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ON THE DADE CHARACTER CORRESPONDENCE AND ISOTYPIES BETWEEN BLOCKS OF FINITE GROUPS

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Abstract

In [3] Dade generalized the Glauberman character correspondence. In [13] Tasaka showed that the Dade correspondence induces an isotypy between blocks of finite groups under some assumptions. In this paper we obtain a generalization of [13], Theorem 5.5.

1. Introduction

Let p be a prime and $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system such that $\mathcal{K}$ is a splitting filed for all finite groups which we consider in this paper. Let $\mathcal{S}$ denote $\mathcal{O}$ or k. For a finite abelian group F, we denote by $\hat{F}$ the character group of F and by $\hat{F}_q$ the subgroup of $\hat{F}$ of order q for $q \in \pi(F)$ where $\pi(F)$ is the set of all primes dividing the order $|F|$ of F. Let G be a finite group and N a normal subgroup of G. We denote by $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G and $\mathrm{Irr}^G(N)$ be the set of G-invariant irreducible characters of N. For $\phi \in \mathrm{Irr}(N)$, we denote by $\mathrm{Irr}(G|\phi)$ the set of irreducible characters χ of G such that ϕ is a constituent of the restriction χ_N of χ to N.

HYPOTHESIS 1. G is a finite group which is a normal subgroup of a finite group E such that the factor group $F = E/G$ is a cyclic group of order r. λ is a generator of $\hat{F}$. $E_0 = \{x \in E \mid \bar{x}$ is a generator of $F\}$ where $\bar{x} = xG$. E' is a subgroup of E such that $E'G = E$, $G' = G \cap E'$ and $E'_0 = E' \cap E_0$. Moreover $(E'_0)^\tau \cap E'_0$ is the empty set, for all $\tau \in E - E'$.

Under the above hypothesis, in [3], E.C. Dade constructed a bijection between $\mathrm{Irr}^E(G)$ and $\mathrm{Irr}^{E'}(G')$ which is a generalization of the cyclic case of the Glauberman correspondence in [4].

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Theorem 1 ([3], Theorems 6.8 and 6.9). *Assume* Hypothesis 1 *and* $|F| \neq 1$. *For each prime $q \in \pi(F)$, we choose some non-trivial character $\lambda_q \in \hat{F}_q$. There is a bijection*

$$\rho(E, G, E', G'):\ \mathrm{Irr}^E(G) \to \mathrm{Irr}^{E'}(G') \quad (\phi \mapsto \phi' = \phi_{(G')})$$

which satisfies the following conditions. If r is odd, then there are a unique integer $\epsilon_\phi = \pm 1$ and a unique bijection $\psi \mapsto \psi_{(E')}$ of $\mathrm{Irr}(E|\phi)$ *onto* $\mathrm{Irr}(E'|\phi')$ *such that*

$$\left(\prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \psi\right)_{E'} = \epsilon_\phi \prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \psi_{(E')}, \tag{1.1}$$

for any $\psi \in \mathrm{Irr}(E|\phi)$. If r is even, and we choose $\epsilon_\phi = \pm 1$ arbitrarily, then there is a unique bijection $\psi \mapsto \psi_{(E')}$ of $\mathrm{Irr}(E|\phi)$ *onto* $\mathrm{Irr}(E'|\phi')$ *such that* (1.1) *holds for all $\psi \in \mathrm{Irr}(E|\phi)$. In both cases we have*

$$(\lambda\psi)_{(E')} = \lambda\psi_{(E')}$$

for any $\lambda \in \hat{F}$ and and $\psi \in \mathrm{Irr}(E|\phi)$. Furthermore, the resulting bijection is independent of the choice of the non-trivial character $\lambda_q \in \hat{F}_q$, for any $q \in \pi(F)$.

Assume Hypothesis 1. If $|F| = 1$, then $E = E'$. We call $\rho(E, G, E', G')$ the Dade correspondence, where $\rho(E, G, E', G')$ denote the identity map of $\mathrm{Irr}^E(G)$ when $|F| = 1$. Following [13], for $\phi' \in \mathrm{Irr}^{E'}(G)$, we set $\phi'_{(G)} = \rho(E, G, E', G')^{-1}(\phi')$, and for $\psi \in \mathrm{Irr}(E|\phi)$ and $\psi' \in \mathrm{Irr}(E'|\phi')$, we set $\psi'_{(E)} = \psi$ if $\psi' = \psi_{(E')}$. From (1.1) ψ' is a constituent of $(\lambda\psi)_{E'}$ for some $\lambda \in \hat{F}$, hence $\phi_{(G')}$ is a constituent of $\phi_{G'}$. In particular if ϕ is the trivial character of G, then $\phi_{(G')}$ is the trivial character of G'. From the above theorem we have the following also.

Proposition 1. *Assume* Hypothesis 1. *Let $\phi \in \mathrm{Irr}^E(G)$ and $\phi' \in \mathrm{Irr}^{E'}(G')$. Then $\phi' = \phi_{(G')}$ if and only if there exist $\psi \in \mathrm{Irr}(E|\phi)$, $\psi' \in \mathrm{Irr}(E'|\phi')$ and $\epsilon = \pm 1$ such that*

$$\psi(x) = \epsilon\psi'(x) \quad (\forall x \in E'_0).$$

THE GENERALIZED GLAUBERMAN CASE Let G and A be finite groups such that A is cyclic, A acts on G via automorphism and that $(|C_G(A)|, |A|) = 1$. We set $E = G \rtimes A$, $G' = C_G(A)$ and $E' = G' \times A \leq E$. By [3], Lemma 7.5, E, G, E' and G' satisfy Hypothesis 1. Moreover by [3], Proposition 7.8, in the Glauberman case, that is, if $(|A|, |G|) = 1$, then the Glauberman correspondence coincides with the Dade correspondence.

In the generalized Glauberman case, suppose that $p \nmid |A|$ and $p \nmid |G : C_G(A)|$. Then in [8], H. Horimoto proved that there is an isotypy between $b(G)$ and $b(C_G(A))$ induced by the Dade correspondence where $b(G)$ is the principal block of G. Isotypy is a notion defined in [1].

HYPOTHESIS 2. Assume Hypothesis 1. $(p, r) = 1$. b is an E-invariant block of G covered by r distinct blocks of E.

Assume Hypothesis 2 and that r is a prime power. Moreover let b' be a block of G' containing $\phi_{(G')}$ for some $\phi \in \mathrm{Irr}(b)$. In [13], F. Tasaka proved that if r is odd, or $r = 2$ or $b = b(G)$, and if b' is covered by r blocks of E', then there is an isotypy between b and b' induced by the Dade correspondence ([13], Theorem 5.5). In this paper we prove that the arguments in [13] can be extended to the general case (see Theorem 6 in §5). Theorem 6 is a generalization of Theorem 5 in [16]. We also show that the Brauer correspondent of b and that of b' are Puig equivalent (see Theorem 8 in §6).

NOTATIONS. We follow the notations in [13], [12] and [15]. Let G be a finite group. We denote by $G_0(\mathcal{K}G)$ the Grothendieck group of the group algebra $\mathcal{K}G$. If L is a $\mathcal{K}G$-module, then let $[L]$ denote the element in $G_0(\mathcal{K}G)$ determined by the isomorphism class of L. For $\phi \in \mathrm{Irr}(G)$, we denote by $\check{\phi}$, e_ϕ and L_ϕ, the dual character of ϕ, the centrally primitive idempotent of $\mathcal{K}G$ corresponding to ϕ and a $\mathcal{K}G$-module affording ϕ respectively. We also denote by ω_ϕ the linear character of the center $Z(\mathcal{K}G)$ of $\mathcal{K}G$ corresponding to ϕ. Let H be a subgroup of G. We denote by $(\mathcal{S}G)^H$ the set of H-fixed elements of $\mathcal{S}G$. We denote by Pr_H^G the $\mathcal{S}$-linear map from $\mathcal{S}G$ to $\mathcal{S}H$ defined by $\mathrm{Pr}_H^G\left(\sum_{x\in G} a_x x\right) = \sum_{h\in H} a_h h$ and by Tr_H^G the trace map from $(\mathcal{S}G)^H$ to $Z(\mathcal{S}G)$. For $\alpha \in \mathcal{O}$, we denote by α^* the canonical image of α in k. For $a \in \mathcal{O}G$, we denote by a^* the canonical image of a in kG. For a p-subgroup P of G, we denote by $\mathrm{Br}_P^{\mathcal{S}G}$ the Brauer homomorphism from $(\mathcal{S}G)^P$ onto $kC_G(P)$. Also let $G_{p'}$ denote the set of p-regular elements of G.

