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ON THE COMBINATORIAL CUSPIDALIZATION OF HYPERBOLIC CURVES

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Abstract

In this paper, we continue our study of the *pro-Σ fundamental groups of configuration spaces* associated to a hyperbolic curve, where Σ is either the set of all prime numbers or a set consisting of a single prime number, begun in an earlier paper. Our main result may be regarded either as a *combinatorial*, *partially bijective generalization of an injectivity theorem* due to *Matsumoto* or as a *generalization to arbitrary hyperbolic curves of injectivity and bijectivity results for genus zero curves* due to *Nakamura* and *Harbater–Schneps*. More precisely, we show that if one restricts one's attention to outer automorphisms of such a pro-Σ fundamental group of the configuration space associated to a(n) affine (respectively, proper) hyperbolic curve which are compatible with certain "*fiber subgroups*" (i.e., groups that arise as kernels of the various natural projections of a configuration space to lower-dimensional configuration spaces) as well as with certain *cuspidal inertia subgroups*, then, as one lowers the *dimension* of the configuration space under consideration from $n+1$ to $n \geq 1$ (respectively, $n \geq 2$), there is a *natural injection* between the resulting groups of such outer automorphisms, which is a *bijection* if $n \geq 4$. The key tool in the proof is a *combinatorial version of the Grothendieck conjecture* proven in an earlier paper by the author, which we apply to construct certain *canonical sections*.

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Introduction

Topological motivation. From a classical topological point of view, one way to understand the *starting point* of the theory of the present paper is via the *Dehn–Nielsen–Baer theorem* (cf., e.g., [13], Theorem 2.9.B) to the effect that if $\mathcal{X}$ is a *topological surface of type* (g, r) (i.e., the complement of r distinct points in a compact oriented topological surface of genus g), then every automorphism α of its (usual topological)

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fundamental group $\pi_1^{\mathrm{top}}(\mathcal{X})$ that stabilizes the conjugacy classes of the inertia groups arising from the r missing points arises from a *homeomorphism* $\alpha_{\mathcal{X}}: \mathcal{X} \xrightarrow{\sim} \mathcal{X}$.

For $n \geq 1$, let us write $\mathcal{X}_n$ for the complement of the diagonals in the direct product of n copies of $\mathcal{X}$. Then one important consequence of the Dehn–Nielsen–Baer theorem, from the point of view of the present paper (cf., e.g., the proof of Corollary 5.1, (ii)), is that α *extends to a compatible automorphism of* $\pi_1^{\mathrm{top}}(\mathcal{X}_n)$. Indeed, this follows immediately from the fact that $\alpha_{\mathcal{X}}$ induces a homeomorphism $\alpha_{\mathcal{X}_n}: \mathcal{X}_n \xrightarrow{\sim} \mathcal{X}_n$. Note, moreover, that such an argument is *not* possible if one only knows that $\alpha_{\mathcal{X}}$ is a *homotopy equivalence*. That is to say, although a homotopy equivalence $\mathcal{X} \xrightarrow{\sim} \mathcal{X}$ is, for instance, if $r = 0$, necessarily surjective, it is *not necessarily injective*. This possible failure of injectivity means that it is not necessarily the case that such a homotopy equivalence $\mathcal{X} \to \mathcal{X}$ induces a homotopy equivalence $\mathcal{X}_n \to \mathcal{X}_n$.

Put another way, one group-theoretic approach to understanding the Dehn–Nielsen–Baer theorem is to think of this theorem as a solution to the *existence portion* of the following problem:

THE DISCRETE COMBINATORIAL CUSPIDALIZATION PROBLEM (DCCP). Does there *exist* a natural functorial way to reconstruct $\pi_1^{\mathrm{top}}(\mathcal{X}_n)$ from $\pi_1^{\mathrm{top}}(\mathcal{X})$? Is such a reconstruction *unique*?

At a more philosophical level, since the key property of interest of $\alpha_{\mathcal{X}}$ is its injectivity—i.e., the fact that it *separates points*—one may think of this problem as the problem of "*reconstructing the* points *of* $\mathcal{X}$, *equipped with their natural topology, group-theoretically from the group* $\pi_1^{\mathrm{top}}(\mathcal{X})$". Formulated in this way, this problem takes on a somewhat *anabelian* flavor. That is to say, one may think of it as a sort of problem in "*discrete combinatorial anabelian geometry*".

Anabelian motivation. The author was also motivated in the development of the theory of the present paper by the following *naive question* that often occurs in *anabelian geometry*. Let X be a *hyperbolic curve* over a perfect field k; $U \subseteq X$ a *nonempty open subscheme* of X. Write "$\pi_1(-)$" for the *étale fundamental group* of a scheme.

NAIVE ANABELIAN CUSPIDALIZATION PROBLEM (NACP). Does there *exist* a natural functorial "group-theoretic" way to reconstruct $\pi_1(U)$ from $\pi_1(X)$? Is such a reconstruction *unique*?

For $n \geq 1$, write X_n for the *n-th configuration space* associated to X (i.e., the open subscheme of the product of n copies of X over k obtained by removing the diagonals—cf. [24], Definition 2.1, (i)). Thus, one has a natural projection morphism $X_{n+1} \to X_n$, obtained by "forgetting the factor labeled $n + 1$". One may think of this morphism

$X_{n+1} \to X_n$ as parametrizing a sort of "*universal family of curves obtained by removing an effective divisor of degree n from X*". Thus, consideration of the above NACP ultimately leads one to consider the following problem.

UNIVERSAL ANABELIAN CUSPIDALIZATION PROBLEM (UACP). Does there *exist* a natural functorial "group-theoretic" way to reconstruct $\pi_1(X_n)$ from $\pi_1(X)$? Is such a reconstruction *unique*?

The UACP was solved for proper X over *finite fields* in [21], when $n = 2$, and in [7], when $n \geq 3$. Moreover, when k is a finite extension of $\mathbb{Q}_p$ (i.e., the field of p-adic numbers for some prime number p), it is shown in [22], Corollary 1.11, (iii), that the solution of the UACP for $n = 3$ when X is proper or for $n = 2$ when X is affine is precisely the obstacle to verifying the "*absolute p-adic version of Grothendieck conjecture*"—i.e., roughly speaking, realizing the functorial reconstruction of X from $\pi_1(X)$. Here, we recall that for such a p-adic k, the absolute Galois group G_k of k admits *automorphisms that do not arise from scheme theory* (cf. [30], the closing remark preceding Theorem 12.2.7). Thus, the expectation inherent in this "absolute p-adic version of Grothendieck conjecture" is that somehow the property of being *coupled* (i.e., within $\pi_1(X)$) with the geometric fundamental group $\pi_1(X \times_k \bar{k})$ (where $\bar{k}$ is an algebraic closure of k) has the property of *rigidifying* G_k. This sort of result is obtained, for instance, in [21], Corollary 2.3, for X "*of Belyi type*". Put another way, if one thinks of the *ring structure* of k—which, by class field theory, may be thought of as a structure on the various abelianizations of the open subgroups of G_k—as a *certain structure on* G_k which is *not necessarily preserved by automorphisms of* G_k (cf. the theory of [15]), then this expectation may be regarded as amounting to the idea that

> *this* "ring structure *on* G_k" *is somehow* encoded *in the* "gap" *that lies between* $\pi_1(X_n)$ *and* $\pi_1(X)$.

This is precisely the idea that lay behind the development of theory of [22], §1.

By comparison to the NACP, the UACP is closer to the DCCP discussed above. In particular, consideration of the UACP in this context ultimately leads one to the following question. Suppose further that Σ is a *set of prime numbers* which is either of cardinality one or equal to the set of all prime numbers, and that k is an *algebraically closed field of characteristic zero*. Write "$\pi_1^\Sigma(-)$" for the maximal pro-Σ quotient of "$\pi_1(-)$". Note that (unlike the case for more general k) in this case, $\pi_1^\Sigma(X_n)$, $\pi_1^\Sigma(X)$ are *independent of the moduli of* X (cf., e.g., [24], Proposition 2.2, (v)). Thus, in this context, it is natural to write $\Pi_n \stackrel{\text{def}}{=} \pi_1^\Sigma(X_n)$.

PROFINITE COMBINATORIAL CUSPIDALIZATION PROBLEM (PCCP). Does there *exist* a natural functorial "group-theoretic" way to reconstruct Π_n from Π_1? Is such a reconstruction *unique*?

Here, it is important to note that although the PCCP is *entirely independent of k* (and hence, in particular, of any *Galois group actions*), an affirmative answer to PCCP implies an affirmative answer to UACP (and hence to NACP). That is to say:

> *Despite the apparently* purely combinatorial *nature of the PCCP, our discussion above of "ring structures on G_k" suggests that there is* quite substantial arithmetic *content in the PCCP.*

This *anabelian* approach to understanding the arithmetic content of the apparently combinatorial PCCP is interesting in light of the point of view of research on the *Grothendieck–Teichmüller group* (cf., e.g., [5])—which is also concerned with issues similar to the PCCP (cf. the OPCCP below) and their relationship to *arithmetic*, but from a somewhat *different* point of view (cf. the discussion of "canonical splittings and cuspidalization" below for more on this topic).

From a more concrete point of view—motivated by the goal of proving "Grothendieck conjecture-style results to the effect that $\pi_1(-)$ is fully faithful" (cf. Remark 4.1.4)—one way to think of the PCCP is as follows.

OUT-VERSION OF THE PCCP (OPCCP). Does there exist a natural subgroup

$$\mathrm{Out}^*(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$$

of the group of outer automorphisms of the profinite group Π_n such that there exists a natural homomorphism $\mathrm{Out}^*(\Pi_n) \to \mathrm{Out}^*(\Pi_{n-1})$ (hence, by composition, a natural homomorphism $\mathrm{Out}^*(\Pi_n) \to \mathrm{Out}^*(\Pi_1)$) which is *bijective*?

From the point of view of the DCCP, one natural approach to defining "Out*" is to consider the condition of "*quasi-speciality*" as is done by many authors (cf. Remarks 4.1.2, 4.2.1), i.e., a condition to the effect that the *conjugacy classes of certain inertia subgroups* are preserved. In the theory of the present paper, we take a slightly different, but related approach. That is to say, we consider the condition of "*FC-admissibility*", which, at first glance, appears *weaker* than the condition of quasi-speciality, but is, in fact, almost *equivalent* to the condition of quasi-speciality (cf. Proposition 1.3, (vii), for more details). The apparently weaker nature of FC-admissibility renders FC-admissibility *easier to verify* and hence *easier to work with* in the development of theory. By adopting this condition of FC-admissibility, we are able to show that a certain natural homomorphism $\mathrm{Out}^*(\Pi_n) \to \mathrm{Out}^*(\Pi_{n-1})$ as in the OPCCP is *bijective* if $n \geq 5$, *injective* if $n \geq 3$ when X is *arbitrary*, and *injective* if $n \geq 2$ when X is *affine* (cf. Theorem A below).

Main result. Our *main result* is the following (cf. Corollary 1.10, Theorem 4.1 for more details). For more on the relation of this result to earlier work ([10], [29], [32]) in the *pro-l* case, we refer to Remark 4.1.2; for more on the relation of this result to earlier work ([14], [26], [5]) in the *profinite* case, we refer to Remarks 4.1.3, 4.2.1.

Theorem A (Partial profinite combinatorial cuspidalization). *Let*

$$U \to S$$

be a hyperbolic curve of type (g, r) (*cf.* §0) *over* $S = \mathrm{Spec}(k)$, *where* k *is an algebraically closed field of characteristic zero. Fix a* set of prime numbers Σ *which is either of cardinality one or equal to the set of all prime numbers. For integers* $n \geq 1$, *write* U_n *for the* n-th configuration space *associated to* U (*i.e., the open subscheme of the product of* n *copies of* U *over* k *obtained by removing the diagonals—cf.* [24], Definition 2.1, (i));

$$\Pi_n \overset{\mathrm{def}}{=} \pi_1^{\Sigma}(U_n)$$

for the maximal pro-Σ quotient of the fundamental group *of* U_n;

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$$

for the subgroup of "FC-admissible" (*cf.* Definition 1.1, (ii), *for a detailed definition*; Proposition 1.3, (vii), *for the relationship to* "quasi-speciality") *outer automorphisms* α*—i.e.,* α *that satisfy certain conditions concerning the* fiber subgroups *of* Π_n (*cf.* [24], Definition 2.3, (iii)) *and the* cuspidal inertia groups *of certain subquotients of these fiber subgroups. If* U *is* affine, *then set* $n_0 \overset{\mathrm{def}}{=} 2$; *if* U *is* proper *over* k, *then set* $n_0 \overset{\mathrm{def}}{=} 3$. *Then*:

(i) *The natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$$

induced by the projection obtained by "forgetting the factor labeled n*" is* injective *if* $n \geq n_0$ *and* bijective *if* $n \geq 5$.

(ii) *By permuting the various factors of* U_n, *one obtains a natural inclusion*

$$\mathfrak{S}_n \hookrightarrow \mathrm{Out}(\Pi_n)$$

of the symmetric group on n *letters into* $\mathrm{Out}(\Pi_n)$ *whose image* commutes *with* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *if* $n \geq n_0$ *and* normalizes $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *if* $r = 0$ *and* $n = 2$.

(iii) *Write* Π^{tripod} *for the maximal pro-*Σ *quotient of the fundamental group of a* tripod (*i.e., the projective line minus three points*) *over* k; $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *for the subgroup of outer automorphisms which determine outer automorphisms of the quotient* $\Pi_n \twoheadrightarrow \Pi_1$ (*obtained by "forgetting the factors of* U_n *with labels* > 1") *that induce the* identity permutation *of the set of conjugacy classes of* cuspidal inertia groups *of* Π_1. *Let* $n \geq n_0$; x *a* cusp *of the geometric generic fiber of the morphism* $U_{n-1} \to U_{n-2}$ (*which we think of as the projection obtained by "forgetting the factor labeled* $n-1$"), *where we take* $U_0 \overset{\mathrm{def}}{=} \mathrm{Spec}(k)$. *Then* x *determines, up to* Π_n*-conjugacy, an* isomorph $\Pi_{E_x} \subseteq \Pi_n$ *of* Π^{tripod}. *Furthermore, this* Π_n*-conjugacy class is* stabilized *by any* $\alpha \in$

$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$; *the* commensurator *and* centralizer *of* Π_{E_x} *in* Π_n *satisfy the relation* $C_{\Pi_n}(\Pi_{E_x}) = Z_{\Pi_n}(\Pi_{E_x}) \times \Pi_{E_x}$. *In particular, one obtains a* natural outer homomorphism

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \to \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tripod}})$$

associated to the cusp x.

Here, we note in passing that, by combining the "*group-theoreticity of the isomorph of the tripod fundamental group*" given in Theorem A, (iii), with the *injectivity* of Theorem A, (i), one obtains an alternative proof of [14], Theorem 2.2—cf. Remark 4.1.3.

In §1, we discuss various *generalities* concerning étale fundamental groups of configuration spaces, including Theorem A, (iii) (cf. Corollary 1.10). Also, we prove a certain special case of the *injectivity* of Theorem A, (i), in the case of a *tripod* (i.e., a projective line minus three points)—cf. Corollary 1.12, (ii). In §2, we generalize this injectivity result to the case of *degenerating affine curves* (cf. Corollary 2.3, (ii)). In §3, we show that similar techniques allow one to obtain a corresponding *surjectivity* result (cf. Corollary 3.3), under certain *conditions*, for affine curves with *two moving cusps*. In §4, we combine the results shown in §1, §2, §3 to prove the remaining portion of Theorem A (cf. Theorem 4.1) and discuss how the theory of the present paper is related to earlier work (cf. Corollary 4.2; Remarks 4.1.2, 4.1.3, 4.2.1). Finally, in §5, we observe that a somewhat stronger analogue of Theorem 4.1 can be shown for the corresponding *discrete* (i.e., usual topological) *fundamental groups* (cf. Corollary 5.1).

Canonical splittings and cuspidalization. We continue to use the notation of the discussion of the PCCP. In some sense, the fundamental issue involved in the PCCP is the issue of *how to bridge the gap between* Π_2 *and* $\Pi_1 \times \Pi_1$. Here, we recall that there is a natural surjection $\Pi_2 \twoheadrightarrow \Pi_1 \times \Pi_1$. If we consider *fibers* over Π_1, then the fundamental issue may be regarded as the issue of bridging the gap between $\Pi_{2/1} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_2 \twoheadrightarrow \Pi_1)$ (where the surjection is the surjection obtained by projection to the *first* factor; thus, the projection to the *second* factor yields a surjection $\Pi_{2/1} \twoheadrightarrow \Pi_1$) and Π_1 (i.e., relative to the surjection $\Pi_{2/1} \twoheadrightarrow \Pi_1$).

If one thinks of $\Pi_{2/1}$ as $\pi_1^{\Sigma}(X \setminus \{\xi\})$ for some closed point $\xi \in X(k)$, then there is *no natural splitting* of the surjection $\Pi_{2/1} \twoheadrightarrow \Pi_1$. On the other hand, suppose that X is an *affine hyperbolic curve*, and one takes "$X \setminus \{\xi\}$" to be the *pointed stable log curve* $Z^{\log}$ (over, say, a log scheme $S^{\log}$ obtained by equipping $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k)$ with the pro-fs log structure determined by the monoid $\mathbb{Q}_{\geq 0}$ of nonnegative rational numbers together with the zero map $\mathbb{Q}_{\geq 0} \to k$—cf. §0) obtained as the "*limit*" $\xi \to x$, where x is a *cusp* of X. Thus, Z consists of *two irreducible components*, E and F, where F may be identified with the *canonical compactification* of X (so $X \subseteq F$ is an open subscheme), E is a copy of the *projective line* joined to F at a single node ν, and the *marked points* of Z consist of the points $\neq \nu$ of $F \setminus X$ and the two marked points $\neq \nu$ of E. Write $U_E \subseteq E$, $(X =)\ U_F \subseteq F$ for the open subschemes obtained as the complement of the nodes and cusps; $Y^{\log}$ for the pointed stable log curve obtained from $Z^{\log}$ by forgetting the marked point of $E \subseteq Z$ determined by the "limit of ξ" (so we obtain a natural map $Z^{\log} \to Y^{\log}$; X may be identified with the complement of the marked points of Y). Thus, by working with *logarithmic fundamental groups* (cf. §0), one may identify the surjection "$\Pi_{2/1} \twoheadrightarrow \Pi_1$" with the surjection $\pi_1^{\Sigma}(Z^{\log}) \twoheadrightarrow \pi_1^{\Sigma}(Y^{\log}) \cong \pi_1^{\Sigma}(X)$. Then the *technical starting point* of the theory of the present paper may be seen in the following observation:

> The natural outer homomorphism
>
> $$\Pi_1 = \pi_1^{\Sigma}(X) \cong \pi_1^{\Sigma}(U_F) \cong \pi_1^{\Sigma}(U_F \times_Z Z^{\log}) \to \pi_1^{\Sigma}(Z^{\log}) = \Pi_{2/1}$$
>
> determines a "*canonical splitting*" of the surjection $\pi_1^{\Sigma}(Z^{\log}) = \Pi_{2/1} \twoheadrightarrow \pi_1^{\Sigma}(Y^{\log}) \cong \pi_1^{\Sigma}(X) = \Pi_1$.

Put another way, from the point of view of "*semi-graphs of anabelioids*" determined by pointed stable curves (cf. the theory of [20]), this canonical splitting is the splitting determined by the "*verticial subgroup*" $(\pi_1^{\Sigma}(U_F) \cong)\ \Pi_F \subseteq \pi_1^{\Sigma}(Z^{\log}) = \Pi_{2/1}$ corresponding to the irreducible component $F \subseteq Z$. From this point of view, one sees immediately that $\Pi_{2/1}$ is generated by Π_F and the verticial subgroup $(\pi_1^{\Sigma}(U_E) \cong)\ \Pi_E \subseteq \Pi_{2/1}$ determined by E. Thus:

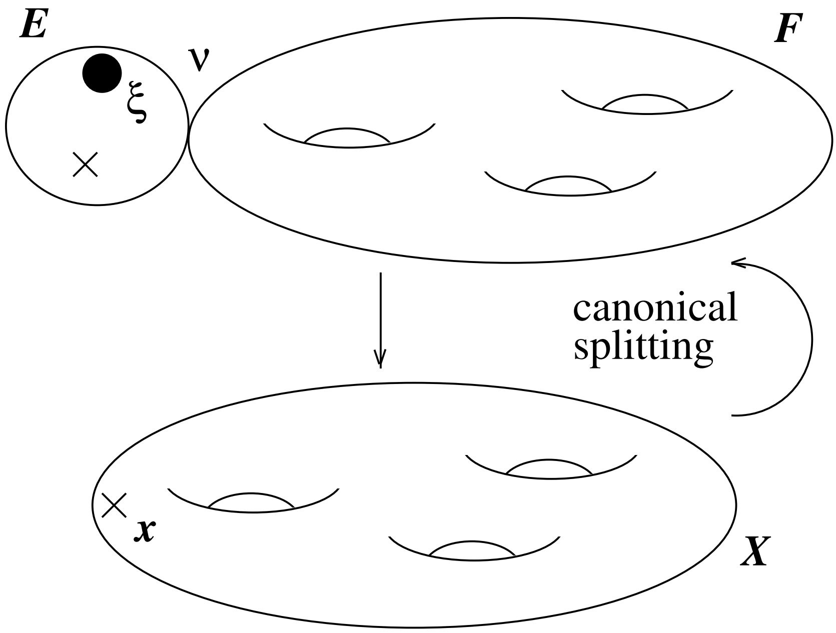
> The study of automorphisms of $\Pi_{2/1}$ that *preserve* Π_E, Π_F, are compatible with the *projection* $\Pi_{2/1} \twoheadrightarrow \Pi_1$ (which induces an isomorphism $\Pi_F \xrightarrow{\sim} \Pi_1$), and induce the *identity* on Π_1 may be reduced to the study of automorphisms of Π_E.

Moreover, by the "*combinatorial version of the Grothendieck conjecture*"—i.e., "combGC"—of [20], it follows that one *sufficient* condition for the preservation of (the conjugacy classes of) Π_E, Π_F is the compatibility of the automorphisms of $\Pi_{2/1}$ under consideration with the *outer action* of the *inertia group* that arises from the degeneration "$\xi \to x$". On the other hand, since this inertia group is none other than the inertia group of the cusp x in Π_1, and the automorphisms of $\Pi_{2/1}$ under consideration arise from automorphisms of Π_2, hence are compatible with the outer action of Π_1 on $\Pi_{2/1}$ determined by the natural exact sequence $1 \to \Pi_{2/1} \to \Pi_2 \to \Pi_1 \to 1$, it thus follows that the automorphisms of $\Pi_{2/1}$ that we are interested in *do indeed preserve* (the conjugacy classes of) Π_E, Π_F, hence are *relatively easy to analyze*. Thus, in a word:

> The theory of the present paper may be regarded as an *interesting application of the* combGC of [20].

This state of affairs is *notable* for a number of reasons—which we shall discuss below—but in particular since at the time of writing, the author is not aware of *any* other applications of "Grothendieck conjecture-type" results.

In light of the central importance of the "canonical splitting determined by the combGC" in the theory of the present paper, it is interesting to *compare* the approach of the present paper with the approaches of other authors. To this end, let us first ob-



serve that since the canonical splitting was *originally constructed via scheme theory*, it stands to reason that if, instead of working with "*arbitrary automorphisms*" as in the OPCCP, one restricts one's attention to *automorphisms that arise from scheme theory*, then one does not need to apply the combGC. This, in effect, is the situation of [14]. That is to say:

> *The* "canonical splitting determined by the combGC" *takes the place of—i.e., may be thought of as a sort of* "combinatorial substitute" *for—the property of* "arising from scheme theory".

Here, it is important to note that it is precisely in situations motivated by problems in *anabelian geometry* that one must contend with "*arbitrary automorphisms that do not necessarily arise from scheme theory*". As was discussed above, it was this sort of situation—i.e., the issue of studying the extent to which the *ring structure* of the base field is somehow *group-theoretically encoded in the* "*gap*" that lies between Π_n and Π_1—that motivated the author to develop the theory of the present paper.

Next, we observe that the "canonical splitting determined by the combGC" is *not necessary* in the theory of [5], precisely because the automorphisms studied in [5] are assumed to satisfy a certain *symmetry condition* (cf. Remark 4.2.1, (iii)). This symmetry condition is sufficiently strong to eliminate the need for reconstructing the canonical splitting via the combGC. Here, it is interesting to note that this symmetry condition that occurs in the theory of the *Grothendieck–Teichmüller group* is motivated by the goal of "*approximating the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ via group theory*". On the other hand, in situations motivated by anabelian geometry—for instance, involving hyperbolic curves of *arbitrary genus*—such symmetry properties are typically *unavailable*. That is to say, although both the point of view of the theory of the *Grothendieck–Teichmüller group*, on the one hand, and the *absolute anabelian* point of view of the present paper, on the other, have the common goal of "*unraveling deep arithmetic properties of arithmetic fields* (*such as* $\mathbb{Q}$, $\mathbb{Q}_p$) *via their absolute Galois groups*", these two points of view may be regarded as going in *opposite directions* in the sense that:

> Whereas the *former* point of view starts with the rational number field $\mathbb{Q}$ "as a *given*" and has as its goal the *explicit construction* and *documentation* of group-theoretic *conditions* (on $\mathrm{Out}(\Pi_1)$, when $(g, r) = (0, 3)$) that *approximate* $G_{\mathbb{Q}}$, the *latter* point of view starts with the ring structure of $\mathbb{Q}_p$ "as an *unknown*" and has as its goal the study of the extent to which the "*ring structure on* $G_{\mathbb{Q}_p}$ *may be recovered from an arbitrary group-theoretic situation which is* not subject to any restricting conditions".

Finally, we conclude by observing that, in fact, the idea of "*applying anabelian results to construct canonical splittings that are of use in solving various cuspidalization problems*"—i.e.,

$$\boxed{\begin{array}{c}\textit{Grothendieck}\\ \textit{conjecture-type result}\end{array}} \rightsquigarrow \boxed{\begin{array}{c}\textit{canonical}\\ \textit{splitting}\end{array}} \rightsquigarrow \boxed{\begin{array}{c}\textit{application to}\\ \textit{cuspidalization}\end{array}}$$

—is *not so surprising*, in light of the following earlier developments (all of which relate to the *first* "$\rightsquigarrow$"; the second and third (i.e., (A2), (A3)) of which relate to the *second* "$\rightsquigarrow$"):

(A1) *Outer actions on center-free groups*: If $1 \to H \to E \to J \to 1$ is an exact sequence of groups, and H is *center-free*, then E may be recovered from the induced outer action of J on H as "$H \overset{\mathrm{out}}{\rtimes} J$"—i.e., as the pull-back via the resulting homomorphism $J \to \mathrm{Out}(H)$ of the natural exact sequence $1 \to H \to \mathrm{Aut}(H) \to \mathrm{Out}(H) \to 1$ (cf. §0). That is to say, the center-freeness of H—which may be thought of as the *most primitive example*, i.e., as a sort of "*degenerate version*", of the property of being "*anabelian*"—gives rise to a sort of "*anabelian semi-simplicity*" in the form of the isomorphism $E \xrightarrow{\sim} H \overset{\mathrm{out}}{\rtimes} J$. This "anabelian semi-simplicity" contrasts sharply with the situation that occurs when H *fails to be center-free*, in which case there are *many possible isomorphism classes* for the extension E. Perhaps the simplest example of this phenomenon—namely, the extensions

$$1 \to p \cdot \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 1$$

and

$$1 \to p \cdot \mathbb{Z} \to (p \cdot \mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z}) \to \mathbb{Z}/p\mathbb{Z} \to 1$$

(where p is a prime number)—suggests strongly that this phenomenon of "anabelian semi-simplicity" has *substantial arithmetic content* (cf., e.g., the discussion of [19], Remark 1.5.1)—i.e., it is as if, by working with *center-free groups* (such as free or pro-Σ free groups), one is afforded with "*canonical splittings of the analogue of the extension* $1 \to p \cdot \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 1$"!

(A2) *Elliptic and Belyi cuspidalizations* (cf. [22], §3): In this theory one constructs cuspidalizations of a hyperbolic curve X by interpreting either a "*multiplication by* n" endomorphism of an elliptic curve or a *Belyi map* to a projective line minus three points as, roughly speaking, an *open immersion* $Y \hookrightarrow X$ of a finite étale covering $Y \to X$ of X. This diagram $X \hookleftarrow Y \to X$ may be thought of as a sort of "*canonical section*"; moreover, this canonical section is constructed *group-theoretically* in loc. cit. precisely by *applying the main* (*anabelian*) *result of* [16].

(A3) *Cuspidalization over finite fields*: Anabelian results such as the main result of [16] have often been referred to as "*versions of the Tate conjecture* (*concerning abelian varieties*) *for hyperbolic curves*". Over finite fields, the "Tate conjecture" is closely related to the "*Riemann hypothesis*" for abelian varieties over finite fields, which is, in turn, closely related to various *semi-simplicity* properties of the Tate module (cf. the theory of [25]). Moreover, such semi-simplicity properties arising from the "Riemann hypothesis" for abelian varieties play a *key role*—i.e., in the form of *canonical splittings via weights*—in the construction of *cuspidalizations over finite fields* in [21], [7].

(A4) *The mono-anabelian theory of* [23]: If one thinks of "canonical splittings" as "*canonical liftings*", then the idea of "*applying anabelian geometry to construct canonical liftings*" permeates the theory of [23] (cf., especially, the discussion of Introduction to [23]).

0. Notations and conventions

Topological groups. If G is a *center-free* topological group, then we have a *natural exact sequence*

$$1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

—where $\mathrm{Aut}(G)$ denotes the group of automorphisms of the topological group G; the injective (since G is center-free!) homomorphism $G \to \mathrm{Aut}(G)$ is obtained by letting G act on G by inner automorphisms; $\mathrm{Out}(G)$ is defined so as to render the sequence exact. If $J \to \mathrm{Out}(G)$ is a homomorphism of groups, then we shall write

$$G \overset{\mathrm{out}}{\rtimes} J \overset{\mathrm{def}}{=} \mathrm{Aut}(G) \times_{\mathrm{Out}(G)} J$$

for the "*outer semi-direct product of* J *with* G". Thus, we have a natural exact sequence: $1 \to G \to G \overset{\mathrm{out}}{\rtimes} J \to J \to 1$.

If $H \subseteq G$ is a closed subgroup of a topological group G, then we shall use the notation $Z_G(H)$, $N_G(H)$, $C_G(H)$ to denote, respectively, the *centralizer*, the *normalizer*, and *commensurator* of H in G (cf., e.g., [20], §0). If $H = N_G(H)$ (respectively, $H =$

$C_G(H)$), then we shall say that H is *normally terminal* (respectively, *commensurably terminal*) in G.

Log schemes. When a *scheme* appears in a diagram of log schemes, the scheme is to be understood as a log scheme equipped with the *trivial log structure*. If $X^{\log}$ is a log scheme, then we shall denote its *interior*—i.e., the largest open subscheme over which the log structure is trivial—by U_X. Fiber products of (pro-)fs log schemes are to be understood as fiber products taken in the category of (pro-)fs log schemes.

The étale fundamental group of a log scheme. Throughout the present paper, we shall often consider the *étale fundamental group* of a connected fs noetherian log scheme (cf. [11]; [6], Appendix B), which we shall denote "$\pi_1(-)$"; we shall denote the *maximal pro-*Σ *quotient* of "$\pi_1(-)$" by "$\pi_1^{\Sigma}(-)$". The theory of the "$\pi_1(-)$" of a connected fs noetherian log scheme extends immediately to *connected pro-fs noetherian log schemes*; thus, we shall apply this routine extension in the present paper without further mention.

