mathbb{Z}^n}\|\eta^\sharp\cdot\Phi : W^M_2\|^q\cdot\|\langle k\rangle^s T_k\eta(D)[M_k\psi * f] : L^W_p\|^q\right)^{1/q}.\end{aligned} \tag{26}$$

Here for (26) we have invoked Lemma 2.2. Now by using

$$M_k\psi * T_k\eta(D)f = M_k\psi * f - (M_k\psi * f - M_k\psi * T_k\eta(D)f)$$

we obtain

$$\begin{aligned}\|f : M^{s,W}_{p,q}\| &\le cK^{M+n}\left(\sum_{k\in\mathbb{Z}^n}\|\langle k\rangle^s T_k\eta(D)[M_k\psi * f] : L^W_p\|^q\right)^{1/q}\\ &\le cK^{M+n}\left(\sum_{k\in\mathbb{Z}^n}\|\langle k\rangle^s M_k\psi * f : L^W_p\|^q\right)^{1/q}\\ &\quad + cK^{M+n}\left(\sum_{k\in\mathbb{Z}^n}\|\langle k\rangle^s M_k\psi * [(1-T_k\eta(D))f] : L^W_p\|^q\right)^{1/q}.\end{aligned} \tag{27}$$

Our strategy for the proof is to establish that the second term of (27) can be made small enough, if we take K sufficiently large. Recall that we have proved (25), that is, for every $g \in \mathcal{S}'$

$$\| \langle k \rangle^s M_k \psi * g : L_p^W \|^q \le c \sum_{m \in \mathbb{Z}^n} \langle k - m \rangle^{-Nq} \cdot \| \langle m \rangle^s T_m \phi(D) g : L_p^W \|^q.$$

If we apply the above inequality with $g = (1 - T_k \eta(D)) f$, then we obtain

$$\begin{aligned} &\| \langle k \rangle^s M_k \psi * (1 - T_k \eta(D)) f : L_p^W \|^q \\ &\le c \sum_{m \in \mathbb{Z}^n} \langle k - m \rangle^{-Nq} \cdot \| \langle m \rangle^s T_m \phi(D) (1 - T_k \eta(D)) f : L_p^W \|^q. \end{aligned}$$

Taking into account the support condition of η again, we are led to

$$\begin{aligned} &\| \langle k \rangle^s M_k \psi * (1 - T_k \eta(D)) f : L_p^W \|^q \\ &\le c \sum_{\substack{m \in \mathbb{Z}^n \\ |k-m| \ge K-2}} \langle k - m \rangle^{-Nq} \cdot \| \langle m \rangle^s T_m \phi(D) f : L_p^W \|^q. \end{aligned}$$

This inequality is summable over $k \in \mathbb{Z}^n$ to $cK^{-Nq+n} \| f : M_{p,q}^s \|^q$. If we insert this estimate to (27), then we obtain

$$\| f : M_{p,q}^{s,W} \| \le cK^{M+n} \| \langle k \rangle^s M_k \psi * f : l_q(L_p^W) \| + cK^{M+n+n/q-N} \| f : M_{p,q}^{s,W} \|. \tag{28}$$

By assumption, we have $f \in M_{p,q}^{s,W}$. Consequently, if we fix N so large that $N > M + n + n/q$ and then choose K large enough, then we can bring the second term of the right-hand side in (28) to the left-hand side. The result is

$$\| f : M_{p,q}^{s,W} \| \le c \| \langle k \rangle^s M_k \psi * f : l_q(L_p^W) \|,$$

proving (20). □

2.3. Proof of Theorem 1.3. First we verify that the sum converges in $\mathcal{S}'$.

Lemma 2.7. *Let $s \in \mathbb{R}$. Assume $\Lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n} \in m_{\infty,\infty}^W = m_{\infty,\infty}$ and that a family of functions $M = \{mol_{ml}^s\}_{m,l \in \mathbb{Z}^n}$ belongs to $\mathcal{M}^s$. Then the series*

$$\sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot mol_{ml}^s$$

is convergent unconditionally in $\mathcal{S}'$.