Let b be a block of G. We denote by $\mathcal{R}_\mathcal{K}(G, b)$ the additive group of generalized characters belonging to b, by $\mathrm{CF}(G, b; \mathcal{K})$ the subspace with a basis $\mathrm{Irr}(b)$ of the $\mathcal{K}$-vector space of the $\mathcal{K}$-valued central functions of $\mathcal{K}G$, and by $\mathrm{CF}_{p'}(G, b; \mathcal{K})$ the subspace containing the elements of $\mathrm{CF}(G, b; \mathcal{K})$ which vanish on p-singular elements of G, where $\mathrm{Irr}(b)$ is the set of ordinary irreducible characters belonging to b. Let (u, b_u) be a b-Brauer element. We denote by $d_G^{(u,b_u)}$ the decomposition map from $\mathrm{CF}(G, b; \mathcal{K})$ onto $\mathrm{CF}_{p'}(C_G(u), b_u; \mathcal{K})$. For $\gamma \in \mathrm{CF}(G, b; \mathcal{K})$ and $c \in C_G(u)_{p'}$, we have $d_G^{(u,b_u)}(\gamma)(c) = \gamma(ucb_u)$. We also denote by ω_b the central character of $Z(\mathcal{O}Gb)$ and by $\mathrm{Bl}(C_G(P), b)$ the set of blocks of $C_G(P)$ associated with b where P is a p-subgroup of G. Let N be a normal subgroup of G. For $\phi \in \mathrm{Irr}(N)$, we denote by $I_G(\phi)$ the inertial group of ϕ in G. For a block $\mathbf{b}$ of N, we denote by $I_G(\mathbf{b})$ the inertial group of $\mathbf{b}$ in G. For a subgroup H and a block $\mathbf{c}$ of H, if $\mathbf{c}$ is associated with a block B of G, then B is denoted by $\mathbf{c}^G$.

2. Preliminaries

In this section we assume Hypothesis 1. For $x \in E$ (resp. $x \in E'$), we denote by $C(x)$ (resp. $C(x)'$) the conjugacy class of E (resp. E') containing x. For $X \subseteq E$, we set $\hat{X} = \sum_{x \in X} x \in \mathcal{S}E$.

Lemma 1. *Let $s \in E'_0$ and let Q, R be subgroups of G'centralized by s. Let $a \in G$. If $Q^a = R$, then $a \in C_G(Q)G'$. In particular $N_G(Q) = C_G(Q)N_{G'}(Q)$.*

Proof. By the assumption, $s^a \in C_E(R) \cap E_0$. By [13], Lemmas 3.9 and 2.4, there exists $y \in C_E(R)$ such that $s^{ay} \in C_{E'}(R)$. Since s^{ay}, $s \in E'_0$, $ay \in E'$. Set $z = ay$. Then $Q^z = R$, hence $a = (zy^{-1}z^{-1})z \in C_E(Q)E'$. Since $C_E(Q) = C_G(Q)\langle s\rangle$ and $E' = \langle s\rangle G'$, $a \in C_G(Q)G'\langle s\rangle$ and hence $a \in C_G(Q)G'$. □

Proposition 2 (see [13], Proposition 3.7). *Let $x \in E'_0$, $\phi \in \mathrm{Irr}^E(G)$ and $\phi' \in \mathrm{Irr}^{E'}(G')$. Then we have the following.*
(i) $\mathrm{Pr}^E_{E'}(\widehat{C(x)}e_\phi) = \widehat{C(x)'}e_{\phi_{(G')}}$.
(ii) $\mathrm{Tr}^E_{E'}(\widehat{C(x)'}e_{\phi'}) = \widehat{C(x)}e_{\phi'_{(G)}}$.

Proof. Let ψ be an extension of ϕ to E. $\widehat{C(x)}e_\phi$ is a $\mathcal{K}$-linear combination of the elements in xG. Hence we have

$$\widehat{C(x)}e_\phi = \frac{|C(x)|}{|E|} \sum_{y \in xG} r\psi(x)\psi(y^{-1})y.$$

From Theorem 1, (1.1), $\psi(z) = \epsilon_\phi \psi_{(E')}(z)$ for any $z \in E'_0$. Therefore we have

$$\begin{aligned} \widehat{C(x)'}e_{\phi_{(G')}} &= \frac{|C(x)'|}{|E'|} \sum_{z \in xG'} r\psi_{(E')}(x)\psi_{(E')}(z^{-1})z \\ &= \frac{|C(x)'|}{|E'|} \sum_{z \in xG'} r\psi(x)\psi(z^{-1})z. \end{aligned}$$

From [13], 2.4, we have (i) and (ii). □

3. The Dade correspondence and blocks

Assume Hypothesis 1 and $p \nmid r$. If an element $s \in E'_0$ centralizes a Sylow p-subgroup of G, then the principal block $b(G)$ satisfies Hypothesis 2.

HYPOTHESIS 3. Assume Hypothesis 1. $(p, r) = 1$. b' is an E'-invariant block of G' covered by r distinct blocks of E'.

Assume Hypotheses 2 and 3 and assume that $\phi_{(G')} \in \mathrm{Irr}(b')$ for some $\phi \in \mathrm{Irr}(b)$. In this section we show the Dade correspondence $\rho(E, G, E', G')$ induces a bijection between $\mathrm{Irr}(b)$ and $\mathrm{Irr}(b')$, and the Brauer categories $\mathbf{B}_G(b)$ and $\mathbf{B}_{G'}(b')$ are equivalent.

Theorem 2 (see [13], Proposition 3.5, (1) and (2)). (i) *Assume* Hypothesis 2. *Then* $\{\phi_{(G')} \mid \phi \in \mathrm{Irr}(b)\}$ *is contained in a block* $b_{(G')}$ *of* G'.
(ii) *Assume* Hypothesis 3. *Then* $\{\phi'_{(G)} \mid \phi' \in \mathrm{Irr}(b')\}$ *is contained in a block* $b'_{(G)}$ *of* G.

Proof. (i) Let $\phi_1, \phi_2 \in \mathrm{Irr}(b)$ and set $\phi'_i = \phi_{i\,(G')}$ for $i = 1, 2$. We show ϕ'_1 and ϕ'_2 belong to a same block of G'. We may assume at least one of these characters is of height 0. Let $\hat{b}$ be a block of G covering b and for $i = 1, 2$, let $\hat{\phi}_i$ be a unique extension of ϕ_i to E belonging to $\hat{b}$ recalling Hypothesis 2. Note $\hat{b}$ and b are isomorphic by restriction. Set $(\hat{\phi}_i)' = (\hat{\phi}_i)_{(E')}$ for $i = 1, 2$. By [12], Chapter III, Lemma 6.34, we have the following for a non-trivial linear character λ of F,

$$\sum_{x \in E_{p'}} \hat{\phi}_1(x)\hat{\phi}_2(x^{-1}) \neq 0, \qquad \sum_{x \in E_{p'}} \hat{\phi}_1(x)\lambda(x^{-1})\hat{\phi}_2(x^{-1}) = 0. \tag{3.1}$$

For each $q \in \pi(F)$, let λ_q be a non-trivial linear character in $\hat{F}_q$. Set $(E_0)_{p'} = E_0 \cap E_{p'}$ and $(E'_0)_{p'} = E'_0 \cap E_{p'}$. We have

$$\begin{aligned}
&\sum_{x \in E_{p'}} \hat{\phi}_1(x)\left(\prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \hat{\phi}_2\right)(x^{-1}) \\
&= \sum_{y \in (E_0)_{p'}} \hat{\phi}_1(y)\left(\prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \hat{\phi}_2\right)(y^{-1})
\end{aligned}$$

by [13], Lemma 2.4,

$$= \frac{|E|}{|E'|} \sum_{z \in (E'_0)_{p'}} \hat{\phi}_1(z)\left(\prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \hat{\phi}_2\right)(z^{-1})$$

by Theorem 1,

$$\begin{aligned}
&= \epsilon_{\phi_1}\epsilon_{\phi_2}\frac{|E|}{|E'|} \sum_{z \in (E'_0)_{p'}} (\hat{\phi}_1)'(w)\left(\prod_{q \in \pi(F)} (1 - \lambda_q) \cdot (\hat{\phi}_2)'\right)(w^{-1}) \\
&= \epsilon_{\phi_1}\epsilon_{\phi_2}\frac{|E|}{|E'|} \sum_{u \in (E')_{p'}} (\hat{\phi}_1)'(u)\left(\prod_{q \in \pi(F)} (1 - \lambda_q) \cdot (\hat{\phi}_2)'\right)(u^{-1}),
\end{aligned}$$

that is,

$$\begin{aligned}\sum_{x\in E_{p'}} \hat{\phi}_1(x)\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot\hat{\phi}_2\right)(x^{-1})\\ = \epsilon_{\phi_1}\epsilon_{\phi_2}\frac{|E|}{|E'|}\sum_{u\in(E')_{p'}}(\hat{\phi}_1)'(u)\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot(\hat{\phi}_2)'\right)(u^{-1}).\end{aligned} \tag{3.2}$$

From (3.1) there exists $\lambda \in \prod_{q\in\pi(F)} \hat{F}_q$ such that

$$\sum_{u\in(E')_{p'}}(\hat{\phi}_1)'(u)(\lambda(\hat{\phi}_2)')(u^{-1}) \neq 0.$$

Then $(\hat{\phi}_1)'$ and $\lambda(\hat{\phi}_2)'$ belong to a same block of E'. Hence ϕ_1' and ϕ_2' belong to a same block of G'. (ii) follows from (3.2) and the above arguments. □

Assume Hypothesis 2. We denote by $\hat{b}_0$ a block of E covering b. For each $\phi \in \mathrm{Irr}(b)$, we denote by $\hat{\phi}$ a unique extension of ϕ which belongs to $\hat{b}_0$. For any $i \in \mathbf{Z}$, we denote by $\hat{b}_i$ the block of E which contains $\lambda^i\hat{\phi}$ where $\phi \in \mathrm{Irr}(b)$. For the block b, $\hat{b}_i$ is fixed throughout this paper. Let $\hat{b}_0 = \sum_{x\in E}\alpha_x x$. Then $\hat{b}_i = \sum_{x\in E}\lambda^i(x^{-1})\alpha_x x$. Moreover we note that for any $t \in E$, $\sum_{x\in Gt}\alpha_x^* x \neq 0$ because $\{(\hat{b}_0)^*, (\hat{b}_1)^*, \dots, (\hat{b}_{r-1})^*\}$ are linearly independent. This fact is used implicitly in the proof of Proposition 5 below.