Recall that if $X^{\log}$ is a *log regular*, *connected* log scheme *of characteristic zero* (i.e., there exists a morphism $X \to \mathrm{Spec}(\mathbb{Q})$), then the *log purity theorem* of Fujiwara–Kato asserts that there is a natural isomorphism

$$\pi_1(X^{\log}) \xrightarrow{\sim} \pi_1(U_X)$$

(cf., e.g., [11]; [17], Theorem B).

Let $S_\circ^{\log}$ be a *log regular* log scheme such that $S_\circ = \mathrm{Spec}(R_\circ)$, where $R_\circ$ is a *complete noetherian local ring of characteristic zero* with *algebraically closed residue field* $k_\circ$. Write $K_\circ$ for the quotient field of $R_\circ$. Let K be a *maximal* algebraic extension of $K_\circ$ among those algebraic extensions that are *unramified* over $R_\circ$. Write $R \subseteq K$ for the integral closure of $R_\circ$ in K; $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(R)$. Then by considering the integral closure of $R_\circ$ in the various finite extensions of $K_\circ$ in K, one obtains a log structure on S such that the resulting log scheme $S^{\log}$ may be thought of as a *pro-fs log scheme* corresponding to a projective system of *log regular* log schemes in which the transition morphisms are (by the log purity theorem) *finite Kummer log étale*. Write k for the *residue field* of R (so $k \cong k_\circ$); $s_\circ^{\log} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k_\circ) \times_{S_\circ} S_\circ^{\log}$; $s^{\log} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k) \times_S S^{\log}$.

Next, let

$$X_\circ^{\log} \to S_\circ^{\log}$$

be a *proper*, *log smooth* morphism; write

$$X^{\log} \stackrel{\mathrm{def}}{=} X_\circ^{\log} \times_{S_\circ^{\log}} S^{\log} \to S^{\log};$$

$$X_{\circ s}^{\log} \stackrel{\mathrm{def}}{=} X_\circ^{\log} \times_{S_\circ^{\log}} s_\circ^{\log} \to s_\circ^{\log}; \quad X_s^{\log} \stackrel{\mathrm{def}}{=} X_\circ^{\log} \times_{S_\circ^{\log}} s^{\log} \to s^{\log}$$

for the result of base-changing via the morphisms $S^{\log} \to S_\circ^{\log}$, $s_\circ^{\log} \to S_\circ^{\log}$, $s^{\log} \to S_\circ^{\log}$. Then by [33], Théorème 2.2, (a) (in the case where $S_\circ$ is a *trait*); [6], Corollary 1 (for the general case), we have a natural "*specialization isomorphism*" $\pi_1(X_{\circ s}^{\log}) \xrightarrow{\sim} \pi_1(X_\circ^{\log})$. We shall also refer to the composite isomorphism $\pi_1(X_{\circ s}^{\log}) \xrightarrow{\sim} \pi_1(X_\circ^{\log}) \xrightarrow{\sim} \pi_1(U_{X_\circ})$ (where the second isomorphism arises from the *log purity theorem*) as the "*specialization isomorphism*". By applying these specialization isomorphisms to the result of base-changing $X_\circ^{\log} \to S_\circ^{\log}$ to the various log regular log schemes that appear in the projective system (discussed above) associated to the pro-fs log scheme $S^{\log}$, we thus obtain "*specialization isomorphisms*"

$$\pi_1(X_s^{\log}) \xrightarrow{\sim} \pi_1(X^{\log}) \xrightarrow{\sim} \pi_1(U_X)$$

for $X^{\log} \to S^{\log}$. Here, we note that if $\overline{K}$ is any algebraic closure of K, and the restriction of $X_\circ^{\log} \to S_\circ^{\log}$ to $U_{S_\circ}$ is a *log configuration space* associated to some family of hyperbolic curves over $U_{S_\circ}$ (cf. [24], Definition 2.1, (i)), then we have a *natural isomorphism*

$$\pi_1(U_X) \xrightarrow{\sim} \pi_1(U_X \times_K \overline{K})$$

(cf. [24], Proposition 2.2, (iii)). We shall also refer to the composite isomorphism $\pi_1(X_s^{\log}) \xrightarrow{\sim} \pi_1(U_X \times_K \overline{K})$ as the "*specialization isomorphism*".

Curves. We shall use the terms *hyperbolic curve*, *cusp*, *stable log curve*, and *smooth log curve* as they are defined in [20], §0. Thus, the interior of a smooth log curve over a scheme determines a family of hyperbolic curves over the scheme. A smooth log curve or family of hyperbolic curves of type $(0, 3)$ will be referred to as a *tripod*. We shall use the terms *n-th configuration space* and *n-th log configuration space* as they are defined in [24], Definition 2.1, (i). If g, r are positive integers such that $2g - 2 + r > 0$, then we shall write $\overline{\mathcal{M}}_{g,r}^{\log}$ for the *moduli stack* $\overline{\mathcal{M}}_{g,r}$ *of pointed stable curves of type* (g, r) over (the ring of rational integers) $\mathbb{Z}$ equipped with the log structure determined by the *divisor at infinity*. Here, we assume the marking sections of the pointed stable curves to be *ordered*. The *interior* of $\overline{\mathcal{M}}_{g,r}^{\log}$ will be denoted $\mathcal{M}_{g,r}$.

1. Generalities and injectivity for tripods

In the present §1, we begin by discussing various generalities concerning the various *log configuration spaces* associated to a hyperbolic curve. This discussion leads naturally to a proof of a certain special case (cf. Corollary 1.12, (ii)) of our main result (cf. Theorem 4.1 below) for *tripods* (cf. §0).

Let $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k)$, where k is an *algebraically closed field of characteristic zero*, and

$$X^{\log} \to S$$

a *smooth log curve of type* (g, r) (cf. §0). Fix a *set of prime numbers* Σ which is either of cardinality one or equal to the set of all prime numbers.

DEFINITION 1.1. Let $n \geq 1$ be an integer.
(i) Write $X_n^{\log}$ for the *n-th log configuration space* associated to (the family of hyperbolic curves determined by) $X^{\log}$ (cf. §0); $X_0^{\log} \stackrel{\text{def}}{=} S$. We shall think of the factors of $X_n^{\log}$ as *labeled by the indices* $1, \ldots, n$. Write

$$X_n^{\log} \to X_{n-1}^{\log} \to \cdots \to X_m^{\log} \to \cdots \to X_2^{\log} \to X_1^{\log}$$

for the projections obtained by forgetting, successively, the factors labeled by indices $> m$ (as m ranges over the positive integers $\leq n$). Write

$$\Pi_n \stackrel{\text{def}}{=} \pi_1^{\Sigma}(X_n^{\log})$$

for the *maximal pro-*Σ *quotient of the fundamental group* of the log scheme $X_n^{\log}$ (cf. §0; the discussion preceding [24], Definition 2.1, (i)). Thus, we obtain a sequence of surjections

$$\Pi_n \twoheadrightarrow \Pi_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \Pi_m \twoheadrightarrow \cdots \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1$$

—which we shall refer to as *standard*. If we write $K_m \stackrel{\text{def}}{=} \mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_m)$, $\Pi_0 \stackrel{\text{def}}{=} \{1\}$, then we obtain a filtration of subgroups

$$\{1\} = K_n \subseteq K_{n-1} \subseteq \cdots \subseteq K_m \subseteq \cdots \subseteq K_2 \subseteq K_1 \subseteq K_0 = \Pi_n$$

—which we shall refer to as the *standard fiber filtration on* Π_n. Also, for nonnegative integers $a \leq b \leq n$, we shall write

$$\Pi_{b/a} \stackrel{\text{def}}{=} K_a/K_b$$

—so we obtain a natural injection $\Pi_{b/a} \hookrightarrow \Pi_n/K_b \cong \Pi_b$. Thus, if m is a positive integer $\leq n$, then we shall refer to $\Pi_{m/m-1}$ as a *standard-adjacent subquotient* of Π_n. The standard-adjacent subquotient $\Pi_{m/m-1}$ may be naturally identified with the maximal pro-Σ quotient of the étale fundamental group of the geometric generic fiber of the morphism on interiors $U_{X_m} \to U_{X_{m-1}}$. Since this geometric generic fiber is a *hyperbolic curve of type* $(g, r + m - 1)$, it makes sense to speak of the *cuspidal inertia groups*—each of which is (noncanonically!) isomorphic to the maximal pro-Σ quotient $\hat{\mathbb{Z}}^{\Sigma}$ of $\hat{\mathbb{Z}}$—of a standard-adjacent subquotient.
(ii) Let

$$\alpha \colon \Pi_n \xrightarrow{\sim} \Pi_n$$

be an *automorphism* of the topological group Π_n. Let us say that α is *C-admissible* (i.e., "cusp-admissible") if $\alpha(K_a) = K_a$ for every subgroup appearing in the standard fiber filtration, and, moreover, α induces a *bijection* of the collection of *cuspidal inertia groups* contained in each standard-adjacent subquotient of the standard fiber filtration. Let us say that α is *F-admissible* (i.e., "fiber-admissible") if $\alpha(H) = H$ for every *fiber subgroup* $H \subseteq \Pi_n$ (cf. [24], Definition 2.3, (iii), as well as Remark 1.1.2 below). Let us say that α is *FC-admissible* (i.e., "fiber-cusp-admissible") if α is F-admissible and C-admissible. If $\alpha\colon \Pi_n \xrightarrow{\sim} \Pi_n$ is an FC-admissible automorphism, then let us say that α is a *DFC-admissible* (i.e., "diagonal-fiber-cusp-admissible") if α induces the *same* automorphism of Π_1 relative to the various quotients $\Pi_n \twoheadrightarrow \Pi_1$ by *fiber subgroups of co-length* 1 (cf. [24], Definition 2.3, (iii)). If $\alpha\colon \Pi_n \xrightarrow{\sim} \Pi_n$ is a DFC-admissible automorphism, then let us say that α is an *IFC-admissible automorphism* (i.e., "identity-fiber-cusp-admissible") if α induces the *identity* automorphism of Π_1 relative to the various quotients $\Pi_n \twoheadrightarrow \Pi_1$ by *fiber subgroups of co-length* 1. Write $\mathrm{Aut}(\Pi_n)$ for the group of automorphisms of the topological group Π_n;

$$\mathrm{Aut}^{\mathrm{IFC}}(\Pi_n) \subseteq \mathrm{Aut}^{\mathrm{DFC}}(\Pi_n) \subseteq \mathrm{Aut}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Aut}^{\mathrm{F}}(\Pi_n) \subseteq \mathrm{Aut}(\Pi_n) \supseteq \mathrm{Inn}(\Pi_n)$$

for the subgroups of F-admissible, FC-admissible, DFC-admissible, IFC-admissible, and inner automorphisms;

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)/\,\mathrm{Inn}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Aut}^{\mathrm{F}}(\Pi_n)/\,\mathrm{Inn}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$$

for the corresponding outer automorphisms. Thus, we obtain a *natural exact sequence*

$$1 \to \mathrm{Aut}^{\mathrm{IFC}}(\Pi_n) \to \mathrm{Aut}^{\mathrm{DFC}}(\Pi_n) \to \mathrm{Aut}(\Pi_1)$$

induced by the standard surjection $\Pi_n \twoheadrightarrow \Pi_1$ of (i).
(iii) Write

$$i_n \subseteq \Pi_n$$

for the *intersection* of the various *fiber subgroups of co-length* 1. Thus, we obtain a natural inclusion

$$i_n \hookrightarrow \mathrm{Aut}^{\mathrm{IFC}}(\Pi_n)$$

induced by the inclusion $i_n \subseteq \Pi_n \xrightarrow{\sim} \mathrm{Inn}(\Pi_n) \subseteq \mathrm{Aut}(\Pi_n)$ (cf. Remark 1.1.1 below).
(iv) By permuting the various factors of $X_n^{\log}$, one obtains a natural inclusion

$$\mathfrak{S}_n \hookrightarrow \mathrm{Out}(\Pi_n)$$

of the *symmetric group on n letters* into $\mathrm{Out}(\Pi_n)$. We shall refer to the elements of the image of this inclusion as the *permutation outer automorphisms* of Π_n, and to elements of $\mathrm{Aut}(\Pi_n)$ that lift permutation outer automorphisms as *permutation automorphisms* of Π_n. Write

$$\mathrm{Out}^{\mathrm{FCP}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

for the subgroup of outer automorphisms that *commute* with the permutation outer automorphisms.

(v) We shall append the *superscript* "cusp" to the various groups of FC-admissible (outer) automorphisms discussed in (ii), (iv) to denote the subgroup of FC-admissible (outer) automorphisms that determine (via the standard surjection $\Pi_n \twoheadrightarrow \Pi_1$ of (i)) an (outer) automorphism of Π_1 that induces the *identity permutation* of the set of conjugacy classes of cuspidal inertia groups of Π_1.

(vi) When $(g, r) = (0, 3)$, we shall write $\Pi^{\mathrm{tripod}} \stackrel{\mathrm{def}}{=} \Pi_1$, $\Pi_n^{\mathrm{tripod}} \stackrel{\mathrm{def}}{=} \Pi_n$. Suppose that $(g, r) = (0, 3)$, and that the cusps of $X^{\log}$ are *labeled* a, b, c. Here, we regard the symbols $\{a, b, c, 1, 2, \ldots, n\}$ as equipped with the ordering $a < b < c < 1 < 2 < \cdots < n$. Then, as is well-known, there is a *natural isomorphism* over k

$$X_n^{\log} \xrightarrow{\sim} \left(\overline{\mathcal{M}}_{0,n+3}^{\log}\right)_k$$

—where we write $\left(\overline{\mathcal{M}}_{0,n+3}^{\log}\right)_k$ for the moduli scheme over k of *pointed stable curves of type* $(0, n+3)$, equipped with its natural log structure (cf. §0). (Here, we assume the marking sections of the pointed stable curves to be *ordered*.) In particular, there is a natural action of the *symmetric group on* $n+3$ *letters* on $\left(\overline{\mathcal{M}}_{0,n+3}^{\log}\right)_k$, hence also on $X_n^{\log}$. We shall denote this symmetric group—regarded as a group acting on $X_n^{\log}$—by $\mathfrak{S}_{n+3}^{\mathcal{M}}$. In particular, we obtain a *natural homomorphism*

$$\mathfrak{S}_{n+3}^{\mathcal{M}} \to \mathrm{Out}(\Pi_n^{\mathrm{tripod}})$$

the elements of whose image we shall refer to as *outer modular symmetries*. (Thus, the permutation outer automorphisms are the outer modular symmetries that occur as elements of the image of the inclusion $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+3}^{\mathcal{M}}$ obtained by considering permutations of the subset $\{1, \ldots, n\} \subseteq \{a, b, c, 1, \ldots, n\}$.) We shall refer to elements of $\mathrm{Aut}(\Pi_n^{\mathrm{tripod}})$ that lift outer modular symmetries as *modular symmetries* of Π_n^{tripod}. Write

$$\mathrm{Out}^{\mathrm{FCS}}(\Pi_n^{\mathrm{tripod}}) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tripod}})$$

for the subgroup of elements that *commute with the outer modular symmetries*;

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tripod}})^{\mathrm{S}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tripod}})$$

for the inverse image of the subgroup $\mathrm{Out}^{\mathrm{FCS}}(\Pi_1^{\mathrm{tripod}}) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\mathrm{tripod}})$ via the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tripod}}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\mathrm{tripod}})$ induced by the standard surjection $\Pi_n^{\mathrm{tripod}} \twoheadrightarrow \Pi_1^{\mathrm{tripod}}$ of (i). Thus, we have inclusions

$$\mathrm{Out}^{\mathrm{FCS}}(\Pi_n^{\mathrm{tripod}}) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tripod}})^{\mathrm{S}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n^{\mathrm{tripod}})^{\mathrm{cusp}}$$

and an equality $\mathrm{Out}^{\mathrm{FCS}}(\Pi_1^{\mathrm{tripod}}) = \mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\mathrm{tripod}})^{\mathrm{S}}$. Here, the second displayed inclusion follows by considering the induced permutations of the conjugacy classes of the cuspidal inertia groups of Π_1^{tripod}, in light of the fact that $\mathfrak{S}_3$ is *center-free*.

REMARK 1.1.1. We recall in passing that, in the notation of Definition 1.1, Π_n is *slim* (cf. [24], Proposition 2.2, (ii)). In particular, we have a natural isomorphism $\Pi_n \xrightarrow{\sim} \mathrm{Inn}(\Pi_n)$.

REMARK 1.1.2. We recall in passing that, in the notation of Definition 1.1, when $(g, r) \notin \{(0, 3); (1, 1)\}$, it holds that for any $\alpha \in \mathrm{Aut}(\Pi_n)$ and any fiber subgroup $H \subseteq \Pi_n$, $\alpha(H)$ is a fiber subgroup of Π_n (though it is not necessarily the case that $\alpha(H) = H$!). Indeed, this follows from [24], Corollary 6.3.

REMARK 1.1.3. If $\alpha \in \mathrm{Aut}(\Pi_n)$ satisfies the condition that $\alpha(K_a) = K_a$ for $a = 1, \ldots, n$, then often—e.g., in situations where there is a "sufficiently nontrivial" *Galois action* involved—it is possible to verify the *C-admissibility* of α by applying [20], Corollary 2.7, (i), which allows one to conclude "*group-theoretic cuspidality*" from "*l-cyclotomic full-ness*".

REMARK 1.1.4. In the context of Definition 1.1, (vi), we observe that if, for instance, $n = 2$, then one verifies immediately that the outer modular symmetry determined by the permutation $\sigma \stackrel{\mathrm{def}}{=} (a\ b)(c\ 1)$ yields an example of a *C-admissible* element of $\mathrm{Out}(\Pi_2^{\mathrm{tripod}})$ (since conjugation by σ preserves the set of transpositions $\{(a\ 2), (b\ 2), (c\ 2), (1\ 2)\}$) which is *not F-admissible* (since conjugation by σ *switches* the transpositions $(c\ 2)$, $(1\ 2)$—cf. the argument of the final portion of Remark 1.1.5 below). On the other hand, whereas *every* element of $\mathrm{Out}(\Pi_1^{\mathrm{tripod}})$ is *F-admissible*, it is easy to construct (since Π_1^{tripod} is a *free pro-*Σ *group*) examples of elements of $\mathrm{Out}(\Pi_1^{\mathrm{tripod}})$ which are *not C-admissible*. Thus, in general, neither of the two properties of C- and F-admissibility implies the other.

REMARK 1.1.5. Let $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$. Then observe that α necessarily induces the *identity permutation* on the set of *conjugacy classes of cuspidal inertia groups* of *every* standard-adjacent subquotient of Π_n (i.e., not just Π_1). Indeed, by applying the *interpretation* of the various $\Pi_{b/a}$ as "Π_{b-a}'s" for appropriate "$X^{\log}$" (cf. [24], Proposition 2.4, (i)), we reduce immediately to the case $n = 2$. But then the cuspidal inertia group $\subseteq \Pi_{2/1}$ associated to the *unique new cusp* that appears may be characterized by the property that it is contained in Ξ_2 (which, in light of the *F-admissibility* of α, is clearly preserved by α).

Proposition 1.2 (First properties of admissibility). *In the notation of* Definition 1.1, (ii), *let* $\alpha \in \mathrm{Aut}(\Pi_n)$. *Then*:
(i) *Suppose that* $\alpha(\Xi_n) = \Xi_n$. *Then there exists a* permutation automorphism $\sigma \in \mathrm{Aut}(\Pi_n)$ *such that* $\alpha \circ \sigma$ *is* F-admissible. *In particular, if* α *is* C-admissible, *then it follows that* α *is* FC-admissible.
(ii) *Suppose that* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$. *Let* $\rho\colon \Pi_n \twoheadrightarrow \Pi_m$ *be the quotient of* Π_n *by a* fiber subgroup *of co-length* $m \le n$ (*cf.* [24], Definition 2.3, (iii)). *Then* α *induces*, *relative to* ρ, *an element* $\alpha_\rho \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_m)$. *If, moreover,* $\alpha \in \mathrm{Aut}^{\mathrm{DFC}}(\Pi_n)$ (*respectively,* $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_n)$), *then* $\alpha_\rho \in \mathrm{Aut}^{\mathrm{DFC}}(\Pi_m)$ (*respectively,* $\alpha_\rho \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_m)$).
(iii) *Suppose that* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$. *Then there exist* $\beta \in \mathrm{Aut}^{\mathrm{DFC}}(\Pi_n)$, $\iota \in \mathrm{Inn}(\Pi_n)$ *such that* $\alpha = \beta \circ \iota$.

Proof. First, we consider assertion (i). Since $\alpha(i_n) = i_n$, it follows that α induces an automorphism of the quotient $\Pi_n \twoheadrightarrow \Pi_1 \times \cdots \times \Pi_1$ (i.e., onto the direct product of n copies of Π_1) determined by the various fiber subgroups of co-length 1. Moreover, by [24], Corollary 3.4, this automorphism of $\Pi_1 \times \cdots \times \Pi_1$ is necessarily *compatible* with the direct product decomposition of this group, up to some permutation of the factors. Thus, by replacing α by $\alpha \circ \sigma$ for some permutation automorphism σ, we may assume that the induced automorphism of $\Pi_1 \times \cdots \times \Pi_1$ *stabilizes* each of the direct factors. Now let us observe that this stabilization of the direct factors is sufficient to imply that $\alpha(H) = H$ for any fiber subgroup $H \subseteq \Pi_n$. Indeed, without loss of generality, we may assume (by possibly re-ordering the indices) that $H = K_a$ for some K_a as in Definition 1.1, (i). By applying the same argument to α^{-1}, it suffices to verify that $\alpha(K_a) \subseteq K_a$. Thus, let us suppose that $\alpha(K_a) \subseteq K_b$ for some $b < a$, but $\alpha(K_a) \not\subseteq K_{b+1}$. On the other hand, the image of $\alpha(K_a)$ in $\Pi_{b+1/b} = K_b/K_{b+1}$ is *normal*, *closed*, *topologically finitely generated*, and *of infinite index* (since, in light of the stabilization of direct factors observed above, this image maps to $\{1\}$ via the natural projection $K_b/K_{b+1} \twoheadrightarrow \Pi_1$). Thus, by [24], Theorem 1.5—i.e., essentially the theorem of *Lubotzky–Melnikov–van den Dries*—we conclude that this image is *trivial*, a contradiction. This contradiction completes the proof of assertion (i).

Assertion (ii) is immediate from the definitions. Next, we consider assertion (iii). For positive integers $m \le n$, write $\phi_m\colon \Pi_n \twoheadrightarrow \Pi_1$ for the quotient of Π_n by the *fiber subgroup* whose *co-profile* is equal to $\{m\}$ (cf. [24], Definition 2.3, (iii)). Thus, by assertion (ii), we obtain various $\alpha_m \overset{\mathrm{def}}{=} \alpha_{\phi_m} \in \mathrm{Aut}(\Pi_1)$, with images $[\alpha_m] \in \mathrm{Out}(\Pi_1)$. Then let us observe that to complete the proof of assertion (iii), it suffices to verify the following *claim*:

$$[\alpha_m] \in \mathrm{Out}(\Pi_1) \quad \text{is independent of} \quad m.$$

To verify this *claim*, we reason as follows: By applying assertion (ii) to the surjection $\rho\colon \Pi_n \twoheadrightarrow \Pi_2$ for which $\mathrm{Ker}(\rho)$ has co-profile $\{1, m\}$ for $m \neq 1$, we reduce immediately to the case where $n = 2$. Then observe that it follows immediately from the "*uniqueness of a cusp associated to a given cuspidal inertia group*" (cf. [20], Proposition 1.2, (i)) that the *decomposition groups* $\subseteq \Pi_2$ (all of which are Π_2-conjugate to one another) associated to the *diagonal* divisor in X_2 may be reconstructed as the *normalizers* of the various cuspidal inertia groups of $\Pi_{2/1}$ that lie in Ξ_2. In particular, it follows immediately that α induces a *bijection* of the collection of decomposition groups of Π_2 associated to the diagonal divisor in X_2 (all of which are Π_2-conjugate to one another). Thus, the automorphism of $\Pi_1 \times \Pi_1$ induced by α relative to the quotient $(\phi_1, \phi_2)\colon \Pi_2 \twoheadrightarrow \Pi_1 \times \Pi_1$ maps the diagonal $\Pi_1 \subseteq \Pi_1 \times \Pi_1$ (which is the image of a decomposition group associated to the diagonal divisor in X_2) to some $(\Pi_1 \times \Pi_1)$-conjugate of the diagonal $\Pi_1 \subseteq \Pi_1 \times \Pi_1$. But then it follows formally that $[\alpha_1] = [\alpha_2]$. This completes the proof of the *claim*, and hence of assertion (iii). $\square$

Proposition 1.3 (Decomposition and inertia groups). *Let $n \geq 1$. Write $\mathcal{D}_n$ for the set of* irreducible divisors *contained in the complement of the interior $X_n \setminus U_{X_n}$ of $X_n^{\log}$;*

$$\mathbb{I}_\delta \subseteq \mathbb{D}_\delta \subseteq \Pi_n$$

for the inertia *and* decomposition *groups, well-defined (as a pair) up to Π_n-conjugacy, associated to $\delta \in \mathcal{D}_n$; $\psi^{\log}\colon X_n^{\log} \to X_{n-1}^{\log}$ for the projection obtained by "forgetting the factor labeled n"; $\phi^{\log}\colon X_n^{\log} \to X_1^{\log}$ for the projection obtained by "forgetting the factors with labels $\neq n$"; $\rho_\psi\colon \Pi_n \twoheadrightarrow \Pi_{n-1}$, $\rho_\phi\colon \Pi_n \twoheadrightarrow \Pi_1$ for the surjections determined by $\psi^{\log}$, $\phi^{\log}$. Also, we recall the notation "$Z_{(-)}(-)$", "$N_{(-)}(-)$", "$C_{(-)}(-)$" reviewed in §0. Then:*

(i) *$\mathcal{D}_n$ may be* decomposed *as a union of two disjoint subsets*

$$\mathcal{D}_n = \mathcal{D}_n^{\mathrm{hor}} \cup \mathcal{D}_n^{\mathrm{ver}}$$

—where $\mathcal{D}_n^{\mathrm{hor}}$ is the set of divisors which are horizontal *with respect to $\psi^{\log}$ (i.e., the* cusps *of the geometric generic fiber of $\psi^{\log}$); $\mathcal{D}_n^{\mathrm{ver}}$ is the set of divisors $\mathcal{D}_n^{\mathrm{ver}}$ which are* vertical *with respect to $\psi^{\log}$ (so $n \geq 2$, and $\psi_n(\delta) \in \mathcal{D}_{n-1}$ for $\delta \in \mathcal{D}_n^{\mathrm{hor}}$).*

(ii) *Let $n \geq 2$; $\epsilon \in \mathcal{D}_{n-1}$. Then the log structure on $X^{\log}$ determines on the fiber $(X_n)_\epsilon$ of $\psi^{\log}$ over the generic point of ϵ a structure of* pointed stable curve*; $(X_n)_\epsilon$ consists of* precisely two irreducible components *(which may be thought of as elements of $\mathcal{D}_n^{\mathrm{ver}}$) joined by a single* node *ν. One of these two irreducible components, which we shall denote $\delta_F \in \mathcal{D}_n^{\mathrm{ver}}$, maps* isomorphically *to $X_1 = X$ via ϕ; the other, which we shall denote $\delta_E \in \mathcal{D}_n^{\mathrm{ver}}$, maps to a* cusp *of $X_1 = X$ via ϕ.*

(iii) *In the situation of* (ii), *let $\zeta \in \{\delta_F, \delta_E\}$; suppose that the various conjugacy classes have been chosen so that $\rho_\psi(\mathbb{D}_\zeta) = \mathbb{D}_\epsilon$. Write*

$$\Pi_{n,\epsilon} \stackrel{\mathrm{def}}{=} \rho_\psi^{-1}(\mathbb{I}_\epsilon) \subseteq \Pi_n; \quad \mathbb{D}_\zeta^{\mathbb{I}} \stackrel{\mathrm{def}}{=} \mathbb{D}_\zeta \cap \Pi_{n,\epsilon} \subseteq \Pi_{n,\epsilon}; \quad \Pi_\zeta \stackrel{\mathrm{def}}{=} \mathbb{D}_\zeta \cap \Pi_{n/n-1}$$

and $\Pi_\nu \subseteq \Pi_{\delta_F} \cap \Pi_{\delta_E} \subseteq \Pi_{n/n-1}$ *for the* decomposition group *of* ν *in* $\Pi_{n/n-1}$. *Then*:

(a) ρ_ϕ *induces an* isomorphism $\Pi_{\delta_F} \xrightarrow{\sim} \Pi_1$;

(b) ρ_ϕ *maps* Π_{δ_E} *onto a* cuspidal inertia group *of* Π_1;

(c) Π_ζ, Π_ν *are* commensurably terminal *in* $\Pi_{n/n-1}$;

(d) ρ_ψ *induces an* isomorphism $\mathbb{I}_\zeta \xrightarrow{\sim} \mathbb{I}_\epsilon$;

(e) *the inclusions* $\mathbb{I}_\zeta, \Pi_\zeta \subseteq \Pi_{n,\epsilon}$ *induce an* isomorphism $\mathbb{I}_\zeta \times \Pi_\zeta \xrightarrow{\sim} \mathbb{D}^{\mathbb{I}}_\zeta$;

(f) $\mathbb{D}^{\mathbb{I}}_\zeta = C_{\Pi_{n,\epsilon}}(\Pi_\zeta)$;

(g) $\mathbb{I}_\zeta = Z_{\Pi_{n,\epsilon}}(\Pi_\zeta)$.

(iv) *In the situation of* (ii), *let* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$; $\theta \in \{\delta_F, \delta_E, \nu\}$; $\epsilon, \epsilon' \in \mathcal{D}_{n-1}$. (*Thus, we obtain "primed versions"* $\delta'_F, \delta'_E \in \mathcal{D}^{\mathrm{hor}}_n$, ν', θ' *corresponding to* ϵ' *of the data constructed in* (ii), (iii) *for* ϵ.) *Suppose that the automorphism of* Π_{n-1} *induced via* ρ_ψ *by* α *stabilizes* $\mathbb{I}_\epsilon \subseteq \Pi_{n-1}$ (*respectively, maps* $\mathbb{I}_\epsilon \subseteq \Pi_{n-1}$ *to* $\mathbb{I}_{\epsilon'} \subseteq \Pi_{n-1}$). *Then* α *maps the* $\Pi_{n/n-1}$-conjugacy (*respectively,* Π_n-conjugacy) *class of* Π_θ *to itself* (*respectively, to the* Π_n*-conjugacy class of* $\Pi_{\theta'}$). *If* $\theta \in \{\delta_F, \delta_E\}$ (*so* $\theta' \in \{\delta'_F, \delta'_E\}$), *then a similar statement holds with* "Π_θ", "$\Pi_{\theta'}$" *replaced by* "$\mathbb{D}^{\mathbb{I}}_\theta$", "$\mathbb{D}^{\mathbb{I}}_{\theta'}$" *or* "$\mathbb{I}_\theta$", "$\mathbb{I}_{\theta'}$".

(v) *The assignment* $\delta \mapsto \mathbb{I}_\delta$ *determines an* injection *of* $\mathcal{D}_n$ *into the set of* Π_n*-conjugacy classes of subgroups of* Π_n *that are isomorphic to the maximal pro-*Σ *quotient* $\hat{\mathbb{Z}}^\Sigma$ *of* $\hat{\mathbb{Z}}$.

(vi) *Every* $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$ stabilizes the Π_n-conjugacy class *of the inertia group* $\mathbb{I}_\delta$, *for* $\delta \in \mathcal{D}_n$.