Proof. Fix a test function $\phi \in \mathcal{S}$ and set $\Phi_{ml}(x) := e^{-im\cdot x}\, mol^s_{ml}(x)$, $m, l \in \mathbb{Z}^n$ for the sake of brevity. Then $\{\Phi_{ml}\}_{m,l\in\mathbb{Z}^n} \subset C^K$ fulfills the following differential inequality

$$\sup_{x\in\mathbb{R}^n} \langle x-l\rangle^N |\partial^\alpha \Phi_{ml}(x)| \le c\langle m\rangle^{-s}$$

for all $m, l \in \mathbb{Z}^d$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le K$, where c is independent of m, l and α. Therefore we have

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(x)\, mol^s_{ml}(x)\, dx &= \int_{\mathbb{R}^n} \phi(x)\Phi_{ml}(x) \exp(im\cdot x)\, dx \\ &= \langle m\rangle^{-2K_0} \int_{\mathbb{R}^n} [(1-\Delta)^{K_0}(\phi(x)\Phi_{ml}(x))] \exp(im\cdot x)\, dx.\end{aligned}$$

Here $K_0 := [K/2]$. Therefore it follows that

$$\left| \int_{\mathbb{R}^n} \phi(x)\, mol^s_{ml}(x)\, dx \right| \le \frac{1}{\langle m\rangle^{2K_0}} \int_{\mathbb{R}^n} |[(1-\Delta)^{K_0}(\phi(x)\Phi_{ml}(x))]|\, dx \le \frac{c\langle m\rangle^{-2K_0-s}}{\langle l\rangle^{2K_0}}.$$

From this and (2) we can readily deduce the desired convergence. □

Lemma 2.8. *Suppose that* $0 < p, q \le \infty$ *and* $s \in \mathbb{R}$. *Any* $(s; m, l)$*-molecule belongs to* $M^{s,W}_{p,q}$, *provided* K *and* N *in* Definition 1.1 *are large enough.*

Proof. Let M be an $(s; m, l)$-molecule. Then we have

$$\begin{aligned}[\mathcal{F}^{-1}T_m\phi] * M(x) &= e^{im\cdot x} \int_{\mathbb{R}^n} e^{-im\cdot y} \mathcal{F}\phi(x-y) M(y)\, dy \\ &= \frac{e^{im\cdot x}}{(1+|m|^2)^{K_0}} \int_{\mathbb{R}^n} [(1-\Delta)^{K_0/2} e^{-im\cdot y}] \mathcal{F}\phi(x-y) M(y)\, dy \\ &= \frac{e^{im\cdot x}}{(1+|k|^2)^{K_0}} \int_{\mathbb{R}^n} e^{-im\cdot y} (1-\Delta)^{K_0/2} [\mathcal{F}\phi(x-y) M(y)]\, dy \\ &= \frac{e^{im\cdot x}}{(1+|k|^2)^{K_0}} \int_{\mathbb{R}^n} e^{-im\cdot(x-y)} (1-\Delta)^{K_0/2} [\mathcal{F}\phi(y) M(x-y)]\, dy,\end{aligned}$$

where $K_0 = [K/2]$. Note that

$$|(1-\Delta)^{K_0/2}[\mathcal{F}\phi(y)M(x-y)]| \le c\langle y\rangle^{-N-n-1}\langle x-y\rangle^{-N} \le c\langle y\rangle^{-n-1}\langle x\rangle^{-N}.$$

Hence, it follows that

$$|[\mathcal{F}^{-1}T_m\phi] * M(x)| \le \frac{c}{(1+|m|^2)^{K_0}} \langle x\rangle^{-N}.$$

As a result, we obtain

$$\|[\mathcal{F}^{-1}T_m\phi] * M : L^W_p\| \le \frac{c}{(1+|m|^2)^{K_0}}$$

because W is an A_P-weight. This inequality is summable and we obtain

$$\|M : M^{s,W}_{p,q}\| < \infty.$$

Thus, the proof is complete. □

With these lemmas in mind, we prove the remaining part of Theorem 1.3.