Proposition 3 (see [13], Proposition 3.5, (3)). *Assume* Hypotheses 2 *and* 3, *and assume* $b' = b_{(G')}$ *using the notation in* Theorem 2. *Then there exists a block* $(\hat{b}_0)_{(E')}$ *of* E' *such that* $\mathrm{Irr}((\hat{b}_0)_{(E')}) = \{(\hat{\phi})_{(E')} \mid \phi \in \mathrm{Irr}(b)\}$. *If* r *is odd, then* $(\hat{b}_0)_{(E')}$ *is uniquely determined, and if* r *is even, we have exactly two choices for* $(\hat{b}_0)_{(E')}$.

Proof. Let $\phi_1, \phi_2 \in \mathrm{Irr}(b)$ and suppose that ϕ_1 is of height 0. Assume $(\hat{\phi}_1)_{(E')}$ belongs to a block $(\hat{b}_0)_{(E')}$ of E'. Here we note that we have two choices for $(\hat{\phi}_1)_{(E')}$ when r is even by Theorem 1, and hence we have two choices for $(\hat{b}_0)_{(E')}$. By the proof of Theorem 2 and by our assumption, there is a unique linear character $\nu \in \hat{F}$ such that $\nu(\hat{\phi}_2)_{(E')}$ belongs to $(\hat{b}_0)_{(E')}$ and that $\nu = 1$ or ν is a product of some elements of $\{\lambda_q \mid q \in \pi(F)\}$. Hence if r is odd, then $\nu = 1$ because λ_q can be replaced by another non-trivial linear character in $\hat{F}_q$. If r is even, $\nu = 1$ or $\nu = \lambda_2$, hence $(\hat{\phi}_2)_{(E')}$ belongs to $(\hat{b}_0)_{(E')}$ by replacing ϵ_{ϕ_2} by $-\epsilon_{\phi_2}$ if necessary. This combined with Theorem 1 completes the proof. □

With the notation in the above proposition, we denote by $(\hat{b}_i)_{(E')}$ the block of E' containing $\lambda^i(\hat{\phi})_{(E')}$ ($\phi \in \mathrm{Irr}(b)$) for $i \in \mathbf{Z}$. Moreover, when r is even, we fix one of two $(\hat{b}_0)_{(E')}$, and hence $(\hat{b}_i)_{(E')}$ are fixed.

Lemma 2 (see [13], Lemma 3.3). *Assume* Hypothesis 2. *We have the following holds.*
(i) *There exists* $s \in E_0$ *such that* $(\omega_{\hat{b}_i}(\widehat{C(s)}))^* \neq 0$ *for all* $i \in \mathbf{Z}$.
(ii) *For* s *in* (i), $\widehat{C(s)}b \in Z(\mathcal{O}Eb)^\times$, *that is,* $\widehat{C(s)}b$ *is invertible in* $Z(\mathcal{O}Eb)$.

Proof. (i) By the assumption and [12], Chapter III, Theorem 6.24, for any $q \in \pi(F)$, there exists $s(q) \in E$ such that $(\omega_{\hat{b}_i}(\widehat{C(s(q))}))^* \neq 0$ and that $s(q)G$ is a generator of the Sylow q-subgroup of F. Then $\left(\omega_{\hat{b}_i}\left(\prod_{q\in\pi(F)} \widehat{C(s(q))}\right)\right)^* \neq 0$. This implies that there exists $s \in E_0$ such that $(\omega_{\hat{b}_i}(\widehat{C(s)}))^* \neq 0$.

(ii) From (i) $\widehat{C(s)}\hat{b}_i \in Z(\mathcal{O}E\hat{b}_i)^\times$ for any i because $Z(\mathcal{O}E\hat{b}_i)$ is local. Hence $\widehat{C(s)}b \in Z(\mathcal{O}Eb)^\times$. □

Assume Hypothesis 2. By the above lemma and [13], Lemma 2.4, there exists an element $s \in E_0'$ such that $\widehat{C(s)}b \in Z(\mathcal{O}Eb)^\times$. Hence there exists a defect group D of b centralized by s, and hence contained in G' (see [13], Lemma 3.10). Let $P \leq D$. Then by [13], Lemma 3.9, $C_E(P)$, $C_G(P)$, $C_{E'}(P)$ and $C_{G'}(P)$ satisfy Hypothesis 1. Moreover we note $F \cong C_E(P)/C_G(P)$. Let $e \in \mathrm{Bl}(C_G(P), b)$. Then we see that $\mathrm{Br}_P^{\mathcal{O}E}(\widehat{C(s)}b)e^* \in (Z(kC_E(P)e^*))^\times$. This implies that e is covered by r blocks of $C_E(P)$. Similarly assume Hypothesis 3. Let D' be a defect group of b' and $e' \in \mathrm{Bl}(C_{G'}(P'), b')$ for a subgroup P' of D'. Then e' is covered by r blocks of $C_{E'}(P')$.

Theorem 3 (see [13], Proposition 3.11). *Using the same notations as in* Theorem 2 *we have the following.*
(i) *Assume* Hypothesis 2. *Let* D *be a defect group of* b *obtained in the above and let* $P \leq D$. *Let* $e \in \mathrm{Bl}(C_G(P), b)$. *Then* $e_{(C_{G'}(P))} \in \mathrm{Bl}(C_{G'}(P), b_{(G')})$. *In particular,* $b_{(G')}$ *has a defect group containing* D.
(ii) *Assume* Hypothesis 3. *Let* D' *be a defect group of* b' *and let* $P' \leq D'$. *Let* $e' \in \mathrm{Bl}(C_{G'}(P'), b')$. *Then* $e'_{(C_G(P'))} \in \mathrm{Bl}(C_G(P'), b'_{(G)})$. *In particular,* $b'_{(G)}$ *has a defect group containing* D'.

Proof. See the proof of [13], Proposition 3.11. □

Assume Hypotheses 2 and 3, and assume $b' = b_{(G')}$ where $b_{(G')}$ is the block determined by Theorem 2. We have

$$\mathrm{Irr}(b') = \{\phi_{(G')} \mid \phi \in \mathrm{Irr}(b)\}$$

by Theorem 2. Let D be a common defect group of b and b', and let $P \leq D$. Such a defect group exists by the above theorem. Let (D, b_D) be maximal b-Brauer pair and let (P, b_P) be a b-Brauer pair contained in (D, b_D). By the above theorem, $(D, (b_D)_{(C_{E'}(D))})$

is a maximal b'-Brauer pair and $(P, (b_P)_{(C_{E'}(P))})$ is a b'-Brauer pair. We set

$$(b_P)' = (b_P)_{(C_{E'}(P))}$$

and

$$(b_P^*)' = ((b_P)')^*.$$

For any $u \in C_{E'}(P)$, we denote by $C(u)_{(P)}$ the conjugacy class of $C_E(P)$ containing u, and by $C(u)'_{(P)}$ the conjugacy class of $C_{E'}(P)$ containing u.

Theorem 4 (see [13], Theorem 5.2). *Assume* Hypotheses 2 *and* 3, *and assume* $b' = b_{(G')}$ *where* $b_{(G')}$ *is the block determined by* Theorem 2. *Then the Brauer categories* $\mathbf{B}_G(b)$ *and* $\mathbf{B}_{G'}(b')$ *are equivalent.*

Proof. Our proof is essentially the same as the proof of [13], Theorem 5.2. Let D be a common defect group of b and b', and let $P \leq D$. There is an element $t \in C_E(P) \cap E'_0$ such that $\widehat{C(t)_{(P)}} b_P^* \in (Z(kC_E(P))b_P^*)^\times$. By Lemma 2, such an element exists. For any $a \in G'$ we have the following using Proposition 2 and Theorem 2.

$$\widehat{C(t^a)'_{(P^a)}}((b_P^*)')^a = \mathrm{Pr}^{C_E(P^a)}_{C_{E'}(P^a)}(\widehat{C(t^a)_{(P^a)}}(b_P^*)^a) \neq 0. \tag{3.3}$$

In fact we have

$$\begin{aligned}\widehat{C(t^a)'_{(P^a)}}((b_P^*)')^a &= (\widehat{C(t)'_{(P)}}(b_P^*)')^a \\ &= \left(\mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}(\widehat{C(t)_{(P)}} b_P^*)\right)^a = \mathrm{Pr}^{C_E(P^a)}_{C_{E'}(P^a)}(\widehat{C(t^a)_{(P^a)}}(b_P^*)^a) \neq 0.\end{aligned}$$