(vii) *Write* P_n *for the product* $X \times_k \cdots \times_k X$ *of* n *copies of* X *over* k; $\mathcal{D}^*_n \subseteq \mathcal{D}_n$ *for the subset consisting of the strict transforms in* X_n *of the various irreducible divisors in the complement of the image of the natural open immersion* $U_{X_n} \hookrightarrow P_n$;

$$\mathrm{Out}^{\mathrm{QS}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$$

—where "QS" stands for "quasi-special"*—for the subgroup of outer automorphisms that stabilize the conjugacy class of each inertia group* $\mathbb{I}_\delta$, *for* $\delta \in \mathcal{D}^*_n$. *Then* $\mathrm{Out}^{\mathrm{QS}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$.

Proof. We apply *induction* on n. Thus, in the following, we may assume that Proposition 1.3 has been verified for "smaller n" than the "n under consideration". Assertion (i) is immediate from the definitions. Assertion (ii) follows from the well-known geometry of $X^{\log}_n$, $X^{\log}_{n-1}$, by thinking of $X^{\log}_{n-1}$ as a certain "*moduli space of pointed stable curves*" and $\psi^{\log}$ as the "*tautological pointed stable curve over this moduli space*". Next, we consider assertion (iii). First, we observe that by applying the *specialization isomorphisms* (cf. §0) associated to the restriction of $\psi^{\log}\colon X^{\log}_n \to X^{\log}_{n-1}$ to the completion of X_{n-1} along the generic point of ϵ, we conclude that the pointed stable curve structure on $(X_n)_\epsilon$ (cf. assertion (ii)) determines a "*semi-graph of anabelioids of pro-*Σ *PSC-type*" as discussed in [20], Definition 1.1, (i) (cf. also the discussion of [18], Appendix) whose associated "*PSC-fundamental group*" may be identified with $\Pi_{n/n-1}$. From this point of view, Π_ζ forms a "*verticial subgroup*" (cf. [20], Definition 1.1, (ii)); Π_ν forms a(n) (nodal) "*edge-like subgroup*" (cf. [20], Definition 1.1, (ii)). In particular, Π_ζ is *center-free* (cf., e.g., [20], Remark 1.1.3). Now (a), (b) follow from the description of δ_F, δ_E given in assertion (ii); (c) follows from [20], Proposition 1.2, (ii). To verify (d), observe that by general considerations, the inertia group $\mathbb{I}_\zeta$ is isomorphic to some quotient of $\hat{\mathbb{Z}}^\Sigma$; on the other hand, by the *induction hypothesis*, $\mathbb{I}_\epsilon$ is *isomorphic to* $\hat{\mathbb{Z}}^\Sigma$ (cf. assertion (v) for "$n-1$"); thus, since $(X_n)_\epsilon$ is *reduced* at its two generic points (which correspond to δ_F, δ_E), it follows that the homomorphism $(\hat{\mathbb{Z}}^\Sigma \twoheadrightarrow)\ \mathbb{I}_\zeta \to \mathbb{I}_\epsilon\ (\cong \hat{\mathbb{Z}}^\Sigma)$ is *surjective*, hence an *isomorphism*. Now (e) follows immediately from (d); (f) follows from (c), (d), and (e); since, as observed above, $\mathbb{I}_\epsilon$ is *abelian*, (g) follows from (d), (e), (f), and the fact that Π_ζ is *center-free*. This completes the proof of assertion (iii). Next, we observe that since α induces a *bijection* of the collection of *cuspidal inertia groups* $\subseteq \Pi_{n/n-1}$ (a fact which renders it possible to apply the theory of [20] in the *noncuspidal* case), assertion (iv) for Π_θ, $\Pi_{\theta'}$ follows immediately from [20], Corollary 2.7, (iii); assertion (iv) for "$\mathbb{D}^{\mathbb{I}}_\theta$", "$\mathbb{D}^{\mathbb{I}}_{\theta'}$" or "$\mathbb{I}_\theta$", "$\mathbb{I}_{\theta'}$" follows from assertion (iv) for Π_θ, $\Pi_{\theta'}$ by applying (f), (g) of assertion (iii).

Next, we consider assertions (v), (vi). When $n = 1$, assertions (v), (vi) follow, respectively, from the "*uniqueness of a cusp associated to a given cuspidal inertia group*" (cf. [20], Proposition 1.2, (i)), and the fact that $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$. Thus, we may assume that $n \geq 2$. The fact that α stabilizes the conjugacy classes of the $\mathbb{I}_\delta$ for $\delta \in \mathcal{D}^{\mathrm{hor}}_n$ follows immediately from the fact that α is *C-admissible* (cf. also Remark 1.1.5). Now let $\zeta \in \mathcal{D}^{\mathrm{ver}}_n$, $\epsilon \in \mathcal{D}_{n-1}$ be as in assertion (iii). By the *induction hypothesis*, $\mathbb{I}_\epsilon$ is *isomorphic to* $\hat{\mathbb{Z}}^\Sigma$ and determines a Π_{n-1}-conjugacy class that is *distinct* from the Π_{n-1}-conjugacy classes of the "$\mathbb{I}_{(-)}$" of elements of $\mathcal{D}_{n-1}$ that are $\neq \epsilon$; moreover, the outer automorphism $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})^{\mathrm{cusp}}$ induced by α via ρ_ψ *stabilizes* the conjugacy class of $\mathbb{I}_\epsilon$. In particular, by (d) of assertion (iii), it follows that $\mathbb{I}_\zeta$ is *isomorphic to* $\hat{\mathbb{Z}}^\Sigma$, hence that the "$\mathbb{I}_{(-)}$" of elements of $\mathcal{D}^{\mathrm{hor}}_n$ may be *distinguished* from those of $\mathcal{D}^{\mathrm{ver}}_n$ by the property that they lie in $\Pi_{n/n-1} = \mathrm{Ker}(\rho_\psi)$ and from one another by [20], Proposition 1.2, (i). Thus, to complete the proof of assertions (v), (vi), it suffices to verify assertions (v), (vi) with "$\mathcal{D}_n$" replaced by "the subset $\{\delta_F, \delta_E\} \subseteq \mathcal{D}_n$". But then assertion (vi) follows from the *resp'd* case of assertion (iv); moreover, by the *non-resp'd* case of assertion (iv), if $\mathbb{I}_{\delta_E}$, $\mathbb{I}_{\delta_F}$ are Π_n*-conjugate*, then they are $\Pi_{n/n-1}$*-conjugate*.

Thus, to complete the proof of assertion (v), it suffices to derive a contradiction under the *assumption* that $\mathbb{I}_{\delta_E} = \gamma \cdot \mathbb{I}_{\delta_F} \cdot \gamma^{-1}$, where $\gamma \in \Pi_{n/n-1}$. Note that by (e) of assertion (iii), this assumption implies that $\mathbb{I}_{\delta_E}$ *commutes* with Π_{δ_E}, $\gamma \cdot \Pi_{\delta_F} \cdot \gamma^{-1}$. Next, observe that by projecting to the various maximal pro-l quotients for some $l \in \Sigma$, we may assume without loss of generality that $\Sigma = \{l\}$. Then one verifies immediately that the images of Π_{δ_E}, Π_{δ_F} in the *abelianization* $\Pi^{\mathrm{ab}}_{n/n-1}$ of $\Pi_{n/n-1}$ generate $\Pi^{\mathrm{ab}}_{n/n-1}$, hence (since $\Pi_{n/n-1}$ is a *pro-l group*—cf., e.g., [31], Proposition 7.7.2) that $\Pi_{n/n-1}$ is generated by Π_{δ_E} and *any single* $\Pi_{n/n-1}$*-conjugate* of Π_{δ_F}. Thus, in summary, we conclude that $\mathbb{I}_{\delta_E}$ *commutes* with $\Pi_{n/n-1}$, i.e., that the outer action of $\mathbb{I}_\epsilon$ on $\Pi_{n/n-1}$

is *trivial*. On the other hand, since the nodal curve $(X_n)_\epsilon$ is *not smooth*, we obtain a *contradiction*, for instance, from [20], Proposition 2.6. This completes the proof of assertion (v).

Finally, we consider assertion (vii). The fact that $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{QS}}(\Pi_n)$ follows immediately from assertion (vi). Next, let us observe that by applying "*Zariski–Nagata purity*" (i.e., the classical non-logarithmic version of the "log purity theorem" discussed in §0) to the product of n copies of U_X over k, it follows that the subgroup $\Xi_n \subseteq \Pi_n$ is *topologically normally generated* by the $\mathbb{I}_\delta$, for the $\delta \in \mathcal{D}_n^*$ that arise as strict transforms of the various *diagonals* in P_n. Thus, the fact that $\mathrm{Out}^{\mathrm{QS}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$ follows immediately from the definition of "$\mathrm{Out}^{\mathrm{QS}}(-)$" and Proposition 1.2, (i). This completes the proof of assertion (vii). $\square$

REMARK 1.3.1. The theory of *inertia* and *decomposition groups* such as those discussed in Proposition 1.3 is developed in greater detail in [22], §1.

For $i = 1, 2$, write

$$\mathrm{pr}_i^{\log} : X_2^{\log} \to X_1^{\log}$$

for the projection to the factor labeled i, $\mathrm{pr}_i : X_2 \to X_1$ for the underlying morphism of schemes, and $p_i : \Pi_2 \to \Pi_1$ for the surjection induced by $\mathrm{pr}_i^{\log}$.

DEFINITION 1.4. Let $x \in X(k)$ be a *cusp* of $X^{\log}$.

(i) Observe that the log structure on $X_2^{\log}$ determines on the fiber $(X_2)_x$ of the morphism $\mathrm{pr}_1 : X_2 \to X_1$ over x a structure of *pointed stable curve*, which consists of *two* irreducible components, one of which—which we shall denote F_x—*maps isomorphically* to X via $\mathrm{pr}_2 : X_2 \to X_1 = X$, the other of which—which we shall denote E_x—*maps to the point* $x \in X(k)$ via pr_2; F_x, E_x are joined at a single *node* ν_x (cf. Proposition 1.3, (ii)). Let us refer to F_x as the *major cuspidal component at* x, to E_x as the *minor cuspidal component at* x, and to ν_x as the *nexus at* x. Thus, the complement in F_x (respectively, E_x) of the nodes and cusps (relative to the pointed stable curve structure on $(X_2)_x$) of F_x (respectively, E_x)—which we shall refer to as the *interior* U_{F_x} of F_x (respectively, U_{E_x} of E_x)—determines a *hyperbolic curve* U_{F_x} (respectively, *tripod* U_{E_x}). Moreover, pr_2 induces (compatible) *isomorphisms* $U_{F_x} \xrightarrow{\sim} U_X$, $F_x \xrightarrow{\sim} X$.

(ii) As discussed in Proposition 1.3, (iii), and its proof, the major and minor cuspidal components at x, together with the nexus at x, determine (conjugacy classes of) *verticial* and *edge-like subgroups* (cf. [20], Definition 1.1, (ii))

$$\Pi_{F_x}, \Pi_{E_x}, \Pi_{\nu_x} \subseteq \Pi_{2/1}$$

—which we shall refer to, respectively, as *major verticial*, *minor verticial*, and *nexus* *subgroups*. Thus, (cf. Proposition 1.3, (iii), (a), (b)) the morphism $p_2\colon \Pi_2 \to \Pi_1$ determines an *isomorphism*

$$\Pi_{F_x} \xrightarrow{\sim} \Pi_1$$

—i.e., the major verticial subgroups may be thought of as defining *sections of the projection* $p_2\colon \Pi_2 \twoheadrightarrow \Pi_1$; p_2 maps Π_{E_x} onto a *cuspidal inertia group* of Π_1 associated to x. For *suitable choices* within the various conjugacy classes involved, we have *natural inclusions*

$$\Pi_{E_x} \supseteq \Pi_{\nu_x} \subseteq \Pi_{F_x}$$

(inside $\Pi_{2/1}$).

Proposition 1.5 (First properties of major and minor verticial subgroups). *In the notation of* Definition 1.4:
(i) Π_{ν_x}, Π_{F_x}, *and* Π_{E_x} *are* commensurably terminal *in* $\Pi_{2/1}$.
(ii) *Suppose that one* fixes $\Pi_{\nu_x} \subseteq \Pi_{2/1}$ *among its various* $\Pi_{2/1}$*-conjugates. Then the* condition *that there exist inclusions*

$$\Pi_{\nu_x} \subseteq \Pi_{E_x}; \quad \Pi_{\nu_x} \subseteq \Pi_{F_x}$$

completely determines Π_{E_x} *and* Π_{F_x} *among their various* $\Pi_{2/1}$*-conjugates.*
(iii) *In the notation of* (ii), *the compatible inclusions* $\Pi_{\nu_x} \subseteq \Pi_{E_x} \subseteq \Pi_{2/1}$, $\Pi_{\nu_x} \subseteq \Pi_{F_x} \subseteq \Pi_{2/1}$ *determine an* isomorphism

$$\varinjlim(\Pi_{E_x} \hookleftarrow \Pi_{\nu_x} \hookrightarrow \Pi_{F_x}) \xrightarrow{\sim} \Pi_{2/1}$$

—*where the* inductive limit *is taken in the category of* pro-Σ groups.

Proof. Assertion (i) follows from [20], Proposition 1.2, (ii) (cf. Proposition 1.3, (iii), (c)). Assertion (ii) follows from the fact that "every nodal edge-like subgroup is contained in *precisely two verticial subgroups*" (cf. [20], Proposition 1.5, (i)). Assertion (iii) may be thought of as a consequence of the "*van Kampen theorem*" in elementary algebraic topology. At a more *combinatorial* level, one may reason as follows: It follows immediately from the *simple structure of the dual graph of the pointed stable curves* considered in Definition 1.4 that there is a natural *equivalence of categories* (arising from the parenthesized inductive system in the statement of assertion (iii)) between
(a) the category of finite sets E with continuous $\Pi_{2/1}$-action (and $\Pi_{2/1}$-equivariant morphisms) and
(b) the category of finite sets equipped with continuous actions of Π_{F_x}, Π_{E_x} which restrict to the *same* action on $\Pi_{\nu_x} \subseteq \Pi_{F_x}$, $\Pi_{\nu_x} \subseteq \Pi_{E_x}$ (and Π_{F_x}-, Π_{E_x}-equivariant morphisms).

The isomorphism between $\Pi_{2/1}$ and the inductive limit of the parenthesized inductive system of assertion (iii) now follows *formally* from this equivalence of categories. $\square$

REMARK 1.5.1. The technique of "*van Kampen-style gluing*" of fundamental groups that appears in Proposition 1.5, (iii), will play an important role in the present paper. Similar methods involving isomorphs of the fundamental group of a tripod (cf. Corollary 1.10, (iii), below; Theorem A, (iii), of the Introduction) may be seen in the arguments of [27], [28].

Proposition 1.6 (Inertia groups and symmetry). *In the notation of the discussion preceding* Definition 1.4, *write*

$$\Pi_{1\backslash 2} \stackrel{\text{def}}{=} \mathrm{Ker}(p_2\colon \Pi_2 \twoheadrightarrow \Pi_1)$$

(*cf.* $\Pi_{2/1} = \mathrm{Ker}(p_1\colon \Pi_2 \twoheadrightarrow \Pi_1)$). *Thus, each cusp of the family of hyperbolic curves* $\mathrm{pr}_2|_{U_{X_2}}\colon U_{X_2} \to U_{X_1}$ *gives rise to a well-defined, up to* $\Pi_{1\backslash 2}$*-conjugacy,* cuspidal inertia group $\subseteq \Pi_{1\backslash 2}$. *Then*:

(i) *Write* δ *for* diagonal *divisor in* X_2. *Let* $\mathbb{I}_\delta \subseteq \mathbb{D}_\delta$ *be a pair of* inertia *and* decomposition groups *associated to* δ. *Then*:

(a) *the cuspidal inertia groups* $\subseteq \Pi_{1\backslash 2}$ *corresponding to the cusp determined by* δ *are contained in* $\Xi_2 = \Pi_{1\backslash 2} \cap \Pi_{2/1}$ *and coincide with the cuspidal inertia groups* $\subseteq \Pi_{2/1}$ *corresponding to the cusp determined by* δ, *as well as with the* Π_2*-conjugates of* $\mathbb{I}_\delta$;

(b) *either* p_1 *or* p_2 *determines* (*the final nontrivial arrow in*) *an exact sequence* $1 \to \mathbb{I}_\delta \to \mathbb{D}_\delta \to \Pi_1 \to 1$;

(c) *we have* $\mathbb{D}_\delta = C_{\Pi_2}(\mathbb{I}_\delta)$.

(ii) *Let* $x \in X_1(k) = X(k)$ *be a* cusp *of* $X^{\log}$. *Let us think of* x, F_x *as elements of* $\mathcal{D}_1$, $\mathcal{D}_2^{\mathrm{ver}}$, *respectively* (*cf.* Proposition 1.3, (i)). *Then*:

(a) *the major cuspidal component* F_x *at* x *is equal to the closure in* X_2 *of the divisor of* U_{X_2} *determined by* $\mathrm{pr}_1^{-1}(x)$;

(b) $\mathbb{I}_x = \mathbb{D}_x$;

(c) $\mathbb{I}_{F_x}$ *is a* cuspidal inertia group $\subseteq \Pi_{1\backslash 2}$ *associated to the cusp* U_{F_x} *of the family of hyperbolic curves* $\mathrm{pr}_2|_{U_{X_2}}\colon U_{X_2} \to U_{X_1}$;

(d) $\mathbb{D}_{F_x} = \mathbb{D}^{\mathbb{I}}_{F_x}$;

(e) $\mathbb{D}_{F_x} \cap \Pi_{1\backslash 2} = \mathbb{I}_{F_x}$;

(f) $\mathbb{D}_{F_x} = C_{\Pi_2}(\mathbb{D}_{F_x})$.

(iii) *Let* σ *be a* non-inner permutation *automorphism of* Π_2, $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$. *Then* $\alpha_\sigma \stackrel{\text{def}}{=} \sigma \circ \alpha \circ \sigma^{-1} \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$.

Proof. The content of (a), (b) of assertion (i) follows immediately from the definitions involved; (c) follows immediately from (b), together with the fact that $\mathbb{I}_\delta$ is *commensurably terminal* in either $\Pi_{2/1}$ or $\Pi_{1\backslash 2}$ (cf. [20], Proposition 1.2, (i)). Next, we consider assertion (ii). First, let us observe that (a), (b) are immediate from the definitions; (c) follows immediately from the definitions and (a); (d) follows immediately from (b) (cf. Proposition 1.3, (iii)). To verify (e), let us first observe that it follows immediately from the geometry of the morphism $\mathrm{pr}_2^{\log}\colon X_2^{\log} \to X_1^{\log}$ that $p_2(\mathbb{I}_{F_x}) = \{1\}$; thus, (e) follows (in light of (d)) from Proposition 1.3, (iii), (a), (e). Finally, since $\mathbb{I}_x$ is *commensurably terminal* in Π_1 (cf. [20], Proposition 1.2, (ii)), (f) follows immediately from (d) and Proposition 1.3, (iii), (d), (e), (f). This completes the proof of assertion (ii). Finally, we consider assertion (iii). It is immediate from the definitions that $\alpha_\sigma \in \mathrm{Aut}(\Pi_2)$ is *F-admissible*. Moreover, it follows immediately from Proposition 1.2, (iii), together with the *C-admissibility* of α, that α_σ induces a bijection of the collection of cuspidal inertial groups of the quotient $p_1\colon \Pi_2 \twoheadrightarrow \Pi_1$. Thus, it suffices to verify that α_σ induces a bijection of the collection of cuspidal inertial groups of $\Pi_{2/1}$, i.e., that α induces a bijection of the collection of cuspidal inertial groups of $\Pi_{1\backslash 2}$. But in light of assertions (i) and (ii), (c), this follows immediately from the FC-admissibility of α and Proposition 1.3, (vi). This completes the proof of assertion (iii). $\square$

Proposition 1.7 (Inertia and decomposition groups of minor cuspidal components). *In the notation of* Proposition 1.6, *suppose further that* $x \in X_1(k) = X(k)$ *is a* cusp *of* $X^{\log}$. *Let us think of* x, E_x *as elements of* $\mathcal{D}_1$, $\mathcal{D}_2^{\mathrm{ver}}$, *respectively* (*cf.* Proposition 1.3, (i)). *Then*:

(a) $\mathbb{D}_{E_x} = \mathbb{D}^{\mathbb{I}}_{E_x}$;

(b) $\mathbb{I}_{E_x} \cap \Pi_{1\backslash 2} = \{1\}$;

(c) $\mathbb{D}_{E_x} = C_{\Pi_2}(\mathbb{D}_{E_x})$;

(d) *for any open subgroup* $J \subseteq \Pi_{E_x}$, $Z_{\Pi_2}(J) = \mathbb{I}_{E_x}$;

(e) $\mathbb{D}_{E_x} = C_{\Pi_2}(\Pi_{E_x})$.

Proof. First, we observe that the equality of (a) (respectively, (c)) follows by a similar argument to the argument applied to prove Proposition 1.6, (ii), (d) (respectively, 1.6, (ii), (f)); (b) follows immediately from the geometric fact that the inverse image via $\mathrm{pr}_2\colon X_2 \to X_1$ of the closed point x contains the divisor E_x with *multiplicity one*. Next, let us consider (d). First, let us observe that, in the notation of Proposition 1.6, (i), the diagonal divisor δ *intersects* E_x *transversely*; in particular, (for appropriate choices of conjugates) we have $\mathbb{I}_\delta \subseteq \Pi_{E_x}$. Thus, $Z_{\Pi_2}(J) \subseteq Z_{\Pi_2}(J \cap \mathbb{I}_\delta) \subseteq C_{\Pi_2}(\mathbb{I}_\delta) = \mathbb{D}_\delta$ (cf. Proposition 1.6, (i), (c)). On the other hand, note that $p_2(\Pi_{E_x})$ is a *cuspidal inertia group*—i.e., "$\mathbb{I}_x$"—of Π_1 associated to x (cf. Proposition 1.3, (iii), (b)), hence *commensurably terminal* in Π_1 (cf. [20], Proposition 1.2, (ii)). Thus, the inclusion $Z_{\Pi_2}(J) \subseteq \mathbb{D}_\delta$ implies (for appropriate choices of conjugates) that $p_1(Z_{\Pi_2}(J)) = p_2(Z_{\Pi_2}(J)) \subseteq \mathbb{I}_x$, so the desired equality $Z_{\Pi_2}(J) = \mathbb{I}_{E_x}$ follows immediately from Proposition 1.3, (iii), (e), (f), together with the fact that Π_{E_x} is *slim* (cf. Remark 1.1.1). This completes the proof of (d). Now it follows immediately from (d) that $C_{\Pi_2}(\Pi_{E_x}) \subseteq N_{\Pi_2}(\mathbb{I}_{E_x})$. Thus, in light of (a), we conclude from Proposition 1.3, (iii), (e), that $C_{\Pi_2}(\Pi_{E_x}) \subseteq C_{\Pi_2}(\mathbb{D}_{E_x})$, so (e) follows immediately from (c). $\square$

For $i, j \in \{1, 2, 3\}$ such that $i < j$, write

$$\underline{\mathrm{pr}}_{ij}^{\log}\colon X_3^{\log} \to X_2^{\log}$$

for the projection to the factors labeled i and j of $X_3^{\log}$—which we think of as corresponding, respectively, to the factors labeled 1 and 2 of $X_2^{\log}$; $\underline{\mathrm{pr}}_{ij}\colon X_3 \to X_2$ for the underlying morphism of schemes; and $\underline{p}_{ij}\colon \Pi_3 \to \Pi_2$ for the surjection induced by $\underline{\mathrm{pr}}_{ij}^{\log}$. Also, for $i \in \{1, 2, 3\}$, write

$$\underline{\mathrm{pr}}_{i}^{\log}\colon X_3^{\log} \to X_1^{\log}$$

for the projection to the factor labeled i of $X_3^{\log}$; $\underline{\mathrm{pr}}_i\colon X_3 \to X_1$ for the underlying morphism of schemes; $\underline{p}_i\colon \Pi_3 \to \Pi_1$ for the surjection induced by $\underline{\mathrm{pr}}_i^{\log}$.

DEFINITION 1.8. Write $U \overset{\mathrm{def}}{=} U_X$; $V \subseteq U \times_k U$ for the diagonal (so we have a natural isomorphism $V \xrightarrow{\sim} U$); $V^{\log}$ for the log scheme obtained by equipping V with the log structure pulled back from $X_2^{\log}$ (where we recall that we have a natural immersion $U \times_k U \hookrightarrow X_2$). Let $P^{\log}$ be a *tripod* over k.
(i) The morphism of log schemes $\underline{\mathrm{pr}}_{12}^{\log}\colon X_3^{\log} \to X_2^{\log}$ determines a structure of *family of pointed stable curves* on the restriction $X_3|_V \to V$ of $\underline{\mathrm{pr}}_{12}$ to V. Moreover, $X_3|_V$ consists of precisely *two irreducible components* F_V, E_V—which we refer to, respectively, as *major cuspidal* and *minor cuspidal*. Here, the intersection $F_V \cap E_V$ is a *node* $\nu_V\colon V \to X_3|_V$; either $\underline{\mathrm{pr}}_{13}$ or $\underline{\mathrm{pr}}_{23}$ induces an isomorphism $F_V \xrightarrow{\sim} V \times_k X$ over V; the natural projection $E_V \to V$ is a $\mathbb{P}^1$-*bundle*; the three sections of $E_V \to V$ given by ν_V and the two cusps of $X_3|_V \to V$ that intersect E_V determine a *unique isomorphism* $E_V \xrightarrow{\sim} V \times_k P$ over V (i.e., such that the three sections of $E_V \to V$ correspond to the cusps of the tripod, which we think of as being "*labeled*" by these three sections). Write $(V \times_k U_P \cong)\ W \subseteq E_V$ for the open subscheme given by the complement of these three sections; $W^{\log}$ for the log scheme obtained by equipping W with the log structure pulled back from $X_3^{\log}$ via the natural inclusion $W \subseteq E_V \subseteq X_3|_V \subseteq X_3$. Thus, we obtain a natural morphism of log schemes $W^{\log} \to V^{\log}$.
(ii) For $x \in U(k)$, denote the fibers relative to $\underline{\mathrm{pr}}_1$ over x by means of a subscript "x"; write $Y^{\log} \to \mathrm{Spec}(k)$ for the smooth log curve determined by the hyperbolic curve $U \setminus \{x\}$, $y \in Y(k)$ for the cusp determined by x. Thus, we have a natural isomorphism $(X_3^{\log})_x \xrightarrow{\sim} Y_2^{\log}$ (cf. [24], Remark 2.1.2); this isomorphism allows one to identify $\Pi_{3/1}$ with the "Π_2" associated to $Y^{\log}$ (cf. [24], Proposition 2.4, (i)). Relative to this isomorphism $(X_3^{\log})_x \xrightarrow{\sim} Y_2^{\log}$, $F_V|_x$, $E_V|_x$ may be identified with the irreducible components "F_y", "E_y" of Definition 1.4, (i), applied to $Y^{\log}$, y (in place of $X^{\log}$, x). In particular, we obtain *major* and *minor verticial subgroups* $\Pi_{F_V} \subseteq \Pi_{3/2}$, $\Pi_{E_V} \subseteq \Pi_{3/2}$

(i.e., corresponding to the "Π_{F_y}", "Π_{E_y}" of Definition 1.4, (ii)).

Proposition 1.9 (Minor cuspidal components in three-dimensional configuration spaces). *In the notation of* Definition 1.8, *let us think of* V, W *as elements of* $\mathcal{D}_2^{\mathrm{hor}}$, $\mathcal{D}_3^{\mathrm{ver}}$, *respectively, and suppose that* $\underline{p}_{12}(\mathbb{D}_W) = \mathbb{D}_V$ (*cf.* Proposition 1.3, (i), (iii)). *Then*:

(i) *Write* $\mathbb{J}_W \stackrel{\mathrm{def}}{=} Z_{\mathbb{D}_W}(\Pi_{E_V})$. *Then*:

(a) $\underline{p}_{12}$ *induces an isomorphism* $\mathbb{J}_W \xrightarrow{\sim} \mathbb{D}_V$;

(b) *the inclusions* $\mathbb{J}_W \hookrightarrow \mathbb{D}_W$, $\Pi_{E_V} \hookrightarrow \mathbb{D}_W$ *induce an* isomorphism $\mathbb{J}_W \times \Pi_{E_V} \xrightarrow{\sim} \mathbb{D}_W$;

(c) $\underline{p}_1$ *determines natural exact sequences* $1 \to \mathbb{I}_W \to \mathbb{J}_W \to \Pi_1 \to 1$, $1 \to \mathbb{I}_V \to \mathbb{D}_V \to \Pi_1 \to 1$, *which are compatible with the isomorphisms* $\mathbb{I}_W \xrightarrow{\sim} \mathbb{I}_V$, $\mathbb{J}_W \xrightarrow{\sim} \mathbb{D}_V$ *induced by* $\underline{p}_{12}$.

(ii) *For any open subgroup* $J \subseteq \Pi_{E_V}$, *we have*: $Z_{\Pi_3}(J) = \mathbb{J}_W$.

(iii) *We have*: $C_{\Pi_3}(\Pi_{E_V}) = \mathbb{D}_W$.

Proof. Since $\Pi_{E_V} \cong \Pi^{\mathrm{tripod}}$ is *center-free* (cf. Remark 1.1.1), assertion (i) follows immediately from the *isomorphism of log schemes* $W^{\log} \xrightarrow{\sim} V^{\log} \times_k U_P$ induced by the isomorphism of schemes $W \xrightarrow{\sim} V \times_k U_P$ and the morphism of natural log schemes $W^{\log} \to V^{\log}$ (cf. Definition 1.8, (i)). Next, we consider assertion (ii). Since $\underline{p}_1$ induces a *surjection* $\mathbb{J}_W \twoheadrightarrow \Pi_1$, and it is immediate that $\mathbb{J}_W \subseteq Z_{\Pi_3}(J)$, it suffices to verify that $\mathbb{J}_W \cap \Pi_{3/1} = Z_{\Pi_3}(J) \cap \Pi_{3/1} = Z_{\Pi_{3/1}}(J)$. But this follows from Proposition 1.7, (d) (cf. the discussion of Definition 1.8, (ii)). In a similar vein, since $\underline{p}_1$ induces a *surjection* $\mathbb{D}_W \twoheadrightarrow \Pi_1$, and it is immediate that $\mathbb{D}_W \subseteq C_{\Pi_3}(\Pi_{E_V})$, in order to verify assertion (iii), it suffices to verify that $\mathbb{D}_W \cap \Pi_{3/1} = C_{\Pi_{3/1}}(\Pi_{E_V})$. But this follows from Proposition 1.7, (e). This completes the proof of Proposition 1.9. $\square$

Corollary 1.10 (Outer actions on minor verticial subgroups). *Suppose that* $n \geq 2$. *Then the subquotient* $\Pi_{n-1/n-2}$ *of* Π_n *may be regarded* (*cf.* [24], Proposition 2.4, (i)) *as the pro-*Σ *fundamental group—i.e.,* "Π_1"*—of the geometric generic fiber* $Z^{\log}$ *of the morphism* $X_{n-1}^{\log} \to X_{n-2}^{\log}$ (*which we think of as the projection obtained by "forgetting the factor labeled* $n-1$"); *the subquotient* $\Pi_{n/n-2}$ *may then be thought of* (*cf.* [24], Proposition 2.4, (i)) *as the pro-*Σ *fundamental group of* 2*-nd log configuration space—i.e.,* "Π_2"*—associated to* $Z^{\log}$. *In particular, any* cusp x *of* $Z^{\log}$ *determines, up to* $\Pi_{n/n-2}$*-conjugacy, a* minor verticial subgroup*—i.e., an isomorph of* Π^{tripod}—$\Pi_{E_x} \subseteq \Pi_{n/n-1}$. *Then*:

(i) *Any* $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$ (*cf.* Definition 1.1, (v)) *stabilizes the* $\Pi_{n/n-2}$-conjugacy class *of* Π_{E_x}.