Proof of Theorem 1.3. Let $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n}$. We define $f = \sum_{ml\in\mathbb{Z}^n} \lambda_{ml}\, mol^s_{ml}$. Then Lemmas 2.7 and 2.8 together with Fatou's lemma reduce the matters to showing

$$\|\langle k\rangle^s M_k\psi * f : l_q(L^W_p)\| \le c\|\lambda : m^W_{p,q}\|, \tag{29}$$

where ψ is a smooth function supported on a small ball $B(r)$ and the elements in λ are zero with finite exceptions. Let $k, l, m \in \mathbb{Z}^n$ be fixed. We estimate

$$M_k\psi * mol^s_{ml}(x) = e^{ik\cdot x}\int_{\mathbb{R}^n} e^{i(m-k)\cdot y}\psi(x-y)\cdot(e^{-im\cdot y}\, mol^s_{ml}(y))\,dy.$$

First insert $(1-\Delta)^{K_0}e^{i(m-k)\cdot y} = \langle m-k\rangle^{2K_0}e^{i(m-k)\cdot y}$ and carry out the integration by parts. Here $K_0 := [K/2]$. Then we obtain

$$M_k\psi * mol^s_{ml}(x) = \frac{e^{ik\cdot x}}{\langle m-k\rangle^{2K_0}}\int_{\mathbb{R}^n}\frac{(1-\Delta_y)^{K_0}\{\psi(x-y)(e^{-im\cdot y}\, mol^s_{ml}(y))\}}{e^{i(k-m)\cdot y}}\,dy.$$

Thus, since $\{mol^s_{ml}\}_{m,l\in\mathbb{Z}^n} \in \mathcal{M}^s$ and ψ is a function supported on a small ball $B(r)$, we are led to

$$|M_k\psi * mol^s_{ml}(x)| \le \frac{c\langle m\rangle^{-s}}{\langle m-k\rangle^{2K_0}}\int_{B(x,r)}\langle y-l\rangle^{-2K_0}\,dy \le \frac{c\langle m\rangle^{-s}}{(\langle m-k\rangle\cdot\langle x-l\rangle)^{2K_0}}.$$

Inserting this estimate, we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^n}\left\{\int_{\mathbb{R}^n}\left(\langle k\rangle^s\sum_{m,l\in\mathbb{Z}^n}|\lambda_{ml}\cdot M_k\psi * mol^s_{ml}(x)|\right)^p W(x)\,dx\right\}^{q/p}\\ &\le c\sum_{k\in\mathbb{Z}^n}\left\{\int_{\mathbb{R}^n}\left(\langle k\rangle^s\sum_{m,l\in\mathbb{Z}^n}\langle m\rangle^{-s}|\lambda_{ml}|\cdot(\langle m-k\rangle\cdot\langle x-l\rangle)^{-2K_0}\right)^p W(x)\,dx\right\}^{q/p}.\end{aligned} \tag{30}$$

To estimate (30), we proceed as follows:

$$\sum_{l\in\mathbb{Z}^n}\frac{|\lambda_{ml}|}{\langle x-l\rangle^{2K_0}} = \sum_{\substack{j\in\mathbb{N},l\in\mathbb{Z}^n\\ 2^{j-1}\le\langle x-l\rangle\le 2^j}}\frac{|\lambda_{ml}|}{\langle x-l\rangle^{2K_0}} \le c\sum_{j\in\mathbb{N}}\frac{1}{2^{2jK_0}}\sum_{l\in\mathbb{Z}^n,\langle x-l\rangle\le 2^j}|\lambda_{ml}|.$$

Now that $0 < \eta < 1$, we have

$$\sum_{l \in \mathbb{Z}^n} \frac{|\lambda_{ml}|}{\langle x-l \rangle^{2K_0}} \le c \sum_{j \in \mathbb{N}} \frac{1}{2^{2jK_0}} \left(\sum_{l \in \mathbb{Z}^n,\ \langle x-l \rangle \le 2^j} \|\lambda_{ml}\|^{\eta} \right)^{1/\eta}.$$

Since K_0 is sufficiently large, we obtain

$$\begin{aligned}
\sum_{l \in \mathbb{Z}^n} |\lambda_{ml}| \cdot \langle x-l \rangle^{-2K_0} &\le c \sum_{j \in \mathbb{N}} \frac{1}{2^{2jK_0 - jn/\eta}} \left(\frac{1}{2^{jn}} \sum_{l \in \mathbb{Z}^n,\ \langle x-l \rangle \le 2^j} |\lambda_{ml}|^{\eta} \right)^{1/\eta} \\
&\le c \sum_{j \in \mathbb{N}} \frac{1}{2^{2jK_0 - jn/\eta}} M^{(\eta)} \left[\sum_{l \in \mathbb{Z}^n} \lambda_{ml} \chi_{Q_l} \right](x) \\
&\le c M^{(\eta)} \left[\sum_{l \in \mathbb{Z}^n} \lambda_{ml} \chi_{Q_l} \right](x).
\end{aligned}$$

