Koiso, N. and Sugimoto, M.
Osaka J. Math.
47 (2010), 787–815

MOTION OF ELASTIC WIRE WITH THICKNESS

NORIHITO KOISO and MITSURU SUGIMOTO

(Received January 6, 2009, revised April 3, 2009)

Abstract

There are several types of equation of motion of elastic wires. In this paper, we treat an equation taking account of the thickness of wire. The equation was introduced by Caflisch and Maddocks on plane curves, and they proved the existence of solutions. We will prove the existence of solutions for any dimensional Euclidean space. Note that, in the case of plane, the equation can be explicitly written in terms of polar coordinates. For higher dimensional case, we use covariant differentiation on the unit sphere.

1. Introduction and results

In the previous work [9], the first author considered the motion of a fixed length elastic wire, governed by the elastic energy. Let $\gamma = \gamma(x)$ be a curve in the N-dimensional Euclidean space $\mathbf{R}^N$ with arc length parameter $x \in [0, 1]$, i.e., $|\gamma'(x)| \equiv 1$. We denote its motion by $\gamma = \gamma(x, t)$. We denote by $(* \mid *)$ (resp. $| * |$, $\langle * \mid * \rangle$, $\| * \|$) the pointwise inner product (resp. pointwise norm, the L_2-inner product with respect to the variable x, the L_2-norm with respect to the variable x). The potential energy of the elastic wire $\gamma(x)$ is defined by the square integral $U = \|\gamma_{xx}\|^2$ of the curvature. Assuming that the wire is infinitely thin, the kinetic energy of the motion of the wire is defined by $E = \|\gamma_t\|^2$.

By Hamilton's principle, the equation of motion is given as critical points of the variational problem defined by the functional

$$F = \int_0^T E - U \, dt = \int_0^T \|\gamma_t\|^2 - \|\gamma_{xx}\|^2 \, dt. \tag{1.1}$$

The equation turns out to be a coupled system of semi-linear 1-dimensional plate equation, where the derivatives of unknown functions up to fourth order are involved.

In [9] and also in A. Burchard and L.E. Thomas [2], the existence of a unique short-time solution satisfying some initial data was proved. The former used a perturbation to a composition of parabolic operators, and latter used the contraction principle and Hasimoto's transformation in the 3-dimensional case. A linear version of such equation can be also found, for example, in R. Courant and D. Hilbert [4].

2000 Mathematics Subject Classification. Primary 35Q72; Secondary 53C44, 53A04.

We may replace Newton's dynamics by the gradient flow equation, where the equation turns out to be a coupled system of semi-linear parabolic equation. The gradient flow equation is treated by the first author [7] and Y. Wen [10] in the Euclidean space, and in [8] in the Riemannian manifolds.

In this article, we consider the case when the wire is thick. Following the argument in R. Caflisch and J. Maddocks [5], we define the kinetic energy of the motion of the wire as $E = \|\gamma_t\|^2 + \|\gamma_{xt}\|^2$, where the term $\|\gamma_{xt}\|^2$ comes from the thickness of the wire. See also [3]. Then the equation of the wire is derived from the functional

$$F = \int_0^T E - U\,dt = \int_0^T \|\gamma_t\|^2 + \|\gamma_{xt}\|^2 - \|\gamma_{xx}\|^2\,dt \tag{1.2}$$

instead of (1.1). Such equation in the planer case ($N = 2$) was derived by R. Caflisch and J. Maddocks [5], and the global in time existence of the solutions was established under some kind of constrained conditions. The equation can also be obtained as a planar version of the director rod theory due to Antman and Kenney [1]. See also [6].

The objective of this article is to generalize the global existence results in [5] to any space dimension. We remark that, in the case of plane, the equation can be explicitly written in terms of trigonometric functions. For higher dimensional case, it is natural to introduce a geometrical aspect, and we use covariant differentiation on the unit sphere. Exact expression of such equation will be given later in Section 2, but we also remark here that it is a completely different equation from the one for infinitely thin elastic wire case treated in [9] and [2].

We consider several boundary conditions for the minimizing function $\gamma = \gamma(x)$ of the functional (1.2), abbreviating the time variable t. Let P_0 and P_1 be affine subspaces of $\mathbf{R}^N$.

CONDITION 1.1. We assume that (B) the boundary point $\gamma(0)$ belongs to an affine subspace P_0 of $\mathbf{R}^n$ and the vector $\gamma(1) - \gamma(0)$ belongs to an affine subspace P_1. And, we assume that (E) the boundary direction $\gamma_x(0)$ and $\gamma_x(1)$ are free or fixed (4 types), or (P) $\gamma_x(1) = \gamma_x(0)$. We denote by (BE) the combination of (B) and (E), and by (BP) the combination of (B) and (P).

The above condition includes the following special cases: When both P_0 and P_1 are point sets, the boundary point of the curve are fixed. When both P_0 and P_1 are $\mathbf{R}^N$, the boundary point of the curve are free. The motion of closed curves are represented by the condition (BP) with $P_0 = \mathbf{R}^N$ and $P_1 = \{0\}$. [5] treats the planner ($N = 2$) non-closed case (BE) with (1) $P_0 = \{0\}$, $P_1 = \mathbf{R}^2$, $\gamma_x(0)$ is fixed, $\gamma_x(1)$ is free; and (2) $P_0 = \{0\}$, $P_1 = \mathbf{R}^1$, $\gamma_x(0)$ and $\gamma_x(1)$ are free.

We assume that the distance of P_1 and the origin 0 is less than 1. We denote by P_i^T the vector subspace given by translating P_i to the origin, and by $P_i^\perp$ the orthogonal complement vector space to P_i^T. The projection to P_i^T (resp. $P_i^\perp$) is denoted by $*^{P_i^T}$ (resp. $*^{P_i^\perp}$).

The constrained condition 1.1 implies that the initial value $\gamma_0(x) = \gamma(x, 0)$, $\gamma_1(x) = \gamma_t(x, 0)$ have to satisfy the following conditions. We always assume that the initial value satisfies the condition.

CONDITION 1.2 (constrained condition of initial value). (B) $\gamma_0(0) \in P_0$, $\gamma_1(0) \in P_0^T$, $\gamma_0(1) - \gamma_0(0) \in P_1$, $\gamma_1(1) - \gamma_1(0) \in P_1^T$.
(E) $\gamma_1'(0) = 0$ if $\gamma_x(0)$ is fixed. $\gamma_1'(1) = 0$ if $\gamma_x(1)$ is fixed.
(P) $\gamma_0'(1) = \gamma_0'(0)$, $\gamma_1'(1) = \gamma_1'(0)$.

As we will see in Proposition 2.1, Conditions 1.1 (BE), (BP) require that the solution $\gamma(x, t)$ have to satisfy the following compatibility conditions.

CONDITION 1.3. (BE$'$) $\gamma_{xx}(0, t) = 0$ if $\gamma_x(0)$ is free. $\gamma_{xx}(1, t) = 0$ if $\gamma_x(1)$ is free.
(BP$'$) $\gamma_{xx}(1, t) = \gamma_{xx}(0, t)$.

These conditions corresponds to the compatibility condition of initial value:

CONDITION 1.4 (compatibility condition of initial value). (CE$'$) $\gamma_0''(0) = 0$ if $\gamma_x(0)$ is free. $\gamma_0''(1) = 0$ if $\gamma_x(1)$ is free.
(CP$'$) $\gamma_0''(1) = \gamma_0''(0)$.

To state our results, we prepare notations. Let C^n_{pw} be the space of all C^{n-1} functions $\{u(x)\}$ such that $u^{(n)}(x)$ are piecewise continuous. Let $C^n_{\mathrm{pw}\times}$ be the space of all C^{n-1} functions $\{u(x, t)\}$ which are C^n outside of finite number of lines $x \pm t = c_i$. Let $C^n_{\mathrm{pw}+}$ be the space of all C^{n-1} functions $\{u(x, t)\}$ which are C^n outside of finite number of lines $t = t_i$ and $x \in \mathbf{Z}$. Similarly, let $C^n_{\mathrm{pw}\times+}$ be the space of all C^{n-1} functions $\{u(x, t)\}$ which are C^n outside of finite number of lines $x \pm t = c_i$, $t = t_i$ and $x \in \mathbf{Z}$.

We will get the following results. We always have short time solutions.

Theorem 3.13. *If the initial value $\gamma_0(x)$ and $\gamma_1(x)$ satisfy* Condition 1.2 (*constrained condition*) *and if $\gamma_0(x)$, $\gamma_1(x) \in C^0_{\mathrm{pw}}$, then, even if they do not satisfy* Condition 1.4 (*compatibility condition*), *there exists a unique short time solution $\gamma(x, t)$ such that both $\gamma_{xx}(x, t)$ and $\gamma_{xt}(x, t)$ are in $C^0_{\mathrm{pw}\times}$.*

And, the solution exists for infinite time, if the initial value satisfies the compatibility condition.

Theorem 5.3. *If the initial value $\gamma_0(x)$ and $\gamma_1(x)$ are C^1, and they satisfy* Condition 1.1 (*constrained condition*) *and* Condition 1.3 (*compatibility condition*), *then there exists a unique infinite time solution $\gamma(x,t)$ such that $\gamma_{xx}(x,t)$ and $\gamma_{xt}(x,t)$ are continuous.*

Moreover, problems of the following types have infinite time solutions, even if the compatibility condition are not satisfied.

CONDITION 1.5. (a) $P_0 = \mathbf{R}^N$ and ($P_1 = \mathbf{R}^N$ or ($\xi(0)$ and $\xi(1)$ are free)), or
(b) $\xi(0)$ is free and ($P_1 = \mathbf{R}^N$ or $\xi(1)$ is free), or
(c) $P_0 = \mathbf{R}^N$ and $P_1 = \{0\}$ and $\xi(0) = \xi(1)$.

Theorem 5.4. *Suppose that the constrained condition is one of the type* (1.5). *Then, the solution $\gamma(x,t)$ of* Theorem 3.13 *exists infinite time.*

2. Derivation of equation

To derive the equation of motion, we rewrite the functional $F(\gamma)$ as a functional of $\xi := \gamma_x \in S^{N-1}$. Since the variable t always appear, we omit it. The term γ_t can be express as $\gamma(x) = \gamma(0) + \int_0^x \xi(y)\,dy$.

$$\begin{aligned}(2.1)\quad &\left\|\gamma_t(0) + \int_0^x \xi_t(y)\,dy\right\|^2 \\ &= |\gamma_t(0)|^2 + 2\left(\gamma_t(0)\;\middle|\;\int_0^1\int_0^x \xi_t(y)\,dy\,dx\right) + \left\langle \int_0^x \xi_t(y)\,dy \;\middle|\; \int_0^x \xi_t(z)\,dz\right\rangle,\end{aligned}$$

$$\begin{aligned}(2.2)\quad \int_0^1\int_0^x \xi_t(y)\,dy\,dx &= \iint_{0\le y\le x\le 1} \xi_t(y)\,dx\,dy = \int_0^1\int_y^1 \xi_t(y)\,dx\,dy \\ &= \int_0^1 (1-y)\xi_t(y)\,dy,\end{aligned}$$

$$\begin{aligned}(2.3)\quad \left\langle \int_0^x \xi_t(y)\,dy \;\middle|\; \int_0^x \xi_t(z)\,dz\right\rangle &= \int_0^1\int_0^x\int_0^x (\xi_t(y)\mid\xi_t(z))\,dy\,dz\,dx \\ &= \iiint_{0\le y,z\le x\le 1} (\xi_t(y)\mid\xi_t(z))\,dx\,dy\,dz \\ &= \iint_{0\le y,z\le 1}\left\{\int_{\max\{y,z\}}^1 (\xi_t(y)\mid\xi_t(z))\,dx\right\}dy\,dz \\ &= \int_0^1\int_0^1 (1-\max\{y,z\})(\xi_t(y)\mid\xi_t(z))\,dy\,dz.\end{aligned}$$