(ii) *The* commensurator *and* centralizer *of* Π_{E_x} *in* Π_n *satisfy the relation* $C_{\Pi_n}(\Pi_{E_x}) = Z_{\Pi_n}(\Pi_{E_x}) \times \Pi_{E_x}$. *In particular, for any open subgroup* $J \subseteq \Pi_{E_x}$, *we have* $Z_{\Pi_n}(J) = Z_{\Pi_n}(\Pi_{E_x})$.

(iii) *By applying* (i), (ii), *one obtains a* natural homomorphism

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})$$

and hence a natural outer homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \to \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tripod}})$, *associated to the* cusp x *of* $Z^{\log}$.

Proof. In light of the superscript "cusp" and the *FC-admissibility* of α (cf. Remark 1.1.5), assertion (i) follows immediately from the resp'd portion of Proposition 1.3, (iv). Next, we consider assertion (ii). First, let us recall that Π_{E_x} is *commensurably terminal* in $\Pi_{n/n-1}$ (cf. Proposition 1.5, (i)). On the other hand, it is immediate from the definitions that $C_{\Pi_n}(\Pi_{E_x}) \subseteq N_{\Pi_n}(C_{\Pi_{n/n-1}}(\Pi_{E_x}))$. Thus, we conclude that $C_{\Pi_n}(\Pi_{E_x}) = N_{\Pi_n}(\Pi_{E_x})$. In particular, to complete the proof of assertion (ii), it suffices (since Π_{E_x} is *slim*—cf. Remark 1.1.1) to verify that

(∗) the *natural outer action* of $N_{\Pi_n}(\Pi_{E_x})$ on Π_{E_x} is *trivial*.

Now let $j \in \{1, \dots, n-1\}$ be the *smallest* element $m \in \{1, \dots, n-1\}$ such that x corresponds to a cusp of the geometric generic fiber of the morphism $X_m^{\log} \to X_{m-1}^{\log}$ (which we think of as the projection obtained by "forgetting the factor labeled m"). (Here, we write $X_0^{\log} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k)$.) Now if $j = 1$, then by applying the projection $\Pi_n \twoheadrightarrow \Pi_2$ determined by the factors labeled 1, n, we conclude that (∗) follows from Propositions 1.3, (iii), (e); 1.7, (a), (e). In a similar vein, if $j \geq 2$, then by applying the projection $\Pi_n \twoheadrightarrow \Pi_3$ determined by the factors labeled $j-1$, j, n, we conclude that (∗) follows from Proposition 1.9, (i), (b); 1.9, (iii). This completes the proof of assertion (ii).

Finally, we observe that assertion (iii) follows immediately from assertions (i), (ii), by choosing some isomorphism $\Pi_{E_x} \xrightarrow{\sim} \Pi^{\mathrm{tripod}}$ (which is determined only up to composition with an element of $\mathrm{Aut}^{\mathrm{FC}}(\Pi^{\mathrm{tripod}})$) that is compatible with the cuspidal inertia groups. That is to say, if $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$, then by assertion (i), $\alpha_0(\alpha(\Pi_{E_x})) = \Pi_{E_x}$ for some Π_n-inner automorphism α_0 of Π_n. Since α_0 is uniquely determined up to composition with an element of $N_{\Pi_n}(\Pi_{E_x})$, it follows from assertion (ii) that the *outer automorphism* $\alpha_1 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})$ determined by $\alpha_0 \circ \alpha$ is *uniquely determined* by α. Moreover, one verifies immediately that the assignment $\alpha \mapsto \alpha_1$ determines a *homomorphism* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})$, hence an *outer homomorphism* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \to \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tripod}})$, as desired. □

DEFINITION 1.11. (i) In the situation of Definition 1.1, (vi), let us write

$$\mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tripod}})^{\triangle} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FCS}}(\Pi^{\mathrm{tripod}}) = \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tripod}})^{\mathrm{S}}$$

and

$$\mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tripod}})^{\triangle +} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tripod}})^{\triangle}$$

for the *subgroup* given by the *image* of $\mathrm{Out}^{\mathrm{FC}}(\Pi_2^{\mathrm{tripod}})^{\mathrm{S}}$ via the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_2^{\mathrm{tripod}}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\mathrm{tripod}})$ induced by the standard surjection $\Pi_2^{\mathrm{tripod}} \twoheadrightarrow \Pi_1^{\mathrm{tripod}}$.
(ii) Now let us return to the case of *arbitrary* (g, r); suppose that $n \geq 2$. Then let us write

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\triangle+} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\triangle} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$$

for the *subsets* (which are not necessarily subgroups!) given by the unions of the respective *inverse images* of $\mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})^{\triangle+} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})^{\triangle} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})$ via the *natural homomorphism* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})$ associated in Corollary 1.10, (iii), to a cusp x (as in loc. cit.), as x ranges over all cusps as in loc. cit.

REMARK 1.11.1. It is shown in [5] (cf. Corollary 4.2, (i), (ii), below; Remark 4.2.1 below; [5], §0.1, Main Theorem, (b)) that $\mathrm{Out}^{\mathrm{FC}}(\Pi^{\mathrm{tripod}})^{\triangle+}$ may be identified with the *Grothendieck–Teichmüller group*. Thus, one may think of the set $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\triangle+}$ of Definition 1.11, (ii), as the set of outer automorphisms *"of Grothendieck–Teichmüller type"*.

Corollary 1.12 (Injectivity for tripods). *Suppose that* $X^{\log}$ *is a* tripod. *Then*:
(i) *The natural inclusion* $\Xi_2 \hookrightarrow \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$ *is an* isomorphism.
(ii) *The natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

induced by $p_1 \colon \Pi_2 \twoheadrightarrow \Pi_1$ *is* injective.
(iii) *We have*: $\mathrm{Out}^{\mathrm{FCP}}(\Pi_2) = \mathrm{Out}^{\mathrm{FC}}(\Pi_2)$.

Proof. First, we observe that assertion (ii) follows formally from assertion (i) and Proposition 1.2, (iii). Next, we observe that assertion (iii) follows formally from assertion (ii) and Propositions 1.2, (iii); 1.6, (iii). Thus, to complete the proof of Corollary 1.12, it suffices to verify assertion (i). To this end, let $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$. Let us assign the cusps of $X^{\log}$ the *labels* a, b, c. Note that the labels of the cusps of $X^{\log}$ induce labels "a", "b", "c" for three of the cusps of the geometric generic fiber of the morphism $U_{X_2} \to U_{X_1}$ determined by pr_1; assign the fourth cusp of this geometric generic fiber the *label* $*$. Since $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$, it follows that α induces (relative to p_1 or p_2) the *identity permutation* of the conjugacy classes of cuspidal inertia groups of Π_1. Since cuspidal inertia groups associated to $*$ may be characterized by the property that they are contained in Ξ_2, we thus conclude that α induces the *identity permutation* of the conjugacy classes of cuspidal inertia groups of $\Pi_{2/1}$.

Now let us *fix* a cuspidal inertia group $I_a \subseteq \Pi_{2/1}$ associated to the cusp labeled a. Thus, $\alpha(I_a) = \zeta \cdot I_a \cdot \zeta^{-1}$, for some $\zeta \in \Pi_{2/1}$. Since $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$, and $J_a \stackrel{\mathrm{def}}{=} p_2(I_a)$ is *normally terminal* in Π_1 (cf. [20], Proposition 1.2, (ii)), it thus follows that $p_2(\zeta) \in J_a$,

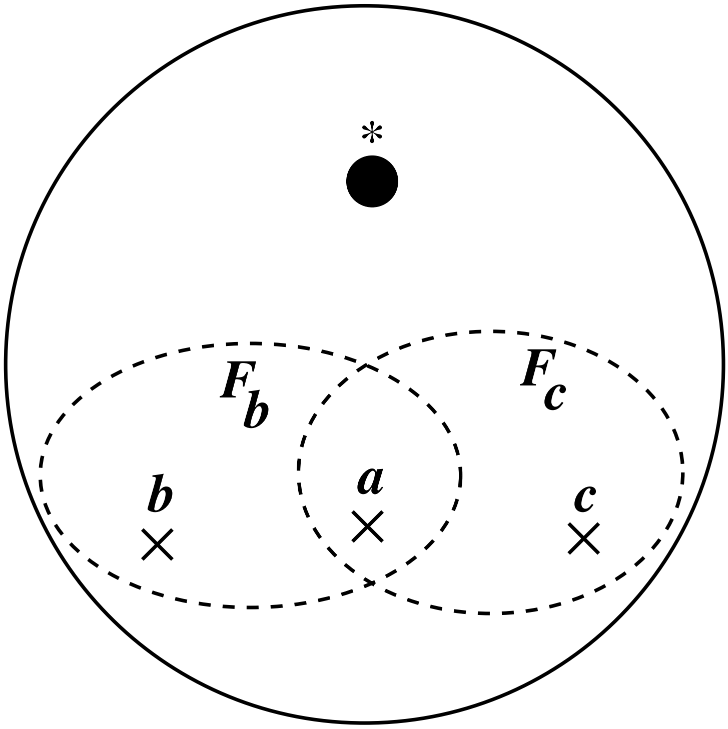


Fig. 1. The geometry of a tripod equipped with a fourth cusp "$*$".

so (by replacing ζ by an appropriate element $\in \zeta \cdot I_a$) we may assume without loss of generality that $\zeta \in \Pi_{2/1} \cap \Pi_{1\backslash 2} = i_2$. Thus, by replacing α by the composite of α with a Ξ_2-inner automorphism, we may assume without loss of generality that $\alpha(I_a) = I_a$. By [20], Proposition 1.5, (i), it follows that there exists a *unique* (i.e., among its $\Pi_{2/1}$-conjugates) *major verticial subgroup* Π_{F_b} at b (respectively, Π_{F_c} at c) such that $I_a \subseteq \Pi_{F_b}$ (respectively, $I_a \subseteq \Pi_{F_c}$). By the *non-resp'd* portion of Proposition 1.3, (iv) (which is *applicable since* $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$!—cf. Remark 1.13.2 below), we thus conclude that $\alpha(\Pi_{F_b}) = \Pi_{F_b}$, $\alpha(\Pi_{F_c}) = \Pi_{F_c}$. Since $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$, and p_2 induces *isomorphisms* $\Pi_{F_b} \xrightarrow{\sim} \Pi_1$, $\Pi_{F_c} \xrightarrow{\sim} \Pi_1$ (cf. Definition 1.4, (ii)), we thus conclude that α is the *identity* on Π_{F_b}, Π_{F_c}. On the other hand, it follows immediately—for instance, by considering the well-known geometry of "loops around cusps" of the *complex plane with three points removed* (cf. Lemma 1.13; Fig. 1 above)—that $\Pi_{2/1}$ is *topologically generated* by Π_{F_b}, Π_{F_c}. Thus, we conclude that α induces the the *identity* on $\Pi_{2/1}$. But since the extension $1 \to \Pi_{2/1} \to \Pi_2 \to \Pi_1 \to 1$ induced by p_1 may be constructed naturally from the resulting outer action of Π_1 on $\Pi_{2/1}$ (i.e., as $\Pi_{2/1} \overset{\mathrm{out}}{\rtimes} \Pi_1$—cf. §0; Remark 1.1.1), we thus conclude that α is the *identity*. This completes the proof of assertion (i). $\square$

The following result is well-known.

Lemma 1.13 (Topological generation by loops around cusps). *In the notation of the proof of* Corollary 1.12, *the compatible inclusions* $I_a \subseteq \Pi_{F_b} \subseteq \Pi_{2/1}$, $I_a \subseteq \Pi_{F_c} \subseteq \Pi_{2/1}$ *determine an* isomorphism

$$\varinjlim(\Pi_{F_b} \hookleftarrow \Pi_{I_a} \hookrightarrow \Pi_{F_c}) \xrightarrow{\sim} \Pi_{2/1}$$

—where the inductive limit *is taken in the category of* pro-Σ groups. *In particular,* $\Pi_{2/1}$ *is topologically generated by* Π_{F_b}, Π_{F_c}.

Proof. In the following, we shall denote the *usual topological fundamental group* by "$\pi_1^{\mathrm{top}}(-)$". We may assume without loss of generality that k is the field $\mathbb{C}$ of complex numbers. Then, as is well-known, the topology of a stable curve may be understood—from the point of view of "*pants decompositions*" (cf., e.g., [1], Chapter 2)—as the result of *collapsing* various "partition curves" on a hyperbolic Riemann surface to points (which form the *nodes* of the stable curve). In particular, in the case of interest, one obtains that $\Pi_{F_b} \subseteq \Pi_{2/1}$, $\Pi_{F_c} \subseteq \Pi_{2/1}$ may be described in the following fashion: Write V for the Riemann surface obtained by removing the points $\{0, 3, -3\}$ from the complex plane $\mathbb{C}$. Write D_+ (respectively, D_-) for the intersection with V of the open disc of radius 3 centered at 1 (respectively, -1). Note that V is equipped with a holomorphic automorphism $\iota\colon V \to V$ given by "multiplication by -1"; $\iota(D_+) = D_-$, $\iota(D_-) = D_+$. Let us think of -3, 0, 3 as corresponding, respectively, to the cusps b, a, c. Then we may think of $\Pi_{2/1}$ as the pro-Σ completion of $\pi_1^{\mathrm{top}}(V)$ and of $\Pi_{F_b} \subseteq \Pi_{2/1}$ as corresponding, at least up to $\Pi_{2/1}$-conjugacy, to the pro-Σ completion of $\pi_1^{\mathrm{top}}(D_-) \subseteq \pi_1^{\mathrm{top}}(V)$. By *transport of structure* via ι, we then obtain that we may think of $\Pi_{F_c} \subseteq \Pi_{2/1}$ as corresponding, at least up to $\Pi_{2/1}$-conjugacy, to the pro-Σ completion of $\pi_1^{\mathrm{top}}(D_+) \subseteq \pi_1^{\mathrm{top}}(V)$. As in the proof of Corollary 1.12, we may *rigidify* the various conjugacy indeterminacies by taking the basepoints of $\pi_1^{\mathrm{top}}(V)$, $\pi_1^{\mathrm{top}}(D_+)$, and $\pi_1^{\mathrm{top}}(D_-)$ to be the point $i \in \mathbb{C}$ and taking $I_a \subseteq \Pi_{2/1}$ to correspond to the subgroup topologically generated by the element of $\pi_1^{\mathrm{top}}(V)$ determined by the circle γ_a of radius 1 centered at a (i.e., 0), oriented counterclockwise (so $\gamma_a \subseteq D_+ \cap D_-$). Thus, if one takes γ_b (respectively, γ_c) to be a *loop in* V, oriented counterclockwise, given by a *slight deformation* of the path obtained by traveling from i to b (respectively, c) and then back to i along the line segment from i to b (respectively, c), then $\gamma_b \subseteq D_-$, $\gamma_c \subseteq D_+$. Moreover, as is well-known from the "*van Kampen theorem*" in elementary algebraic topology (cf. also the more *combinatorial* point of view discussed in the proof of Proposition 1.5, (iii)), $\pi_1^{\mathrm{top}}(V) = \pi_1^{\mathrm{top}}(D_+ \cup D_-)$ is *naturally isomorphic* to the *inductive limit*, in the category of groups, of the diagram

$$\pi_1^{\mathrm{top}}(D_-) \hookleftarrow \pi_1^{\mathrm{top}}(D_+ \cap D_-) \hookrightarrow \pi_1^{\mathrm{top}}(D_+)$$

—where we observe that $\pi_1^{\mathrm{top}}(D_-)$ is generated by γ_a and γ_b, $\pi_1^{\mathrm{top}}(D_+ \cap D_-)$ is generated by γ_a, and $\pi_1^{\mathrm{top}}(D_+)$ is generated by γ_a and γ_c. Thus, Lemma 1.13 follows by passing to pro-Σ completions. $\square$

REMARK 1.13.1. In the notation of Corollary 1.12 and its proof, we observe that the isomorphism of Lemma 1.13 suggests that it may be possible to verify that the *natural injection*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

of Corollary 1.12, (ii), is *surjective* (hence an *isomorphism*) via the following argument: Let $\beta_1 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_1)$. Then it suffices to verify that β_1 arises (via p_1) from an element of $\mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$. Fix a *"rigidified triple"*

$$\Pi_{F_b} \supseteq I_a \subseteq \Pi_{F_c}$$

as in the proof of Corollary 1.12. Let us assume, for simplicity, that $\beta_1(J_a) = J_a$ (where we recall that $J_a = p_2(I_a)$). Next, let us observe that p_2 induces *isomorphisms* $\Pi_{F_b} \xrightarrow{\sim} \Pi_1$, $\Pi_{F_c} \xrightarrow{\sim} \Pi_1$ which *coincide* on $I_a \subseteq \Pi_{F_b}$, $I_a \subseteq \Pi_{F_c}$. Thus, it follows *formally* from the isomorphism of Lemma 1.13 that *there exists a unique automorphism* $\beta_{2/1}$ of $\Pi_{2/1}$ that is *compatible*, relative to p_2, with the automorphism β_1 of Π_1. In particular, $\beta_{2/1}$ constitutes a *natural candidate* for (the restriction to $\Pi_{2/1}$ of) a lifting of β_1 to $\mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$. On the other hand, unfortunately, it is *not clear* whether or not $\beta_{2/1}$, constructed in this way, *stabilizes the* $\Pi_{2/1}$*-conjugacy class of the cuspidal inertia groups associated to the cusp* $*$. In particular, this argument alone is *not sufficient* to construct a lifting of β_1 to $\mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$ from $\beta_{2/1}$.

REMARK 1.13.2. Another (perhaps more fundamental!) problem with the approach proposed in Remark 1.13.1 is the following. If one already knows that $\beta_1 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_1)$ arises (via p_1) from some $\beta_2 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$, then one wishes for the *explicit construction* of $\beta_{2/1}$ that is applied to give rise to the outer automorphism of $\Pi_{2/1}$ obtained by restricting β_2 to $\Pi_{2/1}$. For instance, if β_1 is *inner*, then it arises from a $\beta_2 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$ which is *inner*. Moreover, in order to pass from the $\beta_{2/1}$ constructed from an *arbitrary* $\beta_1 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_1)$ by applying the natural isomorphism $\Pi_2 \xrightarrow{\sim} \Pi_{2/1} \overset{\mathrm{out}}{\rtimes} \Pi_1$ (cf. §0; Remark 1.1.1), it is *of crucial importance* for the *explicit construction* $\beta_1 \rightsquigarrow \beta_{2/1}$ to be a *homomorphism* which yields the restriction to $\Pi_{2/1}$ of an inner lifting to $\mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$ when applied to an inner β_1. On the other hand, if β_1 is a *non-trivial inner* automorphism of Π_1, then (as is easily verified) there do *not* exist cuspidal inertia groups $J_b, J_c \subseteq \Pi_1^{\mathrm{tripod}}$ corresponding to the cusps labeled b, c such that $\beta_1(J_a) = J_a$, $\beta_1(J_b) = J_b$, $\beta_1(J_c) = J_c$. In particular, in the case of such an arbitrary inner β_1, one may not apply the *non-resp'd* portion of Proposition 1.3, (iv), to conclude that the $\Pi_{2/1}$-*conjugacy* classes of major and minor verticial subgroups or nexus subgroups of $\Pi_{2/1}$ are preserved by an inner lifting β_2. Instead, one may only apply the *resp'd* portion of Proposition 1.3, (iv), to conclude that the Π_2-*conjugacy* classes of such subgroups are preserved by β_2—which is *insufficient* for the execution of the construction of Remark 1.13.1 (i.e., of the proof of Corollary 1.12).

Corollary 1.14 (Modular symmetries of tripods). *Suppose that* $X^{\log}$ *is a* tripod. *Let* $n \geq 2$. *Then*:
(i) *The* outer modular symmetries $\in \mathrm{Out}(\Pi_n)$ normalize $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$. *If, moreover, the natural homomorphism* $\mathrm{Out}^{\mathrm{FC}}(\Pi_m) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{m-1})$ *induced by the standard surjection* $\Pi_m \twoheadrightarrow \Pi_{m-1}$ *is* injective *for all integers m such that* $2 \leq m \leq n$, *then we have* $\mathrm{Out}^{\mathrm{FCP}}(\Pi_n) \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{S}} = \mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$.
(ii) *Let x be as in* Corollary 1.10. *Write* $\pi : \Pi_n \twoheadrightarrow \Pi_1$ *for the standard surjection. Then there exists an* outer modular symmetry $\sigma \in \mathrm{Out}(\Pi_n)$ *such that the restriction of* $\pi \circ \sigma : \Pi_n \twoheadrightarrow \Pi_1$ *to* $\Pi_{E_x} \subseteq \Pi_n$ *determines an* outer isomorphism $\Pi_{E_x} \xrightarrow{\sim} \Pi_1$ *that is independent of the choice of* Π_{E_x} *among its* Π_n*-conjugates.*
(iii) *Suppose that we are in the situation of* (ii). *Let* $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$; $\alpha|_{E_x} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})$ *the result of applying the displayed homomorphism of* Corollary 1.10, (iii), *to* α; $\alpha^\sigma \stackrel{\mathrm{def}}{=} \sigma \cdot \alpha \cdot \sigma^{-1} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$ (*cf.* (i)); $\alpha_1^\sigma \in \mathrm{Out}^{\mathrm{FC}}(\Pi_1)^{\mathrm{cusp}}$ *the outer automorphism of* Π_1 *induced by* α^σ *via* π. (*Thus,* $\alpha = \alpha^\sigma$ *whenever* $\alpha \in \mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$.) *Then* $\alpha|_{E_x}$ *and* α_1^σ *are* compatible *with the outer isomorphism* $\Pi_{E_x} \xrightarrow{\sim} \Pi_1$ *of* (ii). *In particular, if* $\alpha|_{E_x} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})^{\mathrm{S}}$, *then* $\alpha^\sigma \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{S}}$.
(iv) *We have*: $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\triangle+}$.

Proof. First, we consider assertion (i). We apply *induction* on n. First, let us observe that relative to the *natural isomorphism* $X_n^{\log} \xrightarrow{\sim} \left(\overline{\mathcal{M}}_{0,n+3}^{\log}\right)_k$ (cf. Definition 1.1, (vi)), the divisors of X_n that belong to $\mathcal{D}_n^*$ (cf. Proposition 1.3, (vii)) are precisely the *divisors at infinity* of $\left(\overline{\mathcal{M}}_{0,n+3}^{\log}\right)_k$ whose generic points parametrize stable curves of genus zero with precisely two components, one of which contains *precisely two cusps*. (Indeed, this follows immediately from the well-known geometry of $\left(\overline{\mathcal{M}}_{0,n+3}^{\log}\right)_k$.) In particular, the automorphisms of $\left(\overline{\mathcal{M}}_{0,n+3}^{\log}\right)_k$ arising from the permutations of the ordering of the cusps *permute* the divisors that belong to $\mathcal{D}_n^*$. Thus, we conclude that the *outer modular symmetries* $\in \mathrm{Out}(\Pi_n)$ *normalize* $\mathrm{Out}^{\mathrm{QS}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$ (cf. Proposition 1.3, (vii)). Now let $\tau \in \mathrm{Out}(\Pi_n)$ be an outer modular symmetry that arises from a *permutation of the subset* $\{a, b, c, 1, 2, \ldots, n-1\} \subseteq \{a, b, c, 1, 2, \ldots, n-1, n\}$ (cf. the notation of Definition 1.1, (vi)); $\alpha \in \mathrm{Out}^{\mathrm{FCP}}(\Pi_n) \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{S}} \subseteq \mathrm{Out}^{\mathrm{QS}}(\Pi_n)$ (cf. Proposition 1.3, (vii)); $\alpha_\tau \stackrel{\mathrm{def}}{=} \tau^{-1} \circ \alpha \circ \tau \in \mathrm{Out}^{\mathrm{QS}}(\Pi_n)$. Then since τ is *compatible* with the standard surjection $\Pi_n \twoheadrightarrow \Pi_{n-1}$, it follows from the *induction hypothesis* that α, α_τ map to the *same* element $\in \mathrm{Out}^{\mathrm{QS}}(\Pi_{n-1})$ via the natural homomorphism $\mathrm{Out}^{\mathrm{QS}}(\Pi_n) \to \mathrm{Out}^{\mathrm{QS}}(\Pi_{n-1})$ induced by this surjection. Thus, we conclude from the *injectivity condition* in the statement of assertion (i) (cf. also Proposition 1.3, (vii)) that $\alpha = \alpha_\tau$. Since the group of *all* permutations of the set $\{a, b, c, 1, 2, \ldots, n-1, n\}$ is generated by the subgroups of permutations of the subsets $\{a, b, c, 1, 2, \ldots, n-1\} \subseteq \{a, b, c, 1, 2, \ldots, n-1, n\}$ and $\{1, 2, \ldots, n-1, n\} \subseteq \{a, b, c, 1, 2, \ldots, n-1, n\}$, we thus conclude that $\alpha \in \mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$. This completes the proof that $\mathrm{Out}^{\mathrm{FCP}}(\Pi_n) \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{S}} \subseteq \mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$; the opposite inclusion follows immediately from the definitions. This completes the proof of assertion (i).

In light of Corollary 1.10, (ii), assertions (ii) and (iii) follow immediately from the definitions and the well-known geometry of $X_n^{\log}$ (i.e., $\left(\overline{\mathcal{M}}_{0,n+3}^{\log}\right)_k$). Finally, we consider assertion (iv). By assertion (iii), it follows that the image of the restriction $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})$ to $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$ of the natural homomorphism of Corollary 1.10, (iii), lies in $\mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})^{\triangle}$. Write $\pi' \colon \Pi_n \twoheadrightarrow \Pi_2$, $\pi'' \colon \Pi_2 \twoheadrightarrow \Pi_1$ (so $\pi = \pi'' \circ \pi'$) for the standard surjections. Then the existence of the factorization $\pi \circ \sigma = \pi'' \circ (\pi' \circ \sigma) \colon \Pi_n \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1$—which is *compatible* with elements of $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$—implies that the image of the homomorphism $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})$ in fact lies in $\mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})^{\triangle+}$. This implies the desired inclusion $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\triangle+}$ and hence completes the proof of assertion (iv). $\square$

2. Injectivity for degenerating affine curves

In the present §2, we generalize (cf. Corollary 2.3, (ii)) the *injectivity* asserted in Corollary 1.12, (ii), to the case of *arbitrary* $X^{\log}$ such that U_X is *affine*, by considering what happens when we allow $X^{\log}$ to *degenerate*.

Let

- $k_\circ \stackrel{\mathrm{def}}{=} k$ be as in §1;
- $R_\circ \stackrel{\mathrm{def}}{=} k_\circ[[t]]$—i.e., the *ring of power series* with coefficients in $k_\circ$;
- $K_\circ$ the *quotient field* of $R_\circ$;
- K an *algebraic closure* of $K_\circ$; $\eta \stackrel{\mathrm{def}}{=} \mathrm{Spec}(K)$;
- R the integral closure of $R_\circ$ in K;
- $S_\circ^{\log}$, $S^{\log}$ the log schemes obtained by equipping $S_\circ \stackrel{\mathrm{def}}{=} \mathrm{Spec}(R_\circ)$, $S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(R)$, respectively, with the log structures determined by the nonzero regular functions;
- $s_\circ^{\log} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k_\circ) \times_{S_\circ} S_\circ^{\log}$;
- $s^{\log} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k) \times_S S^{\log}$.

Here, we wish to think of k as the *residue field of* R.

Next, let

$$X_\circ^{\log} \to S_\circ^{\log}$$

be a *stable log curve of type* (g, r) (whose restriction to $U_{S_\circ}$ is a *smooth log curve*);

$$X^{\log} \stackrel{\mathrm{def}}{=} X_\circ^{\log} \times_{S_\circ^{\log}} S^{\log} \to S^{\log};$$

$$X_{\circ s}^{\log} \stackrel{\mathrm{def}}{=} X_\circ^{\log} \times_{S_\circ^{\log}} s_\circ^{\log} \to s_\circ^{\log}; \quad X_s^{\log} \stackrel{\mathrm{def}}{=} X_\circ^{\log} \times_{S_\circ^{\log}} s^{\log} \to s^{\log}$$

for the result of base-changing via the morphisms $S^{\log} \to S_{\circ}^{\log}$, $s_{\circ}^{\log} \to S_{\circ}^{\log}$, $s^{\log} \to S_{\circ}^{\log}$. Thus, we are in a situation as discussed in §0. By ordering the cusps of $X_{\circ}^{\log}$, we obtain a *classifying* (1-)*morphism* $S_{\circ}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$. If n is a positive integer, then by pulling back the natural (1-)morphism $\overline{\mathcal{M}}_{g,r+n}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ obtained by "forgetting the last n points" via this classifying morphism, we thus obtain a "*log configuration space*"

$$X_{n\circ}^{\log} \to S_{\circ}^{\log}$$

—i.e., whose restriction to $U_{S_{\circ}}$ is a "log configuration space" as in [24], Definition 2.1, (i). We shall write

$$X_n^{\log} \to S^{\log}; \quad X_{n\circ s}^{\log} \to s_{\circ}^{\log}; \quad X_{n,s}^{\log} \to s^{\log}$$

for the result of base-changing $X_{n\circ}^{\log} \to S_{\circ}^{\log}$ to $S^{\log}$, $s_{\circ}^{\log}$, or $s^{\log}$. Thus, we may apply the discussion of §0 to $X_n^{\log} \to S^{\log}$ for arbitrary n. Also, we may apply the theory of §1 by taking

$$X_{n,\eta}^{\log} \stackrel{\text{def}}{=} X_n^{\log} \times_S \eta \to \eta$$

to be the "$X_n^{\log} \to S$" of §1; this results in a "Π_n" of the form

$$\Pi_n \stackrel{\text{def}}{=} \pi_1^{\Sigma}(X_{n,\eta}^{\log})$$

—to which we may apply the *specialization isomorphisms* discussed in §0.

For $i = 1, 2$, write

$$\mathrm{pr}_i^{\log}\colon X_2^{\log} \to X_1^{\log}$$

for the projection to the factor labeled i, $\mathrm{pr}_i\colon X_2 \to X_1$ for the underlying morphism of schemes, and $p_i\colon \Pi_2 \to \Pi_1$ for the surjection induced by $\mathrm{pr}_i^{\log}$.

DEFINITION 2.1. Let $i_X \in \{1, 2\}$. Suppose that X_s is *singular* and has i_X *irreducible components*, one of which we shall denote T; if $i_X = 2$, then we shall write Q for the other irreducible component of X_s. Write $U_T \subseteq T$ (respectively, (when $i_X = 2$) $U_Q \subseteq Q$) for the complement in T (respectively, (when $i_X = 2$) Q) of the nodes and cusps of X_s relative to the log structure of $X_s^{\log}$. Suppose further that U_T is a *tripod*. Let $x \in X(S)$ be a *cusp* of $X^{\log}$ whose restriction $x_s \in X_s(s) \subseteq X(k)$ to s lies in T ($\subseteq X_s$) (cf. Remark 2.1.1 below).