If we insert this to (30), then we obtain

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}^n} \left\{ \int_{\mathbb{R}^n} \left(\langle k \rangle^s \sum_{m,l \in \mathbb{Z}^n} |\lambda_{ml} \cdot M_k \psi * mol_{ml}^s(x)| \right)^p W(x)\,dx \right\}^{q/p} \\
&\le c \sum_{k \in \mathbb{Z}^n} \left\{ \int_{\mathbb{R}^n} \left(\sum_{m \in \mathbb{Z}^n} M^{(\eta)} \left[\sum_{l \in \mathbb{Z}^n} \lambda_{ml} \chi_{Q_l} \right](x) \cdot \langle m-k \rangle^{-2K_0 + |s|} \right)^p W(x)\,dx \right\}^{q/p}.
\end{aligned}$$

Assuming K_0 sufficiently large, we are in the position of using Lemma 2.4 with

$$F_m(x) = M^{(\eta)} \left[\sum_{l \in \mathbb{Z}^n} \lambda_{ml} \chi_{Q_l} \right](x)$$

and $N = 2K_0 - |s|$. Using Lemma 2.4 and the maximal inequality, we obtain

$$\begin{aligned}
&\| \langle k \rangle^s M_k \psi * f : l_q(L_p^W) \| \\
&\le \sum_{k \in \mathbb{Z}^n} \left\{ \int_{\mathbb{R}^n} \left(\langle k \rangle^s \sum_{m,l \in \mathbb{Z}^n} |\lambda_{ml} \cdot M_k \psi * mol_{ml}^s(x)| \right)^p W(x)\,dx \right\}^{q/p} \\
&\le c \left\{ \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} M^{(\eta)} \left[\sum_{l \in \mathbb{Z}^n} \lambda_{ml} \chi_{Q_l} \right](x)^p W(x)\,dx \right\}^{q/p} \\
&\le c \left\{ \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \left| \sum_{l \in \mathbb{Z}^n} \lambda_{ml} \chi_{Q_l}(x) \right|^p W(x)\,dx \right\}^{q/p} \\
&= c \| \lambda : m_{p,q}^W \|^q,
\end{aligned}$$

which is exactly the result (29) we wish to prove. □

3. Pseudo-differential operators

In this section, as an application of Theorem 1.3, we prove the boundedness of the pseudo-differential operators.

Given $a \in S^m_{\rho,\delta}$, $m \in \mathbb{R}$, $0 \le \delta, \rho \le 1$, we define

$$a(x, D) f(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} a(x, \xi)\mathcal{F} f(\xi) \exp(ix \cdot \xi)\, d\xi, \tag{31}$$

for $f \in \mathcal{S}$. Following [28], we denote $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. As is easily seen by carrying out the integration by parts, $a(x, D)$ is a continuous linear operator on $\mathcal{S}$. If we define $a^\sharp(x, D)$, the adjoint operator of $a(x, D)$ by

$$a^\sharp(x, D)g(x) := (2\pi)^{-n} \iint_{\mathbb{R}^n\times\mathbb{R}^n} a(y, \xi)g(y)e^{i(y\cdot\xi - x\cdot\xi)}\, dy\, d\xi \tag{32}$$

in the sense of oscillatory integral, then we see that $a^\sharp(x, D)$ is also a continuous linear operator on $\mathcal{S}$. Therefore, we can extend $a(x, D)$ to a continuous linear operator on $\mathcal{S}'$ by defining, for $f \in \mathcal{S}'$

$$\langle a(x, D) f, \phi\rangle := \langle f, a^\sharp(x, D)\phi\rangle, \quad \phi \in \mathcal{S}. \tag{33}$$

3.1. Symbol class $S^0_{0,0}$. In this section we shall prove $M^{s,W}_{p,q}$-boundedness by means of molecular decomposition of pseudo-differential operators with symbols in $S^0_{0,0}$.