We put $\kappa(x, y) := 1 - \max\{x, y\}$, and get

(2.4)
$$F = \int_0^T \Big\{ |\gamma_t(0)|^2 + 2\Big(\gamma_t(0) \Big| \int_0^1 (1-y)\xi_t(y)\,dy \Big) + \int_0^1\int_0^1 \kappa(x, y)(\xi_t(x) \mid \xi_t(y))\,dx\,dy + \|\xi_t\|^2 - \|\xi_x\|^2 \Big\}\,dt.$$

The first variation with respect to an infinitesimal variation $(\breve{\gamma}, \breve{\xi})$ is given by

(2.5)
$$\begin{aligned}
\frac{\breve{F}}{2} &= \int_0^T \Big\{ (\gamma_t(0) \mid \breve{\gamma}_t(0)) + \Big(\breve{\gamma}_t(0) \mid \int_0^1 (1-y)\xi_t(y)\,dy \Big) \\
&\qquad + \Big(\gamma_t(0) \Big| \int_0^1 (1-y)\breve{\xi}_t(y)\,dy \Big) + \int_0^1\int_0^1 \kappa(x, y)(\breve{\xi}_t(x) \mid \xi_t(y))\,dx\,dy \\
&\qquad + \langle \breve{\xi}_t \mid \xi_t \rangle - \langle \breve{\xi}_x \mid \xi_x \rangle \Big\}\,dt \\
&= -\int_0^T \Big\{ (\gamma_{tt}(0) \mid \breve{\gamma}(0)) \Big(\breve{\gamma}(0) \Big| \int_0^1 (1-y)\xi_{tt}(y)\,dy \Big) \\
&\qquad + \Big(\gamma_{tt}(0) \Big| \int_0^1 (1-y)\breve{\xi}(y)\,dy \Big) + \int_0^1\int_0^1 \kappa(x, y)(\breve{\xi}(x) \mid \xi_{tt}(y))\,dx\,dy \\
&\qquad + \langle \breve{\xi} \mid \xi_{tt} \rangle + [(\breve{\xi} \mid \xi_x)]_{x=0}^1 - \langle \breve{\xi} \mid \xi_{xx} \rangle \Big\}\,dt \\
&= -\int_0^T \Big(\breve{\gamma}(0) \Big| \gamma_{tt}(0) + \int_0^1 (1-y)\xi_{tt}(y)\,dy \Big)\,dt - \int_0^T [(\breve{\xi} \mid \xi_x)]_{x=0}^1\,dt \\
&\qquad - \int_0^T \Big\langle \breve{\xi} \Big| (1-x)\gamma_{tt}(0) + \int_0^1 \kappa(x, y)\xi_{tt}(y)\,dy + \xi_{tt} - \xi_{xx} \Big\rangle\,dt.
\end{aligned}$$

Since $\breve{\gamma}(0)$ can move in P_0^T and $\gamma_{tt}(0) \in P_0^T$, we have

(2.6)
$$\gamma_{tt}(0) = -\Big(\int_0^1 (1-y)\xi_{tt}(y)\,dy \Big)^{P_0^T}$$

by eliminating $\breve{\gamma}(0)$, where $*^{P_0^T}$ means the projection. Since the conditions concerning to $\breve{\xi}$ are $(\breve{\xi} \mid \xi) = 0$ and $\int_0^1 \breve{\xi}\,dx \in P_1^T$, we have

(2.7)
$$(1-x)\gamma_{tt}(0) + \int_0^1 \kappa(x, y)\xi_{tt}(y)\,dy + \xi_{tt} - \xi_{xx} = u\xi + v, \quad v \in P_1^\perp$$

by eliminating $\breve{\xi}$. Here, $u = u(x,t)$ is an unknown function and $v = v(t)$ is an unknown vector valued function. We consider the term $[(\breve{\xi} \mid \xi_x)]_{x=0}^1$ later.

Put $\varphi = \xi_{tt} - \xi_{xx}$.

$$\begin{aligned}\int_0^1 (1-y)\xi_{tt}(y)\,dy &= \int_0^1 (1-y)\varphi(y)\,dy + \int_0^1 (1-y)\xi_{xx}(y)\,dy \\ &= \int_0^1 (1-y)\varphi(y)\,dy + [(1-y)\xi_x(y)]_{y=0}^1 + \int_0^1 \xi_x(y)\,dy \\ &= \int_0^1 (1-y)\varphi(y)\,dy - \xi_x(0) + [\xi]_0^1,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\int_0^1 \kappa(x,y)\xi_{tt}(y)\,dy &= \int_0^1 \kappa(x,y)\varphi(y)\,dy + \int_0^1 \kappa(x,y)\xi_{xx}(y)\,dy \\ &= \int_0^1 \kappa(x,y)\varphi(y)\,dy + [\kappa(x,y)\xi_x(y)]_{y=0}^1 + \int_x^1 \xi_x(y)\,dy \\ &= \int_0^1 \kappa(x,y)\varphi(y)\,dy - (1-x)\xi_x(0) + \xi(1) - \xi.\end{aligned} \tag{2.9}$$

Therefore,

$$\begin{aligned}&\varphi + \int_0^1 \kappa(x,y)\varphi(y)\,dy - \left\{\int_0^1 (1-x)(1-y)\varphi(y)\,dy\right\}^{P_0^T} \\ &= (u+1)\xi + v + (1-x)\xi_x(0)^{P_0^\perp} - \xi(1) + (1-x)[\xi]_0^{1\,P_0^T}.\end{aligned} \tag{2.10}$$

We put

$$L_{P_0}(f) := \int_0^1 \kappa(x,y) f(y)\,dy - \left\{\int_0^1 (1-x)(1-y) f(y)\,dy\right\}^{P_0^T} \tag{2.11}$$

and $\phi := \varphi^\top$, where $*^\top$ means the tangential component to the unit sphere, i.e., $\varphi^\top = \varphi - (\varphi \mid \xi)\xi$. Then,

$$\varphi^\top + L_{P_0}(\varphi)^\top = v^\top - \xi(1)^\top + (1-x)\{\xi_x(0)^{P_0^\perp} + [\xi]_0^{1\,P_0^T}\}^\top, \tag{2.12}$$

$$\begin{aligned}&\phi + L_{P_0}(\phi)^\top - v^\top \\ &= -\xi(1)^\top + (1-x)\{\xi_x(0)^{P_0^\perp} + [\xi]_0^{1\,P_0^T}\}^\top + L_{P_0}((|\xi_t|^2 - |\xi_x|^2)\xi)^\top.\end{aligned} \tag{2.13}$$

The constrained condition $\gamma(0) \in P_0$ is eliminated. Under the condition for initial value, the constrained condition $\gamma(1) - \gamma(0) \in P_1$ is equivalent to $\int_0^1 \xi_{tt}\,dx \in P_1^T$, and can be written as $\left\{\int_0^1 \xi_{tt}\,dx\right\}^{P_1^\perp} = 0$. Therefore, using

$$\int_0^1 \xi_{tt}\,dx = \int_0^1 \varphi\,dx + \int_0^1 \xi_{xx}\,dx = \int_0^1 \phi\,dx - \int_0^1 (|\xi_t|^2 - |\xi_x|^2)\xi\,dx + [\xi_x]_0^1, \tag{2.14}$$

the constrained condition $\gamma(1)-\gamma(0)\in P_1$ is equivalent to

$$\left\{\int_0^1 \phi\, dx\right\}^{P_1^\perp} = \left\{\int_0^1 (|\xi_t|^2-|\xi_x|^2)\xi\, dx\right\}^{P_1^\perp} - [\xi_x]_0^{1\,P_1^\perp}. \tag{2.15}$$

Hence, we have

$$\begin{aligned}
&\phi + L_{P_0}(\phi)^\top - v^\top \\
&= -\xi(1)^\top + (1-x)\{\xi_x(0)^{P_0^\perp} + [\xi]_0^{1\,P_0^T}\}^\top + L_{P_0}((|\xi_t|^2-|\xi_x|^2)\xi)^\top, \\
&\left\{\int_0^1 \phi\, dx\right\}^{P_1^\perp} = \left\{\int_0^1 (|\xi_t|^2-|\xi_x|^2)\xi\, dx\right\}^{P_1^\perp} - [\xi_x]_0^{1\,P_1^\perp}.
\end{aligned} \tag{2.16}$$

Now, we consider the condition $[(\breve{\xi} \mid \xi_x)]_0^1 = 0$. If $\xi(x_0) = \gamma_x(x_0)$ $(x_0 = 0, 1)$ is fixed, then $\breve{\xi}(x_0) = 0$ and we have no more equations (Dirichlet condition). If $\xi(x_0)$ is free, then we have a new equation $\xi_x(x_0) = 0$ (Neumann condition). If the condition is periodic: $\xi(1) = \xi(0)$, we have a new equation $\xi_x(1) = \xi_x(0)$ (periodic condition). Note that $[(\xi_t \mid \xi_x)]_0^1 = 0$ is required in all cases.

We proved the following

Proposition 2.1. *The equation of motion defined by the functional* (1.2) *is given by* (2.16). *Corresponding to the constrained condition, the boundary condition is given by*:
(E) *If* $\gamma_x(x_0)$ $(x_0 = 0, 1)$ *is fixed then* $\xi(x_0)$ *is fixed. If* $\gamma_x(x_0)$ *is free then* $\xi_x(x_0) = 0$.
(P) $\xi(1) = \xi(0)$, $\xi_x(1) = \xi_x(0)$.

3. Existence of solutions

We solve (2.16) by the successive approximation. We have to solve

$$\phi + L_{P_0}(\phi)^\top - v^\top = a^\top, \quad \left\{\int_0^1 \phi\, dx\right\}^{P_1^\perp} = b, \tag{3.1}$$

$$\nabla_t \xi_t - \nabla_x \xi_x = c, \quad \{\text{boundary conditions at } x = 0, 1\}. \tag{3.2}$$

Here, $a(x,t)$, $b(t)\in P_1^\perp$, $c(x,t)$ are known functions, and $\phi(x,t)$, $v(t)\in P_1^\perp$, $\xi(x,t)$ are unknown functions. Note that the operator $*^\top$ depends on ξ. ∇ means the covariant differentiation on S^{N-1}, i.e., the tangential component of the derivatives.

3.1. Estimation for equation (3.1) of $\boldsymbol{\phi}$.

To solve (3.1), we consider operator

$$\mathcal{L}_{\xi,P_0,P_1} : (\phi, v) \mapsto \left(\phi + L_{P_0}(\phi)^\top - v^\top, -\left\{\int_0^1 \phi\, dx\right\}^{P_1^\perp}\right), \tag{3.3}$$

where ϕ is a tangential vector field along ξ on S^{N-1}, and v is a vector in $P_1^{\perp}$. We decompose the kernel function of the integral operator L_{P_0} into P_0^T component and $P_0^{\perp}$ component. The two components becomes $p(x, y) := 1 - \max\{x, y\} - (1-x)(1-y)$ and $\kappa(x, y) = 1 - \max\{x, y\}$, and they are symmetric and the ranges are contained in the interval $[0, 1]$. Therefore, the operator L_{P_0} is a self-adjoint Hilbert–Schmidt operator, and satisfies $\|L_{P_0}(\phi)^{\top}\| \leq \|\phi\|$.