(i) Observe that the log structure on $X_2^{\log}$ determines on the fiber $(X_2)_{x_s}$ of the morphism $\mathrm{pr}_1\colon X_2 \to X_1$ ($= X$) over $x_s \in X(k)$ a structure of *pointed stable curve*, which consists of $i_X + 1$ irreducible components, i_X of which—which we shall denote $\ddot{T}$ and (when $i_X = 2$) $\ddot{Q}$—*map isomorphically* to $T \subseteq X_s$ and (when $i_X = 2$) $Q \subseteq X_s$, respectively, via $\mathrm{pr}_2\colon X_2 \to X_1 = X$, the $(i_X + 1)$-th of which—which we shall denote

$\ddot{E}_x$—*maps to the point* $x_s \in X_s(s)$ via pr_2. Let us refer to $\ddot{T}$ and (when $i_X = 2$) $\ddot{Q}$ as the *sub-major cuspidal components at* x_s and to $\ddot{E}_x$ as the *sub-minor cuspidal component at* x_s. Thus, the complement in $\ddot{T}$ (respectively, (when $i_X = 2$) $\ddot{Q}$; $\ddot{E}_x$) of the nodes and cusps (relative to the pointed stable curve structure on $(X_2)_{x_s}$) of $\ddot{T}$ (respectively, (when $i_X = 2$) $\ddot{Q}$; $\ddot{E}_x$)—which we shall refer to as the *interior* $U_{\ddot{T}}$ of $\ddot{T}$ (respectively, (when $i_X = 2$) $U_{\ddot{Q}}$ of $\ddot{Q}$; $U_{\ddot{E}_x}$ of $\ddot{E}_x$)—determines a *tripod* $U_{\ddot{T}}$ (respectively, (when $i_X = 2$) *hyperbolic curve* $U_{\ddot{Q}}$; *tripod* $U_{\ddot{E}_x}$). Moreover, pr_2 induces *isomorphisms* $U_{\ddot{T}} \xrightarrow{\sim} U_T$, (when $i_X = 2$) $U_{\ddot{Q}} \xrightarrow{\sim} U_Q$; we have a diagram (cf. also Fig. 2 below)

$$\ddot{E}_x \ni \ddot{\nu}_x \in \ddot{T} \ni \ddot{\mu}_x \in \ddot{Q}$$

—where the final "$\in \ddot{Q}$" is to be *omitted* if $i_X = 1$; we refer to the unique node $\ddot{\nu}_x$ of $(X_2)_{x_s}$ that lies over $x_s \in X_s(s)$ (via pr_2) as the *sub-nexus* at x_s and to each of the remaining (one or two) nodes $\ddot{\mu}_x$ of $(X_2)_{x_s}$ as the *internal nodes* at x.

(ii) On the other hand, by applying Definition 1.4 to $X^{\log}_{n,\eta} \to \eta$, we obtain *major* and *minor cuspidal components* at x_η (i.e., the restriction $x_\eta \in X(\eta)$ of x to η), as well as a *nexus* at x_η—which we shall denote $F_x, E_x \subseteq (X_2)_{x_\eta}, \nu_x$. Write $\overline{F}_x, \overline{E}_x, \overline{\nu}_x$ for the closures of F_x, E_x, ν_x in $(X_2)_x \overset{\mathrm{def}}{=} X_2 \times_{X_1} S$ (where the fiber product is taken with respect to the morphisms $\mathrm{pr}_1 \colon X_2 \to X_1$, $x \colon S \to X_1 = X$). Thus, we have $\ddot{T} \subseteq \overline{F}_x$, (when $i_X = 2$) $\ddot{Q} \subseteq \overline{F}_x$, $\ddot{E}_x \subseteq \overline{E}_x$, $\ddot{\nu}_x \subseteq \overline{\nu}_x$. Write

$$U_{\overline{F}_x} \subseteq \overline{F}_x; \quad U_{\overline{E}_x} \subseteq \overline{E}_x$$

for the open subschemes given by the complements of the closures of the nodes and cusps of F_x, E_x. Thus, $U_{\overline{E}_x}$ is a family of *tripods* over S; pr_2 determines an *open immersion*

$$U_{\overline{F}_x} \hookrightarrow X$$

whose image is the complement of the cusps of X (relative to the log structure of $X^{\log}$).

(iii) Write $\tilde{T} \to T$ for the *normalization* of T; $\tilde{T}^{\log}$ for the log scheme obtained by equipping $\tilde{T}$ with the log structure determined by the closed points of $\tilde{T}$ that map to points of $T \setminus U_T$. Thus, $U_{\tilde{T}}$ is a *tripod* over s; we have a natural isomorphism ($\tilde{T} \supseteq$) $U_{\tilde{T}} \xrightarrow{\sim} U_T$ ($\subseteq T \subseteq X_s$). Write $\tilde{T}^{\log}_n \to s$ for the *n-th log configuration space* associated to $U_{\tilde{T}}$ (cf. §0). Thus, we have a *natural commutative diagram*

$$\begin{array}{ccc} \tilde{T}_2 & \longrightarrow & X_{2,s} \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \tilde{T} & \longrightarrow & X_s \end{array}$$

—where, by abuse of notation, we write $\mathrm{pr}_i \colon \tilde{T}_2 \to \tilde{T}_1 = \tilde{T}$ for the projection to the factor labeled i (for $i = 1, 2$); we write $\mathrm{pr}_i \colon X_{2,s} \to X_{1,s} = X_s$ for the restriction to the fibers over s of $\mathrm{pr}_i \colon X_2 \to X_1$ (for $i = 1, 2$); the horizontal arrows restrict to *immersions* on $U_{\tilde{T}_2}$, $U_{\tilde{T}}$; the lower horizontal arrow is compatible with the natural isomorphism $(\tilde{T} \supseteq)\ U_{\tilde{T}} \xrightarrow{\sim} U_T\ (\subseteq T \subseteq X_s)$. Write $(\tilde{T}_2)_{x_s}$ for the fiber of $\mathrm{pr}_1 \colon \tilde{T}_2 \to \tilde{T}_1$ over the point x_s, where, by abuse of notation, we write x_s for the point $\in \tilde{T}(s)$ determined by $x_s \in X_s(s)$. Then $(\tilde{T}_2)_{x_s}$ has *precisely two* irreducible components which map isomorphically to $\ddot{E}_x \subseteq (X_2)_{x_s}$, $\ddot{T} \subseteq (X_2)_{x_s}$—so $(\tilde{T}_2)_{x_s}$ may be thought of as consisting of a diagram

$$\ddot{E}_x \ni \ddot{\nu}_x \in \ddot{T}$$

—via the natural morphism $\tilde{T}_2 \to X_{2,s}$. By abuse of notation, we shall also use the notation $\ddot{E}_x$, $\ddot{T}$ for the corresponding irreducible components of $(\tilde{T}_2)_{x_s}$. Write $\Pi_n^{\mathrm{tripod}} \stackrel{\mathrm{def}}{=} \pi_1^{\Sigma}(\tilde{T}_n^{\log})$.

(iv) By applying the *specialization isomorphisms* (cf. §0) associated to the restriction of $\mathrm{pr}_1^{\log} \colon X_2^{\log} \to X_1^{\log}$ to the result of base-changing via $S^{\log} \to S_{\circ}^{\log}$ the completion of $X_{1\circ} = X_{\circ}$ along the cusp of $X_{\circ}$ determined by x, we conclude that the pointed stable curve structure on $(X_2)_{x_s}$ (cf. (i)) determines a "*semi-graph of anabelioids of pro-*Σ *PSC-type*" as discussed in [20], Definition 1.1, (i) (cf. also the discussion of [18], Appendix) whose associated "*PSC-fundamental group*" may be identified with $\Pi_{2/1}$. In particular, we obtain (conjugacy classes of) subgroups (cf. [20], Definition 1.1, (ii))

$$\Pi_{\ddot{T}},\ \Pi_{\ddot{Q}},\ \Pi_{\ddot{E}_x},\ \Pi_{\ddot{\nu}_x},\ \Pi_{\ddot{\mu}_x} \subseteq \Pi_{2/1}$$

(where $\Pi_{\ddot{Q}}$ is to be omitted if $i_X = 1$) corresponding to the sub-major and sub-minor cuspidal components, as well as to the sub-nexus and the internal node(s)—which we shall refer to as *sub-major verticial*, *sub-minor verticial*, *sub-nexus*, and *internal nodal*, respectively. In a similar (but simpler) vein, by applying the *specialization isomorphisms* (cf. §0) associated to $X^{\log} \to S^{\log}$, we obtain (conjugacy classes of) subgroups

$$\Pi_T,\ \Pi_Q \subseteq \Pi_1$$

(where Π_Q is to be omitted if $i_X = 1$)—such that the morphism $p_2 \colon \Pi_2 \to \Pi_1$ determines *isomorphisms*

$$\Pi_{\ddot{T}} \xrightarrow{\sim} \Pi_T; \quad \Pi_{\ddot{Q}} \xrightarrow{\sim} \Pi_Q$$

(where the second isomorphism is to be omitted if $i_X = 1$)—i.e., the sub-major verticial subgroups may be thought of as defining *sections of the projection* $p_2 \colon \Pi_2 \twoheadrightarrow \Pi_1$ over Π_T, (when $i_X = 2$) Π_Q. On the other hand, p_2 maps $\Pi_{\ddot{E}_x}$ onto a *cuspidal inertia group* of Π_1 associated to x; in particular, $p_2(\Pi_{\ddot{E}_x})$ is *abelian*. Finally, we observe that for *suitable choices* within the various conjugacy classes involved, we have *natural inclusions*

$$\Pi_{\ddot{E}_x} \supseteq \Pi_{\ddot{v}_x} \subseteq \Pi_{\ddot{T}} \supseteq \Pi_{\ddot{\mu}_x} \subseteq \Pi_{\ddot{Q}}$$

(where $\Pi_{\ddot{Q}}$ is to be omitted if $i_X = 1$) inside $\Pi_{2/1}$.
(v) On the other hand, by applying Definition 1.4 to $X^{\log}_{n,\eta} \to \eta$, we obtain (conjugacy classes of) subgroups

$$\Pi_{F_x}, \Pi_{E_x}, \Pi_{\nu_x} \subseteq \Pi_{2/1}$$

associated to F_x, E_x, ν_x (cf. (ii)) such that p_2 determines an *isomorphism* $\Pi_{F_x} \xrightarrow{\sim} \Pi_1$. For *suitable choices* within the various conjugacy classes involved, we have *natural inclusions*

$$\Pi_{E_x} \supseteq \Pi_{\nu_x} \subseteq \Pi_{F_x};$$

(inside $\Pi_{2/1}$), as well as *natural inclusions*

$$\Pi_{\ddot{T}}, \Pi_{\ddot{Q}} \subseteq \Pi_{F_x}$$

induced by the natural immersions $U_{\ddot{T}} \hookrightarrow U_{\overline{F}_x}$, $U_{\ddot{Q}} \hookrightarrow U_{\overline{F}_x}$ (where "$\Pi_{\ddot{Q}}$", "$U_{\ddot{Q}} \hookrightarrow U_{\overline{F}_x}$" are to be omitted if $i_X = 1$) by applying the isomorphisms

$$\pi_1^{\Sigma}((U_{\overline{F}_x} \times_X X^{\log}) \times_S s) \xrightarrow{\sim} \pi_1^{\Sigma}(X_s^{\log}) \xrightarrow{\sim} \pi_1^{\Sigma}(X^{\log}) \xrightarrow{\sim} \pi_1^{\Sigma}(U_{\overline{F}_x} \times_X X^{\log})$$

(arising from the *log purity theorem* and the *specialization isomorphism* for $X^{\log} \to S^{\log}$), together with the isomorphisms $\pi_1^{\Sigma}(U_{\overline{F}_x} \times_X X^{\log}) \xrightarrow{\sim} \pi_1^{\Sigma}(U_{F_x}) \xrightarrow{\sim} \Pi_{F_x}$ (the first of which arises from the log purity theorem). In a similar (but simpler) vein, we have *equalities* (of $\Pi_{2/1}$-conjugacy classes of subgroups of $\Pi_{2/1}$)

$$\Pi_{\ddot{E}_x} = \Pi_{E_x}; \quad \Pi_{\ddot{v}_x} = \Pi_{\nu_x}$$

induced by the natural immersion $U_{\ddot{E}_x} \hookrightarrow U_{\overline{E}_x}$ by applying the isomorphism $\pi_1^{\Sigma}(U_{\overline{E}_x} \times_S s) \xrightarrow{\sim} \pi_1^{\Sigma}(U_{\overline{E}_x})$ (arising from the *log purity theorem* and the *specialization isomorphism* for the smooth log curve determined, up to unique isomorphism, by the family of tripods $U_{\overline{E}_x} \to S$), together with the isomorphisms $\pi_1^{\Sigma}(U_{\overline{E}_x}) \xrightarrow{\sim} \pi_1^{\Sigma}(U_{E_x}) \xrightarrow{\sim} \Pi_{E_x}$ (the first of which arises from the log purity theorem).
(vi) One verifies immediately that the natural commutative diagram of (iii) determines a *natural morphism of exact sequences of profinite groups*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{2/1}^{\text{tripod}} & \longrightarrow & \Pi_2^{\text{tripod}} & \longrightarrow & \Pi_1^{\text{tripod}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi_{2/1} & \longrightarrow & \Pi_2 & \longrightarrow & \Pi_1 & \longrightarrow & 1
\end{array}$$

—where the vertical arrows are *injective outer homomorphisms*; the image of the vertical morphism on the right is equal to Π_T. By abuse of notation, we shall write $\Pi_{2/1}^{\text{tripod}}$ (respectively, Π_2^{tripod}; Π_1^{tripod}) for the subgroup, well-defined up to $\Pi_{2/1}$- (respectively, Π_2-; Π_1-) conjugacy, determined by the image of the left-hand (respectively, middle; right-hand) vertical arrow. Thus, for *suitable choices* within the various conjugacy classes involved, we have *natural inclusions*

$$\Pi_{\ddot{E}_x},\ \Pi_{\ddot{T}},\ \Pi_{\ddot{\nu}_x} \subseteq \Pi_{2/1}^{\text{tripod}}$$

(inside $\Pi_{2/1}$).

REMARK 2.1.1. One verifies immediately that data as in Definition 2.1 exists for *arbitrary* (g, r) *such that* $(g, r) \neq (0, 3)$ *and* $r \geq 1$. Moreover, the case $i_X = 1$ corresponds precisely to the case where $(g, r) = (1, 1)$.

Proposition 2.2 (First properties of sub-major and sub-minor verticial subgroups). *In the notation of* Definition 2.1:
(i) $\Pi_{\ddot{T}}$, (*when* $i_X = 2$) $\Pi_{\ddot{Q}}$, $\Pi_{\ddot{E}_x}$, $\Pi_{\ddot{\nu}_x}$, $\Pi_{\ddot{\mu}_x}$, Π_{F_x}, Π_{E_x}, Π_{ν_x}, $\Pi_{2/1}^{\text{tripod}}$ *are* commensurably terminal *in* $\Pi_{2/1}$; Π_T, (*when* $i_X = 2$) Π_Q *are commensurably terminal in* Π_1.
(ii) *Suppose that one* fixes $\Pi_{\nu_x} \subseteq \Pi_{2/1}$ *among its various* $\Pi_{2/1}$*-conjugates. Then the* condition *that there exist inclusions/equalities*

$$\Pi_{\nu_x} \subseteq \Pi_{E_x}; \quad \Pi_{\nu_x} = \Pi_{\ddot{\nu}_x} \subseteq \Pi_{\ddot{T}} \subseteq \Pi_{F_x};$$
$$\Pi_{\ddot{E}_x} = \Pi_{E_x}; \quad \Pi_{\ddot{E}_x},\ \Pi_{\ddot{T}} \subseteq \Pi_{2/1}^{\text{tripod}}$$

completely determines Π_{E_x}, $\Pi_{\ddot{\nu}_x}$, $\Pi_{\ddot{T}}$, Π_{F_x}, $\Pi_{\ddot{E}_x}$, *and* $\Pi_{2/1}^{\text{tripod}}$ *among their various* $\Pi_{2/1}$*-conjugates.*
(iii) *In the notation of* (ii), *the compatible inclusions* $\Pi_{\ddot{\nu}_x} \subseteq \Pi_{\ddot{E}_x} \subseteq \Pi_{2/1}^{\text{tripod}}$, $\Pi_{\ddot{\nu}_x} \subseteq \Pi_{\ddot{T}} \subseteq \Pi_{2/1}^{\text{tripod}}$, $\Pi_{\nu_x} \subseteq \Pi_{E_x} \subseteq \Pi_{2/1}$, $\Pi_{\nu_x} \subseteq \Pi_{F_x} \subseteq \Pi_{2/1}$, *determine* isomorphisms

$$\varinjlim(\Pi_{\ddot{E}_x} \hookleftarrow \Pi_{\ddot{\nu}_x} \hookrightarrow \Pi_{\ddot{T}}) \xrightarrow{\sim} \Pi_{2/1}^{\text{tripod}};$$
$$\varinjlim(\Pi_{E_x} \hookleftarrow \Pi_{\nu_x} \hookrightarrow \Pi_{F_x}) \xrightarrow{\sim} \Pi_{2/1}$$

—*where the* inductive limits *are taken in the category of* pro-Σ groups.

Proof. Assertion (i) follows from [20], Proposition 1.2, (ii). Assertion (ii) follows from the fact that "every nodal edge-like subgroup is contained in *precisely two verticial subgroups*" (cf. [20], Proposition 1.5, (i)), together with the fact that $\Pi_{2/1}^{\text{tripod}}$ is *topologically generated* by $\Pi_{\ddot{E}_x}$, $\Pi_{\ddot{T}}$ (cf. assertion (iii)). Assertion (iii) follows by a similar argument to the argument applied in the proof of Proposition 1.5, (iii). $\square$

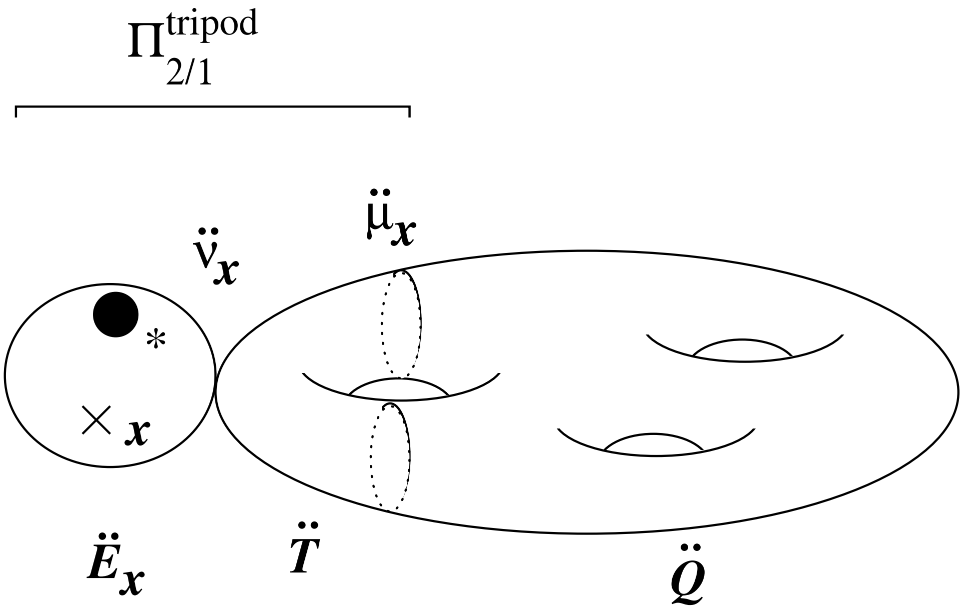


Fig. 2. A degenerating affine curve equipped with an extra cusp "$*$".

Corollary 2.3 (Injectivity for non-tripod degenerating affine curves). *In the notation of* Definition 2.1 (*cf. also* Definition 1.1; Remark 2.1.1):

(i) *The natural inclusion* $\Xi_2 \hookrightarrow \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$ *is an* isomorphism.

(ii) *The natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$$

induced by $p_1\colon \Pi_2 \twoheadrightarrow \Pi_1$ *is* injective.

(iii) *We have*: $\mathrm{Out}^{\mathrm{FCP}}(\Pi_2) = \mathrm{Out}^{\mathrm{FC}}(\Pi_2)$.

Proof. First, we observe that assertion (ii) follows formally from assertion (i) and Proposition 1.2, (iii). Next, we observe that assertion (iii) follows formally from assertion (ii) and Propositions 1.2, (iii); 1.6, (iii). Thus, to complete the proof of Corollary 2.3, it suffices to verify assertion (i). To this end, let $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$. Let us *fix* some $\Pi_{\nu_x} \subseteq \Pi_{2/1}$ among its various $\Pi_{2/1}$-conjugates; let Π_{E_x}, $\Pi_{\ddot{\nu}_x}$, $\Pi_{\ddot{T}}$, Π_{F_x}, $\Pi_{\ddot{E}_x}$, and $\Pi_{2/1}^{\mathrm{tripod}}$ be as in Proposition 2.2, (ii).

Since $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$, it follows that α induces (relative to p_1 or p_2) an automorphism of Π_1 that *stabilizes* every cuspidal inertia group of Π_1. Thus, by the non-resp'd portion of Proposition 1.3, (iv), we conclude that α stabilizes the $\Pi_{2/1}$-conjugacy classes of $\Pi_{\nu_x} = \Pi_{\ddot{\nu}_x}$, Π_{F_x}, $\Pi_{E_x} = \Pi_{\ddot{E}_x}$. In particular, $\alpha(\Pi_{\nu_x}) = \zeta \cdot \Pi_{\nu_x} \cdot \zeta^{-1}$, for some $\zeta \in \Pi_{2/1}$. Since $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$, and $p_2(\Pi_{\nu_x})$ is a cuspidal inertia group of Π_1 associated to x, hence *normally terminal* in Π_1 (cf. [20], Proposition 1.2, (ii)), it thus follows that $p_2(\zeta) \in p_2(\Pi_{\nu_x})$, so (by replacing ζ by an appropriate element $\in \zeta \cdot \Pi_{\nu_x}$) we may assume without loss of generality that $\zeta \in \Pi_{2/1} \cap \Pi_{1\backslash 2} = \Xi_2$. Thus, by replacing α by the composite of α with a Ξ_2-inner automorphism, we may assume without loss of generality that $\alpha(\Pi_{\nu_x}) = \Pi_{\nu_x}$. By Proposition 2.2, (ii), we thus conclude that $\alpha(\Pi_{F_x}) = \Pi_{F_x}$, $\alpha(\Pi_{E_x}) = \Pi_{E_x}$. Since $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$, and p_2 induces an *isomorphism* $\Pi_{F_x} \xrightarrow{\sim} \Pi_1$ (cf. Definition 2.1, (v)), we thus conclude that α restricts to the *identity* on Π_{F_x}. In particular, it follows that α *stabilizes* and restricts to the *identity* on $\Pi_{\ddot{T}}$. Since $\Pi_{2/1}^{\mathrm{tripod}}$ is *topologically generated* by $\Pi_{\ddot{E}_x} = \Pi_{E_x}$, $\Pi_{\ddot{T}}$ (cf. Proposition 2.2, (iii)), we thus conclude that $\alpha(\Pi_{2/1}^{\mathrm{tripod}}) = \Pi_{2/1}^{\mathrm{tripod}}$.

Now since $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$, and $\Pi_{2/1}^{\mathrm{tripod}}$ is *normally terminal* in $\Pi_{2/1}$ (cf. Proposition 2.2, (i)), we thus conclude from the commutative diagram of Definition 2.1, (vi) (i.e., by applying the natural isomorphism $\Pi_2^{\mathrm{tripod}} \xrightarrow{\sim} \Pi_{2/1}^{\mathrm{tripod}} \overset{\mathrm{out}}{\rtimes} \Pi_1^{\mathrm{tripod}}$—cf. §0; Remark 1.1.1), that the automorphism of $\Pi_{2/1}^{\mathrm{tripod}}$ induced by α arises from an automorphism $\alpha^{\mathrm{tripod}} \in \mathrm{Aut}(\Pi_2^{\mathrm{tripod}})$, which is easily verified to be *F-admissible* (cf. Proposition 1.2, (i)). Next, observe that since $\Pi_{\ddot{E}_x}$ is *normally terminal* in $\Pi_{2/1}$ (cf. Proposition 2.2, (i)), it follows immediately from [20], Proposition 1.5, (i), that every cuspidal inertia group of $\Pi_{2/1}$ that is *contained* in $\Pi_{\ddot{E}_x}$ and $\Pi_{2/1}$-*conjugate* to a cuspidal inertia group associated to a cusp of $U_{\ddot{E}_x}$ is, in fact, *equal* to a cuspidal inertia group associated to a cusp of $U_{\ddot{E}_x}$. Since α is C-admissible, and $\alpha \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2)$ restricts to the identity on $\Pi_{\ddot{T}}$, we thus conclude that α^{tripod} is *IFC-admissible*, i.e., $\alpha^{\mathrm{tripod}} \in \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2^{\mathrm{tripod}})$.

On the other hand, by Corollary 1.12, (i), it follows that α^{tripod} lies in the image of the natural inclusion $\Xi_2^{\mathrm{tripod}} \hookrightarrow \mathrm{Aut}^{\mathrm{IFC}}(\Pi_2^{\mathrm{tripod}})$ (where we write Ξ_2^{tripod} for the analogue of "Ξ_2" for Π_2^{tripod}). In particular, we conclude that α induces an *inner automorphism* of $\Pi_{2/1}^{\mathrm{tripod}}$. Since α restricts to the identity on $\Pi_{\ddot{T}}$, which is *center-free* (cf. Remark 1.1.1) and *normally terminal* in $\Pi_{2/1}^{\mathrm{tripod}}$ (cf. Proposition 2.2, (i)), it thus follows that α restricts to the *identity* on $\Pi_{2/1}^{\mathrm{tripod}}$, hence also on $\Pi_{\ddot{E}_x} = \Pi_{E_x}$. Since $\Pi_{2/1}$ is *topologically generated* by Π_{E_x}, Π_{F_x} (cf. Proposition 2.2, (iii)), we thus conclude that α restricts to the *identity* on $\Pi_{2/1}$, hence (by applying the natural isomorphism $\Pi_2 \xrightarrow{\sim} \Pi_{2/1} \overset{\mathrm{out}}{\rtimes} \Pi_1$—cf. §0; Remark 1.1.1) that α is the *identity*. This completes the proof of assertion (i). ☐

Before proceeding, we recall the following well-known result.

Lemma 2.4 (FC-admissible permutations of cusps). *There exist elements* $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *that induce, relative to the standard surjection* $\Pi_n \twoheadrightarrow \Pi_1$, arbitrary permutations *of the set of conjugacy classes of cuspidal inertia groups of* Π_1 (*i.e., the set of cusps of* $X^{\log}$).

Proof. One way to verify Lemma 2.4 is by thinking of Π_n as the pro-Σ completion of the topological fundamental group of the n-th configuration space associated to (i.e., the complement of the various diagonals in the product of n copies of) a *topological surface* $\mathcal{X}$ of type (g, r) (cf. the theory of [24], §7). Then it is easy to construct a *homeomorphism* of $\mathcal{X}$ that induces an *arbitrary permutation* of the cusps; one then verifies immediately that such a homeomorphism induces a homeomorphism of the n-th configuration space associated to $\mathcal{X}$ that gives rise to an element $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ satisfying the conditions in the statement of Lemma 2.4.

Alternatively, one may give a more *log scheme-theoretic* proof by means of the objects introduced in the discussion preceding Definition 2.1 as follows. If $r \leq 1$, then there is nothing to show. Thus, we suppose that $r \geq 2$. Then (by applying the *specialization isomorphisms* of §0) it suffices to verify the existence of automorphisms of $X_s^{\log}$ over $s^{\log}$ that induce *arbitrary transpositions* (i.e., permutations that switch two elements and leave the remaining elements fixed) of the set of cusps of $X_s^{\log}$. If $(g, r) = (0, 3)$ (i.e., $X_s^{\log}$ is a *tripod*), then the existence of such automorphisms of $X_s^{\log}$ (over $s^{\log}$) follows immediately from the well-known structure of tripods. Thus, we may assume that $(g, r) \neq (0, 3)$. This assumption implies (cf. Remark 2.1.1) that we may suppose that we are in the situation of Definition 2.1, and that *precisely two* of the cusps of the *tripod* U_T arise from cusps a, b of $X_s^{\log}$. Then (by the case where $(g, r) = (0, 3)$, which has already been verified) U_T admits an automorphism (over s) that switches the two cusps of U_T corresponding to a, b and leaves the remaining cusp of U_T fixed. Moreover, one verifies immediately that such an automorphism of U_T extends to an automorphism of $X_s^{\log}$ (over $s^{\log}$) that switches a and b and restricts to the identity on Q (hence leaves the remaining cusps of $X_s^{\log}$ fixed). This completes the proof of Lemma 2.4. □

3. Conditional surjectivity for affine curves

In the present §3, we prove a certain special case (cf. Corollary 3.3) of the *surjectivity* portion of our main result (cf. Theorem 4.1 below) for affine hyperbolic curves. The key observation is that the technical obstacles observed, relative to verifying surjectivity, in Remarks 1.13.1, 1.13.2 may be circumvented if one *replaces* "$\Pi_2 \twoheadrightarrow \Pi_1$" by "$\Pi_3 \twoheadrightarrow \Pi_2$" and *works with the subset* "$\triangle+$" of Definition 1.11, (ii).

We return to the notation of §1 (cf. especially the notation of Definition 1.4 and of the discussion preceding Definition 1.8).

DEFINITION 3.1. Let $x \in X(k)$ be a *cusp* of $X^{\log}$. Write $\underline{x} \in X_2(k)$ for the *nexus* ν_x (cf. Definition 1.4, (i)).

(i) Observe that the log structure on $X_3^{\log}$ determines on the fiber $(X_3)_{\underline{x}}$ of the morphism $\underline{\mathrm{pr}}_{12}$: $X_3 \to X_2$ over the point $\underline{x} \in X_2(k)$ a structure of *pointed stable curve*, which consists of *three* irreducible components. Of these three irreducible components, there is a unique irreducible component $\underline{F}_{\underline{x}}$—which we shall refer to as the *quasi-major cuspidal* *component* of $(X_3)_{\underline{x}}$—that *maps isomorphically* to X via $\underline{\mathrm{pr}}_3\colon X_3 \to X_1 = X$; there is a unique irreducible component $\underline{L}_x$—which we shall refer to as the *link cuspidal component* of $(X_3)_{\underline{x}}$—that intersects $\underline{F}_x$ at a single point; there is a unique irreducible component $\underline{E}_x$—which we shall refer to as the *quasi-minor cuspidal component* of $(X_3)_{\underline{x}}$—that intersects $\underline{L}_x$ at a single point. (Thus, $\underline{L}_x$, $\underline{E}_x$ map to the point $x \in X(k)$ via $\underline{\mathrm{pr}}_3$.) The complement in $\underline{F}_x$ (respectively, $\underline{L}_x$; $\underline{E}_x$) of the nodes and cusps (relative to the pointed stable curve structure on $(X_3)_{\underline{x}}$) of $\underline{F}_x$ (respectively, $\underline{L}_x$; $\underline{E}_x$)—which we shall refer to as the *interior* $U_{\underline{F}_x}$ of $\underline{F}_x$ (respectively, $U_{\underline{L}_x}$ of $\underline{L}_x$; $U_{\underline{E}_x}$ of $\underline{E}_x$)—determines a *hyperbolic curve* $U_{\underline{F}_x}$ (respectively, *tripod* $U_{\underline{L}_x}$; *tripod* $U_{\underline{E}_x}$). Moreover, $\underline{\mathrm{pr}}_3$ induces *isomorphisms* $U_{\underline{F}_x} \xrightarrow{\sim} U_X$, $\underline{F}_x \xrightarrow{\sim} X$.