Theorem 3.1. *Suppose that $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Let $a \in S^0_{0,0}$, namely, assume that $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the differential inequalities*

$$\sup_{x,\xi\in\mathbb{R}^n} |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| < \infty$$

for all $\alpha, \beta \in \mathbb{N}_0{}^n$. Then, the operator $a(x, D)$, defined initially on $\mathcal{S}$ by (31), *can be extended continuously to a bounded linear operator on $M^{s,W}_{p,q}$.*

By Theorem 1.3, Theorem 3.1 essentially is reduced to establishing the following.

Lemma 3.2. *Suppose that $s \in \mathbb{R}$. Let $\kappa \in \mathcal{S}$ be a compactly supported function. We define $mol^s_{ml} \in \mathcal{S}$ for $m, l \in \mathbb{Z}^n$ by setting $mol^s_{ml}(x) := \langle m\rangle^{-s} T_l M_m[\mathcal{F}^{-1}\kappa](x)$. Assume in addition that $a \in S^0_{0,0}$. Then we have $\{a(x, D)\, mol^s_{ml}\}_{m,l\in\mathbb{Z}^n} \in \mathcal{M}^s$.*

Proof. To prove this, we write $a(x, D)\, mol^s_{ml}$ out in full. As is easily verified, we have $\mathcal{F}\, mol^s_{ml} = \langle m\rangle^{-s} M_{-l} T_m \kappa$ and hence

$$a(x, D)\, mol^s_{ml}(x) = (2\pi)^{-n/2}\langle m\rangle^{-s}\int_{\mathbb{R}^n} a(x, \xi)e^{-il\cdot\xi}\kappa(\xi - m)e^{i\xi\cdot x}\, d\xi$$
$$= (2\pi)^{-n/2}\langle m\rangle^{-s}\int_{\mathbb{R}^n} a(x, \xi + m)e^{i(\xi+m)\cdot(x-l)}\kappa(\xi)\, d\xi.$$

Therefore, what we have to estimate is the following function:

$$e^{-im\cdot x}a(x, D)\, mol^s_{ml}(x) = (2\pi)^{-n/2}\langle m\rangle^{-s}e^{-im\cdot l}\int_{\mathbb{R}^n} a(x, \xi + m)e^{i\xi\cdot(x-l)}\kappa(\xi)\, d\xi. \tag{34}$$

By using $(1 - \Delta_\xi)^N e^{i\xi\cdot(x-l)} = \langle x - l\rangle^{2N} e^{i\xi\cdot(x-l)}$, it is not so hard to see

$$|e^{-im\cdot x}a(x, D)\, mol^s_{ml}(x)| \le c\langle m\rangle^{-s}\langle x - l\rangle^{-2N}.$$

Since a similar argument works for any partial derivative of $e^{-im\cdot x}a(x, D)\, mol^s_{ml}(x)$ in view of (34), the proof of this lemma is now complete. □

Having proved Lemma 3.2, we turn to the proof of Theorem 3.1.

Proof. Given $f \in M^{s,W}_{p,q} \subset M^{s,W}_{\infty,\infty}$, we expand it again according to (14) along with the coefficient condition:

$$f = (2\pi)^{-n/2}\sum_{m\in\mathbb{Z}^n}\left(\sum_{l\in\mathbb{Z}^n} T_m\psi(D) f(l)\cdot T_l M_m[\mathcal{F}^{-1}\kappa]\right),$$
$$\|\{\langle m\rangle^s T_m\psi(D) f(l)\}_{m,l\in\mathbb{Z}^n} : m^W_{p,q}\| \le c\| f : M^{s,W}_{p,q}\|. \tag{35}$$

With this formula in mind, we define

$$a(x, D) f := (2\pi)^{-n/2}\sum_{m\in\mathbb{Z}^n}\left(\sum_{l\in\mathbb{Z}^n} T_m\psi(D) f(l)\cdot a(x, D)[T_l M_m[\mathcal{F}^{-1}\kappa]]\right). \tag{36}$$