We estimate the norm of the operator L_{P_0}.

Lemma 3.1. *The operator L_{P_0} is bounded from below*:

$$\langle L_{P_0}(\phi)^{\top} \mid \phi \rangle \geq -\frac{1}{4}\|\phi\|^2.$$

Proof. We define two operators, using kernel functions p and κ.

$$L_{P_0}^T : f \mapsto \int_0^1 p(x, y) f(y)\, dy, \quad L_{P_0}^{\perp} : f \mapsto \int_0^1 \kappa(x, y) f(y)\, dy. \tag{3.4}$$

The operator L_{P_0} is decomposed into $L_{P_0}^T + L_{P_0}^{\perp}$. Since

$$\begin{aligned} p(x, y) &= 1 - \max\{x, y\} - (1-x)(1-y) = \min\{x, y\}(1 - \max\{x, y\}) \\ &\leq \min\{x, y\}(1 - \min\{x, y\}) \leq \frac{1}{4}, \end{aligned} \tag{3.5}$$

we have

$$\begin{aligned} |\langle L_{P_0}^T(f) \mid f \rangle| &= \left| \int_0^1 \int_0^1 p(x, y)(f(x) \mid f(y))\, dx\, dy \right| \\ &\leq \frac{1}{4} \int_0^1 \int_0^1 |f(x)|\, |f(y)|\, dx\, dy \leq \frac{1}{4}\|f^2\|. \end{aligned} \tag{3.6}$$

And, since $\kappa(x, y) = p(x, y) + (1-x)(1-y)$, we have

$$\begin{aligned} \langle L_{P_0}^{\perp}(f) \mid f \rangle &= \int_0^1 \int_0^1 \kappa(x, y)(f(x) \mid f(y))\, dx\, dy \\ &= \langle L_{P_0}^T(f) \mid f \rangle + \int_0^1 \int_0^1 (1-x)(1-y) f(x) f(y)\, dx\, dy \\ &\geq -\frac{1}{4}\|f^2\|. \end{aligned} \tag{3.7}$$

We have $\langle L_{P_0}(\phi)^{\top} \mid \phi \rangle \geq -(1/4)\|\phi\|^2$, by combining these two inequalities. $\square$

The operator $\mathcal{L}_{\xi,P_0,P_1}$ is also self adjoint, because

$$\langle \phi_1 + L_{P_0}(\phi_1)^\top - v_1^\top \mid \phi_2\rangle + \left(-\left\{\int_0^1 \phi_1\,dx\right\}^{P_1^\perp} \,\middle|\, v_2\right) \tag{3.8}$$
$$= \langle \phi_1 \mid \phi_2 + L_{P_0}(\phi_2)^\top\rangle - \langle v_1 \mid \phi_2\rangle - \langle \phi_1 \mid v_2\rangle.$$

To see that the operator $\mathcal{L}_{\xi,P_0,P_1}$ is invertible, we prepare a lemma.

Lemma 3.2. *For any non-negative number $D < 1$ and positive number C, there exists a positive constant K with the following property: if the distance between P_1 and the origin is less than D and if the elastic energy $\|\xi_x\|^2$ is less than C^2, then all eigenvalues λ of $\mathcal{L}_{\xi,P_0,P_1}$ satisfy $|\lambda| \geq K^{-1}$.*

Proof. Let λ ($|\lambda| < 1/4$) be an eigenvalue of $\mathcal{L}_{\xi,P_0,P_1}$ and (ϕ, v) be an eigenvector such that $\|\phi\|^2 + |v|^2 = 1$. Since $\phi + L_{P_0}(\phi)^\top - v^\top = \lambda\phi$, we see

$$\lambda = \langle \mathcal{L}_{\xi,P_0,P_1}(\phi, v) \mid (\phi, v)\rangle = \langle \phi + L_{P_0}(\phi)^\top \mid \phi\rangle - \langle v^\top \mid \phi\rangle + \lambda|v|^2$$
$$= \langle \phi + L_{P_0}(\phi)^\top \mid \phi\rangle - \left(v \,\middle|\, \int_0^1 \phi(x)\,dx\right) + \lambda|v|^2 \tag{3.9}$$
$$= \langle \phi + L_{P_0}(\phi)^\top \mid \phi\rangle + 2\lambda|v|^2.$$

Therefore, $\langle \phi + L_{P_0}(\phi)^\top \mid \phi\rangle = \lambda(1 - 2|v|^2)$. Since $\langle L_{P_0}(\phi)^\top \mid \phi\rangle \geq -(1/4)\|\phi\|^2$, we have $|\lambda(1-2|v|^2)| = |\|\phi\|^2 + \langle L_{P_0}(\phi)^\top \mid \phi\rangle| \geq (3/4)\|\phi\|^2$.

Using this, the norm $\|v^\top\|$ is estimated by λ as

$$\|\phi\|^2 \leq \frac{4}{3}|\lambda| \leq 2|\lambda|, \tag{3.10}$$
$$\|v^\top\|^2 = \|(1-\lambda)\phi + L_{P_0}(\phi)^\top\|^2 \leq (2 + |\lambda|)^2\|\phi\|^2 \leq \frac{27}{4}|\lambda| \leq 8|\lambda|.$$

Since $v^\top = v - (v \mid \xi)\xi$, we have

$$(v^\top)_x = -(v \mid \xi_x)\xi - (v \mid \xi)\xi_x, \quad |(v^\top)_x|^2 \leq 2|v|^2|\xi_x|^2 \leq 2|\xi_x|^2, \tag{3.11}$$

$$\max|v^\top|^2 \leq 2\|v^\top\|(\|v^\top\| + \|(v^\top)_x\|) \leq 2\sqrt{8|\lambda|}(\sqrt{8|\lambda|} + \sqrt{2}\|\xi_x\|) \tag{3.12}$$
$$\leq 16|\lambda| + 8C\sqrt{|\lambda|} \leq 8(1+C)\sqrt{|\lambda|},$$

$$(v \mid \xi)^2 = |v|^2 - |v^\top|^2 \geq 1 - 4|\lambda|^2 - 8(1+C)\sqrt{|\lambda|} \tag{3.13}$$
$$\geq 1 - (9+8C)\sqrt{|\lambda|}.$$

It implies that if $|\lambda| < (9+8C)^{-2}$ then $(v \mid \xi)$ does not change its sign, and

$$\left|\left(v \,\middle|\, \int_0^1 \xi\,dx\right)\right| = \left|\int_0^1 (v \mid \xi)\,dx\right| \geq \sqrt{1 - (9+8C)\sqrt{|\lambda|}}. \tag{3.14}$$

On the other hand, we know that $\left|\left(v \mid \int_0^1 \xi\, dx\right)\right| \le |v| \cdot D \le D$, because $v \in P_1^{\perp}$. Therefore, $|\lambda| \ge \{(1-D^2)/(9+8\|\xi_x\|)\}^2$. □

By this lemma, we can solve (3.1) with an L_2 function.

Proposition 3.3. *For any $a \in L_\infty$ and b, equation* (3.1) *has a unique solution* (ϕ, v). *Moreover, the norm of the solution is estimated as*

$$\begin{aligned} &\|\phi\|,\ |v| \le K(\|a\| + |b|),\\ &\max|\phi| \le K(\max|a| + |b|), \end{aligned} \tag{3.15}$$

where K is a positive constant depending only on the energy $\|\xi_x\|^2$.

Proof. By Lemma 3.2, we see $\|\phi\|, |v| \le C_1(\|a\| + |b|)$. Since the solution satisfies $\phi = -L_{P_0}(\phi)^\top + v^\top + a^\top$, we have $\max|\phi| \le \|\phi\| + |v| + \max|a| \le 3C_1(\max|a| + |b|)$. □

Also, the solution ϕ is continuous in the following sense.

Lemma 3.4. *In* Proposition 3.3, *if $\xi(x)$ and $a(x)$ are continuous, then $\phi(x)$ is continuous.*

Proof. Put $(\delta f)(x) := f(x+\varepsilon) - f(x)$.

$$\begin{aligned} &\frac{d}{dx} L_{P_0}(f) = -\int_0^x (1-y) f(y)\, dy - \left(\int_0^1 (1-y) f(y)\, dy\right)^{P_0^T},\\ &|\delta(f^\top)| \le C_1\{|\delta f| + |\delta\xi|\}. \end{aligned} \tag{3.16}$$

□

Proposition 3.5. *Consider equations*

$$\phi_i + L_{P_0}(\phi_i)^{\top_i} - v_i^{\top_i} = a_i^{\top_i}, \quad \left\{\int_0^1 \phi_i\, dx\right\}^{P_1^\perp} = b_i^{P_1^\perp}, \tag{3.17}$$

where $*^{\top_i}$ means the orthogonal component to ξ_i. There exists a positive constant C depending on $\|\xi_x\|$, $\sup|a|$ *and* $|b|$, *such that*

$$|\delta\phi| \le C\{\|\delta\xi\| + \|\delta a\| + |\delta b| + |\delta\xi| + |\delta a|\}, \tag{3.18}$$

where δ means the difference of solutions, e.g., $\delta\phi := \phi_2 - \phi_1$.

Proof. We know that $\sup|\phi_i|, |v_i| \le C_1$. We write $X^{\top_i} := X - (X \mid \xi_i)\xi_i$. This is an extension of the notation $*^{\top_i}$. From

$$X^{\top_2} - X^{\top_1} = -(X \mid \xi_2)\xi_2 + (X \mid \xi_1)\xi_1 = -(X \mid \delta\xi)\xi_1 - (X \mid \xi_1)\delta\xi - (X \mid \delta\xi)\delta\xi, \tag{3.19}$$

we have $|X^{\top_2} - X^{\top_1}| \leq 4|\delta\xi|\,|X|$. Therefore,

$$|\{L_{P_0}(\phi_2)^{\top_2} - v_2^{\top_2} - a_2^{\top_1}\} - \{L_{P_0}(\phi_2)^{\top_1} - v_2^{\top_1} - a_2^{\top_1}\}| \leq C_2|\delta\xi|. \tag{3.20}$$

And,

$$|\delta\phi + L_{P_0}(\delta\phi)^{\top_1} - \delta v^{\top_1}| \leq C_2\{|\delta\xi| + |\delta a|\}, \tag{3.21}$$

$$\left|\left\{\int_0^1 \delta\phi\, dx\right\}^{P_1^\perp}\right| \leq C_2\{|\delta\xi| + |\delta b|\}. \tag{3.22}$$

Since $\delta\phi$ is not orthogonal to ξ_1, we decompose it as $\delta\phi = \delta\phi^{\top_1} + (\delta\phi \mid \xi_1)\xi_1$. Then,

$$(\delta\phi \mid \xi_1) = (\phi_2 - \phi_1 \mid \xi_1) = (\phi_2 \mid \xi_1) = (\phi_2 \mid \xi_2 - \delta\xi) = -(\phi_2 \mid \delta\xi), \tag{3.23}$$

and $|\delta\phi - \delta\phi^{\top_1}| \leq C_3|\delta\xi|$. Hence,

$$\begin{aligned} &|\delta\phi^{\top_1} + L_{P_0}(\delta\phi^{\top_1})^{\top_1} - \delta v^{\top_1}| \leq C_4\{|\delta\xi| + |\delta a|\}, \\ &\left|\left\{\int_0^1 \delta\phi^{\top_1}\, dx\right\}^{P_1^\perp}\right| \leq C_5\{|\delta\xi| + |\delta b|\}. \end{aligned} \tag{3.24}$$