In particular, if $(P, b_P)^a = (P, b_P)$, then $(P, (b_P)')^a = (P, (b_P)')$. □

Now for $P \leq R \leq D$, we prove $(P, (b_P)') \leq (R, (b_R)')$. We may assume $P \trianglelefteq R$. From (3.3) R fixes $(b_P)'$ because R fixes b_P. Now let $s \in E'_0$ be such that $\widehat{C(s)} b \in Z(\mathcal{O}Eb)^\times$. Then $\widehat{C(s) \cap C_{E'}(P)}(b_P)'$ is fixed by R. Moreover $\widehat{C(s) \cap C_E(P)} b_P^*$ is invertible in $(Z(kC_E(P)b_P^*))^R$. Hence $\mathrm{Br}^{kC_E(P)}_{R/P}(\widehat{C(s) \cap C_E(P)} b_P^*) b_R^*$ is invertible in $Z(kC_E(R))b_R^*$ where $\mathrm{Br}^{kC_E(P)}_{R/P}$ is the restriction to $(kC_E(P))^R$ of the Brauer homomorphism Br^{kE}_R. In particular it does not vanish. Hence we have from Proposition 2

$$\begin{aligned}&\mathrm{Br}^{kC_{E'}(P)}_{R/P}(\widehat{C(s) \cap C_{E'}(P)}(b_P^*)')(b_R^*)' \\ &= \mathrm{Br}^{kC_{E'}(P)}_{R/P}\left(\mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}(\widehat{C(s) \cap C_E(P)} b_P^*)\right)(b_R^*)' \\ &= \mathrm{Pr}^{C_E(R)}_{C_{E'}(R)}\left(\mathrm{Br}^{kC_E(P)}_{R/P}(\widehat{C(s) \cap C_E(P)} b_P^*)\right)(b_R^*)' \\ &= \mathrm{Pr}^{C_E(R)}_{C_{E'}(R)}\left(\mathrm{Br}^{kC_E(P)}_{R/P}(\widehat{C(s) \cap C_E(P)} b_P^*) b_R^*\right) \neq 0.\end{aligned}$$

The last inequality follows from [13], Lemmas 3.9 and 2.4. Therefore

$$\mathrm{Br}_{R/P}^{kC_{E'}(P)}((b_P^*)')(b_R^*)' \neq 0.$$

This implies $(P, (b_P)') \trianglelefteq (R, (b_R)')$.

For a subgroup T of D and $a \in G$, suppose that $(P, b_P)^a \leq (T, b_T)$. We show that there is an element $e \in C_G(P)$ such that $ea \in G'$ and $(P, (b_P)')^{ea} \leq (T, (b_T)')$. By Lemma 1, we may assume $a \in G'$. Since we have $(P, b_P)^a = (P^a, b_{P^a})$, $(b_P)^a = b_{P^a}$. From (3.3), $((b_P)')^a = (b_{P^a})'$, hence $(P, (b_P)')^a = (P^a, (b_{P^a})') \leq (T, (b_T)')$. Conversely for $c \in G'$, suppose that $(P, (b_P)')^c \leq (T, (b_T)')$. Then we have $((b_P)')^c = (b_{P^c})'$. By (3.3) again, $b_{P^c} = (b_P)^c$, so $(P, b_P)^c = (P^c, b_{P^c}) \leq (T, b_T)$. This implies that the categories $\mathbf{B}_G(b)$ and $\mathbf{B}_{G'}(b')$ are equivalent. This completes the proof.

4. Perfect isometry induced by the Dade correspondence

In Sections 4, 5 and 6, we assume Hypotheses 2 and 3, and $b' = b_{(G')}$ using the notation in Theorem 2. In this section we show b and b' are perfect isometric in the sense of Broué [1]. Moreover we use notations in §3. In particular, we recall that $\mathrm{Irr}((\hat{b}_i)_{(E')}) = \{\lambda^i(\hat{\phi})_{(E')} \mid \phi \in \mathrm{Irr}(b)\}$. Now we have $b = \sum_{i=0}^{r-1} \hat{b}_i$, and $b' = \sum_{i=0}^{r-1} (\hat{b}_i)_{(E')}$, and hence we have

$$b'b = \sum_{i=0}^{r-1} \sum_{l=0}^{r-1} (\hat{b}_l)_{(E')} \hat{b}_{l+i}.$$

We put

$$b_i = \sum_{l=0}^{r-1} (\hat{b}_l)_{(E')} \hat{b}_{l+i} \quad (\forall i \in \mathbf{Z}). \tag{4.1}$$

Then $(b_i)^2 = b_i$ and $b_i \in (\mathcal{O}Gbb')^{E'}$ for each i because

$$b_i = \sum_{y \in E'} \sum_{x \in E} \sum_{l=0}^{r-1} \lambda^l(y^{-1})\lambda^l(x^{-1})\lambda^i(x^{-1})\beta_y \alpha_x yx \in \mathcal{O}G$$

by the orthogonality relations where $\hat{b}_0 = \sum_{x \in E} \alpha_x x$ and $(\hat{b}_0)_{(E')} = \sum_{y \in E'} \beta_y y$ $(\alpha_x, \beta_y \in \mathcal{O})$. For each prime $q \in \pi(F)$, let $\lambda_q \in \hat{F}_q$ be a non-trivial character as in Theorem 1. Set $l = |\pi(F)|$. Of course we may assume $l > 0$ for our purpose. Moreover we can write for t $(t \leq l)$ distinct primes $q_1, q_2, \ldots, q_t \in \pi(F)$

$$\lambda_{q_1} \cdots \lambda_{q_t} = \lambda^{m_{\{q_1,\ldots,q_t\}}} \quad (m_{\{q_1,\ldots,q_t\}} \in \mathbf{Z})$$

recalling λ is a generator of $\hat{F}$. Then we have

$$\prod_{q\in\pi(F)}(1-\lambda_q) = 1 + \sum_{t=1}^{l}(-1)^t \sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)} \lambda^{m_{\{q_1,\dots,q_t\}}} \tag{4.2}$$

where $\{q_1, \dots, q_t\}$ runs over the set of t-element subsets of $\pi(F)$.

Proposition 4 (see [13], Proposition 4.4). *With the above notations we have*

$$\begin{aligned}&[b_0\mathcal{K}G] + \sum_{t=1}^{l}(-1)^t \sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)} [b_{m_{\{q_1,\dots,q_t\}}}\mathcal{K}G]\\ &= \sum_{\phi\in\mathrm{Irr}(b)} \epsilon_\phi[L_{\phi_{(G')}} \otimes_{\mathcal{K}} L_{\check{\phi}}]\end{aligned}$$

in $G_0(\mathcal{K}(G'\times G))$.

Proof. Our proof is essentially the same as the proof of [13], Proposition 4.4. Let $\phi\in\mathrm{Irr}(b)$. In $G_0(\mathcal{K}E')$ we have the following from (4.1), (4.2) and (1.1)

$$\begin{aligned}&[b_0\mathcal{K}E\otimes_{\mathcal{K}E} L_{\hat{\phi}}] + \sum_{t=1}^{l}(-1)^t \sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)} [b_{m_{\{q_1,\dots,q_t\}}}\mathcal{K}E\otimes_{\mathcal{K}E} L_{\lambda^{m_{\{q_1,\dots,q_t\}}}\hat{\phi}}]\\
&= [b_0(L_{\hat{\phi}})_{E'}] + \sum_{t=1}^{l}(-1)^t \sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)} [b_{m_{\{q_1,\dots,q_t\}}}(L_{\lambda^{m_{\{q_1,\dots,q_t\}}}\hat{\phi}})_{E'}]\\
&\underset{(4.1)}{=} [(\hat{b}_0)_{(E')}(L_{\hat{\phi}})_{E'}] + \sum_{t=1}^{l}(-1)^t \sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)} [(\hat{b}_0)_{(E')}(L_{\lambda^{m_{\{q_1,\dots,q_t\}}}\hat{\phi}})_{E'}]\\
&\underset{(4.2),\ (1.1)}{=} \epsilon_\phi\left([(\hat{b}_0)_{(E')}L_{(\hat{\phi})_{(E')}}] + \sum_{t=1}^{l}(-1)^t \sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)} [(\hat{b}_0)_{(E')}L_{\lambda^{m_{\{q_1,\dots,q_t\}}}(\hat{\phi})_{(E')}}]\right)\\
&\underset{(4.1)}{=} \epsilon_\phi[L_{(\hat{\phi})_{(E')}}].\end{aligned}$$

This implies that in $G_0(\mathcal{K}G')$

$$[b_0\mathcal{K}G\otimes_{\mathcal{K}G} L_\phi] + \sum_{t=1}^{l}(-1)^t \sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)} [b_{m_{\{q_1,\dots,q_t\}}}\mathcal{K}G\otimes_{\mathcal{K}G} L_\phi] = \epsilon_\phi[L_{\phi_{(G')}}].$$

Since $b_ib = b_i$ for any $i\in\mathbf{Z}$, the proof is complete. □

Theorem 5 (see [13], Theorem 4.5). *Assume* Hypotheses 2 *and* 3, *and that* $b' = b_{(G')}$. *Set* $\mu = \sum_{\phi\in\mathrm{Irr}(b)}\epsilon_\phi\phi_{(G')}\phi$. *Then* μ *induces a perfect isometry* $R_\mu\colon \mathcal{R}_{\mathcal{K}}(G,b)\to\mathcal{R}_{\mathcal{K}}(G',b')$ *which satisfies* $R_\mu(\phi) = \epsilon_\phi\phi_{(G')}$.