Since (35) is valid for $f \in \mathcal{S}$, (36) is an extension of $a(x, D)$ from $\mathcal{S}$ to $M^{s,W}_{p,q}$. By virtue of (33) and the convergence of (35) and (36) in $M^{s,W}_{p,q}$, we see that the extension is unique. Now we are in the position of using the synthesis part of Theorem 1.3. As we have verified in Lemma 3.2, we have $\{a(x, D)[T_l M_m[\mathcal{F}^{-1}\kappa]]\}_{m,l\in\mathbb{Z}^n} \in \mathcal{M}^s$. Thus, the estimate of the coefficients yields that $f \mapsto a(x, D) f$ is a continuous operator on $M^{s,W}_{p,q}$. □

REMARK 3.3. It is worth pointing out that we can say more. Let $0 < p, q \le \infty$. Then there is a large integer N_0, which depends on p and q, so that the pseudo-differential operator $a(x, D)$ is bounded on $M^{s,W}_{p,q}$ whenever a is a C^{N_0}-function satisfying

$$\||a|\|_{N_0} := \sup_{\substack{x,\xi\in\mathbb{R}^n \\ \alpha,\beta\in\mathbb{N}_0{}^n \\ |\alpha|,|\beta|\le N_0}} |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| < \infty.$$

Reexamine Definition 1.2 and the proof of Theorem 3.1 together with Lemma 3.2. Then we see $\|a(x, D)\|_{M^{s,W}_{p,q}} := \sup_{f\in M^{s,W}_{p,q}\setminus\{0\}} \|a(x, D) f : M^{s,W}_{p,q}\|/\|f : M^{s,W}_{p,q}\| \le c\||a|\|_{N_0}$, if N_0 is large enough.

3.2. Symbol class $M^0_{\infty,\min(1,p,q)}$. In this section we deal with the symbol class $M^0_{\infty,\min(1,p,q)}$, which contains $S^0_{0,0}$ strictly. The crux of the proof is the decomposition result we have obtained in Section 2. As is easily shown, $M^0_{\infty,\min(1,p,q)}$ can be embedded into L_∞. In general we have

$$M^0_{p,\min(p,p')} \subset L_p \subset M^0_{p,\max(p,p')}, \quad 1 \le p \le \infty.$$

Meanwhile $M^0_{\infty,1}$ is known to contain non-smooth functions. Thus, we can say Theorem 3.1 can be widely extended to the theorem below.

Theorem 3.4. *Suppose that* $0 < p, q \le \infty$. *Let* $a \in M^0_{\infty,\min(1,p,q)}(\mathbb{R}^n \times \mathbb{R}^n)$. *Then, the operator* $a(x, D)$, *defined initially on* $\mathcal{S}$ *by* (31), *can be extended continuously to* $M^{0,W}_{p,q}$. *Furthermore, we have*

$$\|a(x, D)\|_{M^{0,W}_{p,q}\to M^{0,W}_{p,q}} \le c\|a : M^0_{\infty,\min(1,p,q)}(\mathbb{R}^n \times \mathbb{R}^n)\|.$$

Proof. Let $a \in M^0_{\infty,\min(1,p,q)}(\mathbb{R}^n \times \mathbb{R}^n)$. As we have discussed in Theorem 1.3, we take an auxiliary function $\kappa : \mathbb{R}^n \to \mathbb{R}$ satisfying $\chi_{Q(3)} \le \kappa \le \chi_{Q(3+1/100)}$. In order to apply Theorem 1.3, we shall adopt an auxiliary function κ^* of tensored type. Speaking precisely, we replace κ with κ^* given by $\kappa^*(x, \xi) := \kappa(x)\kappa(\xi)$: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The fact that κ is of tensored type gives us

$$a(x, \xi) = \sum_{\alpha,\beta,m,l\in\mathbb{Z}^n} \lambda_{\alpha,\beta,m,l} \cdot T_\beta M_\alpha[\mathcal{F}^{-1}\kappa](x) T_l M_m[\mathcal{F}^{-1}\kappa](\xi) \tag{37}$$

with the coefficient condition

$$\left(\sum_{m,\alpha\in\mathbb{Z}^n}\left(\sup_{l,\beta\in\mathbb{Z}^n}|\lambda_{\alpha,\beta,m,l}|\right)^{\min(1,p,q)}\right)^{1/\min(1,p,q)} \le c\|a : M^0_{\infty,\min(1,p,q)}(\mathbb{R}^n \times \mathbb{R}^n)\|. \tag{38}$$