Therefore we have

$$\|\delta\phi\|, |\delta v| \leq C_6\{\|\delta\xi\| + \|\delta a\| + |\delta b|\}, \tag{3.25}$$

and, from (3.21),

$$\begin{aligned} |\delta\phi| &\leq C_2\{|\delta\xi| + |\delta a|\} + |L_{P_0|(\delta\phi)^{\top_1}} + |\delta v^{\top_1}| \\ &\leq C_2\{|\delta\xi| + |\delta a|\} + \|\delta\phi\| + |\delta v| \\ &\leq C_6\{\|\delta\xi\| + \|\delta a\| + |\delta b| + |\delta\xi| + |\delta a|\}. \end{aligned} \tag{3.26}$$

□

Lemma 3.6. *If ξ belongs to $C^1_{\mathrm{pw}\times}$, then the solution ϕ of* (2.16) *belongs to $C^0_{\mathrm{pw}+}$.*

Proof. By Proposition 3.5, the variation of ϕ with respect to t is bounded by $\sup|\delta a|$ and $\sup|\delta b|$, which are bounded by $\|\delta\xi\|^2$, $|\xi_x(0)|$ and $|\xi_x(1)|$. Therefore, if $\xi_x(x, t)$ is continuous in neighbourhoods of $(0, t_0)$ and $(1, t_0)$, then $\phi(x, t)$ is continuous in a neighbourhoods of $t = t_0$. □

3.2. Estimation for equation (3.2) of ξ. Next, we consider equation (3.2) $\nabla_t\xi_t - \nabla_x\xi_x = c$. Here, c is a tangent vector field on S^{N-1}. We have to extract only vector information of c and ignore point information, or this equation is meaningless. The condition that c is a vector field along ξ is recovered by (2.16).

To do it, we use the stereographic projection $\pi\colon S^{N-1} \to \mathbf{R}^{N-1}$. We introduce a new unknown $u := \pi \circ \xi$, rewrite $\pi_* c$ by c, and consider equation

$$u^i_{tt} - u^i_{xx} + \Gamma_j{}^i{}_k(u)(u^j_t u^k_t - u^j_x u^k_x) = c^i, \tag{3.27}$$

where $\Gamma_j{}^i{}_k$ is Christoffel's symbol of the Riemannian metric on $\mathbf{R}^{N-1}$ induced by π. Note that we treat c as an $\mathbf{R}^{N-1}$-valued function.

We have to choose the center of the stereographic projection so that the distance of the center and the curve ξ is bounded below by a positive constant.

Lemma 3.7. *For any positive number C, there exists a positive number r with following property: If the energy of the curve ξ on S^{N-1} is less than K, then there is a point A on S^{N-1} such that the distance $d(A, \xi)$ of A and ξ is greater than r.*

Proof. Since $\|\xi_x\| \le K$, the length $\|\xi_x\|_{L_1}$ is less than K. Let B_ε be the set of all points $\{A \in S^{N-1}\}$ such that $d(A, \xi) < \varepsilon$. The $(N-1)$-dimensional volume of B_ε is less than $2(K + 2\varepsilon)\varepsilon$. We choose r so that $2(K + 2r)r$ is less than the volume of S^{N-1}. Then the set $S^{N-1} \setminus B_\varepsilon$ is non-empty. □

To solve the wave equation (3.27), we extend x on $\mathbf{R}$ as follows.

(P) [periodic condition] If the boundary condition is $u(1) = u(0)$, we extend u as a periodic function.

(N) [Neumann condition] If $u(0)$ is free, we extend u as an even function: $u(-y) = u(y)$ $(0 \le y \le 1)$.

(D) [Dirichlet condition] If $u(0)$ is fixed we extend u as an odd function: $u(-y) = 2u(0) - u(y)$ $(0 \le y \le 1)$. In this case, we also need revision $\tilde{\Gamma}_j{}^i{}_k$ of $\Gamma_j{}^i{}_k$. Put

$$\Gamma^{*i}_{j\ k}(w) = -\Gamma_j{}^i{}_k(2u(0) - w), \tag{3.28}$$

$\tilde{\Gamma}_j{}^i{}_k = \Gamma_j{}^i{}_k$ $(0 \le x \le 1)$ and $\tilde{\Gamma}_j{}^i{}_k = \Gamma^{*i}_{j\ k}$ $(-1 \le x \le 0)$. (Except this case, we put $\tilde{\Gamma} = \Gamma$.)

We define similar extension at $x = 1$. The initial values $a^i(x) = u^i(x, 0)$, $b^i(x) = u^i_t(x, 0)$ and c^i are extended similarly:

(P) $a(-y) = a(1 - y)$, $b(-y) = b(1 - y)$, $c(-y, t) = c(1 - y, t)$,

(N) $a(-y) = a(y)$, $b(-y) = -b(y)$, $c(-y, t) = c(y, t)$,

(D) $a(-y) = -a(y)$, $b(-y) = b(y)$, $c(-y, t) = -c(y, t)$.

Note that the regularity of the extended functions are lower than the original functions. In particular, $\tilde{\Gamma}$ and c are only piecewise continuous.

We convert the extended differential equation

$$u^i_{tt} - u^i_{xx} + \tilde{\Gamma}_j{}^i{}_k(x, u)(u^j_t u^k_t - u^j_x u^k_x) = c^i \tag{3.29}$$

to an integral equation

$$\begin{aligned} 2u^i(x, t) &= \{a^i(x+t) + a^i(x-t)\} + \int_{x-t}^{x+t} b^i(y)\, dy + W(f^i)(x, t), \\ W(f)(x, t) &:= \int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f^i(y, \tau)\, dy\, d\tau, \\ f^i &= \Phi^i(u) + c^i, \quad \Phi^i(u) := -\tilde{\Gamma}_j{}^i{}_k(x, u)(u^j_t u^k_t - u^j_x u^k_x), \end{aligned} \tag{3.30}$$

and will solve this equation. We denote by $2I(a^i, b^i, f^i)$ the right hand side of the first line.

Lemma 3.8. *Let $u = I(a, b, f)$ be the above integral, where a, b, f are extended functions. Then, we have*

$$\overline{m}(u_x, T) + \overline{m}(u_t, T) \le 2\overline{m}(u_x, 0) + 2\overline{m}(u_t, 0) + 2\overline{m}(f, T)T, \tag{3.31}$$

where $\overline{m}(v, T) := \sup_{0\le x\le 1,\, 0\le t\le T} |v|$.

Proof.

$$\begin{aligned} 2u_x &= \{a'(x+t) + a'(x-t)\} + \{b(x+t) - b(x-t)\} \\ &\quad + \int_0^t f(x+(t-\tau), \tau) - f(x-(t-\tau), \tau)\, d\tau, \\ 2u_t &= \{a'(x+t) - a'(x-t)\} + \{b(x+t) + b(x-t)\} \\ &\quad + \int_0^t f(x+(t-\tau), \tau) + f(x-(t-\tau), \tau)\, d\tau. \end{aligned} \tag{3.32}$$

□

Lemma 3.9. *If $a \in C^1_{\mathrm{pw}}$, $b \in C^0_{\mathrm{pw}}$ and $f \in C^0_{\mathrm{pw}\times+}$, then the integral $u = I(a, b, f)$ belongs to $C^1_{\mathrm{pw}\times}$.*

Proof. Put $u^\pm := u_x \pm u_t$. From (3.32),

$$u^\pm(x, t) = a'(x \pm t) \pm b(x \pm t) \pm \int_0^t f(x \pm (t-\tau), \tau)\, d\tau. \tag{3.33}$$

Therefore, it suffices to show that

$$v^\pm(x, t) := u^\pm(x \mp t, t) = a'(x) \pm b(x) \pm \int_0^t f(x \mp \tau, \tau)\, d\tau \tag{3.34}$$

is continuous. Since f is bounded, $v^{\pm}(x,t)$ is continuous with respect to t. To check the continuity of v^{+} with respect to x, we define

$$\alpha(x,t,\varepsilon) := \int_0^t f(x+\varepsilon-\tau,\tau) - f(x-\tau,\tau)\,d\tau. \tag{3.35}$$

Let K be the maximum value of f. The effect to $\alpha(x,t,\varepsilon)$ of the discontinuity of the function f on the lines $x+t=c_i$, $t=t_i$ and $x\in\mathbf{Z}$ is bounded by $K\varepsilon$. Therefore, $\alpha(x,t,\varepsilon)$ is bounded by ε except on the line $x-t=c_i$. Similar estimation holds for v^{-}. □

The Riemannian metric tensor g_{ij} on $\mathbf{R}^{N-1}=\{(z^i)\}$ induced by the stereographic projection π and its Christoffel's symbol are given as follows.

$$\begin{aligned} &g_{ij}(z) = 4(1+|z|^2)^{-2}\delta_{ij},\\ &\Gamma_j{}^i{}_k(z) = -2(1+|z|^2)^{-1}(\delta^i_k z^j + \delta^i_j z^k - \delta_{jk}z^i), \end{aligned} \tag{3.36}$$

where $|*|$ is the Euclidean norm. Note that they converges to 0 when $|z|\to\infty$, and in particular, they are bounded on $\mathbf{R}^{N-1}$.

We will solve (3.30) by successive approximation. We suppose that that the initial value satisfies $a\in C^1_{\mathrm{pw}}$, $b\in C^0_{\mathrm{pw}}$ and $c\in C^0_{\mathrm{pw}+}$. Then, each extended initial condition belongs to the corresponding space. We put $M(u,T):=\overline{m}(u_x,T)+\overline{m}(u_t,T)$. The norm of $f^i=\Phi^i(u)+c^i$ is bounded by $C_1\{\overline{m}(c,T)+M(u,T)^2\}$, where C_1 is an absolute constant. Therefore, the integral $v^i=I(a^i,b^i,f^i)$ satisfies

$$M(v,T)=\overline{m}(v_t,T)+\overline{m}(v_x,T)\le C_2\{M(u,0)+(\overline{m}(c,T)+M(u,T)^2)T\}, \tag{3.37}$$

where C_2 is an absolute constant and the value $M(u,0)$ depends only on the initial value. Set $K:=C_2\{M(u,0)+\overline{m}(c,T)T\}$, and choose a positive number $T_0\le T$ such that $C_2\{M(u,0)+(\overline{m}(c,T)+K^2)T_0\}\le K$. Let S be the set of all functions $\{u\}$ on $[0,1]\times[0,T_0]$ satisfying the initial condition and the symmetric condition such that $M(u,T_0)\le K$. Since the function $v:=I(a,b,c)$ satisfies $M(v,T)\le C_2\{M(u,0)+\overline{m}(c,T)T\}$, the set S is non-empty.

For any $u\in S$, we put $f=\Phi(u)+c$ and $v^i:=I(a^i,b^i,f^i)$. Then v satisfies

$$M(v,T_0)\le C_2\{M(u,0)+(\overline{m}(c,T)+M(u,T_0)^2)T_0\}\le K. \tag{3.38}$$

Therefore, the correspondence $\Psi\colon u\mapsto v$ is a map from S to S. For $u_1, u_2\in S$, we put $v_1:=\Psi(u_1)$, $v_2:=\Psi(u_2)$, $\delta u=u_2-u_1$, $\delta v=v_2-v_1$. Then,

$$\begin{aligned} &\delta v = I(0,0,\Phi(u_2)-\Phi(u_1)),\\ &|\Phi(u_2)-\Phi(u_1)|\le C_3(|\delta u|+|\delta u_x|+|\delta u_t|), \end{aligned} \tag{3.39}$$

where C_3 depends only on K because the derivative of $\tilde{\Gamma}_j{}^i{}_k(x, z)$ with respect to z is bounded by an absolute constant.