(ii) By applying the *specialization isomorphisms* (cf. §0) associated to the restriction of $\underline{\mathrm{pr}}_{12}^{\log}\colon X_3^{\log} \to X_2^{\log}$ to the completion of X_2 along $\underline{x}$, we conclude that the pointed stable curve structure on $(X_3)_{\underline{x}}$ (cf. (i)) determines a "*semi-graph of anabelioids of pro-Σ PSC-type*" as discussed in [20], Definition 1.1, (i) (cf. also the discussion of [18], Appendix) whose associated "*PSC-fundamental group*" may be identified with $\Pi_{3/2}$. In particular, the quasi-major, link, and quasi-minor cuspidal components determine (conjugacy classes of) *verticial subgroups* (cf. [20], Definition 1.1, (ii))

$$\Pi_{\underline{F}_x},\ \Pi_{\underline{L}_x},\ \Pi_{\underline{E}_x} \subseteq \Pi_{3/2}$$

—which we shall refer to as *quasi-major*, *link*, and *quasi-minor*, respectively. Thus, the morphism $\underline{p}_3\colon \Pi_3 \to \Pi_1$ determines an *isomorphism*

$$\Pi_{\underline{F}_x} \xrightarrow{\sim} \Pi_1$$

—i.e., the quasi-major verticial subgroups may be thought of as defining *sections of the projection* $\underline{p}_3\colon \Pi_3 \twoheadrightarrow \Pi_1$. On the other hand, $\underline{p}_3$ maps $\Pi_{\underline{L}_x}$, $\Pi_{\underline{E}_x}$ onto *cuspidal inertia groups* of Π_1 associated to x; in particular, $\underline{p}_3(\Pi_{\underline{L}_x})$, $\underline{p}_3(\Pi_{\underline{E}_x})$ are *abelian*. Finally, let us refer to the node $\underline{\nu}_x \in \underline{E}_x \cap \underline{L}_x$ (respectively, $\underline{\mu}_x \in \underline{L}_x \cap \underline{F}_x$) of $(X_2)_x$ as the *$\underline{x}$-minor-nexus* (respectively, *$\underline{x}$-major-nexus*) (of $(X_3)_{\underline{x}}$)—so (cf. Fig. 3 below)

$$\underline{E}_x \ni \underline{\nu}_x \in \underline{L}_x \ni \underline{\mu}_x \in \underline{F}_x$$

—and to the (nodal) *edge-like subgroup* (cf. [20], Definition 1.1, (ii))

$$\Pi_{\underline{\nu}_x} \subseteq \Pi_{3/2} \quad \text{(respectively, } \Pi_{\underline{\mu}_x} \subseteq \Pi_{3/2}\text{)}$$

determined up to conjugacy by $\underline{\nu}_x$ (respectively, $\underline{\mu}_x$) as an *$\underline{x}$-minor-nexus* (respectively, *$\underline{x}$-major-nexus*) subgroup. Thus, for *suitable choices* within the various conjugacy classes involved, we have *natural inclusions*

$$\Pi_{\underline{E}_{\underline{x}}} \supseteq \Pi_{\underline{\nu}_{\underline{x}}} \subseteq \Pi_{\underline{L}_{\underline{x}}} \supseteq \Pi_{\underline{\mu}_{\underline{x}}} \subseteq \Pi_{\underline{F}_{\underline{x}}}$$

(inside $\Pi_{3/2}$).
(iii) We shall refer to

$$\underline{B}_{\underline{\nu}} \stackrel{\text{def}}{=} \underline{E}_{\underline{x}} \cup \underline{L}_{\underline{x}} \quad \text{(respectively, } \underline{B}_{\underline{\mu}} \stackrel{\text{def}}{=} \underline{L}_{\underline{x}} \cup \underline{F}_{\underline{x}}\text{)}$$

as the $\underline{\nu}$-*bridge* (respectively, $\underline{\mu}$-*bridge*) of $(X_3)_{\underline{x}}$. If the various choices within conjugacy classes are made so that the *natural inclusions* of (ii) hold, then we shall refer to the subgroup (well-defined up to $\Pi_{3/2}$-conjugacy)

$$\Pi_{\underline{B}_{\underline{\nu}}} \subseteq \Pi_{3/2} \quad \text{(respectively, } \Pi_{\underline{B}_{\underline{\mu}}} \subseteq \Pi_{3/2}\text{)}$$

topologically generated by $\Pi_{\underline{E}_{\underline{x}}}$ and $\Pi_{\underline{L}_{\underline{x}}}$ (respectively, by $\Pi_{\underline{L}_{\underline{x}}}$ and $\Pi_{\underline{F}_{\underline{x}}}$) as the $\underline{\nu}$-*bridge subgroup* (respectively, $\underline{\mu}$-*bridge subgroup*).
(iv) Recall the subgroups $\mathbb{I}_{F_x} \subseteq \mathbb{D}_{F_x} \subseteq \Pi_2$ (respectively, $\mathbb{I}_{E_x} \subseteq \mathbb{D}_{E_x} \subseteq \Pi_2$) of Proposition 1.6 (respectively, 1.7). By applying the *specialization isomorphisms* of §0 first over the completion of F_x (respectively, E_x) along $\underline{x}$, and then over the completion of X_2 along the generic point of U_{F_x} (respectively, U_{E_x}), we conclude that the outer action of $\mathbb{D}_{F_x}$ (respectively, $\mathbb{D}_{E_x}$) on $\Pi_{3/2}$ *stabilizes* the $\Pi_{3/2}$-conjugacy classes of $\Pi_{\underline{E}_{\underline{x}}}$, $\Pi_{\underline{\nu}_{\underline{x}}}$, and $\Pi_{\underline{B}_{\underline{\mu}}}$ (respectively, of $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$). Since, moreover, $\Pi_{\underline{E}_{\underline{x}}}$, $\Pi_{\underline{\nu}_{\underline{x}}}$, and $\Pi_{\underline{B}_{\underline{\mu}}}$ (respectively, of $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$) are *commensurably terminal* in $\Pi_{3/2}$ (cf. Proposition 3.2, (i), below), it follows that this outer action determines outer actions of $\mathbb{D}_{F_x}$ (respectively, $\mathbb{D}_{E_x}$) on $\Pi_{\underline{E}_{\underline{x}}}$, $\Pi_{\underline{\nu}_{\underline{x}}}$, and $\Pi_{\underline{B}_{\underline{\mu}}}$ (respectively, of $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$), whose restriction to $\mathbb{I}_{F_x}$ (respectively, $\mathbb{I}_{E_x}$) is *trivial* (cf. the theory of specialization isomorphisms reviewed in §0). Thus, we obtain *outer actions* of $\mathbb{D}_{F_x}/\mathbb{I}_{F_x} \xrightarrow{\sim} \Pi_{F_x}$ (respectively, $\mathbb{D}_{E_x}/\mathbb{I}_{E_x} \xrightarrow{\sim} \Pi_{E_x}$) on $\Pi_{\underline{E}_{\underline{x}}}$, $\Pi_{\underline{\nu}_{\underline{x}}}$, and $\Pi_{\underline{B}_{\underline{\mu}}}$ (respectively, of $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$). Since the irreducible component of $X_3|_{U_{F_x}}$ (respectively, $X_3|_{U_{E_x}}$) (where "$|$" is taken with respect to $\underline{\text{pr}}_{12}\colon X_3 \to X_2$) determined by $\underline{E}_{\underline{x}}$ (respectively, $\underline{F}_{\underline{x}}$) *descends* from U_{F_x} (respectively, U_{E_x}) to k —i.e., is naturally isomorphic to $U_{F_x} \times_k \underline{E}_{\underline{x}}$ (respectively, $U_{E_x} \times_k \underline{F}_{\underline{x}}$)—we thus conclude that the outer action of Π_{F_x} (respectively, Π_{E_x}) on $\Pi_{\underline{E}_{\underline{x}}}$ (respectively, on $\Pi_{\underline{F}_{\underline{x}}}$) is *trivial*.
(v) On the other hand, the outer action of Π_{F_x} on $\Pi_{\underline{B}_{\underline{\mu}}}$ may be made more explicit, as follows. Write $x^{\log} \stackrel{\text{def}}{=} X^{\log} \times_X x$. Recall that the geometric fibers of $\underline{\text{pr}}_1^{\log}\colon X_3^{\log} \to X_1^{\log} = X^{\log}$ over points of U_X may be regarded as 2-*nd log configuration spaces* associated to the smooth log curves determined by the corresponding fibers of $\text{pr}_1^{\log}\colon X_2^{\log} \to X_1^{\log} = X^{\log}$ (cf. [24], Remark 2.1.2). In a similar way, even though the fiber $(X_2^{\log})_{x^{\log}}$ of $\text{pr}_1^{\log}$ over $x^{\log}$ is a *non-smooth* stable log curve, we may think of the fiber $(X_3^{\log})_{x^{\log}}$ of $\underline{\text{pr}}_{12}^{\log}$ over $x^{\log}$ as the "2-*nd log configuration space*" associated to $(X_2^{\log})_{x^{\log}}$—i.e., in the

sense that it may be obtained as the pull-back of the (1-)morphism $\overline{\mathcal{M}}^{\log}_{g,r+3} \to \overline{\mathcal{M}}^{\log}_{g,r+1}$ (determined by forgetting the last two sections) via the classifying (1-)morphism $x^{\log} \to \overline{\mathcal{M}}^{\log}_{g,r+1}$. If we forget the various log structures involved, then it follows from this point of view that the natural inclusion $X \xrightarrow{\sim} F_x \hookrightarrow (X_2)_x$ fits into a *natural commutative diagram*

$$\begin{array}{ccc} X_2 & \hookrightarrow & (X_3)_x \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \underline{\mathrm{pr}}_{12}} \\ X & \hookrightarrow & (X_2)_x \end{array}$$

—where (by abuse of notation) we use the notation "$\underline{\mathrm{pr}}_{12}$" to denote the appropriate restriction of $\underline{\mathrm{pr}}_{12}$. Now one verifies immediately (cf. Definition 2.1, (vi)) that this commutative diagram determines a *natural morphism of exact sequences of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{2/1} & \longrightarrow & \Pi_2 & \longrightarrow & \Pi_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{3/2} & \longrightarrow & \Pi_{3/1} & \longrightarrow & \Pi_{2/1} & \longrightarrow & 1 \end{array}$$

—where the vertical arrows are *injective outer homomorphisms*; the image of the vertical morphism on the left is equal to $\Pi_{\underline{B}_\mu}$; the image of the vertical morphism on the right is equal to Π_{F_x}. In particular, this commutative diagram of profinite groups allows one to *identify the outer action of* Π_{F_x} *on* $\Pi_{\underline{B}_\mu}$ *with the outer action of* Π_1 *on* $\Pi_{2/1}$.

(vi) In a similar vein, the outer action of Π_{E_x} on a$\Pi_{\underline{B}_\nu}$ may be made more explicit, as follows. Write $T^{\log}$ for the smooth log curve over k determined by the *tripod* E_x; $T^{\log}_n$ for the corresponding *n-th log configuration space* (where $n \geq 1$ is an integer); $\Pi^{\mathrm{tripod}}_n \stackrel{\mathrm{def}}{=} \pi_1^{\Sigma}(T^{\log}_n)$. Then just as in (v), we obtain a *natural commutative diagram*

$$\begin{array}{ccc} T_2 & \hookrightarrow & (X_3)_x \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \underline{\mathrm{pr}}_{12}} \\ T & \hookrightarrow & (X_2)_x \end{array}$$

—where we use the notation "$\underline{\mathrm{pr}}_{12}$" as in (v). Moreover, just as in (v) (cf. also Definition 2.1, (vi)), this commutative diagram determines a *natural morphism of exact sequences of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi^{\mathrm{tripod}}_{2/1} & \longrightarrow & \Pi^{\mathrm{tripod}}_2 & \longrightarrow & \Pi^{\mathrm{tripod}}_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{3/2} & \longrightarrow & \Pi_{3/1} & \longrightarrow & \Pi_{2/1} & \longrightarrow & 1 \end{array}$$

—where the vertical arrows are *injective outer homomorphisms*; the image of the vertical morphism on the left is equal to $\Pi_{\underline{B}_\nu}$; the image of the vertical morphism on the right is equal to Π_{E_x}. In particular, this commutative diagram of profinite groups allows one to *identify the outer action of* Π_{E_x} *on* $\Pi_{\underline{B}_\nu}$ *with the outer action of* Π_1^{tripod} *on* $\Pi_{2/1}^{\text{tripod}}$.

Proposition 3.2 (First properties of quasi-major, link, and quasi-minor verticial subgroups). *In the notation of* Definition 3.1:

(i) $\Pi_{\underline{\nu}_x}$, $\Pi_{\underline{\mu}_x}$, $\Pi_{\underline{E}_x}$, $\Pi_{\underline{L}_x}$, $\Pi_{\underline{F}_x}$, $\Pi_{\underline{B}_\nu}$, *and* $\Pi_{\underline{B}_\mu}$, *are* commensurably terminal *in* $\Pi_{3/2}$.

(ii) *Suppose that one* fixes $\Pi_{\underline{\nu}_x} \subseteq \Pi_{3/2}$ (*respectively,* $\Pi_{\underline{\mu}_x} \subseteq \Pi_{3/2}$) *among its various* $\Pi_{3/2}$*-conjugates. Then the* condition *that there exist inclusions*

$$\Pi_{\underline{\nu}_x} \subseteq \Pi_{\underline{E}_x}; \quad \Pi_{\underline{\nu}_x} \subseteq \Pi_{\underline{L}_x}; \quad \Pi_{\underline{\nu}_x} \subseteq \Pi_{\underline{B}_\mu}$$
$$(\textit{respectively, } \Pi_{\underline{\mu}_x} \subseteq \Pi_{\underline{B}_\nu};\ \Pi_{\underline{\mu}_x} \subseteq \Pi_{\underline{L}_x};\ \Pi_{\underline{\mu}_x} \subseteq \Pi_{\underline{F}_x})$$

completely determines $\Pi_{\underline{E}_x}$, $\Pi_{\underline{L}_x}$, $\Pi_{\underline{B}_\nu}$, *and* $\Pi_{\underline{B}_\mu}$ (*respectively,* $\Pi_{\underline{B}_\nu}$, $\Pi_{\underline{B}_\mu}$, $\Pi_{\underline{L}_x}$, *and* $\Pi_{\underline{F}_x}$) *among their various* $\Pi_{3/2}$*-conjugates.*

(iii) *In the notation of* (ii), *the compatible inclusions* $\Pi_{\underline{\nu}_x} \subseteq \Pi_{\underline{E}_x} \subseteq \Pi_{\underline{B}_\nu} \subseteq \Pi_{3/2}$, $\Pi_{\underline{\nu}_x} \subseteq \Pi_{\underline{L}_x} \subseteq \Pi_{\underline{B}_\nu} \subseteq \Pi_{3/2}$, $\Pi_{\underline{\mu}_x} \subseteq \Pi_{\underline{L}_x} \subseteq \Pi_{\underline{B}_\mu} \subseteq \Pi_{3/2}$, $\Pi_{\underline{\mu}_x} \subseteq \Pi_{\underline{F}_x} \subseteq \Pi_{\underline{B}_\mu} \subseteq \Pi_{3/2}$, *determine* isomorphisms

$$\varinjlim\left(\Pi_{\underline{E}_x} \hookleftarrow \Pi_{\underline{\nu}_x} \hookrightarrow \Pi_{\underline{L}_x}\right) \xrightarrow{\sim} \Pi_{\underline{B}_\nu},$$
$$\varinjlim\left(\Pi_{\underline{E}_x} \hookleftarrow \Pi_{\underline{\nu}_x} \hookrightarrow \Pi_{\underline{B}_\mu}\right) \xrightarrow{\sim} \Pi_{3/2},$$
$$\varinjlim\left(\Pi_{\underline{L}_x} \hookleftarrow \Pi_{\underline{\mu}_x} \hookrightarrow \Pi_{\underline{F}_x}\right) \xrightarrow{\sim} \Pi_{\underline{B}_\mu},$$
$$\varinjlim\left(\Pi_{\underline{B}_\nu} \hookleftarrow \Pi_{\underline{\mu}_x} \hookrightarrow \Pi_{\underline{F}_x}\right) \xrightarrow{\sim} \Pi_{3/2}$$

—*where the* inductive limits *are taken in the category of* pro-Σ groups.

(iv) *The operation of restriction to the various subgroups involved determines a* bijection *between*

> *the set of* outer automorphisms *of* $\Pi_{3/2}$ *that* stabilize *the* $\Pi_{3/2}$*-conjugacy classes of* $\Pi_{\underline{\nu}_x}$, $\Pi_{\underline{\mu}_x}$, $\Pi_{\underline{E}_x}$, $\Pi_{\underline{L}_x}$, $\Pi_{\underline{F}_x}$, $\Pi_{\underline{B}_\nu}$, *and* $\Pi_{\underline{B}_\mu}$

and

> *the set of* pairs $\alpha_{\underline{\nu}} \in \mathrm{Out}(\Pi_{\underline{B}_\nu})$, $\alpha_{\underline{\mu}} \in \mathrm{Out}(\Pi_{\underline{B}_\mu})$

such that:

(a) $\alpha_{\underline{\nu}}$ (*respectively,* $\alpha_{\underline{\mu}}$) stabilizes *the* $\Pi_{\underline{B}_\nu}$- (*respectively,* $\Pi_{\underline{B}_\mu}$-) *conjugacy classes of* $\Pi_{\underline{E}_x}$, $\Pi_{\underline{\nu}_x}$, $\Pi_{\underline{L}_x}$, *and* $\Pi_{\underline{\mu}_x}$ (*respectively, of* $\Pi_{\underline{\nu}_x}$, $\Pi_{\underline{L}_x}$, $\Pi_{\underline{\mu}_x}$, *and* $\Pi_{\underline{F}_x}$);

(b) $\alpha_{\underline{\nu}}$ *and* $\alpha_{\underline{\mu}}$ *induce* (*cf.* (a); (i)) *the* same *element* $\in \mathrm{Out}(\Pi_{\underline{L}_x})$.

Proof. Assertions (i), (ii), (iii) follow from precisely the same arguments applied to prove assertions (i), (ii), and (iii) of Proposition 1.5. In light of assertions (i), (ii), (iii), assertion (iv) follows, in a straightforward manner, from the fact that $\Pi_{\underline{L}_x}$ is *center-free* (cf. Remark 1.1.1), together with the fact "every nodal edge-like subgroup is contained in *precisely two verticial subgroups*" (cf. [20], Proposition 1.5, (i); [20], Proposition 1.2, (i)), which one applies, when verifying (a) for $\alpha_{\underline{\nu}}$ (respectively, $\alpha_{\underline{\mu}}$), first to $\Pi_{\underline{\mu}_x}$ (respectively, $\Pi_{\underline{\nu}_x}$), and then to $\Pi_{\underline{\nu}_x}$ (respectively, $\Pi_{\underline{\mu}_x}$). $\square$

Corollary 3.3 (Conditional surjectivity for affine curves). *Suppose that $X^{\log}$ is of* type (g, r), *where $r \geq 1$. Then* $\mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{\triangle+} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_2)$ *is contained in the image of the natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_3) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_2)$$

induced by $\underline{p}_{12}\colon \Pi_3 \twoheadrightarrow \Pi_2$.

Proof. Let $\beta_2 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{\triangle+}$; $\alpha_2 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$ an automorphism that lifts β_2. To complete the proof of Corollary 3.3, it suffices to construct an $\alpha_3 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_3)$ that lifts α_2. Write $x \in X(k)$ for the *cusp* that exhibits β_2 as an element of $\mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{\triangle+}$ (cf. Definition 1.11, (ii)).

Next, let us *fix* $\Pi_{\nu_x}, \Pi_{E_x}, \Pi_{F_x} \subseteq \Pi_{2/1}$ as in Proposition 1.5, (ii). By the non-resp'd portion of Proposition 1.3, (iv), we may assume without loss of generality that α_2 *stabilizes* Π_{ν_x}, Π_{E_x}, and Π_{F_x}. Write $\alpha_{2/1} \stackrel{\mathrm{def}}{=} \alpha_2|_{\Pi_{2/1}} \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_{2/1})$, $\alpha^E_{2/1} \stackrel{\mathrm{def}}{=} \alpha_2|_{\Pi_{E_x}} \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_{E_x})$, $\alpha^F_{2/1} \stackrel{\mathrm{def}}{=} \alpha_2|_{\Pi_{F_x}} \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_{F_x})$ for the respective restrictions of α_2 to $\Pi_{2/1}$, Π_{E_x}, Π_{F_x}; $\beta_{2/1} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{2/1})$, $\beta^E_{2/1} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})^{\triangle+}$, $\beta^F_{2/1} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{F_x})$ for the resulting outer automorphisms.

Next, let us recall the *outer isomorphisms* $\Pi_{2/1} \xrightarrow{\sim} \Pi_{\underline{B}_\mu}$, $\Pi_1^{\mathrm{tripod}} \xrightarrow{\sim} \Pi_{E_x}$, $\Pi_{2/1}^{\mathrm{tripod}} \xrightarrow{\sim} \Pi_{\underline{B}_\nu}$ implicit (cf. Propositions 1.5, (i); 3.2, (i)) in the *natural morphisms of exact sequences* of Definition 3.1, (v), (vi). Here, we note that it follows from the definitions that in fact, we have an *equality* $\Pi_1^{\mathrm{tripod}} = \Pi_{E_x}$ (i.e., without any indeterminacy with respect to composition with an inner automorphism). By conjugating $\beta_{2/1}$, $\beta^E_{2/1}$, respectively, by the first two of these outer isomorphisms, we thus obtain elements $\beta^{\mu}_{3/2} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{\underline{B}_\mu})$, $\beta_1^{\mathrm{tripod}} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\mathrm{tripod}})^{\triangle+}$, together with a particular lifting $\alpha_1^{\mathrm{tripod}} \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_1^{\mathrm{tripod}})$ of $\beta_1^{\mathrm{tripod}}$. By the definition of $\mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\mathrm{tripod}})^{\triangle+}$ (cf. Definition 1.11, (i)), it follows that $\beta_1^{\mathrm{tripod}}$ lifts to a *unique* (cf. Corollary 1.12, (ii)) element $\beta_2^{\mathrm{tripod}} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_2^{\mathrm{tripod}})^{\mathrm{S}}$. Write $\beta_{2/1}^{\mathrm{tripod}} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{2/1}^{\mathrm{tripod}})$ for the restriction "$\beta_2^{\mathrm{tripod}}|_{\Pi_{2/1}^{\mathrm{tripod}}}$" determined by the *lifting* $\alpha_1^{\mathrm{tripod}}$; $\beta^{\nu}_{3/2} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{\underline{B}_\nu})$ for the result of conjugating $\beta_{2/1}^{\mathrm{tripod}}$ by the outer isomorphism $\Pi_{2/1}^{\mathrm{tripod}} \xrightarrow{\sim} \Pi_{\underline{B}_\nu}$.

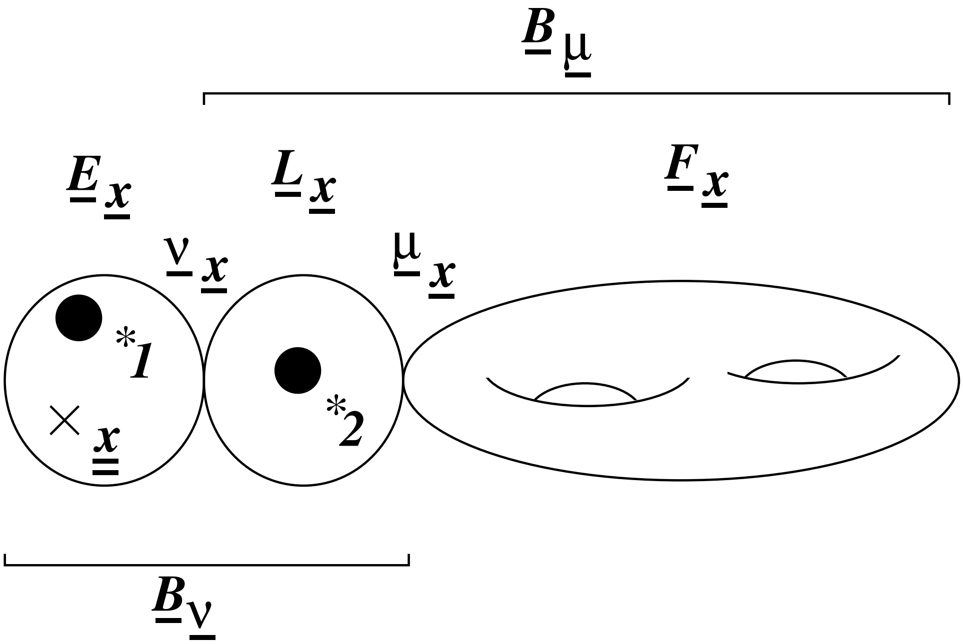


Fig. 3. An affine curve equipped with two extra cusps "$*_1$", "$*_2$".
($\underline{\underline{x}}$ is the cusp that corresponds to x)

Next, let us observe that since $\alpha_{2/1}$ *stabilizes* $\Pi_{\nu_x} \subseteq \Pi_{E_x}$ (where we note that, from the point of view of Π_{E_x}, the subgroup Π_{ν_x} is the *cuspidal inertia group* associated to one of the cusps of the tripod U_{E_x}), it follows from the non-resp'd portion of Proposition 1.3, (iv), applied to the outer automorphism β_2^{tripod} of Π_2^{tripod} (cf. also the lifting α_1^{tripod}), that $\beta_{3/2}^{\nu}$ *stabilizes* the $\Pi_{\underline{B}_{\underline{\nu}}}$-conjugacy classes of $\Pi_{\underline{E}_{\underline{x}}}$, $\Pi_{\underline{L}_{\underline{x}}}$, $\Pi_{\underline{\nu}_{\underline{x}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$ hence (cf. Proposition 3.2, (i)) induces elements $\beta_{3/2}^{E} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{\underline{E}_{\underline{x}}})$, $\beta_{3/2}^{L} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{\underline{L}_{\underline{x}}})$. Moreover, it follows from Proposition 1.2, (iii), in the case of $\beta_{3/2}^{E}$, and from Corollaries 1.12, (ii), (iii); 1.14, (i), (iii), in the case of $\beta_{3/2}^{L}$ (where we note that from the point of view of the situation of Corollary 1.14, (iii), $\underline{L}_{\underline{x}}$ that corresponds to the *minor* cuspidal component, while $\underline{E}_{\underline{x}}$ corresponds to the *major* cuspidal component), that, for any outer isomorphisms $\Pi_1^{\text{tripod}} \xrightarrow{\sim} \Pi_{\underline{E}_{\underline{x}}}$, $\Pi_1^{\text{tripod}} \xrightarrow{\sim} \Pi_{\underline{L}_{\underline{x}}}$ that arise *scheme-theoretically* (i.e., from isomorphisms of k-schemes $U_T \xrightarrow{\sim} U_{\underline{E}_{\underline{x}}}$, $U_T \xrightarrow{\sim} U_{\underline{L}_{\underline{x}}}$), the result of conjugating $\beta_{3/2}^{E}$, $\beta_{3/2}^{L}$, respectively, by these outer isomorphisms yields elements $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\text{tripod}})$ both of which are *equal to* β_1^{tripod}. (Here, we note that it is of *crucial importance* that we know that $\beta_1^{\text{tripod}} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\text{tripod}})^{\triangle}$—i.e., not just $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\text{tripod}})$!—since this *symmetry* of β_1^{tripod} allows one to ignore the issue of "*precisely which cusp is sent to* *which*" by the various scheme-theoretic isomorphisms of tripods that appear.) In particular, it follows from the definition of $\beta^{\mu}_{3/2}$ and $\beta_1^{\mathrm{tripod}}$ that *the restriction of* $\beta^{\mu}_{3/2}$ *to* $\Pi_{\underline{L}_x}$ (cf. Proposition 3.2, (i)) *is equal to* $\beta^{L}_{3/2}$. Thus, it makes sense to *glue* $\beta^{\mu}_{3/2} \in \mathrm{Out}^{\mathrm{FC}}\big(\Pi_{\underline{B}_\mu}\big)$, $\beta^{\nu}_{3/2} \in \mathrm{Out}^{\mathrm{FC}}\big(\Pi_{\underline{B}_\nu}\big)$ along $\Pi_{\underline{L}_x}$ so as to obtain an element

$$\beta_{3/2} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{3/2})$$

as in Proposition 3.2, (iv), that *restricts* to $\beta^{\mu}_{3/2}$ on $\Pi_{\underline{B}_\mu}$ and to $\beta^{\nu}_{3/2}$ on $\Pi_{\underline{B}_\nu}$.

Next, we consider the extent to which $\beta_{3/2}$ is *compatible*, relative to $\alpha_{2/1}$, with the *natural outer action* of $\Pi_{2/1}$ on $\Pi_{3/2}$. In particular, let us consider the following *assertion*:

$(*)$ $\beta_{3/2} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{3/2})$ is *compatible*, relative to $\alpha_{2/1}$, with the *natural outer actions* of Π_{E_x} $(\subseteq \Pi_{2/1})$ and Π_{F_x} $(\subseteq \Pi_{2/1})$ on $\Pi_{3/2}$.

Now I *claim* that to complete the proof of Corollary 3.3, *it suffices to verify* $(*)$. Indeed, since $\Pi_{2/1}$ is *topologically generated* by Π_{E_x}, Π_{F_x} (cf. Proposition 1.5, (iii)), it follows from $(*)$ that $\beta_{3/2} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{3/2})$ is *compatible*, relative to $\alpha_{2/1}$, with the *natural outer action* of $\Pi_{2/1}$. Thus, by applying the *natural isomorphism* $\Pi_{3/1} \xrightarrow{\sim} \Pi_{3/2} \overset{\mathrm{out}}{\rtimes} \Pi_{2/1}$ (cf. §0; Remark 1.1.1), we conclude that $\beta_{3/2}$, $\alpha_{2/1}$ determine an element $\beta_{3/1} \in \mathrm{Out}(\Pi_{3/1})$. It is immediate from the construction of $\beta_{3/1}$ that $\beta_{3/1}$ is *C-admissible*. Since $\beta_{3/1}$ preserves the conjugacy class of inertia groups associated to the *diagonal* divisor in the geometric generic fiber of $\underline{\mathrm{pr}}_1 \colon X_3 \to X_1$ (cf. the argument applied in the proof of Proposition 1.3, (vii)), it follows from Proposition 1.2, (i), that $\beta_{3/1}$ is *FC-admissible*, i.e., $\beta_{3/1} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_{3/1})$. Next, let us write $\alpha_1 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ for the automorphism induced by α_3 via $\underline{p}_1 \colon \Pi_3 \twoheadrightarrow \Pi_1$. Since the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{3/1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{2/1})$ is *injective* by Corollary 2.3, (ii), we thus conclude (from the fact that $\beta_{2/1}$ is manifestly compatible, relative to α_1, with the natural outer action of Π_1 on $\Pi_{2/1}$) that $\beta_{3/1}$ is *compatible*, relative to α_1, with the *natural outer action* of Π_1 on $\Pi_{3/1}$. In particular, by applying the *natural isomorphism* $\Pi_3 \xrightarrow{\sim} \Pi_{3/1} \overset{\mathrm{out}}{\rtimes} \Pi_1$ (cf. §0; Remark 1.1.1), we conclude that $\beta_{3/1}$, α_1 determine an element $\beta_3 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_3)$ (cf. Proposition 1.2, (i)) that lifts β_2, as desired. This completes the proof of the *claim*.