Keeping (37) and (38) in mind, we define

$$a_{m,\alpha}(x,\xi) := \sum_{\beta,l\in\mathbb{Z}^n} \lambda_{\alpha,\beta,m,l} \cdot T_\beta M_\alpha[\mathcal{F}^{-1}\kappa](x) T_l M_m[\mathcal{F}^{-1}\kappa](\xi).$$

Then we have

$$a_{m,\alpha}(x,\xi) = e^{i\alpha\cdot x}\left(\sum_{\beta,l\in\mathbb{Z}^n} e^{-i(l\cdot m+\alpha\cdot\beta)} \cdot \lambda_{\alpha,\beta,m,l} \cdot T_\beta[\mathcal{F}^{-1}\kappa](x) T_l[\mathcal{F}^{-1}\kappa](\xi)\right) e^{im\cdot\xi}.$$

Thus, if we set

$$\begin{aligned}
a^{(1)}_{m,\alpha}(x,\xi) &:= e^{i\alpha\cdot x},\\
a^{(2)}_{m,\alpha}(x,\xi) &:= \sum_{\beta,l\in\mathbb{Z}^n} e^{-i(l\cdot m+\alpha\cdot\beta)} \cdot \lambda_{\alpha,\beta,m,l} \cdot T_\beta[\mathcal{F}^{-1}\kappa](x) T_l[\mathcal{F}^{-1}\kappa](\xi),\\
a^{(3)}_{m,\alpha}(x,\xi) &:= e^{im\cdot\xi},
\end{aligned}$$

then the pseudo-differential operator is factorized into three pseudo-differential operators:

$$a_{m,\alpha}(x,D) = a^{(1)}_{m,\alpha}(x,D) \circ a^{(2)}_{m,\alpha}(x,D) \circ a^{(3)}_{m,\alpha}(x,D).$$

It is easy to see that $a^{(1)}_{m,\alpha}$ is a multiplication operator which is actually an isomorphism on $M^{0,W}_{p,q}$ and that $a^{(3)}_{m,\alpha}$ is a translation operator which is also an isomorphism on $M^{0,W}_{p,q}$. Note that the operator norm is uniformly bounded over m and α. Thus, the matters are reduced to investigating the operator norm of $a^{(2)}_{m,\alpha}$.

Now it is high time to apply Remark 3.3. Assuming $\sup_{l,\beta\in\mathbb{Z}^n}|\lambda_{\alpha,\beta,m,l}| < \infty$, we can easily obtain

$$\||a^{(2)}_{m,\alpha}|\|_{N_0} \le c \sup_{l,\beta\in\mathbb{Z}^n} |\lambda_{\alpha,\beta,m,l}|,$$

provided N_0 is an integer as in Remark 3.3. Thus, we have obtained

$$\|a_{m,\alpha}(x,D)\|_{M^{0,W}_{p,q}\to M^{0,W}_{p,q}} \le c \sup_{l,\beta\in\mathbb{Z}^n} |\lambda_{\alpha,\beta,m,l}|. \tag{39}$$

By $\|f+g : M^{0,W}_{p,q}\|^{\min(1,p,q)} \le \|f : M^{0,W}_{p,q}\|^{\min(1,p,q)} + \|g : M^{0,W}_{p,q}\|^{\min(1,p,q)}$, we obtain

$$\|a(x,D)\|_{M^{0,W}_{p,q}\to M^{0,W}_{p,q}} \le \left(\sum_{m,\alpha} \|a_{m,\alpha}(x,D)\|_{M^0_{p,q}}{}^{\min(1,p,q)}\right)^{1/\min(1,p,q)}.$$

Adding (39) over m and α and using (38), we see that $a(x,D)$ is bounded on $M^{0,W}_{p,q}$. $\square$

4. Dual space

We will apply our decomposition results to specify the dual space of $M^s_{p,q} = M^{s,1}_{p,q}$. We remark that in [19] we have obtained some results even for $0 < p, q < \infty$ and $s = 0$. Our approach here is taking full advantage of Theorem 1.3 to prove the following. Given a $p \in (0, \infty]$, we define $p' := p/(p-1)$ if $p > 1$ and $p' := \infty$ if $p \le 1$.