Therefore, using a norm $M_1(v, T) := \overline{m}(v, T) + \overline{m}(v_t, T) + \overline{m}(v_x, T)$, we have

$$\begin{aligned}&\overline{m}(\delta v_t, T) + \overline{m}(\delta v_x, T) \le 2C_3\overline{m}(|\delta u| + |\delta u_x| + |\delta u_t|, T)T \le 2C_3 M_1(\delta u, T)T,\\ &\overline{m}(\delta v, T) \le 2C_3 M_1(\delta u, T)T^2,\\ &M_1(\delta v, T) \le 4C_3 M_1(\delta u, T)T.\end{aligned} \tag{3.40}$$

Hence, for any positive number $T_1 \le T_0$ satisfying $4C_3T_1^2 < 1$, the map Ψ defined on $[0, 1] \times [0, T_0]$ is a contraction.

We proved

Lemma 3.10. *Equation* (3.30) *has a short time solution u for any initial value. Its existence time T and the norm $M_1(u, T)$ depend only on the initial norm $M(u, 0)$ and $\overline{m}(c, T)$.*

For the continuity, we have

Lemma 3.11. *If the initial value satisfies $a \in C^1_{\mathrm{pw}}$, $b \in C^0_{\mathrm{pw}}$ and $c \in C^0_{\mathrm{pw}+}$, then the solution u of* (3.30) *belongs to $C^1_{\mathrm{pw}\times}$.*

Proof. In Proof of Lemma 3.10, the first approximation $u_{(0)} = I(a, b, c)$ belongs to $C^1_{\mathrm{pw}\times}$ by Lemma 3.9. Since $u \in C^1_{\mathrm{pw}\times}$ implies $\Psi(u) \in C^1_{\mathrm{pw}\times}$, the sequence $u_{(n)} := \Psi^n(u_{(0)})$ belongs to $C^1_{\mathrm{pw}\times}$. This sequence converges with respect to the norm M_1, and the limit belongs to $C^1_{\mathrm{pw}\times}$. □

We estimate the dependence of u on c. Let u_i be the solution for c_i. Then,

$$\begin{aligned}&\delta u = I(0, 0, \Phi(u_2) - \Phi(u_1)) + I(0, 0, \delta c),\\ &|\Phi(u_2) - \Phi(u_1)| \le C_1(|\delta u| + |\delta u_t| + |\delta u_x|).\end{aligned} \tag{3.41}$$

Since $\overline{m}(\delta u, T) \le \overline{m}(\delta u_t, T)T$,

$$M_1(\delta u, T) \le C_2\{M_1(\delta u, T) + \overline{m}(\delta c, T)\}T. \tag{3.42}$$

Therefore, if we choose sufficiently small T, $M_1(\delta u, T)$ is bounded by $\overline{m}(\delta c, T)$. Moreover, since $(d/dt)\|\delta u\| \le \|\delta u_t\|$, we have

$$\sup_{0\le t\le T}\{\|\delta u\| + \|\delta u_t\| + \|\delta u_x\|\} \le C_3 T \sup_{0\le t\le T}\|\delta c\|. \tag{3.43}$$

Lemma 3.12. *For sufficiently small number* $T > 0$, *the solution* u *of* (3.30) *satisfies*

$$M_1(\delta u, T) \le C\overline{m}(\delta c, T),$$
$$\sup_{0\le t\le T}\{\|\delta u\| + \|\delta u_t\| + \|\delta u_x\|\} \le C \sup_{0\le t\le T}\|\delta c\|. \tag{3.44}$$

3.3. Existence of solutions for the coupled equation (2.16). Now, we can prove the following

Theorem 3.13. *Equation* (2.16) *has a unique short time* $C^1_{\mathrm{pw}\times}$ *solution* $\xi(x, t)$ *for any initial data* $\xi_0 \in C^1_{\mathrm{pw}}$, $\xi_1 \in C^0_{\mathrm{pw}}$. *Its existence time depends only on the norm* $m(\xi_t) + m(\xi_x)$ *of the initial data.*

REMARK 3.14. We don't assume the Neumann compatibility condition for the initial data. However, the solution satisfies the compatibility condition for almost all t from the symmetry of the solution.

Proof. Put $A_0 := m(\xi_0{}') + m(\xi_1)$. Since the energy $\|\xi_0{}'\|^2$ is bounded by A_0^2, Lemma 3.7 implies that there is a positive number $r_0 > 0$ depending only on A_0 such that $d(P, \xi_0) > r_0$ for some point $P \in S^{N-1}$. We choose P as the center of stereographic projection π. Let D_0 be the set of all points $\{q\}$ such that $d(q, P) \ge r_0/2$, and R_0 the maximum norm $|d\pi|$ of the derivative of the stereographic projection π on D_0. We put $K_0 := 4A_0 \max\{C_2 R_0, 1\}$ and $T_0 := r_0/(2\max\{A_0, 1\})$, where C_2 is the absolute constant given in (3.37).

Let $a_0(x)$ and $a_1(x)$ be the coordinate expression of the initial value, and let S_0 be the set of all S^{N-1}-valued functions on $[0, 1] \times [0, T_0]$ whose initial value is $\{a_0, a_1\}$ such that $m(\xi_t) + m(\xi_x) < K_0$. Since $K_0 > A_0$, S_0 is non-empty. And, the image of any element of S_0 is contained in the domain D_0. The solution ϕ of (2.16) for the data $\xi \in S_0$ is estimated as $|\phi| \le K_1$, because the right hand side of the equation is bounded only by K_0.

We project the solution ϕ into $\mathbf{R}^{N-1}$. Since the dilatation of the projection π is less than R_0, we have $|\pi_*\phi| < K_1 R_0$. We solve the wave equation of u for the data $\pi_*\phi$. By Lemma 3.10, there is a solution such that $M_1(u) < C_2\{A_0 R_0 + K_1 R_0 T_0\}$ for sufficiently small T_0.

We put $\eta := \pi^{-1} \circ u \in S^{N-1}$. Since the maximum dilatation of π^{-1} is 2, we have

$$m(\eta_t) + m(\eta_x) < 2C_2\{A_0 R_0 + K_1 R_0 T_0\} \le \frac{K_0}{2} + 2C_2 K_1 R_0 T_0. \tag{3.45}$$

Therefore, choosing T_0 such that $2C_2 K_1 R_0 T_0 \le K_0/2$, we have $\eta \in S_0$.

Similarly, we can show that the map $\Psi\colon S \to S$ is a contraction. For $\xi_i \in S_0$, we denote by ϕ_i, u_i and η_i corresponding functions as above. Since ϕ_i, u_i and η_i

are already estimated, we have positive constants K_2, K_3, K_4 depending only on K_0 such that

$$M(\eta_1 - \eta_2) \le K_2 M_1(u_1 - u_2) \le K_3 \overline{m}(\pi_*\phi_1 - \pi_*\phi_2)T \le K_4 M_1(\xi_1 - \xi_2)T \tag{3.46}$$

holds on $[0, T]$. Therefore, Ψ is a contraction if $T < 1/K_4$.

The uniqueness of the solution follows from the construction. Moreover, the sequence $(\xi_{(n)}, \phi_{(n)})$ belongs to $C^1_{\mathrm{pw}\times} \times C^0_{\mathrm{pw}+}$ by Lemma 3.11 and Lemma 3.6, and the limit belongs to the same space. □

4. Regularity of solutions

Let $\xi(x, 0) = a(x)$ and $\xi_t(x, 0) = b(x)$ be the initial data. We assume that: a is C^1, b is C^0, and satisfies the compatibility condition (Dirichlet, Neumann, periodic). It is equivalent to assume that the extension of a (resp. b) over the boundary $x = 0, 1$ is C^1 (resp. C^0).

We will show that the solution ξ is C^1. When both boundary condition are not Dirichlet condition, we can prove it by Proof of Theorem 3.13 replacing piecewise continuity to continuity. When one of boundary condition is Dirichlet condition, we have to give another proof, because the odd extensions of $\tilde{\Gamma}$ and ϕ are not continuous.

Lemma 4.1. *The function $\pi_*\phi(x, t)$ is differentiable with respect to x, and the difference δ_t of the value at $(x, t + \varepsilon)$ and (x, t) satisfies*

$$\sup|\delta_t(\pi_*\phi)| \le C\{\sup|\delta_t\xi| + \sup|\delta_t\xi_x| + \|\delta_t\xi_t\|\}. \tag{4.1}$$

Proof. By Proposition 3.5, we know that

$$\sup|\delta_t\phi| \le C_1\{\sup|\delta_t\xi| + \sup|\delta_t\xi_x| + \|\delta_t\xi_t\|\}. \tag{4.2}$$

Since the stereographic projection π and its derivatives are bounded on the image of ξ, we have $|\delta_t(\pi_*\phi)| \le C_2|\delta_t\phi|$. □

Using this lemma, we prove

Proposition 4.2. *If the initial data (ξ_0, ξ_1) belongs to $C^1 \times C^0$, and if it satisfies the compatibility condition (Dirichlet, Neumann, periodic), then the solution ξ is C^1.*

Proof. The function $v^{\pm}(x, t)$ defined in (3.34)

$$v^{\pm}(x, t) = u^{\pm}(x \mp t, t) = a'(x) \pm b(x) \pm \int_0^t f(x \mp \tau, \tau)\, d\tau \tag{4.3}$$

have bounded derivatives with respect to t. We denote by δ_t the difference of the values at $(x, t+\varepsilon)$ and (x, t), and by δ_x the difference of the values at $(x+\varepsilon, t)$ and (x, t). Then, we have

$$\begin{aligned}
|\delta_x v^+| &\le |\delta_x a'| + |\delta_x b| + \left| \int_0^t f(x+\varepsilon-\tau, \tau)\,d\tau - \int_0^t f(x-\tau, \tau)\,d\tau \right| \\
&\le C_1\varepsilon + \left| \int_{-\varepsilon}^{t-\varepsilon} f(x-\tau, \tau+\varepsilon)\,d\tau - \int_0^t f(x-\tau, \tau)\,d\tau \right| \\
&\le C_2\varepsilon + \left| \int_0^t (\delta_t f)(x-\tau, \tau)\,d\tau \right|.
\end{aligned} \tag{4.4}$$

Here, $f = \Phi(u) + c$ and $c = \pi_*\phi$. We will estimate the norm $|(\delta_t f)(x, t)|$. Note that the odd extension in the case of Dirichlet condition has no effect on the estimation. The term $|\delta_t c|$ can be estimated using the above lemma, and the term $|\delta_t \Phi(u)|$ is bounded by $C_3\{|\delta_t u| + |\delta_t u_x| + |\delta_t u_t|\}$. Hence, $\sup|\delta_t f| \le C_4\{\varepsilon + \sup|\delta_t u_x| + \sup|\delta_t u_t|\}$.

We decompose $\delta_t u_x$ as follows.