Finally, we proceed to *verify the assertion* $(*)$. To this end, let us observe that $\underline{p}_{13} \colon \Pi_3 \twoheadrightarrow \Pi_2$ (respectively, $\underline{p}_{23} \colon \Pi_3 \twoheadrightarrow \Pi_2$) induces a *surjection*

$$\phi_1 \colon \Pi_{3/2} \twoheadrightarrow \Pi_{2/1} \quad (\text{respectively, } \phi_2 \colon \Pi_{3/2} \twoheadrightarrow \Pi_{2/1})$$

whose kernel is topologically normally generated by the cuspidal inertia groups in $\Pi_{3/2}$ that correspond to the cusp parametrized by the factor labeled "2" (respectively, "1") of $X_3^{\log}$. That is to say, ϕ_1 (respectively, ϕ_2) corresponds to the operation of "*forgetting the cusp parametrized by the factor labeled* '2' (*respectively,* '1') *of* $X_3^{\log}$". Note that ϕ_1 (respectively, ϕ_2) induces *isomorphisms* $\Pi_{\underline{E}_x} \xrightarrow{\sim} \Pi_{E_x}$, $\Pi_{\underline{F}_x} \xrightarrow{\sim} \Pi_{F_x}$ (respectively, $\Pi_{\underline{L}_x} \xrightarrow{\sim}$

Π_{E_x}, $\Pi_{\underline{F}_{\underline{x}}} \xrightarrow{\sim} \Pi_{F_x}$, $\Pi_{\underline{B}_{\underline{\mu}}} \xrightarrow{\sim} \Pi_{2/1}$). In the following, if "(–)" is an element of $\Pi_{3/1}$, then let us write $\gamma_{(-)} \in \mathrm{Aut}(\Pi_{3/2})$ for the automorphism induced by conjugation by "(–)".

Next, let us *fix* $\Pi_{\underline{\mu}_{\underline{x}}}$, $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{B}_{\underline{\mu}}}$, $\Pi_{\underline{L}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$ as in the resp'd portion of Proposition 3.2, (ii). Here, we may assume without loss of generality that $\phi_2(\Pi_{\underline{\mu}_{\underline{x}}}) = \Pi_{\nu_x}$. Now let $\sigma_{2/1} \in \Pi_{E_x} \subseteq \Pi_{2/1}$; $\sigma_{3/1} \in \Pi_{3/1}$ a lifting of $\sigma_{2/1}$. Note that $\gamma_{\sigma_{3/1}}$ *stabilizes* the $\Pi_{3/2}$-conjugacy classes of $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$ (cf. the discussion of Definition 3.1, (iv)). In particular, by replacing $\sigma_{3/1}$ by the product of $\sigma_{3/1}$ with an appropriate element of $\Pi_{3/2}$, we may assume without loss of generality that $\gamma_{\sigma_{3/1}}$ *stabilizes* the subgroups $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$ (cf. Proposition 3.2, (ii)). Next, let us observe that (since $\underline{p}_{23}$ induces the natural surjection $\Pi_{2/1} \twoheadrightarrow \Pi_1$; the kernel of this surjection contains $\sigma_{2/1} \in \Pi_{E_x}$) $\gamma_{\sigma_{3/1}}$ induces, relative to ϕ_2, an *inner automorphism* of $\Pi_{2/1}$. Since ϕ_2 is *surjective*, it thus follows that there exists a $\zeta \in \Pi_{3/2}$ such that $\gamma_{\sigma_{3/1}\cdot\zeta}$ induces, relative to ϕ_2, the *identity automorphism* of $\Pi_{2/1}$. On the other hand, since $\phi_2(\Pi_{\underline{\mu}_{\underline{x}}}) = \Pi_{\nu_x}$ is *normally terminal* in $\Pi_{2/1}$ (cf. Proposition 1.5, (i)), it follows that $\phi_2(\zeta) \in \Pi_{\nu_x}$. In particular, by replacing $\sigma_{3/1}$ by the product of $\sigma_{3/1}$ with an appropriate element of $\Pi_{\underline{\mu}_{\underline{x}}}$, we may assume without loss of generality that:

(a) $\gamma_{\sigma_{3/1}}$ *stabilizes* the subgroups $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$;

(b) $\gamma_{\sigma_{3/1}}$ induces, relative to ϕ_2, the *identity automorphism* of $\Pi_{2/1}$. We shall refer to a lifting $\sigma_{3/1}$ of $\sigma_{2/1}$ that satisfies these conditions (a), (b) as *ϕ_2-admissible*.

Now let $\tau_{2/1} \stackrel{\text{def}}{=} \alpha_{2/1}(\sigma_{2/1}) \in \Pi_{2/1}$; $\sigma_{3/1}, \tau_{3/1} \in \Pi_{3/1}$ *ϕ_2-admissible* liftings of $\sigma_{2/1}$, $\tau_{2/1}$; $\alpha_{3/2} \in \mathrm{Aut}(\Pi_{3/2})$ an automorphism that gives rise to $\beta_{3/2}$. Since (by construction) $\beta_{3/2}$ stabilizes the $\Pi_{3/2}$-conjugacy classes of the subgroups $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$ (cf. Proposition 3.2, (iv)), we may assume without loss of generality (cf. Proposition 3.2, (ii)) that $\alpha_{3/2}$ *stabilizes* the subgroups $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$. Now to verify that "$\beta_{3/2}$ is *compatible*, relative to $\alpha_{2/1}$, with the *natural outer action* of Π_{E_x}" (cf. $(*)$), it suffices to verify that:

$(*_E)$ We have: $\gamma_{\tau_{3/1}} = \alpha_{3/2} \circ \gamma_{\sigma_{3/1}} \circ \alpha_{3/2}^{-1}$.

Next, let us recall from Definition 3.1, (iv), that $\gamma_{\tau_{3/1}}$, $\gamma_{\sigma_{3/1}}$ induce the *trivial outer automorphism* on $\Pi_{\underline{F}_{\underline{x}}}$; in particular, the equality of $(*_E)$ holds over $\Pi_{\underline{F}_{\underline{x}}}$, up to composition with an $\Pi_{\underline{F}_{\underline{x}}}$-inner automorphism. Moreover, by the construction of $\beta_{3/2}$, it follows from Definition 3.1, (vi), that the equality of $(*_E)$ holds over $\Pi_{\underline{B}_{\underline{\nu}}}$, up to composition with an $\Pi_{\underline{B}_{\underline{\nu}}}$-inner automorphism. Since $\alpha_{3/2}$, $\gamma_{\tau_{3/1}}$, and $\gamma_{\sigma_{3/1}}$ all *stabilize* $\Pi_{\underline{\mu}_{\underline{x}}}$ (which is *normally terminal* in $\Pi_{3/2}$—cf. Proposition 3.2, (i)), we thus conclude that the equality of $(*_E)$ holds up to composition with some $\delta \in \mathrm{Aut}(\Pi_{3/2})$ that stabilizes the subgroups $\Pi_{\underline{B}_{\underline{\nu}}}$, $\Pi_{\underline{\mu}_{\underline{x}}}$, and $\Pi_{\underline{F}_{\underline{x}}}$, and, moreover, restricts to (possibly distinct!) $\Pi_{\underline{\mu}_{\underline{x}}}$-*inner automorphisms* over $\Pi_{\underline{B}_{\underline{\nu}}}$ (hence over $\Pi_{\underline{L}_{\underline{x}}}$) and $\Pi_{\underline{F}_{\underline{x}}}$. (That is to say, δ is a sort of abstract profinite analogue of a *Dehn twist*!) On the other hand, since $\gamma_{\tau_{3/1}}$, $\gamma_{\sigma_{3/1}}$ induce, relative to ϕ_2, the identity automorphism of $\Pi_{2/1}$, it follows that δ induces, relative to ϕ_2, the *identity automorphism* of $\Pi_{2/1}$. Since ϕ_2 induces *isomorphisms* of *center-free* (cf. Remark 1.1.1) profinite groups $\Pi_{\underline{L}_x} \xrightarrow{\sim} \Pi_{E_x}$, $\Pi_{\underline{F}_x} \xrightarrow{\sim} \Pi_{F_x}$, we thus conclude that δ is the *identity* automorphism. This completes the proof of $(*_E)$.

In a *similar vein*, let us *fix* $\Pi_{\underline{\nu}_x}$, $\Pi_{\underline{B}_\nu}$, $\Pi_{\underline{B}_\mu}$, $\Pi_{\underline{E}_x}$, and $\Pi_{\underline{L}_x}$ as in the non-resp'd portion of Proposition 3.2, (ii). Here, we may assume without loss of generality that $\phi_1(\Pi_{\underline{\nu}_x}) = \Pi_{\nu_x}$. Now let $\sigma_{2/1} \in \Pi_{F_x} \subseteq \Pi_{2/1}$; $\sigma_{3/1} \in \Pi_{3/1}$ a lifting of $\sigma_{2/1}$. Note that $\gamma_{\sigma_{3/1}}$ *stabilizes* the $\Pi_{3/2}$-conjugacy classes of $\Pi_{\underline{E}_x}$, $\Pi_{\underline{\nu}_x}$, and $\Pi_{\underline{B}_\mu}$ (cf. the discussion of Definition 3.1, (iv)). In particular, by replacing $\sigma_{3/1}$ by the product of $\sigma_{3/1}$ with an appropriate element of $\Pi_{3/2}$, we may assume without loss of generality that $\gamma_{\sigma_{3/1}}$ *stabilizes* the subgroups $\Pi_{\underline{E}_x}$, $\Pi_{\underline{\nu}_x}$, and $\Pi_{\underline{B}_\mu}$ (cf. Proposition 3.2, (ii)). Next, let us observe that (since ϕ_1 arises from $\underline{p}_{13}$) $\gamma_{\sigma_{3/1}}$ induces, relative to ϕ_1, an *inner automorphism* of $\Pi_{2/1}$. Since ϕ_1 is *surjective*, it thus follows that there exists a $\zeta \in \Pi_{3/2}$ such that $\gamma_{\sigma_{3/1}\cdot\zeta}$ induces, relative to ϕ_1, the *identity automorphism* of $\Pi_{2/1}$. On the other hand, since $\phi_1(\Pi_{\underline{\nu}_x}) = \Pi_{\nu_x}$ is *normally terminal* in $\Pi_{2/1}$ (cf. Proposition 1.5, (i)), it follows that $\phi_1(\zeta) \in \Pi_{\nu_x}$. In particular, by replacing $\sigma_{3/1}$ by the product of $\sigma_{3/1}$ with an appropriate element of $\Pi_{\underline{\nu}_x}$, we may assume without loss of generality that:

(a) $\gamma_{\sigma_{3/1}}$ *stabilizes* the subgroups $\Pi_{\underline{E}_x}$, $\Pi_{\underline{\nu}_x}$, and $\Pi_{\underline{B}_\mu}$;

(b) $\gamma_{\sigma_{3/1}}$ induces, relative to ϕ_1, the *identity automorphism* of $\Pi_{2/1}$.

We shall refer to a lifting $\sigma_{3/1}$ of $\sigma_{2/1}$ that satisfies these conditions (a), (b) as ϕ_1-*admissible*.

Now let $\tau_{2/1} \stackrel{\text{def}}{=} \alpha_{2/1}(\sigma_{2/1}) \in \Pi_{2/1}$; $\sigma_{3/1}, \tau_{3/1} \in \Pi_{3/1}$ ϕ_1-*admissible* liftings of $\sigma_{2/1}$, $\tau_{2/1}$; $\alpha_{3/2} \in \text{Aut}(\Pi_{3/2})$ an automorphism that gives rise to $\beta_{3/2}$. Since (by construction) $\beta_{3/2}$ stabilizes the $\Pi_{3/2}$-conjugacy classes of the subgroups $\Pi_{\underline{E}_x}$, $\Pi_{\underline{\nu}_x}$, and $\Pi_{\underline{B}_\mu}$ (cf. Proposition 3.2, (iv)), we may assume without loss of generality (cf. Proposition 3.2, (ii)) that $\alpha_{3/2}$ *stabilizes* the subgroups $\Pi_{\underline{E}_x}$, $\Pi_{\underline{\nu}_x}$, and $\Pi_{\underline{B}_\mu}$. Now to verify that "$\beta_{3/2}$ is *compatible*, relative to $\alpha_{2/1}$, with the *natural outer action* of Π_{F_x}" (cf. $(*)$), it suffices to verify that:

$(*_F)$ We have: $$\gamma_{\tau_{3/1}} = \alpha_{3/2} \circ \gamma_{\sigma_{3/1}} \circ \alpha_{3/2}^{-1}.$$

Next, let us recall from Definition 3.1, (iv), that $\gamma_{\tau_{3/1}}$, $\gamma_{\sigma_{3/1}}$ induce the *trivial outer automorphism* on $\Pi_{\underline{E}_x}$; in particular, the equality of $(*_F)$ holds over $\Pi_{\underline{E}_x}$, up to composition with an $\Pi_{\underline{E}_x}$-inner automorphism. Moreover, by the construction of $\beta_{3/2}$, it follows from Definition 3.1, (v), that the equality of $(*_F)$ holds over $\Pi_{\underline{B}_\mu}$, up to composition with an $\Pi_{\underline{B}_\mu}$-inner automorphism. Since $\alpha_{3/2}$, $\gamma_{\tau_{3/1}}$, and $\gamma_{\sigma_{3/1}}$ all *stabilize* $\Pi_{\underline{\nu}_x}$ (which is *normally terminal* in $\Pi_{3/2}$—cf. Proposition 3.2, (i)), we thus conclude that the equality of $(*_F)$ holds up to composition with some $\delta \in \text{Aut}(\Pi_{3/2})$ that stabilizes the subgroups $\Pi_{\underline{E}_x}$, $\Pi_{\underline{\nu}_x}$, and $\Pi_{\underline{B}_\mu}$, and, moreover, restricts to (possibly distinct!) $\Pi_{\underline{\nu}_x}$-*inner automorphisms* over $\Pi_{\underline{E}_x}$ and $\Pi_{\underline{B}_\mu}$. (That is to say, δ is a sort of abstract profinite analogue of a *Dehn twist*!) On the other hand, since $\gamma_{\tau_{3/1}}$, $\gamma_{\sigma_{3/1}}$ induce, relative to ϕ_1, the identity automorphism of $\Pi_{2/1}$, it follows that δ induces, relative to

ϕ_1, the *identity automorphism* of $\Pi_{2/1}$. Since ϕ_1 induces *isomorphisms* of *center-free* (cf. Remark 1.1.1) profinite groups $\Pi_{\underline{E_x}} \xrightarrow{\sim} \Pi_{E_x}$, $\Pi_{\underline{F_x}} \xrightarrow{\sim} \Pi_{F_x}$, we thus conclude that δ is the *identity* automorphism. This completes the proof of $(*_F)$, and hence of Corollary 3.3. $\square$

Corollary 3.4 (Tautological validity of "$\triangle$", "$\triangle+$"). *Suppose that* $X^{\log}$ *is of* type (g, r), *where* $r \geq 0$. *Then*:
(i) *We have*: $\mathrm{Out}^{\mathrm{FCP}}(\Pi_3)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\triangle}$.
(ii) *We have*: $\mathrm{Out}^{\mathrm{FCP}}(\Pi_4)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_4)^{\triangle+}$.
(iii) *Suppose that* $r \geq 1$. *Then* $\mathrm{Out}^{\mathrm{FC}}(\Pi_3)^{\triangle+}$ *contains the inverse image of* $\mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{\triangle}$ *via the natural homomorphism* $\mathrm{Out}^{\mathrm{FC}}(\Pi_3) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_2)$ *induced by* $\underline{p}_{12}$.

Proof. Assertion (i) follows immediately from the definitions, by observing that in the situation of Definition 1.8 and Proposition 1.9, the action of the group of *permutations* (i.e., automorphisms of the set $\{1, 2, 3\}$) on X_3 *preserves* the subscheme $W \subseteq X_3$ of Definition 1.8, (i), and induces the automorphisms of $W \cong V \times_k U_P$ given by *permuting* (over V) the *three cusps* of U_P. Assertion (ii) follows from assertions (i) and (iii) by taking the surjection "$\underline{p}_{12}\colon \Pi_3 \twoheadrightarrow \Pi_2$" that appears in assertion (iii) to be the standard surjection $\Pi_{4/1} \twoheadrightarrow \Pi_{3/1}$. Thus, it remains to verify assertion (iii). To this end, let us assume that we have been given an element $\beta_3 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_3)$ that maps to an element $\beta_2 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{\triangle}$, and that we are in the situation of Definition 3.1, with $x \in X(k)$ taken to be the *cusp* that exhibits β_2 as an element of $\mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{\triangle}$. Let $\alpha_2 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2)$, $\alpha_3 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_3)$ be elements that induce, respectively, β_2, β_3; also, we suppose that α_3 lifts α_2. By Propositions 1.3, (iv) (the resp'd portion); 1.7, (a), we may assume without loss of generality that α_2 *stabilizes* the subgroups $(\Pi_1^{\mathrm{tripod}} \cong)$ Π_{E_x}, $\mathbb{I}_{E_x}$, and $\mathbb{D}_{E_x}$ of Π_2, and that α_2 induces an element $\beta_1^{\mathrm{tripod}} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_1^{\mathrm{tripod}})^{\triangle} \cong \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_x})^{\triangle}$. Thus, it follows from the non-resp'd portion of Proposition 1.3, (iv), that α_3 *stabilizes* the $\Pi_{3/2}$-conjugacy classes of $\Pi_{\underline{B_v}}$, $\Pi_{\underline{F_x}}$ (cf. the discussion of Definition 3.1, (iv), (vi)). In particular, α_3 induces an element $\beta_2^{\mathrm{tripod}} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_2^{\mathrm{tripod}})^{\mathrm{S}}$ that *lifts* $\beta_1^{\mathrm{tripod}}$ (cf. Definition 3.1, (vi)).

Now write $\xi \in X_2(X)$ for the *cusp* of X_2 (relative to $\mathrm{pr}_1^{\log}\colon X_2^{\log} \to X_1^{\log}$) that corresponds to the cusp $x \in X(k)$. Thus, ξ determines—by restricting to the geometric generic fiber of $\underline{\mathrm{pr}}_1^{\log}\colon X_3^{\log} \to X_1^{\log} = X^{\log}$—a *minor verticial subgroup* $\Pi_{E_\xi} \subseteq \Pi_{3/2}$. Moreover, since the restriction of the section $\xi\colon X \to X_2$ to $x \in X(k)$ determines a *cusp* $\underline{\xi}$ of U_{E_x}, it follows that (for suitable choices within the various $\Pi_{3/2}$-conjugacy classes) $\Pi_{E_\xi} \subseteq \Pi_{\underline{B_v}}$, and that this subgroup Π_{E_ξ} of $\Pi_{\underline{B_v}} \cong \Pi_{2/1}^{\mathrm{tripod}}$ forms a *minor verticial subgroup* $\Pi_{E_{\underline{\xi}^{\mathrm{tripod}}}}$ at $\underline{\xi}$ of $\Pi_{2/1}^{\mathrm{tripod}}$. In particular, we conclude from the resp'd portion of Proposition 1.3, (iv), that $\beta_2^{\mathrm{tripod}} \in \mathrm{Out}^{\mathrm{FC}}(\Pi_2^{\mathrm{tripod}})^{\mathrm{S}}$ *stabilizes* the Π_2^{tripod}-conjugacy class of $\Pi_{E_{\underline{\xi}^{\mathrm{tripod}}}}$ and, moreover, induces an element $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_{E_\xi}) \cong \mathrm{Out}^{\mathrm{FC}}\left(\Pi^{\mathrm{tripod}}_{E_{\underline{\xi}}}\right)$ which, by Corollaries 1.12, (ii), (iii); 1.14, (i), (iii), *coincides*—relative to any isomorphism $\Pi^{\text{tripod}}_{E_{\underline{\xi}}} \xrightarrow{\sim} \Pi^{\text{tripod}}_1$ that arises from a k-isomorphism $U_{E_{\underline{\xi}}} \xrightarrow{\sim} U_T$—with $\beta^{\text{tripod}}_1 \in \text{Out}^{\text{FC}}(\Pi^{\text{tripod}}_1)^{\triangle+} \cong \text{Out}^{\text{FC}}(\Pi_{E_x})^{\triangle+}$. Thus, by Definition 1.11, (ii), we conclude that $\beta_3 \in \text{Out}^{\text{FC}}(\Pi_3)^{\triangle+}$, as desired. This completes the proof of assertion (iii), and hence of Corollary 3.4. $\square$

4. The general profinite case

In the present §4, we derive the *main result* (cf. Theorem 4.1) of the present paper from the various partial results obtained in §1, §2, §3.

Theorem 4.1 (Partial profinite combinatorial cuspidalization). *Let*

$$X^{\log} \to S$$

be a smooth log curve of type (g, r) (*cf.* §0) *over* $S = \text{Spec}(k)$, *where* k *is an algebraically closed field of characteristic zero. Fix a* set of prime numbers Σ *which is either of cardinality one or equal to the set of all prime numbers. For* n *a nonnegative integer, write* $X^{\log}_n$ *for the* n-th log configuration space *associated to* $X^{\log}$ (*cf.* [24], Definition 2.1, (i)), *where we take* $X^{\log}_0 \stackrel{\text{def}}{=} \text{Spec}(k)$;

$$\Pi_n \stackrel{\text{def}}{=} \pi^{\Sigma}_1(X^{\log}_n)$$

for the maximal pro-Σ quotient of the fundamental group *of the log scheme* $X^{\log}_n$ (*cf.* §0; *the discussion preceding* [24], Definition 2.1, (i));

$$\text{Out}^{\text{FC}}(\Pi_n) \subseteq \text{Out}(\Pi_n)$$

for the subgroup of outer automorphisms α *that satisfy the following conditions* (1), (2) (*cf.* Definition 1.1, (ii)):

(1) $\alpha(H) = H$ *for every* fiber subgroup $H \subseteq \Pi_n$ (*cf.* Remark 1.1.2; [24], Definition 2.3, (iii)).

(2) *For* m *a nonnegative integer* $\leq n$, *write* $K_m \subseteq \Pi_n$ *for the fiber subgroup that arises as the kernel of the projection obtained by "forgetting the factors of* X_n *with labels* $> m$". *Then* α *induces a* bijection *of the collection of conjugacy classes of* cuspidal inertia groups *contained in each* K_{m-1}/K_m (*where* $m = 1, \ldots, n$) *associated to the various cusps of the geometric generic fiber of the projection* $X^{\log}_m \twoheadrightarrow X^{\log}_{m-1}$ *obtained by "forgetting the factor labeled* m". (*Here, we regard the map* $\Pi_m \cong \Pi_n/K_m \twoheadrightarrow \Pi_n/K_{m-1} \cong \Pi_{m-1}$ *of quotients of* Π_n *as the homomorphism that arises by "forgetting, successively, the factors with labels* $> m$ *and the factors with labels* $> m - 1$".)

If the interior U_X *of* $X^{\log}$ *is* affine (*i.e.*, $r \geq 1$), *then set* $n_0 \stackrel{\text{def}}{=} 2$; *if the interior* U_X *of* $X^{\log}$ *is* proper *over* k (*i.e.*, $r = 0$), *then set* $n_0 \stackrel{\text{def}}{=} 3$. *Then*:

(i) *The natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$$

induced by the projection obtained by "forgetting the factor labeled n" is injective *if* $n \geq n_0$ *and* bijective *if* $n \geq 5$.

(ii) *The* image *of the natural homomorphism* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ *of* (i) *contains the following two subsets* (*cf.* Definition 1.11):

(a) $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})^{\triangle+}$, *when* $n \geq 2$ (*a set which is* well-defined *and* nonempty *only if* $(g, r) = (0, 3)$ *or* $n - 1 \geq n_0$);

(b) *the inverse image in* $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ *via the natural homomorphism* $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-2})$ *of* $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-2})^{\triangle}$, *when* $n \geq 3$ (*a set which is* well-defined *and* nonempty *only if either* $(g, r) = (0, 3)$ *or* $n - 2 \geq n_0$).

(iii) *Let* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ *be as in* (i), *where* $n \geq n_0$. *Let* $\sigma \in \mathrm{Out}(\Pi_n)$ *be an outer automorphism that satisfies the following properties*:

(a) *for every fiber subgroup* $H \subseteq \Pi_n$, $\sigma(H)$ *is a fiber subgroup*;

(b) $\sigma(K_{n-1}) = K_{n-1}$;

(c) σ *induces a* bijection *of the collection of conjugacy classes of* cuspidal inertia groups *contained in* K_{n-1};

(d) *the outer automorphism* $\sigma' \in \mathrm{Out}(\Pi_{n-1})$ *determined by* σ (*cf.* (b)) normalizes (*respectively,* commutes *with*) $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$. *Then* σ normalizes (*respectively,* commutes *with*) $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$.

(iv) *By permuting the various factors of* $X_n^{\log}$, *one obtains a natural inclusion*

$$\mathfrak{S}_n \hookrightarrow \mathrm{Out}(\Pi_n)$$

of the symmetric group on n letters into $\mathrm{Out}(\Pi_n)$ *whose image* commutes *with* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *if* $n \geq n_0$ *and* normalizes $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *if* $r = 0$ *and* $n = 2$.

Proof. First, we consider the *injectivity* portion of assertion (i). Consider the *natural isomorphisms*

$$\Pi_n \xrightarrow{\sim} K_{n-2} \overset{\mathrm{out}}{\rtimes} \Pi_{n-2}; \quad \Pi_{n-1} \xrightarrow{\sim} (K_{n-2}/K_{n-1}) \overset{\mathrm{out}}{\rtimes} \Pi_{n-2}$$

(cf. §0; Remark 1.1.1), together with the *interpretation* of $\Pi_{n/n-2} = K_{n-2} \twoheadrightarrow K_{n-2}/K_{n-1} = \Pi_{n-1/n-2}$ as the "$\Pi_2 \twoheadrightarrow \Pi_1$" (i.e., the projection that arises by forgetting the factor labeled 2) associated to an "$X^{\log}$" of type $(g, r + n - 2)$ (cf. [24], Proposition 2.4, (i)). (Here, we note that one verifies easily that this "interpretation" is compatible with the definition of the various "$\mathrm{Out}^{\mathrm{FC}}(-)$'s" involved.) Now the above *natural isomorphisms* allow one to reduce the injectivity portion of assertion (i) to the case $n = 2$, $r \geq 1$, which follows immediately from Corollaries 1.12, (ii); 2.3, (ii) (cf. also Remark 2.1.1). This completes the proof of the *injectivity* portion of assertion (i).

Next, we consider assertion (iii). Let $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$. Write α' for the image of α in $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$; $\alpha_\sigma \stackrel{\mathrm{def}}{=} \sigma \cdot \alpha \cdot \sigma^{-1}$; $\alpha'_{\sigma'} \stackrel{\mathrm{def}}{=} \sigma' \cdot \alpha' \cdot (\sigma')^{-1}$. Then it follows immediately from property (a) that α_σ is *F-admissible* and from properties (b), (c), (d) that α_σ is *C-admissible*. Thus, $\alpha_\sigma \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$. If, moreover, it holds that $\alpha' = \alpha'_{\sigma'}$, then it follows from the injectivity portion of assertion (i) that $\alpha = \alpha_\sigma$. This completes the proof of assertion (iii).

Next, we consider assertion (iv). When $n = 2$, assertion (iv) follows immediately from Proposition 1.6, (iii); Corollaries 1.12, (iii); 2.3, (iii) (cf. also Remark 2.1.1). Note that when $n \geq 3$, by applying the *natural isomorphism*

$$\Pi_n \xrightarrow{\sim} K_{n-2} \overset{\mathrm{out}}{\rtimes} \Pi_{n-2}$$

(cf. §0; Remark 1.1.1), together with the *interpretation* of $\Pi_{n/n-2} = K_{n-2}$ as the "Π_2" associated to an "$X^{\log}$" of type $(g, r + n - 2)$ (cf. [24], Proposition 2.4, (i)), we thus conclude from "assertion (iv) for $n = 2$" (whose proof has already been completed) that $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *commutes* with the permutation outer automorphism $\sigma \in \mathrm{Out}(\Pi_n)$ that arises from the permutation $((n-1)\ n)$ of $\{1, 2, \ldots, n\}$ (i.e., the permutation that switches n and $n-1$ and fixes all other elements of $\{1, 2, \ldots, n\}$). Now we apply *induction* on n. When U_X is *affine*, let us observe that (by the induction hypothesis) every permutation outer automorphism $\sigma \in \mathrm{Out}(\Pi_n)$ that arises from a permutation of $\{1, 2, \ldots, n\}$ that *fixes* n satisfies the properties (a), (b), (c), (d) of assertion (iii) in the *resp'd case*. Thus, when U_X is affine, the *induction step* (i.e., the derivation of "assertion (iv) for n" from "assertion (iv) for $n - 1$") follows from assertion (iii), together with the fact that the permutation group of $\{1, 2, \ldots, n\}$ is generated by "$((n-1)\ n)$" and the subgroup of permutations that fix n. If U_X is *proper* and $n \geq 4$, then the *induction step* (i.e., the derivation of "assertion (iv) for n" from "assertion (iv) for $n-1$") follows by a similar argument. Thus, it remains to verify the *induction step* when U_X is *proper* and $n = 3$. To this end, let us first observe that, as discussed above, $\mathrm{Out}^{\mathrm{FC}}(\Pi_3)$ *commutes* with (the permutation outer automorphism that arises from the permutation of $\{1, 2, 3\}$ given by) (23). Moreover, by applying assertion (iii) in the *non-resp'd case* to (the permutation outer automorphism that arises from the permutation of $\{1, 2, 3\}$ given by) (12), we conclude that (12) *normalizes* $\mathrm{Out}^{\mathrm{FC}}(\Pi_3)$. Thus, by conjugating by (12), we conclude that $\mathrm{Out}^{\mathrm{FC}}(\Pi_3)$ *commutes* with (13). Now since the group of permutations of $\{1, 2, 3\}$ is generated by (12), (13), we conclude that $\mathrm{Out}^{\mathrm{FC}}(\Pi_3)$ *commutes* with all permutation outer automorphisms. This completes the proof of assertion (iv).

Next, we consider assertion (ii). First, let us observe that when $(g, r) = (0, 3)$ and $n = 2$, assertion (ii) for the subset of (a) is a *tautology* (cf. Definition 1.11, (i)); when $(g, r) = (0, 3)$ and $n = 3$, assertion (ii) for the subset of (b) may be reduced, in light of the inclusion $\mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{\mathrm{S}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{\triangle +}$ (cf. Corollaries 1.12, (ii), (iii); 1.14, (i), (iv)), to assertion (ii) for the subset of (a) when $n = 3$. Next, let us observe that when $n \geq 4$, by the definition of "$\triangle$" (cf. Definition 1.11, (ii)), *every element* $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1/n-4})$ (where we recall that $\Pi_{n-1/n-4}$ is the "Π_3" associated to an "$X^{\log}$"

of type $(g, r+n-4)$) that is induced, relative to the inclusion $\Pi_{n-1/n-4} \hookrightarrow \Pi_{n-1}$, by an element $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ of *the subset of* (b) maps, via the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1/n-4}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-2/n-4})$ (obtained by "forgetting the factor labeled $n-1$"), to an element of $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-2/n-4})^{\triangle}$, hence, by Corollary 3.4, (iii), is contained in $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1/n-4})^{\triangle+}$; but, by the definition of "$\triangle+$" (cf. Definition 1.11, (ii)), this implies that every element of the subset of (b) is contained in $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})^{\triangle+}$. Thus, to complete the proof of assertion (ii), it suffices to verify assertion (ii) for the subset of (a) in the case of $n \geq 3$. On the other hand, when $n \geq 3$, by applying the *natural isomorphisms* $\Pi_n \xrightarrow{\sim} \Pi_{n/n-3} \overset{\text{out}}{\rtimes} \Pi_{n-3}$, $\Pi_{n-1} \xrightarrow{\sim} \Pi_{n-1/n-3} \overset{\text{out}}{\rtimes} \Pi_{n-3}$ (cf. the proof of the injectivity portion of assertion (i)), together with the *injectivity* portion of assertion (i) (which is necessary in order to conclude the *compatibility of liftings*, relative to the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n/n-3}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1/n-3})$, with the respective outer actions of Π_{n-3}), to complete the proof of assertion (ii), we conclude that it suffices to verify assertion (ii) for the subset of (a) in the case of $n = 3$. But this is precisely the content of Corollary 3.3. This completes the proof of assertion (ii).