Proof. We note that $b_j\mathcal{O}G$ is projective as a right $\mathcal{O}G$-module and as a left $\mathcal{O}G'$-module if $b_j \neq 0$. Hence by [1], Proposition 1.2, μ is perfect. This and the above proposition imply the theorem. □

5. Isotypy induced by the Dade correspondence

In this section we show that b and b' are isotypic. Here we set

$$\hat{b}'_i = (\hat{b}_i)_{(E')} \quad (i \in \mathbf{Z}).$$

Then D is a defect group of $\hat{b}'_i$ since $p \nmid r$. Let $P \leq D$ and let $(\hat{b}_P)_i$ be a block of of $C_E(P)$ such that it covers b_P and it is associated with $\hat{b}_i$. By our assumption and Lemma 2, $(\hat{b}_P)_i$ is uniquely determined. Similarly there exists a unique block of $C_{E'}(P)$ such that it covers $(b_P)'$ and it is associated with $\hat{b}'_i$. By applying Proposition 2 for $C_E(P)$, $C_G(P)$ and b_P, let $((\hat{b}_P)_i)_{(C_{E'}(P))}$ be a block of $C_{E'}(P)$ such that $\mathrm{Irr}(((\hat{b}_P)_i)_{(C_{E'}(P))}) = \{\lambda^i(\hat{\phi}_P)_{(C_{E'}(P))} \mid \phi_P \in \mathrm{Irr}(b_P)\}$, where $\hat{\phi}_P \in \mathrm{Irr}((\hat{b}_P)_0)$ is an extension of ϕ_P. Recall that we have two choices for $((\hat{b}_P)_0)_{(C_{E'}(P))}$ when r is even (Proposition 3). Here we set

$$(\hat{b}_P)'_i = ((\hat{b}_P)_i)_{(C_{E'}(P))}$$

and

$$(\hat{b}^*_P)'_i = ((\hat{b}_P)'_i)^* \quad (i \in \mathbf{Z}).$$

Proposition 5 (see [13], Lemma 5.4). *With the above notations, for a subgroup P of D, $(\hat{b}_P)'_i$ is associated with $\hat{b}'_i$ for $i \in \mathbf{Z}$, if we choose appropriately $(\hat{b}_P)'_0$ when r is even.*

Proof. Our proof is essentially the same as the proof of [13], Lemma 5.4. Let $s \in E'_0$. We have

$$\widehat{C(s)}\hat{b}_i = \frac{1}{|C_{E'}(s)|} \sum_{\phi \in \mathrm{Irr}(b)} \left(\sum_{x \in E_0} (\lambda^i \hat{\phi})(s)(\lambda^i \hat{\phi})(x^{-1})x + \sum_{y \in E - E_0} (\lambda^i \hat{\phi})(s)(\lambda^i \hat{\phi})(y^{-1})y \right)$$

since $C_E(s) = C_{E'}(s)$. Similarly we have

$$\widehat{C(s)'}\hat{b}'_i = \frac{1}{|C_{E'}(s)|} \sum_{\phi \in \mathrm{Irr}(b)} \left(\sum_{x \in E'_0} (\lambda^i (\hat{\phi})_{(E')})(s)(\lambda^i (\hat{\phi})_{(E')})(x^{-1})x \right.$$
$$\left. + \sum_{y \in E' - E'_0} (\lambda^i (\hat{\phi})_{(E')})(s)(\lambda^i \hat{\phi})_{(E')})(y^{-1})y \right).$$

Recall that $\hat{\phi}(x) = \epsilon_\phi(\hat{\phi})_{(E')}(x)$ for $x \in E_0'$. The above equalities, the fact $E_0' = E' \cap E_0$ and [13], Lemma 2.4 imply the following.

$$\mathrm{Pr}^E_{E'}(\widehat{C(s)}\hat{b}_i) - \widehat{C(s)'}\hat{b}_i' \in \mathcal{O}[E' - E_0']^{E'} \tag{5.1}$$

where $\mathcal{S}[E' - E_0']^{E'}$ is the $\mathcal{S}$-submodule of $Z(\mathcal{S}E')$ which is spanned by $\{\widehat{C(t)'} \mid t \in E' - E_0'\}$.

In order to prove the proposition, it suffices to show that $(\hat{b}_P)_0'$ is associated with $\hat{b}_0'$, if we choose $(\hat{b}_P)_0'$ appropriately when r is even. Suppose that $(\hat{b}_P)_j'$ is associated with $\hat{b}_0'$ for some j $(0 \leq j \leq r-1)$. We have

$$\begin{aligned}&\mathrm{Pr}^E_{C_{E'}(P)}(\widehat{C(s)}\hat{b}_0)^*(b_P^*)' \\ &= \mathrm{Pr}^{E'}_{C_{E'}(P)}[\mathrm{Pr}^E_{E'}(\widehat{C(s)}\hat{b}_0)]^*(b_P^*)'\end{aligned}$$

from (5.1),

$$\begin{aligned}&= \mathrm{Br}_P^{\mathcal{O}E'}(\widehat{C(s)'}\hat{b}_0' + c)(b_P^*)' \\ &= \mathrm{Br}_P^{\mathcal{O}E'}(\widehat{C(s)'}b'\hat{b}_0' + c)(b_P^*)' \\ &= [\mathrm{Br}_P^{\mathcal{O}E'}(\widehat{C(s)'}b')\,\mathrm{Br}_P^{\mathcal{O}E'}(\hat{b}_0') + \mathrm{Br}_P^{\mathcal{O}E'}(c)](b_P^*)' \\ &= \mathrm{Br}_P^{\mathcal{O}E'}(\widehat{C(s)'}b')(\hat{b}_P^*)_j' + \mathrm{Br}_P^{\mathcal{O}E'}(c)(b_P^*)'\end{aligned}$$

where c is some element of $\mathcal{O}[E' - E_0']^{E'}$. On the other hand, we can see

$$\begin{aligned}&\mathrm{Pr}^E_{C_{E'}(P)}(\widehat{C(s)}\hat{b}_0)^*(b_P^*)' \\ &= \mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}[\mathrm{Pr}^E_{C_E(P)}(\widehat{C(s)}\hat{b}_0)]^*(b_P^*)' \\ &= \mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}[\mathrm{Pr}^E_{C_E(P)}(\widehat{C(s)})^*\,\mathrm{Br}_P^{\mathcal{O}E}(\hat{b}_0)](b_P^*)'\end{aligned}$$

from the argument in the above of Theorem 3 and (5.1) for $C_E(P)$

$$= \mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}[\mathrm{Pr}^E_{C_E(P)}(\widehat{C(s)})^*](\hat{b}_P^*)_0' + d(b_P^*)'$$

and by Theorem 3

$$\begin{aligned}&= \mathrm{Br}_P^{\mathcal{O}E'}[\mathrm{Pr}^E_{E'}(\widehat{C(s)})]\,\mathrm{Br}_P^{\mathcal{O}E'}(b')(\hat{b}_P^*)_0' + d(b_P^*)' \\ &= \mathrm{Br}_P^{\mathcal{O}E'}(\widehat{C(s)'}b')(\hat{b}_P^*)_0' + d(b_P^*)'\end{aligned}$$

where d is some element of $k[C_{E'}(P) - C_{E'_0}(P)]^{C_{E'}(P)}$.

Now we choose an element $s \in C_{E'_0}(P)$ such that

$$\mathrm{Br}_P^{\mathcal{O}E'}(\widehat{C(s)'}b') \in (kC_{E'}(P)\,\mathrm{Br}_P^{\mathcal{O}E'}(b'))^{\times}.$$

Note that $\mathrm{Br}_P^{\mathcal{O}E'}(\widehat{C(s)'}b')$ is a k-linear combination of elements in $sC_{G'}(P)$ because $\widehat{C(s)'}b'$ is an $\mathcal{O}$-linear combination of elements in sG'. By the above equations

$$\mathrm{Br}_P^{\mathcal{O}E'}(\widehat{C(s)'}b')((\hat{b}_P^*)'_j - (\hat{b}_P^*)'_0) \in k[C_{E'}(P) - C_{E'_0}(P)]^{C_{E'}(P)}.$$

Set $\upsilon = (\hat{b}_P^*)'_j - (\hat{b}_P^*)'_0$. The coefficient of any element of $s^{-2}C_{G'}(P)$ in υ is zero. Hence $\lambda^j(s^2) = \lambda^{2j}(s) = 1$. Therefore if r is odd, then $j = 0$. If r is even, $j = 0$ or $j = r/2$. Therefore by replacing ϵ_{ϕ_P} by $-\epsilon_{\phi_P}$ for all $\phi_P \in \mathrm{Irr}(b_P)$ if $j = r/2$, we have $(\hat{b}_P)'_0$ is associated with $\hat{b}'_0$. This completes the proof. □

Let $P \le D$. We note again that for any integer i, $(\hat{b}_P)_i$ covers b_P and it is associated with $\hat{b}_i$. Moreover $(\hat{b}_P)_i$ contains $\lambda^i\hat{\phi}_P$ ($\hat{\phi}_P \in \mathrm{Irr}((\hat{b}_P)_0)$). Let R^P be the perfect isometry between $\mathcal{R}_{\mathcal{K}}(C_G(P), b_P)$ and $\mathcal{R}_{\mathcal{K}}(C_{G'}(P), (b_P)')$ obtained by

$$\rho(C_E(P), C_G(P), C_{E'}(P), C_{G'}(P))$$

(see Theorem 5). Also let $R^P_{p'}$ be the restriction of R^P to $\mathrm{CF}_{p'}(C_G(P), b_P; \mathcal{K})$, where R^P is regarded as a linear isometry from $\mathrm{CF}(C_G(P), b_P; \mathcal{K})$ onto $\mathrm{CF}(C_{G'}(P), (b_P)'; \mathcal{K})$. We set

$$(b_P)_i = \sum_{l=0}^{r-1} (\hat{b}_P)'_l(\hat{b}_P)_{l+i} \in (\mathcal{O}C_G(P)b_P(b_P)')^{C_{E'}(P)}.$$

For $u \in D$ we set

$$b_u = b_{\langle u\rangle}, \quad (b_u)' = (b_{\langle u\rangle})', \quad (\hat{b}_u)'_0 = (\hat{b}_{\langle u\rangle})'_0, \quad (b_u)_i = (b_{\langle u\rangle})_i.$$

Theorem 6 (see [13], Theorem 5.5). *Assume* Hypotheses 2 *and* 3, *and assume* $b' = b_{(G')}$. *With the above notations,* b *and* b' *are isotypic with the local system* $(R^P)_{\{P(\text{cyclic}) \le D\}}$.