(4.5)

$$\begin{aligned}
2\delta_t u_x(x, t) &= 2u_x(x, t+\varepsilon) - 2u_x(x, t) \\
&= u^+(x, t+\varepsilon) + u^-(x, t+\varepsilon) - u^+(x, t) - u^-(x, t) \\
&= v^+(x+t+\varepsilon, t+\varepsilon) + v^-(x-t-\varepsilon, t+\varepsilon) - v^+(x+t, t) - v^-(x-t, t) \\
&= \delta_t v^+(x+t+\varepsilon, t) + \delta_x v^+(x+t, t) \\
&\quad + \delta_t v^-(x-t-\varepsilon, t) + \delta_x v^-(x-t-\varepsilon, t).
\end{aligned}$$

Hence we see that $2\sup|\delta_t u_x| \le C_5\varepsilon + \sup|\delta_x v^+| + \sup|\delta_x v^-|$. Combining it and a similar estimation: $2\sup|\delta_t u_t| \le C_6\varepsilon + \sup|\delta_x v^+| + \sup|\delta_x v^-|$, we get

$$\begin{aligned}
&\sup|\delta_t f| \le C_7\{\varepsilon + \sup|\delta_x v^+| + \sup|\delta_x v^-|\}, \\
&\sup|\delta_x v^+|(t) \le C_8\left\{\varepsilon + \int_0^t \sup|\delta_x v^+|(\tau) + \sup|\delta_x v^-|(\tau)\,d\tau\right\}.
\end{aligned} \tag{4.6}$$

We can estimate $\delta_x v^-$ similarly, and get

$$\sup|\delta_x v^+|(t) + \sup|\delta_x v^-|(t) \le C_9\left\{\varepsilon + \int_0^t \sup|\delta_x v^+|(\tau) + \sup|\delta_x v^-|(\tau)\,d\tau\right\}. \tag{4.7}$$

Therefore, $\sup|\delta_x v^+|(t) + \sup|\delta_x v^-|(t)$ increases at most exponentially, and converges to 0 when $\varepsilon \to 0$. It means that $v^\pm$ is uniformly continuous also with respect to x, and u_x and u_t are continuous with respect to x and t. □

We need more differentiability to prove that the solution is a classical solution.

Proposition 4.3. *If the initial data* (ξ_0, ξ_1) *belongs to* $C^2_{\mathrm{pw}} \times C^1_{\mathrm{pw}}$, *and if it satisfies the compatibility condition* (*Dirichlet, Neumann, periodic*), *then the solution* ξ *belongs to* $C^2_{\mathrm{pw}\times}$.

Proof. As in Proof of Proposition 4.2, we have to show that the function

$$v^{\pm}(x,t) = a'(x) \pm b(x) \pm \int_0^t f(x \mp \tau, \tau)\, d\tau \tag{4.8}$$

belongs to $C^1_{\mathrm{pw}\pm}$. The function $v^{\pm}_t(x,t) = \pm f(x \mp t, t)$ is continuous. To show that $v^{\pm}_x$ belongs to $C^0_{\mathrm{pw}\pm}$, we put $g = \Phi(u)$ and decompose $v^{\pm}_x$ as follows.

$$\begin{aligned} v^{\pm}_x(x,t) &= a''(x) \pm b'(x) \pm \frac{\partial}{\partial x} \int_0^t f(x \mp \tau, \tau)\, d\tau \\ &= a''(x) \pm b'(x) \pm \int_0^t c_x(x \mp \tau, \tau)\, d\tau \\ &\qquad \mp \{g(x \mp t, t) - g(x, 0)\} \pm \int_0^t g_t(x \mp \tau, \tau)\, d\tau. \end{aligned} \tag{4.9}$$

The functions a'', b' are piecewise continuous and $g(x,t)$ is continuous on $x \notin \mathbf{Z}$. Also, the function $c_x(x,t)$ is continuous on $x \notin \mathbf{Z}$, because $c = \pi_*(\phi)$ and ϕ is a solution of the integral equation (2.16).

We have to check $\int_0^t g_t(x \mp \tau, \tau)\, d\tau$. Since $v^{\pm}_t$ is bounded,

$$\begin{aligned} &\sup|g_t| \le C_1\{1 + \sup|u_{xt|} + \sup|u_{tt|}\}, \\ &\sup|u_{tt}|, \sup|u_{xt}| \le C_2\{1 + \sup|v^+_x| + \sup|v^-_x|\}. \end{aligned} \tag{4.10}$$

Therefore, we have

$$\sup|v^+_x|(t) + \sup|v^-_x|(t) \le C_3 \int_0^t \{1 + \sup|v^+_x|(\tau) + \sup|v^-_x|(\tau)\}\, d\tau. \tag{4.11}$$

Hence, $\sup|v^+_x|(t) + \sup|v^-_x|(t)$ increases at most exponentially, and is bounded. It implies that $|g_t|$, $|u_{xt}|$ and $|u_{tt}|$ are bounded.

To prove that $v^{\pm}_x$ belongs to $C^0_{\mathrm{pw}\pm}$, we consider the differences $\delta_t v^{\pm}_x$ and $\delta_x v^{\pm}_x$. Since $|c_x|$ and $|g_t|$ are bounded, (4.9) implies that $\delta_t v^{\pm}_x$ converges to 0 when $\varepsilon \to 0$ except discrete lines $x = x_i$. For $\delta_x v^{\pm}_x$, as the above calculation, we can check that

$$|\delta_x v^{\pm}_x(x,t)| \le C_4 \int_0^t \{O(\varepsilon) + |\delta_x v^+_x(x,\tau)| + |\delta_x v^-_x(x,\tau)|\}\, d\tau, \tag{4.12}$$

except ε-neighbourhood of discrete lines $x = x_i$. This gives the desired estimation. □

Proposition 4.4. *We assume* Condition 1.5 (*boundary condition*). *Assume that the initial data* (ξ_0, ξ_1) *belongs to* $C^2 \times C^1$ *and satisfies the compatibility condition* (*Dirichlet, Neumann, periodic*). *When the boundary condition at* $x = x_0$ *is Dirichlet condition, we assume that* $\nabla_x \xi_{0x}$ $(= (\xi_{0xx})^\top) = 0$, $\nabla_x \xi_1$ $(= (\xi_{1x})^\top) = 0$ *at the point. Then, the solution* ξ *belongs to* C^2.

Proof. By the assumption, the extended initial data (a, b) belongs to $C^2 \times C^1$, and the extended Christoffel symbol $\tilde{\Gamma}$ is continuous. Therefore, we can prove that ξ belongs to C^2 by the method of Proof of Proposition 4.3 without exceptional lines $x = x_i$. $\square$

5. Long time existence

As in Section 3, if the quantity $\max|\xi_t| + \max|\xi_x|$ is bounded, then we can extend the solution. We recall equation (2.16).

$$\begin{aligned}
&\phi + L_{P_0}(\phi)^\top - v^\top \\
&= \xi(1)^\top + (1-x)\{\xi_x(0)^{P_0^\perp} + [\xi]_0^{1\,P_0^T}\}^\top + L_{P_0}((|\xi_t|^2 - |\xi_x|^2)\xi)^\top, \\
&\left\{\int_0^1 \phi\, dx\right\}^{P_1^\perp} = \left\{\int_0^1 (|\xi_t|^2 - |\xi_x|^2)\xi\, dx\right\}^{P_1^\perp} - [\xi_x]_0^{1\,P_1^\perp}.
\end{aligned}$$

By Hamilton's principle, the solution should preserve the total energy:

$$\begin{aligned}
T(\gamma) &= (\|\gamma_t\|^2 + \|\gamma_{xt}\|^2) + \|\gamma_{xx}\|^2 \\
&= |\gamma_t(0)|^2 + 2\left(\gamma_t(0) \,\middle|\, \int_0^1 (1-y)\xi_t(y)\, dy\right) \\
&\quad + \int_0^1\int_0^1 \kappa(x, y)(\xi_t(x) \mid \xi_t(y))\, dx\, dy + \|\xi_t\|^2 + \|\xi_x\|^2.
\end{aligned} \tag{5.1}$$

In fact, we constructed the solution so that the derivative

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}T &= \left(\gamma_t(0) \,\middle|\, \gamma_{tt}(0) + \int_0^1 (1-y)\xi_{tt}(y)\, dy\right) + [(\xi_t \mid \xi_x)]_{x=0}^1 \\
&\quad + \left\langle \xi_t \,\middle|\, (1-x)\gamma_{tt}(0) + \int_0^1 \kappa(x, y)\xi_{tt}(y)\, dy + \xi_{tt} - \xi_{xx}\right\rangle
\end{aligned} \tag{5.2}$$

vanishes.

However, its general proof requires smoothness of the solution. Since our solutions are not smooth, we have to check the energy preserving law. In below, we assume that the initial data (ξ_0, ξ_1) belongs to $C^1 \times C^0$ and satisfies the compatibility condition.

By Proposition 4.2, the solution ξ is of class C^1. As in Proof of Lemma 3.8, the integral $u = I(a, b, f)$ and $u^{\pm} := u_x \pm u_t$ satisfy

$$u^{\pm}(x, t) = a'(x \pm t) \pm b(x \pm t) \pm \int_0^t f(x \pm t \mp \tau, \tau)\, d\tau, \tag{5.3}$$

we have

$$u^{\pm}(x \mp t, t) = a'(x) \pm b(x) \pm \int_0^t f(x \mp \tau, \tau)\, d\tau. \tag{5.4}$$

Therefore, the function $t \mapsto u^{\pm}(x \mp t, t)$ is differentiable.

$$\frac{\partial}{\partial t}\{u^{\pm}(x \mp t, t)\} = \pm f(x \mp t, t). \tag{5.5}$$

From this, we can check that $\xi^{\pm}(x, t) = \xi_x \pm \xi_t$ has similar property as follows. Forgetting the original definition of φ, we re-introduce a tangent vector valued function φ by

$$\varphi(x, t) := \phi(x, t) - (|\xi_t|^2 - |\xi_x|^2)\xi. \tag{5.6}$$

Lemma 5.1. *Let ξ, ϕ be the solution. On the domain $\pm t \le x \le 1 \pm t$ (hence $\tilde{\Gamma} = \Gamma$), it holds that*

$$\nabla_t\{\xi^{\pm}(x \mp t, t)\} = \pm\phi(x \mp t, t), \tag{5.7}$$

$$\frac{\partial}{\partial t}\{\xi^{\pm}(x \mp t, t)\} = \pm\varphi(x \mp t, t). \tag{5.8}$$

Proof.

$$\begin{aligned} &\text{The } i\text{-th component of } \nabla_t\{\xi^{\pm}(x \mp t, t)\} \\ &= \frac{\partial}{\partial t}\{\xi^{\pm i}(x \mp t, t)\} + \Gamma_j{}^i{}_k(\xi(x \mp t, t))\frac{\partial}{\partial t}\{\xi^j(x \mp t, t)\}\xi^{\pm k}(x \mp t, t) \\ &= \pm\{f^i - \Gamma_j{}^i{}_k(\xi)(\xi_x^j \mp \xi_t^j)\xi^{\pm k}\}|_{(x \mp t, t)} \\ &= \pm\{f^i - \Gamma_j{}^i{}_k(\xi)(\xi_x^j \mp \xi_t^j)(\xi_x^k \pm \xi_t^k)\}|_{(x \mp t, t)} \\ &= \pm\{f^i - \Gamma_j{}^i{}_k(\xi)(\xi_x^j\xi_x^k - \xi_t^j\xi_t^k)\}|_{(x \mp t, t)} \\ &= \pm\phi^i(x \mp t, t). \end{aligned} \tag{5.9}$$

Hence,

$$\begin{aligned}
&\frac{\partial}{\partial t}\{\xi^{\pm}(x \mp t, t)\} \\
&= \nabla_t\{\xi^{\pm}(x \mp t, t)\} + \left(\frac{\partial}{\partial t}\{\xi^{\pm}(x \mp t, t)\} \,\middle|\, \xi(x \mp t, t)\right)\xi(x \mp t, t) \\
&= \pm\phi(x \mp t, t) - \left(\xi^{\pm}(x \mp t, t) \,\middle|\, \frac{\partial}{\partial t}\{\xi(x \mp t, t)\}\right)\xi(x \mp t, t) \\
&= \{\pm\phi - (\xi_x \pm \xi_t \mid \mp\xi_x + \xi_t)\xi\}|_{(x\mp t,t)} \\
&= \{\pm\phi \mp (|\xi_t|^2 - |\xi_x|^2)\xi\}|_{(x\mp t,t)} \\
&= \pm\varphi(x \mp t, t).
\end{aligned} \tag{5.10}$$

□

Lemma 5.2. *The solution* (ξ, ϕ) *satisfies*

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 p(x)\xi^{\pm}(x, t)\,dx \\
&= \pm\left\{[p(x)\xi^{\pm}(x, t)]_0^1 + \int_0^1 -p'(x)\xi^{\pm}(x, t) + p(x)\varphi(x, t)\,dx\right\},
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 p(x)\xi_x(x, t)\,dx \\
&= [p(x)\xi_t(x, t)]_0^1 - \int_0^1 p'(x)\xi_t(x, t)\,dx,
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 p(x)\xi_t(x, t)\,dx \\
&= [p(x)\xi_x(x, t)]_0^1 + \int_0^1 -p'(x)\xi_x(x, t) + p(x)\varphi(x, t)\,dx.
\end{aligned} \tag{5.13}$$

Proof. It suffices to prove the first equation. The second and third equations are given by the sum and the difference.