Finally, we consider the *surjectivity* (i.e., bijectivity) portion of assertion (i) for $n \geq 5$. First, let us observe that by Lemma 2.4, to complete the proof of assertion (i), it suffices to verify that the image of the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ of assertion (i) contains the subset $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$. Next, let us observe that by assertion (iv) and Remark 1.1.5, *every element* $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1/n-5})$ (where we recall that $\Pi_{n-1/n-5}$ is the "Π_4" associated to an "$X^{\log}$" of type $(g, r+n-5)$) that is induced, relative to the inclusion $\Pi_{n-1/n-5} \hookrightarrow \Pi_{n-1}$, by an *element* $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})^{\mathrm{cusp}}$ is contained in $\mathrm{Out}^{\mathrm{FCP}}(\Pi_{n-1/n-5})^{\mathrm{cusp}}$, hence, by Corollary 3.4, (ii), in $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1/n-5})^{\triangle+}$. But this implies that $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})^{\mathrm{cusp}} = \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})^{\triangle+}$ (cf. Definition 1.11, (ii)). Thus, in summary, to complete the proof of assertion (i), it suffices to verify that the image of the natural homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ of assertion (i) contains the subset $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})^{\triangle+} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$. But this follows from assertion (ii) (cf. the subset of (a)). This completes the proof of assertion (i). □

REMARK 4.1.1. The argument applied to verify Theorem 4.1, (iv), in the *proper* case suggests that even if one cannot verify the injectivity of the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ in the proper case, it may be possible to verify the injectivity of the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_3) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ (i.e., induced by the projection obtained by "forgetting the factors labeled 2, 3") in the proper case.

REMARK 4.1.2. In the *pro-l case* (i.e., the case where Σ is of cardinality one), a number of results related to Theorem 4.1, (i), have been obtained by various authors.
(i) In [10], Theorem 1 (cf. also [8], which is discussed further in Remark 4.2.1, (ii), below), a similar injectivity result to that of Theorem 4.1, (i), is obtained in the pro-l case for outer automorphisms satisfying certain conditions—i.e., the conditions "(σ1), (σ2)" of [10], Theorem 1. It is immediate (cf. Proposition 1.3, (vii)) that outer auto-

morphisms lying in the kernel of the homomorphism in question which satisfy these conditions "(σ1), (σ2)" are *FC-admissible*. Thus, (at least when the condition of *hyperbolicity* $2g-2+r>0$ is satisfied) [10], Theorem 1, may be obtained as a consequence of Theorem 4.1, (i).
(ii) In [29], a *filtered pro-l injectivity* result (cf. [29], Theorem 4.3) is obtained for a certain filtration on a subgroup $\Gamma_{g,r}^{(n)} \subseteq \mathrm{Out}(\Pi_n)$ (where $\Gamma_{g,r}^{(n)}$ is as in [29], (2.11)—except with "r" and "n" reversed!). It follows immediately from the conditions used to define $\Gamma_{g,r}^{(n)}$ (cf. [29], (2.10), (2.11)) that

$$\Gamma_{g,r}^{(n)} = \mathrm{Out}^{\mathrm{QS}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$$

(cf. Proposition 1.3, (vii)). In particular, the injectivity of Theorem 4.1, (i), in the pro-l case may also be thought of as yielding a new proof of the injectivity that holds as a consequence of the "filtered injectivity" of [29], Theorem 4.3.
(iii) In the context of (ii), *graded pro-l surjectivity* results are obtained in [32]. Related results may be found in [9].

REMARK 4.1.3. The *injectivity* of the restriction of the homomorphism of Theorem 4.1, (i), to an "*image of Galois*" $\subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ that arises from *scheme theory* is precisely the content of [14], Theorem 2.2. Indeed, it was precisely the goal of attaining a more *abstract*, *combinatorial understanding* of the theory of [14] that motivated the author to develop the theory of the present paper. Also, we observe that the remaining portion of [14], Theorem 2.2—involving related *outer actions on* Π^{tripod}—follows immediately from the existence of the *natural outer homomorphism* of Corollary 1.10, (iii).

REMARK 4.1.4. (i) Observe that the various "Π_n" that arise from different "$X^{\log}$'s" of the *same type* (g,r) are always *isomorphic*, in a fashion that is *compatible* with the various *fiber subgroups* and *cuspidal inertia groups* of subquotients. Indeed, this follows immediately (cf. the various "*specialization isomorphisms*" discussed in §0) from the well-known fact (cf., [3]) that the moduli stack $\overline{\mathcal{M}}_{g,r}$ (cf. §0) is *smooth*, *proper*, and *geometrically connected* over $\mathbb{Z}$.
(ii) Although we have formulated Theorem 4.1, (i), in terms of outer *automorphisms*, it is a routine exercise—in light of the observation of (i)—to reformulate Theorem 4.1, (i), in terms of outer *isomorphisms*, as is often of interest in applications to *anabelian geometry*.

REMARK 4.1.5. In [7], a group-theoretic construction is given for the geometrically pro-l arithmetic fundamental groups of configuration spaces of arbitrary dimension from the geometrically pro-l arithmetic fundamental group of a proper hyperbolic curve over a finite field. This construction is performed by considering various *Lie versions* of these arithmetic fundamental groups of configuration spaces of *arbitrary dimension*.

On the other hand, by applying the *injectivity* portion of Theorem 4.1, (i) (cf. the argument involving "$\overset{\text{out}}{\rtimes}$" given in the proof of Theorem 4.1, (ii)), one may *simplify* the argument of [7]: That is to say, instead of working with Lie versions of geometrically pro-l arithmetic fundamental groups of configuration spaces of *arbitrary dimension* (associated to a proper hyperbolic curve over a finite field), one may instead restrict oneself to working with Lie versions of geometrically pro-l arithmetic fundamental groups of *two-dimensional* configuration spaces (associated to a (not necessarily proper) hyperbolic curve over a finite field). (We leave the routine details to the interested reader.) This reduction to the case of Lie algebras associated to two-dimensional configuration spaces results in a *substantial reduction* of the book-keeping involved.

The following result allows one to relate the theory of the present paper to the work of *Nakamura* and *Harbater–Schneps* (cf. [26], [5]).

Corollary 4.2 (Partial profinite combinatorial cuspidalization for tripods). *In the notation of* Theorem 4.1: *Suppose further that* $X^{\log}$ *is a* tripod. *Then, for* $n \geq 1$:
(i) *We have*:

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{S}} = \mathrm{Out}^{\mathrm{FCS}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\triangle} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$$

if $n = 1$;

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{S}} = \mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\triangle+} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$$

if $n \geq 2$ (*cf.* Definitions 1.1, (vi); 1.11, (i), (ii)).
(ii) *The natural homomorphism*

$$\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FCS}}(\Pi_{n-1})$$

induced by the projection obtained by "forgetting the factor labeled n" is injective *if* $n \geq 2$ *and* bijective *if* $n \geq 3$.

Proof. First, we consider assertion (i). When $n = 1$, assertion (i) follows immediately from Definitions 1.1, (vi); 1.11, (i). Thus, we may assume that $n \geq 2$. Then the fact that $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{S}} = \mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$ follows formally from Corollary 1.14, (i); Theorem 4.1, (i), (iv). The fact that $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\triangle+}$ follows from Corollary 1.14, (iv). This completes the proof of assertion (i).

Now the *injectivity* portion of assertion (ii) follows from the injectivity portion of Theorem 4.1, (i); in light of this injectivity, the *bijectivity* portion of assertion (ii) follows from assertion (i) and Theorem 4.1, (ii) (cf. the subset of (a)). This completes the proof of assertion (ii) and hence of Corollary 4.2. □

REMARK 4.2.1. (i) Suppose that we are in the situation of Corollary 4.2, and that Σ is the *set of all prime numbers*. Then various injectivity and bijectivity results are obtained by Nakamura and Harbater–Schneps in [26], [5] concerning the subgroup

$$\mathrm{Out}^{\sharp}_{n+3} \subseteq \mathrm{Out}(\Pi_n)$$

(where $n \geq 1$). This subgroup is defined in [5], §0.1, Definition, by means of two conditions "(i)" (i.e., "*quasi-speciality*"), "(ii)" (i.e., "*symmetry*"). From the point of view of the theory of the present paper, these two conditions amount to the condition on $\alpha \in \mathrm{Out}(\Pi_n)$ that "$\alpha \in \mathrm{Out}^{\mathrm{QS}}(\Pi_n)$, and, moreover, α commutes with all of the outer symmetry permutations"—i.e.,

$$\mathrm{Out}^{\sharp}_{n+3} = \mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$$

(cf. Proposition 1.3, (vii)).
(ii) In [5], it is shown that the natural homomorphism

$$\mathrm{Out}^{\sharp}_{n+3} \to \mathrm{Out}^{\sharp}_{n+2}$$

is *injective* if $n \geq 2$ and *bijective* if $n \geq 3$ (cf. [5], §0.1, Corollary). The injectivity portion of this result of [5] is derived (cf. [5], Proposition 8) from the injectivity obtained in [26], Lemma 3.2.2, and may be regarded as a profinite version of an earlier pro-l result due to *Ihara* (cf. [8])—cf. the discussion of [5], §0.2. On the other hand, unlike the case with [5], the approach of [8] allows one to treat, in essence, the full group $\mathrm{Out}^{\mathrm{QS}}(\Pi_n)$ (i.e., not just $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n) = \mathrm{Out}^{\sharp}_{n+3}$) in the pro-$l$ case. In light of the discussion of (i), the proofs given in the present paper of Theorem 4.1, (i), and Corollary 4.2, (ii), may be regarded as *alternate proofs* of these results of [8] and [5].
(iii) The *strong symmetry assumption* imposed on elements of $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$ suggests that there is a *substantial gap* between injectivity or bijectivity results for $\mathrm{Out}^{\mathrm{FCS}}(\Pi_n)$ and injectivity or bijectivity results for $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$. This gap accounts for the lack of the need to invoke such results as the "*combinatorial version of the Grothendieck conjecture*" (i.e., [20], Corollary 2.7, (iii)) in the proofs of [26], [5].

5. The discrete case

In the present §5, we discuss a *discrete analogue* (cf. Corollary 5.1) of Theorem 4.1. One important aspect of this discrete analogue is that it is a relatively easy consequence of the well-known *theorem of Dehn–Nielsen–Baer* (cf., e.g., [13], Theorem 2.9.B), together with the *injectivity* asserted in Theorem 4.1, (i), that the discrete analogue of the homomorphism of Theorem 4.1, (i), is *surjective*.

In the following, we use the notation "$\pi_1^{\mathrm{top}}(-)$" to denote the (*usual*) *topological fundamental group* of the connected topological space in parentheses.

Corollary 5.1 (Partial discrete combinatorial cuspidalization). *Let* $\mathcal{X}$ *be a* topological surface of type (g, r) (*i.e., the complement of* r *distinct points in a compact oriented topological surface of genus* g). *For integers* $n \geq 1$, *write* $\mathcal{X}_n$ *for the complement of the diagonals in the direct product of* n *copies of* $\mathcal{X}$;

$$\Pi_n \stackrel{\text{def}}{=} \pi_1^{\text{top}}(\mathcal{X}_n)$$

for the (*usual topological*) fundamental group *of* $\mathcal{X}_n$; $\hat{\Pi}_n$ *for the* profinite completion *of* Π_n;

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n) \quad (\textit{respectively}, \ \mathrm{Out}^{\mathrm{F}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n))$$

for the subgroup of outer automorphisms α *that satisfy the following condition*(*s*) (1), (2) (*respectively,* (1)):

(1) $\alpha(H) = H$ *for every* fiber subgroup $H \subseteq \Pi_n$ (*cf.* [24], Definition 7.2, (ii); [24], Corollary 7.4).

(2) *For* m *a nonnegative integer* $\leq n$, *write* $K_m \subseteq \Pi_n$ *for the fiber subgroup that arises as the kernel of the projection obtained by "forgetting the factors of* X_n *with labels by* $> m$"; $\Pi_{b/a} \stackrel{\text{def}}{=} K_a/K_b$ *for* $a, b \in \{0, 1, \ldots, n\}$ *such that* $a \leq b$. *Then* α *induces a* bijection *of the collection of conjugacy classes of* cuspidal inertia groups *contained in each* $\Pi_{m/m-1}$ (*where* $m = 1, \ldots, n$) *associated to the various cusps of the topological surfaces that arise as fibers of the projection* $\mathcal{X}_m \twoheadrightarrow \mathcal{X}_{m-1}$ *obtained by "forgetting the factor labeled* m". (*Here, we regard the map* $\Pi_m \cong \Pi_n/\Pi_{n/m} \twoheadrightarrow \Pi_n/\Pi_{n/m-1} \cong \Pi_{m-1}$ *of quotients of* Π_n *as the homomorphism that arises by "forgetting, successively, the factors with labels* $> m$ *and the factors with labels* $> m-1$".) *We refer to* Definition 5.2 *below for more details on the notion of an "inertia group".*

If $r \geq 1$—*i.e.,* $\mathcal{X}$ *is* non-compact—*then set* $n_0 \stackrel{\text{def}}{=} 2$; *if* $r = 0$—*i.e.,* $\mathcal{X}$ *is* compact—*then set* $n_0 \stackrel{\text{def}}{=} 3$. *Then*:

(i) *The natural homomorphisms*

$$\Pi_n \to \hat{\Pi}_n; \quad \mathrm{Out}^{\mathrm{F}}(\Pi_n) \to \mathrm{Out}^{\mathrm{F}}(\hat{\Pi}_n)$$

are injective *for* $n \geq 1$. *Here, the injectivity of the first homomorphism is equivalent to the assertion that* Π_n *is* residually finite.

(ii) *The natural homomorphism*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$$

induced by the projection obtained by "forgetting the factor labeled n" *is* bijective *if* $n \geq n_0$ *and* surjective *if* $n = 2$.

(iii) *Let* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ *be as in* (ii), $n \geq n_0$. *Let* $\sigma \in \mathrm{Out}(\Pi_n)$ *be an outer automorphism that satisfies the following properties*:

(a) *for every fiber subgroup* $H \subseteq \Pi_n$, $\sigma(H)$ *is a fiber subgroup*;

(b) $\sigma(K_{n-1}) = K_{n-1}$;

(c) σ *induces a* bijection *of the collection of conjugacy classes of* cuspidal inertia groups *contained in* K_{n-1};

(d) *the outer automorphism* $\sigma' \in \mathrm{Out}(\Pi_{n-1})$ *determined by* σ (*cf.* (*b*)) normalizes (*respectively*, commutes *with*) $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$. *Then* σ normalizes (*respectively*, commutes *with*) $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$.

(iv) *By permuting the various factors of* $X_n^{\log}$, *one obtains a natural inclusion*

$$\mathfrak{S}_n \hookrightarrow \mathrm{Out}(\Pi_n)$$

of the symmetric group on n letters into $\mathrm{Out}(\Pi_n)$ *whose image* commutes *with* $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *if* $n \geq n_0$ *and* normalizes $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ *if* $r = 0$ *and* $n = 2$.

Proof. In the following, we shall write

$$\mathrm{Aut}^{\mathrm{FC}}(\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\Pi_n) \times_{\mathrm{Out}(\Pi_n)} \mathrm{Out}^{\mathrm{FC}}(\Pi_n),$$
$$\mathrm{Aut}^{\mathrm{F}}(\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\Pi_n) \times_{\mathrm{Out}(\Pi_n)} \mathrm{Out}^{\mathrm{F}}(\Pi_n)$$

for $n \geq 1$. Now let us consider assertion (i). The fact that Π_n is *residually finite* is well-known (cf., e.g., [24], Proposition 7.1, (ii)). Thus, it remains to verify the injectivity of the natural homomorphism $\mathrm{Out}^{\mathrm{F}}(\Pi_n) \to \mathrm{Out}^{\mathrm{F}}(\hat{\Pi}_n)$. When $n = 1$, the injectivity of the natural homomorphism $\mathrm{Out}(\Pi_1) \to \mathrm{Out}(\hat{\Pi}_1)$ is the content of [2], Lemma 3.2.1, when $\mathcal{X}$ is *non-compact*; when $\mathcal{X}$ is *compact*, the injectivity of this homomorphism is *implicit* in the proofs of [4], Theorems 1, 3. This completes the proof of assertion (i) when $n = 1$. Now "assertion (i) for arbitrary n" follows by applying *induction on* n, together with the *natural isomorphism*

$$\Pi_n \xrightarrow{\sim} K_1 \overset{\mathrm{out}}{\rtimes} \Pi_1$$

(cf. §0; Remark 1.1.1) and the evident discrete analogue of the *interpretation* of $\Pi_{n/1} = K_1$ given in [24], Proposition 2.4, (i), which allows one to apply the *induction hypothesis* to K_1 (as well as to Π_1). Indeed, if $\alpha \in \mathrm{Aut}^{\mathrm{F}}(\Pi_n)$ induces an inner automorphism of $\hat{\Pi}_n$, then the automorphism $\alpha_1 \in \mathrm{Aut}^{\mathrm{F}}(\Pi_1)$ determined by α induces an inner automorphism of $\hat{\Pi}_1$. Thus, by the induction hypothesis, α_1 is inner, so by replacing α with the composite of α with an appropriate inner automorphism, we may assume that α_1 is the identity. Then α induces an automorphism $\alpha_K \in \mathrm{Aut}^{\mathrm{F}}(K_1)$ which is compatible with the outer action of Π_1 on K_1. Moreover, α_K arises (relative to the inclusion $K_1 \subseteq \Pi_n \hookrightarrow \hat{\Pi}_n$) from conjugation by an element $\gamma \in \hat{\Pi}_n$ whose image in $\hat{\Pi}_1$ induces (by conjugation) the *identity* automorphism of Π_1 ($\hookrightarrow \hat{\Pi}_1$), hence also the identity automorphism of $\hat{\Pi}_1$. Since $\hat{\Pi}_1$ is *center-free* (cf. Remark 1.1.1), we thus conclude that γ lies in the closure of the image of K_1 in $\hat{\Pi}_n$ (which is naturally isomorphic to the profinite completion of K_1—cf. [24], Proposition 7.1, (i); [24], Proposition 2.2, (i)). Thus, by applying the induction hypothesis to K_1, we conclude that α_K is *inner*, hence (by applying the natural isomorphism $\Pi_n \xrightarrow{\sim} K_1 \overset{\mathrm{out}}{\rtimes} \Pi_1$) that α is *inner*. This completes the proof of assertion (i).

Next, we consider assertion (ii). First, let us recall that by the well-known *theorem of Dehn–Nielsen–Baer* (cf., e.g., [13], Theorem 2.9.B) every automorphism $\alpha \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_1)$ arises from a *homeomorphism* (or even a *diffeomorphism*!) $\alpha_{\mathcal{X}} : \mathcal{X} \xrightarrow{\sim} \mathcal{X}$. Since $\alpha_{\mathcal{X}}$ then induces a homeomorphism $\mathcal{X}_n \xrightarrow{\sim} \mathcal{X}_n$ for every $n \geq 1$, we thus obtain elements $\alpha_n \in \mathrm{Aut}(\Pi_n)$ that (as is easily verified) belong to $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$ and lift α (relative, say, to the projection $\Pi_n \twoheadrightarrow \Pi_1$ determined by the factor labeled 1). In particular, the corresponding natural homomorphisms $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ are *surjective* for $n \geq 1$.

Next, let us observe that the *injectivity* of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ for $n \geq n_0$ follows *formally* from the injectivity of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\hat{\Pi}_n)$ (cf. assertion (i)) and the injectivity of Theorem 4.1, (i). In light of the *surjectivity* of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$, we thus conclude that if $\mathcal{X}$ is *non-compact* (so $n_0 = 2$), then $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ is *bijective* for $n \geq 2$. This completes the proof of assertion (ii) for *non-compact* $\mathcal{X}$.

Next, let us consider the case where $\mathcal{X}$ is *compact*. Then one may verify the *surjectivity* of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ for $n \geq 3$ by arguing as follows. Let $\beta \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_{n-1})$, where we think of Π_{n-1} as "$\Pi_n/\Pi_{n/n-1} = \Pi_n/K_{n-1}$". Then β determines automorphisms $\beta_K \in \mathrm{Aut}^{\mathrm{FC}}(K_1/K_{n-1})$, $\beta_1 \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_1)$ (where we think of Π_1 as "$\Pi_n/\Pi_{n/1} = \Pi_n/K_1$") which are compatible with the natural outer action of Π_1 on K_1/K_{n-1}. Then by applying assertion (ii) in the *non-compact* case (whose proof has already been completed) to K_1, we conclude that $\mathrm{Out}^{\mathrm{FC}}(K_1) \to \mathrm{Out}^{\mathrm{FC}}(K_1/K_{n-1})$ is *bijective*. Let $\alpha_K \in \mathrm{Aut}^{\mathrm{FC}}(K_1)$ be a lifting of β_K. Note that the *injectivity* of $\mathrm{Out}^{\mathrm{FC}}(K_1) \to \mathrm{Out}^{\mathrm{FC}}(K_1/K_{n-1})$ (together with the compatibility of β_1, β_K with the natural outer action of Π_1 on K_1/K_{n-1}) implies that β_1, α_K are *compatible with the natural outer action of* Π_1 *on* K_1. Thus, by applying the *natural isomorphism* $\Pi_n \xrightarrow{\sim} K_1 \overset{\mathrm{out}}{\rtimes} \Pi_1$ (cf. §0; Remark 1.1.1), we conclude that α_K, β_1 determine an automorphism $\alpha \in \mathrm{Aut}(\Pi_n)$ which (as is easily verified, in light of the *residual finiteness* of assertion (i), by applying Proposition 1.2, (i), (iii), to $\hat{\Pi}_n$) belongs to $\mathrm{Aut}^{\mathrm{FC}}(\Pi_n)$. This completes the proof of the *surjectivity* of $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_{n-1})$ for $n \geq 3$, and hence of assertion (ii).

The proof of assertion (iii) as a consequence of assertion (ii) is entirely similar to the proof of Theorem 4.1, (iii) (as a consequence of Theorem 4.1, (i)). Finally, we consider assertion (iv). When $r = 0$ and $n = 2$, assertion (iv) follows immediately from the evident discrete analogue of Proposition 1.6, (i), (a). Thus, it remains to verify that $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$ *commutes* with the image of $\mathfrak{S}_n$ when $n \geq n_0$. To this end, let $\sigma \in \mathrm{Out}(\Pi_n)$ be an element of the image of $\mathfrak{S}_n$; $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$; $\alpha_\sigma \overset{\mathrm{def}}{=} \sigma \cdot \alpha \cdot \sigma^{-1} \in \mathrm{Out}(\Pi_n)$. Then one verifies immediately that $\alpha_\sigma \in \mathrm{Out}^{\mathrm{F}}(\Pi_n)$. Moreover, by Theorem 4.1, (iv), the images of α and α_σ in $\mathrm{Out}^{\mathrm{F}}(\hat{\Pi}_n)$ *coincide*. Thus, the fact that $\alpha = \alpha_\sigma$ follows from the *injectivity* of $\mathrm{Out}^{\mathrm{F}}(\Pi_n) \to \mathrm{Out}^{\mathrm{F}}(\hat{\Pi}_n)$ (cf. assertion (i)). This completes the proof of assertion (iv). □

REMARK 5.1.1. There is a partial overlap between the content of Corollary 5.1 above and Theorems 1, 2 of [12].

DEFINITION 5.2. Let $n \geq 2$ be an integer.
(i) Write $\mathbb{R}$ for the underlying topological space of the topological field of *real numbers*; $\gamma_2 \subseteq \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ for the *unit circle*; $\gamma_n \subseteq \mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (i.e., the product of n copies of $\mathbb{R}$) for the image of the embedding $\gamma_2 \subseteq \mathbb{R}^2 \hookrightarrow \mathbb{R}^n$ obtained by taking the *first* $n-2$ *coordinates* to be zero.
(ii) Let $\mathcal{M}$ be a *connected topological manifold of dimension* n; $\mathcal{L} \subseteq \mathcal{M}$ a *connected submanifold of dimension* $n-2$; $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{M} \setminus \mathcal{L}$. Thus, for each point $x \in \mathcal{L}$, there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{M}$ of x in $\mathcal{U}$, together with an open immersion $\mathcal{U} \hookrightarrow \mathbb{R}^n$ that maps x to the origin of $\mathbb{R}^n$, contains γ_n in its image, and induces an open immersion $\mathcal{U} \cap \mathcal{L} \hookrightarrow \mathbb{R}^{n-2}$ $(\subseteq \mathbb{R}^n)$ (where we think of $\mathbb{R}^{n-2}$ as the subspace of $\mathbb{R}^n$ whose last two coordinates are zero). In particular, we obtain an immersion $\gamma_n \hookrightarrow \mathcal{P} \subseteq \mathcal{M}$; write

$$\mathbb{I}_{\mathcal{M}} \subseteq \pi_1^{\text{top}}(\mathcal{P})$$

for the image of the homomorphism $(\mathbb{Z} \cong)\ \pi_1^{\text{top}}(\gamma_n) \to \pi_1^{\text{top}}(\mathcal{P})$ induced by this immersion $\gamma_n \hookrightarrow \mathcal{P}$ $(\subseteq \mathcal{M})$. One verifies easily that $\mathbb{I}_{\mathcal{M}}$ is *well-defined up to* $\pi_1^{\text{top}}(\mathcal{P})$-*conjugacy* and *independent* of the choice of x, $\mathcal{U}$, and the open immersion $\mathcal{U} \hookrightarrow \mathbb{R}^n$. We shall refer to $\mathbb{I}_{\mathcal{M}}$ as the *inertia group associated to* $\mathcal{M}$ in $\pi_1^{\text{top}}(\mathcal{P})$.

Corollary 5.3 (Quasi-speciality). *In the situation of* Corollary 5.1: *Suppose that* $\mathcal{X}$ *is obtained as the complement of* r *points—i.e.,* "cusps"*—of a compact oriented topological surface* $\mathcal{Z}$. *Write* $\mathcal{P}_n$ *for the product* $\mathcal{Z} \times \cdots \times \mathcal{Z}$ *of* n *copies of* $\mathcal{Z}$; $\mathfrak{D}_n^*$ *for the set of connected submanifolds of codimension* 2 *of* $\mathcal{P}_n$ *given by the* $n(n-1)/2$ *diagonals and the* $n \cdot r$ *fibers of cusps via the* n *projection maps* $\mathcal{P}_n \to \mathcal{Z}$. *For each* $\delta \in \mathfrak{D}_n^*$, *write*

$$\mathcal{X}_n^{\delta} \stackrel{\text{def}}{=} \mathcal{P}_n \setminus \left(\bigcup_{\epsilon \neq \delta} \epsilon \right) \subseteq \mathcal{P}_n$$

—where the union ranges over elements $\epsilon \neq \delta$ *of* $\mathfrak{D}_n^*$;

$$\mathbb{I}_{\delta} \subseteq \Pi_n$$

for the inertia group (*well-defined up to* Π_n*-conjugacy*) *determined by the submanifold* $\delta \cap \mathcal{X}_n^{\delta} \subseteq \mathcal{X}_n^{\delta}$ (*where we note that* $\mathcal{X}_n = \mathcal{X}_n^{\delta} \setminus (\delta \cap \mathcal{X}_n^{\delta})$). *Write*

$$\text{Out}^{\text{QS}}(\Pi_n) \subseteq \text{Out}(\Pi_n)$$

—*where* "*QS*" *stands for* "quasi-special" (*cf.* Proposition 1.3, (vii))—*for the subgroup of outer automorphisms that stabilize the conjugacy class of each inertia group* $\mathbb{I}_\delta$, *for* $\delta \in \mathcal{D}_n^*$;

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$$

for the subgroup of outer automorphisms that induce, via the surjection $\Pi_n \twoheadrightarrow \Pi_1$ *obtained by "forgetting the factors with labels* > 1", *outer automorphisms of* Π_1 *that* stabilize *each of the conjugacy classes of the inertia groups of the cusps. Then*:

(i) *We have*: $\mathrm{Out}^{\mathrm{QS}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$.

(ii) *The natural homomorphism of* Corollary 5.1, (ii), *restricts to a homomorphism*

$$\mathrm{Out}^{\mathrm{QS}}(\Pi_n) \to \mathrm{Out}^{\mathrm{QS}}(\Pi_{n-1})$$

which is bijective *if* $n \geq n_0$ (*where* n_0 *is as in Corollary* 5.1) *and* surjective *if* $n = 2$.

Proof. First, we consider assertion (i). We begin by observing that it follows immediately from the definitions (together with well-known facts concerning the relationship between topological and étale fundamental groups) that *profinite completion* induces a homomorphism $\mathrm{Out}^{\mathrm{QS}}(\Pi_n) \to \mathrm{Out}^{\mathrm{QS}}(\hat{\Pi}_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\hat{\Pi}_n)$ (cf. Proposition 1.3, (vii)). Thus, it follows immediately from the *residual finiteness* of Corollary 5.1, (i), that $\mathrm{Out}^{\mathrm{QS}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$. In particular, the fact that $\mathrm{Out}^{\mathrm{QS}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$ follows immediately from the definition of "$\mathrm{Out}^{\mathrm{QS}}(-)$" (cf. the proof of Proposition 1.3, (vii)). Now it remains to verify that $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{QS}}(\Pi_n)$. To this end, let us first observe that if $\mathcal{X}$ is *compact*, then every $\mathbb{I}_\delta$ (where $\delta \in \mathfrak{D}_n^*$) lies in the kernel of the surjection $\Pi_n \twoheadrightarrow \Pi_1$ obtained by "forgetting the factors with labels > 1"; in particular, (by thinking of $\mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_1)$ as a "Π_{n-1}" that arises for some topological surface of type $(g, 1)$) we conclude that it suffices to verify the inclusion $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \subseteq \mathrm{Out}^{\mathrm{QS}}(\Pi_n)$ for *non-compact* $\mathcal{X}$. Thus, let us suppose that $\mathcal{X}$ is *non-compact*. Then by Corollary 5.1, (ii), we have a *bijection*

$$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}} \xrightarrow{\sim} \mathrm{Out}^{\mathrm{FC}}(\Pi_1)^{\mathrm{cusp}}$$

—i.e., (cf. the proof of Corollary 5.1, (ii)) every element $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{cusp}}$ arises from a *homeomorphism* $\alpha_{\mathcal{X}}\colon \mathcal{X} \xrightarrow{\sim} \mathcal{X}$. Moreover, it follows immediately from the superscript "cusp" that this homeomorphism extends to a homeomorphism $\alpha_{\mathcal{Z}}\colon \mathcal{Z} \xrightarrow{\sim} \mathcal{Z}$ that *fixes each of the cusps*. In particular, $\alpha_{\mathcal{Z}}$ induces *compatible self-homeomorphisms* of $\mathcal{X}_n \subseteq \mathcal{X}_n^\delta \subseteq \mathcal{P}_n$ for each $\delta \in \mathfrak{D}_n^*$. Thus, it follows immediately from the definitions that $\alpha \in \mathrm{Out}^{\mathrm{QS}}(\Pi_n)$. This completes the proof of assertion (i). Finally, assertion (ii) follows immediately from assertion (i) and Corollary 5.1, (ii). □

REMARK 5.3.1. Suppose that $(g, r) = (0, 3)$. Then the *injectivity* portion of Corollary 5.3, (ii), is (essentially) the content of [8], §1.2, "the injectivity theorem (i)". By applying this injectivity, together with a classical result of *Nielsen* to the effect that

$\mathrm{Out}^{\mathrm{QS}}(\Pi_1) = \{\pm 1\}$ (cf. [8], §6.1; here, the element of $\mathrm{Out}^{\mathrm{QS}}(\Pi_1)$ corresponding to "-1" is the automorphism induced by *complex conjugation*), one obtains that $\mathrm{Out}^{\mathrm{QS}}(\Pi_n) = \{\pm 1\}$ for all $n \geq 2$ (cf. [8], §1.2, "the vanishing theorem").

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