Proof. Our proof is essentially the same as the proof of [13], Theorem 5.5. Let $\gamma \in \mathrm{CF}(G, b; \mathcal{K})$, $u \in D$ and let $c' \in C_{G'}(u)_{p'}$. Let $S(u)$ be the p-section of G containing u. We remark that if $v \in S(u)$, then $\widehat{C(v)}b$ is an $\mathcal{O}$-linear combination of elements of

$S(u)$ by [12], Chapter V, Theorem 4.5. We can see from Proposition 4

$$[(d_{G'}^{(u,(b_u)')}\circ R^{\langle 1\rangle})(\gamma)](c')$$
$$=\frac{1}{|G|}\sum_{g\in G}\left[\sum_{\phi\in\mathrm{Irr}(b)}\left(\phi(uc'(b_u)'b_0)+\sum_{t=1}^{l}(-1)^t\sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)}\phi(uc'(b_u)'b_{m_{\{q_1,\dots,q_t\}}})\right)\phi(g^{-1})\right]\gamma(g)$$
$$=\frac{1}{|G|}\sum_{g\in G}\left[\sum_{\phi\in\mathrm{Irr}(b)}\left(\hat{\phi}(uc'(b_u)'b_0)+\sum_{t=1}^{l}(-1)^t\sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)}\hat{\phi}(uc'(b_u)'b_{m_{\{q_1,\dots,q_t\}}})\right)\hat{\phi}(g^{-1})\right]\gamma(g)$$

from (4.1) and the fact $\hat{\phi}\in\mathrm{Irr}(\hat{b}_0)$

$$=\frac{1}{|G|}\sum_{g\in G}\left[\sum_{\hat{\phi}\in\mathrm{Irr}(\hat{b}_0)}\left(\hat{\phi}(uc'(b_u)'\hat{b}_0')+\sum_{t=1}^{l}(-1)^t\sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)}\hat{\phi}(uc'(b_u)'\hat{b}'_{-m_{\{q_1,\dots,q_t\}}})\right)\hat{\phi}(g^{-1})\right]\gamma(g)$$
$$=\frac{1}{|G|}\sum_{g\in G}\left[\sum_{\hat{\phi}\in\mathrm{Irr}(\hat{b}_0)}\left(\hat{\phi}\left(1+\sum_{t=1}^{l}(-1)^t\sum_{\{q_1,\dots,q_t\}\subseteq\pi(F)}\lambda^{m_{\{q_1,\dots,q_t\}}}\right)\right)(uc'(b_u)'\hat{b}_0')\hat{\phi}(g^{-1})\right]\gamma(g)$$

from (4.2)

$$=\frac{1}{|G|}\sum_{g\in G}\left[\sum_{\hat{\phi}\in\mathrm{Irr}(\hat{b}_0)}\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot\hat{\phi}\right)(uc'(b_u)'\hat{b}_0')\hat{\phi}(g^{-1})\right]\gamma(g)$$

by applying [12], Chapter V, Theorem 4.5 for E and $\hat{b}_0$

$$=\frac{1}{|G|}\sum_{x\in S(u)}\left[\sum_{\hat{\phi}\in\mathrm{Irr}(\hat{b}_0)}\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot\hat{\phi}\right)(uc'(b_u)'\hat{b}_0')\hat{\phi}(x^{-1})\right]\gamma(x)$$
$$=\frac{1}{|C_G(u)|}\sum_{y\in C_G(u)_{p'}}\left[\sum_{\hat{\phi}\in\mathrm{Irr}(\hat{b}_0)}\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot\hat{\phi}\right)(uc'(b_u)'\hat{b}_0')\hat{\phi}(y^{-1}u^{-1})\right]\gamma(uy)$$

by using (1.1) twice, and by Brauer's second main theorem on blocks ([12], Chapter V, Theorem 4.1) and Proposition 5

$$=\frac{1}{|C_G(u)|}\sum_{y\in C_G(u)_{p'}}\left[\sum_{\hat{\phi}\in\mathrm{Irr}(\hat{b}_0)}\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot(\hat{\phi})_{(E')}\right)(uc'(b_u)')\hat{\phi}(y^{-1}u^{-1})\right]\gamma(uy)$$
$$=\frac{1}{|C_G(u)|}\sum_{y\in C_G(u)_{p'}}\left[\sum_{\hat{\phi}\in\mathrm{Irr}(\hat{b}_0)}\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot(\hat{\phi})_{(E')}\right)(uc'(\hat{b}_u)_0')\hat{\phi}(y^{-1}u^{-1})\right]\gamma(uy)$$

$$= \frac{1}{|C_G(u)|} \sum_{y \in C_G(u)_{p'}} \left[\sum_{\hat{\phi} \in \mathrm{Irr}(\hat{b}_0)} \left(\prod_{q \in \pi(F)} (1-\lambda_q) \cdot \hat{\phi} \right) (uc'(\hat{b}_u)'_0) \hat{\phi}(y^{-1}u^{-1}) \right] \gamma(uy)$$

from [12], Chapter V, Theorem 4.11

$$= \frac{1}{|C_G(u)|} \sum_{y \in C_G(u)_{p'}} \left[\sum_{e \in \mathrm{Bl}(C_E(u), \hat{b}_0)} \sum_{\rho \in \mathrm{Irr}(e)} \left(\prod_{q \in \pi(F)} (1-\lambda_q) \cdot \rho \right) (c'(\hat{b}_u)'_0) \rho(y^{-1}) \right] \gamma(uy)$$

from (1.1) for $C_E(u)$

$$= \frac{1}{|C_G(u)|} \sum_{y \in C_G(u)_{p'}} \left[\sum_{e \in \mathrm{Bl}(C_E(u), \hat{b}_0)} \sum_{\rho \in \mathrm{Irr}(e)} \left(\prod_{q \in \pi(F)} (1-\lambda_q) \cdot \rho_{(C_{E'}(u))} \right) (c'(\hat{b}_u)'_0) \rho(y^{-1}) \right] \gamma(uy)$$

recalling $(\hat{b}_u)'_0 = ((\hat{b}_{\langle u \rangle})_0)_{(C_{E'}(u))}$

$$= \frac{1}{|C_G(u)|} \sum_{y \in C_G(u)_{p'}} \left[\sum_{\hat{\xi} \in \mathrm{Irr}((\hat{b}_u)_0)} \left(\prod_{q \in \pi(F)} (1-\lambda_q) \cdot \hat{\xi} \right) (c'(\hat{b}_u)'_0) \hat{\xi}(y^{-1}) \right] \gamma(uy)$$

from (4.2)

$$= \frac{1}{|C_G(u)|} \sum_{y} \left[\sum_{\hat{\xi} \in \mathrm{Irr}((\hat{b}_u)_0)} \left(\hat{\xi}(c'(\hat{b}_u)'_0) + \sum_{t=1}^{l} (-1)^t \sum_{\{q_1,\dots,q_t\}} \hat{\xi}(c'(\hat{b}_u)'_{-m_{\{q_1,\dots,q_t\}}}) \right) \hat{\xi}(y^{-1}) \right] \gamma(uy)$$

from (4.1)

$$= \frac{1}{|C_G(u)|} \sum_{y} \left[\sum_{\xi \in \mathrm{Irr}(b_u)} \left(\xi(c'(b_u)_0) + \sum_{t=1}^{l} (-1)^t \sum_{\{q_1,\dots,q_t\}} \xi(c'(b_u)_{m_{\{q_1,\dots,q_t\}}}) \right) \xi(y^{-1}) \right] \gamma(uy)$$

and from [12], Chapter V, Theorem 4.7

$$= \frac{1}{|C_G(u)|} \sum_{y} \left[\sum_{\xi \in \mathrm{Irr}(b_u)} \left(\xi(c'(b_u)_0) + \sum_{t=1}^{l} (-1)^t \sum_{\{q_1,\dots,q_t\}} \xi(c'(b_u)_{m_{\{q_1,\dots,q_t\}}}) \right) \xi(y^{-1}) \right] \times (d_G^{(u,b_u)}(\gamma))(y)$$

$$= [(R_{p'}^{\langle u \rangle} \circ d_G^{(u,b_u)})(\gamma)](c')$$

recalling the definition of the perfect isometry $R^{\langle u \rangle}$, where y runs over $C_G(u)_{p'}$ and $\{q_1, \dots, q_t\}$ runs over the set of t-element subsets of $\pi(F)$. This and Theorem 4 complete the proof. □