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 p(x)\xi^{\pm}(x, t)\,dx = \frac{d}{dt}\int_{\pm t}^{1\pm t} p(x \mp t)\xi^{\pm}(x \mp t, t)\,dx \\
&= \pm p(1)\xi^{\pm}(1, t) \mp p(0)\xi^{\pm}(0, t) + \int_{\pm t}^{1\pm t} \mp p'(x \mp t)\xi^{\pm}(x \mp t, t)\,dx \\
&\qquad + \int_{\pm t}^{1\pm t} \pm p(x \mp t)\varphi(x \mp t, t)\,dx \\
&= \pm[p(x)\xi^{\pm}(x, t)]_0^1 \mp \int_0^1 p'(x)\xi^{\pm}(x, t)\,dx \pm \int_0^1 p(x)\varphi(x, t)\,dx.
\end{aligned} \tag{5.14}$$

□

We calculate each term $T(\gamma)$ in (5.1) using Lemma 5.1. We re-define the curve γ by

$$\begin{aligned}\gamma_{tt}(0,t) &= -\left\{\int_0^1 (1-x)\varphi(x,t)\,dx - \xi_x(0,t) + [\xi(x,t)]_{x=0}^1\right\}^{P_0^T}, \\ \gamma_x(x,t) &= \xi(x,t).\end{aligned} \tag{5.15}$$

Note that we retain the condition for initial data: $\gamma(0,0) \in P_0$, $\gamma_t(0,0) \in P_0^T$ and $\int_0^1 \xi_t(x,0)dx = [\gamma_t(x,0)]_0^1 \in P_1$. Therefore, the definition of γ implies that $\gamma_t(0,t) \in P_0^T$ and $\gamma(0,t) \in P_0$. Moreover, by Lemma 5.2 and equation of ϕ,

$$\frac{d}{dt}\int_0^1 \xi_t\,dx = [\xi_x]_0^1 + \int_0^1 \varphi\,dx \in P_1^T \tag{5.16}$$

holds, and the constrained condition $[\gamma_t(x,t)]_0^1 \in P_1$ is satisfied.

To simplify notation, we introduce a function $\alpha(t)$ by

$$\alpha(t) := \int_0^1 (1-x)\varphi(x,t)\,dx - \xi_x(0,t) + [\xi(x,t)]_{x=0}^1. \tag{5.17}$$

Since $\gamma_{tt}(0,t) = -\alpha(t)^{P_0^T}$, $\gamma_t(0,t) \in P_0^T$, we have

$$\frac{d}{dt}|\gamma_t(0)|^2 = 2(\gamma_t(0) \mid \gamma_{tt}(0)) = -2(\gamma_t(0) \mid \alpha(t)). \tag{5.18}$$

For the second term of $T(\gamma)$, we apply Lemma 5.2.

$$\begin{aligned}\frac{d}{dt}\int_0^1 (1-x)\xi_t(x,t)\,dx &= [(1-x)\xi_x]_0^1 + \int_0^1 \xi_x + (1-x)\varphi\,dx \\ &= -\xi_x(0) + [\xi]_0^1 + \int_0^1 (1-x)\varphi(x)\,dx \\ &= \alpha.\end{aligned} \tag{5.19}$$

Hence,

$$\begin{aligned}&2\frac{d}{dt}\left(\gamma_t(0) \,\middle|\, \int_0^1 (1-y)\xi_t(y)\,dy\right) \\ &= -2\left(\alpha^{P_0^T} \,\middle|\, \int_0^1 (1-y)\xi_t(y)\,dy\right) + 2(\gamma_t(0) \mid \alpha) \\ &= -2\langle (1-x)\alpha^{P_0^T} \mid \xi_t\rangle + 2(\gamma_t(0) \mid \alpha).\end{aligned} \tag{5.20}$$

For the third term of $T(\gamma)$, we have to check only derivatives of ξ_t. Put $p(x,t) = \int_0^1 \kappa(x,y)\xi_t(y)\,dy$.

$$p(0,t) = \int_0^1 (1-y)\xi_t(y,t)\,dy, \quad p(1,t) = 0, \tag{5.21}$$

$$p_x(x,t) = \int_0^1 \kappa_x(x,y)\xi_t(y)\,dy = -\int_0^x \xi_t(y)\,dy. \tag{5.22}$$

Lemma 5.2 implies that

$$\begin{aligned}
&\frac{d}{dt}\int_0^1\int_0^1 \kappa(x,y)(\xi_t(x) \mid \xi_t(y))\,dx\,dy = \frac{d}{dt}\int_0^1 (\xi_t(x) \mid p(x))\,dx \\
&= 2[(p \mid \xi_x)]_0^1 - 2\langle p_x \mid \xi_x\rangle + 2\langle p \mid \varphi\rangle \\
&= -2\left(\int_0^1 (1-y)\xi_t(y,t)\,dy \,\middle|\, \xi_x(0)\right) + 2\left\langle \int_0^x \xi_t(y)\,dy \,\middle|\, \xi_x \right\rangle + 2\langle p \mid \varphi\rangle \\
&= -2\langle \xi_t \mid (1-x)\xi_x(0)\rangle + 2\left[\left(\int_0^x \xi_t(y)\,dy \,\middle|\, \xi\right)\right]_0^1 - 2\langle \xi_t \mid \xi\rangle + 2\langle p \mid \varphi\rangle \\
&= -2\langle \xi_t \mid (1-x)\xi_x(0)\rangle + 2\left(\int_0^1 \xi_t(y)\,dy \,\middle|\, \xi(1)\right) \\
&\quad + 2\left\langle \xi_t(x) \,\middle|\, \int_0^1 \kappa(x,y)\varphi(y)\,dy \right\rangle \\
&= 2\left\langle \xi_t \,\middle|\, -(1-x)\xi_x(0) + \xi(1) + \int_0^1 \kappa(x,y)\varphi(y)\,dy \right\rangle.
\end{aligned} \tag{5.23}$$

Here, we used the fact that ξ_t is orthogonal to ξ.

For the fourth and fifth term, we see that

$$\begin{aligned}
\|\xi_t\|^2 + \|\xi_x\|^2 &= \frac{1}{2}\{\|\xi^+\|^2 + \|\xi^-\|^2\} \\
&= \frac{1}{2}\int_t^{1+t} |\xi^+(x-t,t)|^2\,dx + \frac{1}{2}\int_{-t}^{1-t} |\xi^+(x+t,t)|^2\,dx
\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}
&\frac{d}{dt}\{\|\xi_t\|^2 + \|\xi_x\|^2\} \\
&= \frac{1}{2}[|\xi^+(x,t)|^2]_0^1 + \int_t^{1+t} (\xi^+(x-t,t) \mid \varphi(x-t,t))\,dx \\
&\quad - \frac{1}{2}[|\xi^-(x,t)|^2]_0^1 - \int_{-t}^{1-t} (\xi^-(x+t,t) \mid \varphi(x+t,t))\,dx \\
&= 2[(\xi_x \mid \xi_t)]_0^1 + 2\int_0^1 (\xi_t \mid \varphi)\,dx \\
&= 2\langle \xi_t \mid \varphi\rangle.
\end{aligned} \tag{5.25}$$

The last equality comes from the extension of the solution over $x = 0, 1$. Note that the initial data should satisfy compatibility condition for Neumann type.

Put all together and $\int_0^1 \xi_t \, dx \in P_1^T$, $v \in P_1^\perp$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}T(\gamma) \\
&= -(\gamma_t(0) \mid \alpha) - \langle (1-x)\alpha^{P_0^T} \mid \xi_t \rangle + (\gamma_t(0) \mid \alpha) \\
&\quad + \left\langle \xi_t \,\middle|\, -(1-x)\xi_x(0) + \xi(1) + \int_0^1 \kappa(x, y)\varphi(y)\, dy \right\rangle \\
&\quad + \langle \xi_t \mid \varphi \rangle \\
&= \left\langle \xi_t \,\middle|\, -(1-x)\int_0^1 (1-y)\varphi(y)\, dy^{P_0^T} + (1-x)\xi_x(0)^{P_0^T} - (1-x)[\xi]_0^{1\,P_0^T} \right. \\
&\qquad\qquad \left. - (1-x)\xi_x(0) + \xi(1) + \int_0^1 \kappa(x, y)\varphi(y)\, dy + \varphi \right\rangle \\
&= \langle \xi_t \mid v^\top \rangle = \langle \xi_t \mid v \rangle = \left(\int_0^1 \xi_t \, dx \,\middle|\, v \right) = 0.
\end{aligned} \tag{5.26}$$

Hence, the total energy $T(\gamma)$ is preserved. Therefore, the quantity $h(t) = \sup|\xi^+| + \sup|\xi^-|$ satisfies $\sup|\phi| \le C_1(1 + h)$ by Proposition 3.3. Moreover, by Lemma 5.1, we see

$$\frac{d}{dt}|\xi^\pm(x \mp t, t)|^2 = \pm 2(\xi^\pm(x \mp t, t) \mid \phi(x \mp t, t)), \tag{5.27}$$

and $(d/dt)|\xi^\pm(x \mp t, t)| \le |\phi(x \mp t, t)| \le \sup|\phi|$. It implies that $(d/dt)\sup|\xi^\pm| \le \sup|\phi|$, and $h'(t) = (d/dt)\sup|\xi^+| + (d/dt)\sup|\xi^-| \le 2\sup|\phi| \le 2C_1(1 + h(t))$. Therefore, the function $h(t)$ increases at most exponentially, and we proved the following

Theorem 5.3. *Equation* (2.16) *has a infinite time solution for any initial data* (ξ_0, ξ_1) *in* $C^1 \times C^0$ *with compatibility condition.*

When the initial data does not satisfy the compatibility condition, we have to approximate the solution by smooth solutions. We denote by $L_x^{m,p}$ the space of all functions whose m-th derivatives belong to L_x^p, and put $L_{x,t}^{m,p} = \{u(x, t) \mid u_x, u_t \in L_x^{m-1,p}\}$. We define the norm of the space by

$$\|u\|_{L_{x,t}^{m,p}} = \sup_t(\|u\|_{L_x^{m,p}} + \|u_t\|_{L_x^{m-1,p}}). \tag{5.28}$$

To eliminate $\xi_x(0)^{P_0^\perp}$ and $[\xi_x]_0^{1\,P_1^\perp}$ from equation (2.16), we assume Condition 1.5.