Theorem 4.1. *Let $0 < p, q < \infty$ and $s \in \mathbb{R}$.*
1. *Let $f \in M^{-s}_{p',q'}$. Then the functional $g \in \mathcal{S} \mapsto \langle f, g\rangle \in \mathbb{C}$ can be extended to a continuous linear functional on $M^s_{p,q}$.*
2. *Conversely any continuous linear functional on $M^s_{p,q}$ can be realized with $f \in M^{-s}_{p',q'}$.*

Proof. The proof of 1 is straightforward and we omit the detail. We shall prove 2 only in the case when $0 < p \le 1 \le q < \infty$, the rest being covered in [19] when $s = 0$. An argument similar to the one below works for the remaining case. Let ζ be a continuous functional on $M^s_{p,q}$. Then we can define the continuous operator R from $m_{p,q}$ to $M^s_{p,q}$ as follows: Let

$$R(\lambda)(x) := \sum_{m,l\in\mathbb{Z}^n} \lambda_{ml} \cdot \langle m\rangle^{-s} T_l M_m[\mathcal{F}^{-1}\kappa](x),$$

where κ is a function appearing in Theorem 1.3. Set $\psi := \zeta \circ R\colon m_{p,q} \to \mathbb{C}$. Then ψ is a continuous functional on $m_{p,q}$. As is well-known, any continuous functional on $m_{p,q}$ can be realized with a coupling, that is, $\psi(\lambda)$ can be expressed as

$$\psi(\lambda) = \sum_{m,l\in\mathbb{Z}^n} \rho_{ml} \cdot \lambda_{ml}, \quad \lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n} \quad \text{with} \quad \|\rho : m_{\infty,q'}\| \le c\|\zeta \circ R\|_*,$$

where $\rho = \{\rho_{ml}\}_{m,l\in\mathbb{Z}^n} \in m_{\infty,q'}$ and $\|\cdot\|_*$ denotes the operator norm. Be reminded that ϕ is a function satisfying (7) to define the norm $\|f : M^s_{p,q}\|$. Setting

$$S(g) := \{\langle m\rangle^s T_m\phi(D)g(l)\}_{l,m\in\mathbb{Z}^n}, \quad g \in M^s_{p,q},$$

we obtain a linear mapping $S\colon M^s_{p,q} \to m_{p,q}$ satisfying

$$\|S(g) : m_{p,q}\| \le c\|g : M^s_{p,q}\|, \quad \zeta = \zeta \circ R \circ S = \psi \circ S.$$

Thus, we have $\zeta(g) = \psi(\{\langle m\rangle^s T_m\phi(D)g(l)\}_{l,m\in\mathbb{Z}^n}) = \sum_{m,l\in\mathbb{Z}^n} \langle m\rangle^s \rho_{ml} \cdot T_m\phi(D)g(l)$ for all $g \in M^s_{p,q}$. Now we set $f := (2\pi)^{-n/2} \sum_{m,l\in\mathbb{Z}^n} \langle m\rangle^s \rho_{ml} \cdot T_l M_{-m}[\mathcal{F}\phi]$. Then Theorem 1.3 gives us

$$f \in M^{-s}_{\infty,q'}, \quad \|f : M^{-s}_{\infty,q'}\| \le c\|\rho : m_{\infty,q'}\| \le c\|\zeta \circ R\|_* \le c\|\zeta\|_*.$$

A simple calculation yields

$$\begin{aligned}\langle f, g\rangle &= (2\pi)^{-n/2} \sum_{m,l\in\mathbb{Z}^n} \langle m\rangle^s \rho_{ml} \cdot \langle T_l M_{-m}[\mathcal{F}\phi], g\rangle \\ &= (2\pi)^{-n/2} \sum_{m,l\in\mathbb{Z}^n} \langle m\rangle^s \rho_{ml} \cdot \langle \mathcal{F}^{-1}[T_m\phi](l-*), g\rangle \\ &= \zeta(g)\end{aligned}$$

for all $g \in \mathcal{S}$. Therefore, 2 is proved. □

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