Corollary 1 ([8]). *Let G and A be finite groups such that A is cyclic, A acts on G via automorphism and that $(|C_G(A)|, |A|) = 1$. If $p \nmid |A|$ and $p \nmid |G : C_G(A)|$, then the Dade correspondence induces an isotypy between $b(G)$ and $b(C_G(A))$.*

Proof. Let s be a generator of A. Let $E = G \rtimes A$, $G' = C_G(A)$ and $E' = G'A$. Then E, G, E' and G' satisfy Hypothesis 1 by [3], Lemma 7.5. By the assumption $\widehat{C(s)}b(E)$ is invertible in $Z(\mathcal{O}Eb(E))$. Also $sb(E')$ is invertible in $Z(\mathcal{O}E'b(E'))$. Hence the corollary follows from Theorem 6. □

EXAMPLE. Suppose $p = 5$, and let $G = Sz(2^{2n+1})$, the Suzuki group, $A = \langle\sigma\rangle$ where σ is the Frobenius automorphism of G with respect to $\mathrm{GF}(2^{2n+1})/\mathrm{GF}(2)$. Set $G' = Sz(2) = C_G(A)$, $E = G \rtimes A$, $E' = G' \times A$. Suppose that $5 \nmid 2n + 1$. Then $(2n+1, |G'|) = (2n+1, 20) = 1$. Moreover a Sylow 5-subgroup of G has order 5. By the above corollary, the Dade correspondence gives an isotypy between $b(G)$ and $b(G')$.

6. Normal defect group case

In the Glauberman correspondence case if the defect group D is normal in G, there is a Puig equivalence (splendidly Morita equivalence) between b and b' which affords the Glauberman correspondence on the character level ([6], [14]). In the Dade correspondence case we show that b and b' are Puig equivalent if D is normal in G. By our assumption, there exist a defect group D of b and b', and an element $s \in E'_0$ such that $s \in C_E(D)$ and $\widehat{C(s)}b \in Z(\mathcal{O}Eb)^\times$. Let $\phi \in \mathrm{Irr}(b)$ be of height 0. From [13], Lemma 2.4 and (1.1) in Theorem 1, we have

$$0 \neq (\omega_{\hat{\phi}}(\widehat{C(s)}))^* = \left(\epsilon_\phi \frac{|E|\phi_{(G')}(1)}{|E'|\phi(1)} \omega_{(\hat{\phi})_{(E')}}(\widehat{C(s)'})\right)^*.$$

Since b and b' have the same defect,

$$\left(\omega_{(\hat{\phi})_{(E')}}(\widehat{C(s)'})\right)^* \neq 0.$$

Hence $\widehat{C(s)'}b' \in Z(\mathcal{O}Eb')^\times$. The element s is used in the next lemma.

Lemma 3. *Let E_1 be a subgroup of $N_E(D)$ containing $C_E(D)$ and set $G_1 = G \cap E_1$, $E'_1 = E' \cap E_1$, and $G'_1 = G' \cap E_1$. Then E_1, G_1, E'_1 and G'_1 satisfy* Hypothesis 1. *Moreover $(b_D)^{G_1}$ satisfies* Hypothesis 2, *$((b_D)')^{G'_1}$ satisfies* Hypothesis 3 *and*

$$((b_D)^{G_1})_{(G'_1)} = ((b_D)')^{G'_1}. \tag{6.1}$$

Proof. By our assumption $E = G\langle s\rangle$, hence we have $E_1 = G_1\langle s\rangle = E'_1G_1$, $G'_1 = G_1 \cap E'_1$. Also $E_1/G_1 \cong E'_1/G'_1 \cong F$. Hence the former is clear. On the other hand, since $\mathrm{Br}_D^{\mathcal{O}E}(\widehat{C(s)}b)b_D^* \in Z(kE_1(b_D)^*)^\times = Z(kE_1((b_D)^{G_1})^*)^\times$ and $\mathrm{Br}_D^{\mathcal{O}E'}(\widehat{C(s)'}b')(b'_D)^* \in Z(kE'_1((b_D)')^*)^\times = Z(kE'_1(((b_D)')^{G'_1})^*)^\times$, $(b_D)^{G_1}$ satisfies Hypothesis 2, and $((b_D)')^{G'_1}$ satisfies Hypothesis 3. By applying Theorem 3, (i) for E_1, G_1 and $(b_D)^{G_1}$, we have (6.1). □

In the above lemma, we set $E_1 = N_E(D)$. Then $(b_D)^{G_1} = (b_D)^{N_G(D)}$ is a Brauer correspondent of b, and $((b_D)')^{N_{G'}(D)}$ is a Brauer correspondent of b'. From now we assume D is *normal* in G. Then D is normal in E.

Lemma 4. *With the notations in* Lemma 3, *suppose that E_1 is normal in E. Let $\xi \in \mathrm{Irr}((b_D)^{G_1})$ and $x' \in E'$. We have $(\xi^{x'})_{(G'_1)} = (\xi_{(G'_1)})^{x'}$ and $(((b_D)^{G_1})^{x'})_{(G'_1)} = (((b_D)')^{G'_1})^{x'}$. In particular $I_E(\xi) \cap E' = I_{E'}(\xi_{(G'_1)})$ and $I_E((b_D)^{G_1}) \cap E' = I_{E'}(((b_D)')^{G'_1})$.*

Proof. Note that $(b_D)^{G_1}$ and $((b_D)^{G_1})^{x'}$ respectively satisfy Hypothesis 2. Let $\hat{\xi} \in \mathrm{Irr}(E_1|\xi)$ and $\xi' = \xi_{(G'_1)}$. By Theorem 1 and (1.1),

$$\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot\hat{\xi}\right)_{E'_1} = \epsilon_\xi \prod_{q\in\pi(F)}(1-\lambda_q)\cdot(\hat{\xi})_{(E'_1)}$$

where $\epsilon_\xi = \pm 1$. Hence we have,

$$\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot(\hat{\xi})^{x'}\right)_{E'_1} = \epsilon_\xi \prod_{q\in\pi(F)}(1-\lambda_q)\cdot((\hat{\xi})_{(E'_1)})^{x'}.$$

Therefore by Theorem 1 we have $(\xi^{x'})_{(G'_1)} = \xi'^{x'}$ because $((\hat{\xi})^{x'})_{G_1} = \xi^{x'}$ and $(((\hat{\xi})_{(E'_1)})^{x'})_{G'_1} = \xi'^{x'}$. This implies the lemma because the Dade correspondence $\rho(E_1, G_1, E'_1, G'_1)$ induces the bijection between $\mathrm{Irr}((b_D)^{G_1})$ and $\mathrm{Irr}(((b_D)')^{G'_1})$ by Lemma 3. □

By Lemma 4 we have $I_E(b_D) \cap E' = I_{E'}((b_D)')$. By Lemma 3 $I_E(b_D)$, $I_G(b_D)$, $I_{E'}((b_D)')$ and $I_{G'}((b_D)')$ satisfy Hypothesis 1. Moreover $(b_D)^{I_G(b_D)}$ satisfies Hypothesis 2, and $((b_D)')^{I_{G'}((b_D)')}$ satisfies Hypothesis 3. Also we have

$$((b_D)^{I_G(b_D)})_{(I_{G'}((b_D)'))} = ((b_D)')^{I_{G'}((b_D)')}. \tag{6.2}$$

By Lemma 3, $DC_E(D)$, $DC_G(D)$, $DC_{E'}(D)$ and $DC_{G'}(D)$ also satisfy Hypothesis 1. Set $K = DC_G(D)$ and $K' = DC_{G'}(D)$. Then $(b_D)^K$ satisfies Hypothesis 2, and $((b_D)')^{K'}$ satisfies Hypothesis 3. Moreover we have

$$((b_D)^K)_{(K')} = ((b_D)')^{K'}.$$

Now suppose that b_D is G-invariant for a while. Then $(b_D)^K$ is G-invariant. Note that as elements of $\mathcal{O}G$, $b = b_D = (b_D)^K$. By Lemma 4, $((b_D)')^{K'}$ is G'-invariant. Since

b is covered by r blocks of E and since $(b_D)^K$ is covered by r blocks of $DC_E(D)$, any block of $DC_E(D)$ covering $(b_D)^K$ is E-invariant. Let $\widehat{(b_D)^K}$ be a block of $DC_E(D)$ covering $(b_D)^K$. In fact the block idempotent of a block of E covering b belongs to $\mathcal{O}DC_E(D)$. If $\xi \in \mathrm{Irr}^G((b_D)^K)$ and $\hat{\xi}$ is an extension of ξ to $DC_E(D)$ belonging to $\widehat{(b_D)^K}$, then G fixes $\hat{\xi}$ and hence E fixes $\hat{\xi}$ because $(b_D)^K$ and $\widehat{(b_D)^K}$ are isomorphic by restriction. Similarly if $\xi' \in \mathrm{Irr}^{G'}(((b_D)')^{K'})$ and $\hat{\xi}'$ is an extension of ξ' to $DC_{E'}(D)$, $\hat{\xi}'$ is E'-invariant. We note that if $\xi \in \mathrm{Irr}^G((b_D)^K)$ then $\xi_{(K')} \in \mathrm{Irr}^{G'}(((b_D)')^{K'})$ by Lemma 4. The following is proved by the analogous way to that of the proof of [10], Lemma 3.2.