Theorem 5.4. *Assume that the boundary condition is one of* Condition 1.5. *If the initial data* $(\xi_0, \xi_1) \in L^{1,\infty}_x \times L^\infty_x$ *satisfies the boundary conditions on* $\gamma(0)$, $\gamma(1)$, $\xi(0)$, $\xi(1)$ (*but may not satisfies the conditions on* $\xi_x(0)$, $\xi_x(1)$), *then the* $L^{1,\infty}_{x,t}$ *solution of* Theorem 3.13 *exists on infinite time.*

Proof. We approximate the initial data by a Cauchy sequence $(\xi_0^{(n)}, \xi_1^{(n)})$. We may assume that each $(\xi_0^{(n)}, \xi_1^{(n)})$ is smooth and satisfies the compatibility condition. In particular, $\sup|\phi^{(n)}|$ is bounded by the constant total energy and that $\|\xi^{(n)}\|_{L^{1,\infty}_{x,t}}$ increases at most linearly by Lemma 3.8.

We denote by $\delta *$ the difference of the solution $\xi^{(n)}$ and $\xi^{(m)}$. By formula $(\partial/\partial t)\{\xi^\pm(x \mp t, t)\} = \pm\varphi(x \mp t, t)$ in Lemma 5.1, we have

(5.29)

$$\begin{aligned}
\frac{d}{dt}\|\delta\xi^\pm\|^2 &= \frac{d}{dt}\int_{\pm t}^{1\pm t} |\delta\xi^\pm(x \mp t, t)|^2\, dx \\
&= \pm[|\delta\xi^\pm(x \mp t, t)|^2]_{x=\pm t}^{1\pm t} \pm 2\int_{\pm t}^{1\pm t} (\delta\xi^\pm(x \mp t, t) \mid \delta\varphi(x \mp t, t))\, dx \\
&= \pm[|\delta\xi^\pm(x, t)|^2]_0^1 \pm 2\langle\delta\xi^\pm(x, t) \mid \delta\varphi(x, t)\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{d}{dt}\{\|\delta\xi^+\|^2 + \|\delta\xi^-\|^2\} &= [|\delta\xi^+|^2 - |\delta\xi^-|^2]_0^1 \pm 2\langle\delta\xi^+ - \delta\xi^- \mid \delta\varphi\rangle \\
&= 4[(\delta\xi_t \mid \delta\xi_x)]_0^1 \pm 2\langle\delta\xi^+ - \delta\xi^- \mid \delta\varphi\rangle.
\end{aligned} \tag{5.30}$$

Here, the boundary term $[(\delta\xi_t \mid \delta\xi_x)]_0^1$ vanishes. Condition 1.5 implies that $\|\delta\phi\|$ is estimated by $\|\delta\xi^+\| + \|\delta\xi^-\| + \sup|\delta\xi|$, and that $\sup|\delta\xi|$ is bounded by $\|\delta\xi_x\| + \|\delta\xi\|$. Hence, we have

$$\frac{d}{dt}\{\|\delta\xi\|^2 + \|\delta\xi^+\|^2 + \|\delta\xi^-\|^2\} \le C\{\|\delta\xi\|^2 + \|\delta\xi^+\|^2 + \|\delta\xi^-\|^2\}. \tag{5.31}$$

The constant C is independent of the approximate solutions. Therefore, the solutions $\xi^{(n)}$ converges in $L^{1,2}_{x,t}$. Moreover, the limit belongs to $L^{1,\infty}_{x,t}$ because $\xi^{(n)}$ are uniformly bounded in $L^{1,\infty}_{x,t}$.

On the other hand, the sequence $\phi^{(n)}$ converges in $L^2_{x,t}$. It means that the convergence is in $L^\infty_{x,t}$, by the boundary condition. Therefore, by Lemma 3.12, $\xi^{(n)}$ converges in $L^{1,\infty}_{x,t}$, and the limit coincides with the solution of Theorem 3.13. □

6. Uniqueness of periodic solutions

When we consider equation for closed curves γ, we choose the origin $x = 0$ and apply Section 5. Therefore, to say that the solution is unique, we have to prove that the solution is independent of the choice of the origin.

Let $\gamma_0(x)$ be a C^2 closed curve: $S^1 = \mathbf{R}/\mathbf{Z} \to \mathbf{R}^3$ with unit tangent vector field $\xi_0 = \gamma_{0x}$, and $\gamma_1(x)$ be a C^1 vector field along $\gamma_0(x)$ such that $\xi_1 = \gamma_{1x}$ is orthogonal to ξ_0 at each point. For each $x = a$, Theorem 3.13 gives a periodic solution for the initial data $(\xi_0(x), \xi_1(x))$ defined on the interval $[a, a+1]$. We denote the solution by $\xi_{(a)}\colon S^1 \times \mathbf{R}_+ \to S^2$.

Theorem 6.1. *The above solution $\xi_{(a)}$ is independent of the choice of a.*

Proof. We denote by (ξ, ϕ, v) the solution for $a = 0$. Since each solution $\xi_{(a)}$ is unique, it suffices to prove that $(\tilde{\xi}(x), \tilde{\phi}(x)) = (\xi(x+a), \phi(x+a))$ is a solution on $-a \le x \le 1-a$.

We put $\kappa_2(x, y) = \kappa(x, y) - (1-x)(1-y)$ and $L(f) = \int_0^1 \kappa_2(x, y) f(y)\, dy$. The equation in the periodic case becomes

$$
\begin{aligned}
&\phi + L(\phi)^\top - v^\top = L((|\xi_t|^2 - |\xi_x|^2)\xi)^\top, \\
&\int_0^1 \phi\, dx = \int_0^1 (|\xi_t|^2 - |\xi_x|^2)\xi\, dx, \\
&\nabla_t \xi_t - \nabla_x \xi_x = \phi, \\
&\xi(1) = \xi(0), \quad \xi_x(1) = \xi_x(0).
\end{aligned}
\tag{6.1}
$$

Here, we eliminated the term $\xi(1)^\top$ by merging into the unknown function $v^\top$.

Since ξ and ϕ are periodic, these equality for $(\tilde{\xi}, \tilde{\phi})$ holds automatically except the first one. The first one is equivalent to $\varphi + L(\varphi)^\top = v^\top$, where $\varphi = \phi - (|\xi_t|^2 - |\xi_x|^2)\xi$. Hence, if there exists a function $\tilde{v}(t)$ such that $\tilde{\varphi} + L(\tilde{\varphi})^{\tilde{\top}} = v^{\tilde{\top}}$, then $(\tilde{\xi}, \tilde{\phi}, \tilde{v}))$ is a solution.

From $\kappa_2(0, y) = 0$ and $\kappa_{2x}(x, y) = \{-y\ (y \le x), 1-y\ (y \ge x)\}$, we see that $L(f)(0) = 0$ and

$$
\begin{aligned}
L(f)'(x) &= \int_0^1 \kappa_{2x}(x, y) f(y)\, dy \\
&= \int_0^x -y f(y)\, dy + \int_x^1 (1-y) f(y)\, dy \\
&= -\int_0^1 y f(y)\, dy + \int_x^1 f(y)\, dy.
\end{aligned}
\tag{6.2}
$$

Hence,

$$
L(f)'(0) = \int_0^1 (1-y) f(y)\, dy, \quad L(f)''(x) = -f(x). \tag{6.3}
$$

We put $d(x) = L(\tilde{\varphi})(x) - L(\varphi)(x+a)$. The function d satisfies $d(0) = L(\tilde{\varphi})(0) - L(\varphi)(a) = -L(\varphi)(a)$ and

$$\begin{aligned} L(\tilde{\varphi})'(0) &= \int_0^1 (1-y)\tilde{\varphi}(y)\,dy = -\int_0^{1-a} y\varphi(y+a)\,dy - \int_{1-a}^1 y\varphi(y+a-1)\,dy \\ &= -\int_a^1 (z-a)\varphi(z)\,dz - \int_0^a (z+1-a)\varphi(z)\,dz \\ &= -\int_0^1 z\varphi(z)\,dz - \int_0^a \varphi(z)\,dz. \end{aligned} \tag{6.4}$$

On the other hand,

$$L(\varphi)'(a) = -\int_0^1 y\varphi(y)\,dy + \int_a^1 \varphi(y)\,dy = L(\tilde{\varphi})'(0), \tag{6.5}$$

and

$$\begin{aligned} &d'(0) = 0, \\ &d''(x) = L(\tilde{\varphi})''(x) - L(\varphi)''(a+x) = -\tilde{\varphi}(x) + \varphi(a+x) = 0. \end{aligned} \tag{6.6}$$

Therefore, $d(x) = -L(\varphi)(a)$ and

$$\begin{aligned} L(\tilde{\varphi})^{\tilde{\top}}(x) &= L(\tilde{\varphi})(x) - (L(\tilde{\varphi})(x) \mid \tilde{\xi}(x)) \\ &= L(\varphi)(a+x) - L(\varphi)(a) - (L(\varphi)(a+x) - L(\varphi)(a) \mid \xi(a+x)) \\ &= L(\varphi)^{\top}(a+x) - L(\varphi)(a)^{\top}(a+x) = \{v - L(\varphi)(a)\}^{\top}(a+x). \end{aligned} \tag{6.7}$$

It implies that $\tilde{v} = v - L(\varphi)(a)$ satisfies the desired equality $\tilde{\varphi} + L(\tilde{\varphi})^{\tilde{\top}} = v^{\tilde{\top}}$. $\square$

References

[1] S.S. Antman and C.S. Kenney: *Large buckled states of nonlinearly elastic rods under torsion, thrust, and gravity*, Arch. Rational Mech. Anal. **76** (1981), 289–338.

[2] A. Burchard and L.E. Thomas: *On the Cauchy problem for a dynamical Euler's elastica*, Comm. Partial Differential Equations **28** (2003), 271–300.

[3] B.D. Coleman, E.H. Dill, M. Lembo, Z. Lu and I. Tobias: *On the dynamics of rods in the theory of Kirchhoff and Clebsch*, Arch. Rational Mech. Anal. **121** (1992), 339–359.

[4] R. Courant and D. Hilbert: Methods of Mathematical Physics, vol. I, Interscience Publishers, Inc., New York, 1953.

[5] R.E. Caflisch and J.H. Maddocks: *Nonlinear dynamical theory of the elastica*, Proc. Roy. Soc. Edinburgh Sect. A **99** (1984), 1–23.

[6] D.J. Dichmann, J.H. Maddocks and R.L. Pego: *Hamiltonian dynamics of an elastica and the stability of solitary waves*, Arch. Rational Mech. Anal. **135** (1996), 357–396.

[7] N. Koiso: *On the motion of a curve towards elastica*; in Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Soc. Math. France, Paris, 403–436, 1996.
[8] N. Koiso: *Convergence towards an elastica in a Riemannian manifold,* Osaka J. Math. **37** (2000), 467–487.
[9] N. Koiso: *On motion of an elastic wire and singular perturbation of a* 1*-dimensional plate equation*, Osaka J. Math. **37** (2000), 905–924.
[10] Y. Wen: *Curve straightening flow deforms closed plane curves with nonzero rotation number to circles*, J. Differential Equations **120** (1995), 89–107.

Norihito Koiso
Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka, 560–0043
Japan

Mitsuru Sugimoto
Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya, 464–8602
Japan