Lemma 5. *Suppose that b_D is G-invariant. Let $\xi \in \mathrm{Irr}^G((b_D)^K)$. Then the factor set α of G/K defined by ξ and the factor set α' of G'/K' defined by $\xi_{(K')}$ are cohomologous when G/K and G'/K' are identified.*

Proof. At first we note again that $G = KG'$ by Lemma 1, $E = DC_E(D)E'$, $E = DC_E(D)G$ and $E' = DC_{E'}(D)G'$. Moreover we have

$$G/K \cong E/DC_E(D) \cong E'/DC_{E'}(D) \cong G'/K'.$$

We may assume $G \neq K$. Let t be a prime dividing $|G : K|$ and let E_t be a subgroup of E containing $DC_E(D)$ such that $E_t/DC_E(D)$ is a Sylow t-subgroup of $E/DC_E(D)$. Set $G_t = G \cap E_t$, $E'_t = E' \cap E_t$ and $G'_t = G' \cap E_t$. By Lemma 3, E_t, G_t, E'_t and G'_t satisfy Hypothesis 1. Moreover $(b_D)^{G_t}$ satisfies Hypothesis 2, $((b_D)')^{G'_t}$ satisfies Hypothesis 3 and that $((b_D)^{G_t})_{(G'_t)} = ((b_D)')^{G'_t}$. Now by a theorem of Gaschütz (see [5], Theorem 15.8.5), we may assume $E = E_t$.

Let $\hat{\xi} \in \mathrm{Irr}(DC_E(D)|\xi)$. From Theorem 1 and (1.1),

$$\left(\left(\prod_{q\in\pi(F)}(1-\lambda_q)\cdot\hat{\xi}\right)_{DC_{E'}(D)}, (\hat{\xi})_{(DC_{E'}(D))}\right) = \pm 1,$$

where the left hand side is the inner product. Hence there exists an extension $\tilde{\xi}$ of ξ to $DC_E(D)$ such that $(\tilde{\xi}_{DC_{E'}(D)}, (\hat{\xi})_{(DC_{E'}(D))})$ is relatively prime to t. As we stated in the above $\tilde{\xi}$ is E-invariant, and $(\hat{\xi})_{(DC_{E'}(D))}$ is E'-invariant because $\xi_{(K')}$ is G'-invariant. By [2], Theorem 4.4, the factor set of $E/DC_E(D)$ defined by $\tilde{\xi}$ and the factor set of $E'/DC_{E'}(D)$ defined by $(\hat{\xi})_{(DC_{E'}(D))}$ are cohomologous when $E/DC_E(D)$ and $E'/DC_{E'}(D)$ are identified. Similarly by [2], Theorem 4.4, since $\tilde{\xi}$ is an extension of ξ, α and the factor set of $E/DC_E(D)$ defined by $\tilde{\xi}$ are cohomologous when G/K and $E/DC_E(D)$ are identified. Further α' and the factor set of $E'/DC_{E'}(D)$ defined by $(\hat{\xi})_{(DC_{E'}(D))}$ are cohomologous when G'/K' and $E'/DC_{E'}(D)$ are identified, because $(\hat{\xi})_{(DC_{E'}(D))}$ is an extension of $\xi_{(K')}$. Hence α and α' are cohomologous. □

In the above lemma we can take as ξ the canonical character of b belonging to $(b_D)^K$. Then $\xi_{(K')}$ is the canonical character of (b') because $\xi_{(K')}$ is a constituent of $\xi_{K'}$, and hence D is contained in the kernel of $\xi_{(K')}$. Moreover $\alpha, \alpha' \in Z^2(G/K, \mathcal{O}^\times)$ since ξ and $\xi_{(K')}$ are respectively characters of a G-invariant $\mathcal{O}K$-lattice and a G'-invariant $\mathcal{O}K'$-lattice. By Lemma 5, we see α and α' are cohomologous.

Generally let G be a finite group, b be a block of G with a normal defect group D, and let $\mathbf{b}$ be a G-invariant block of $C_G(D)$ covered by b. Set $K = DC_G(D)$ and let i be a primitive idempotent of $\mathcal{O}C_G(D)\mathbf{b}$. Then we see that i is primitive in $(\mathcal{O}G)^D$ because D is normal in G and i^* is primitive in $kC_G(D)$, and hence $i\mathcal{O}Gi$ is a source algebra of b. Set $B = i(\mathcal{O}G)i$. Let H be a complement of $DC_G(D)/C_G(D)$ in $G/C_G(D)$. Then H is isomorphic to a subgroup of Aut D. For each $h \in H$, we choose $x_h \in G$ such that $h = C_G(D)x_h$. We set $d^h = d^{x_h}$ for any $d \in D$. Moreover let α be a factor set of H defined by the canonical character χ of b, where H and G/K are identified.

Theorem 7. *With the above notations, B is isomorphic to a twisted group algebra $\mathcal{O}^{\alpha^{-1}}(D \rtimes H)$ of the semi direct product $D \rtimes H$ over $\mathcal{O}$ with the factor set α^{-1} (considered as a factor set of $D \rtimes H$), as interior $\mathcal{O}D$-algebras.*

Proof. For any $h \in H$ we can choose $u_h \in (\mathcal{O}C_G(D)\mathbf{b})^\times$ such that $i^{x_h^{-1}} = i^{u_h}$. Put $v_h = u_h x_h i$. For any $d \in D$, we have

$$v_h^{-1}(id)v_h = id^h \tag{6.3}$$

where v_h^{-1} is the inverse of v_h in B. Then we have

$$B = \bigoplus_{h\in H} i\mathcal{O}Kx_h i = \bigoplus_{h\in H} i\mathcal{O}Kiv_h = \bigoplus_{h\in H}(i\mathcal{O}Di)v_h.$$

Thus B is a crossed product of H over $i\mathcal{O}Di$. As is well known $i\mathcal{O}Di \cong \mathcal{O}D$. Since H is a p'-group, from (6.3) and the proof of Lemma M in [11], B is a twisted group algebra of $D \rtimes H$ over $\mathcal{O}$ with a factor set $\gamma \in Z^2(D \rtimes H, \mathcal{O}^\times)$ which is the inflation of a factor set of H. In fact γ satisfies that

$$v_h v_{h'} = \gamma(h, h')v_{hh'} \quad (\forall h, h' \in H)$$

by replacing v_h by $v_h\delta_h$ for some $\delta_h \in i + iJ(Z(\mathcal{O}D))i$ if necessary. Here $J(Z(\mathcal{O}D))$ is the radical of the center of $\mathcal{O}D$.

For any $a \in \mathcal{O}G$, we denote by $\bar{a}$ the image of a by the natural homomorphism from $\mathcal{O}G$ onto $\mathcal{O}(G/D)$. We set $\bar{G} = G/D$ and $\bar{K} = K/D \leq \bar{G}$. We have

$$\bar{i}\mathcal{O}\bar{G}\bar{i} = \bigoplus_{h\in H}(\mathcal{O}\bar{K}\overline{x_h} \cap (\bar{i}\mathcal{O}\bar{G}\bar{i})) = \bigoplus_{h\in H} \mathcal{O}\overline{v_h}.$$

Also we have

$$\overline{v_h}\,\overline{v_{h'}} = \gamma(h, h')\overline{v_{hh'}}.$$

Since $\bar{i}$ is a primitive idempotent of $\mathcal{O}\bar{G}$ corresponding to χ, $\bar{i}\mathcal{O}\bar{G}\bar{i}$ is a twisted group algebra of $\bar{G}$ over $\mathcal{O}$ with factor set α^{-1}. This implies that γ and α^{-1} are cohomologous. This completes the proof. □

Theorem 8. *Assume* Hypotheses 2 *and* 3, *and* $b' = b_{(G')}$. *Further assume the defect group* D *of* b *and* b' *is normal in* G. *Then* b *and* b' *are Puig equivalent.*

Proof. As is well known b and $(b_D)^{I_G(b_D)}$ are Puig equivalent. Hence by Lemma 4 and (6.2), we may assume that b_D is G-invariant. Then from Lemma 5 and Theorem 7, b and b' are Puig equivalent. This completes the proof. □

By the above theorem, the Brauer correspondent of b and that of b' are Puig equivalent assuming Hypotheses 2 and 3, and $b' = b_{(G')}$.

Corollary 2. *In the above theorem, let* $b = b(G)$. *Then* $a \in \mathcal{O}G'b(G') \mapsto ab(G) \in \mathcal{O}Gb(G)$ *is an algebra isomorphism.*

Proof. Since $\mathcal{O}Gb(G)$ is a source algebra of $b(G)$, $\mathcal{O}G'b(G')$ are $\mathcal{O}Gb(G)$ are isomorphic. Therefore dim $\mathcal{K}Gb(G) =$ dim $\mathcal{K}G'b(G')$, and hence the Dade correspondence from Irr($b(G)$) onto Irr($b(G')$) coincides with restriction, that is, $b(G)$ and $b(G')$ are isomorphic. Hence by [9], Theorem 1 or [7], Theorem 4.1 completes the proof. □

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