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THE DUALITY BETWEEN SINGULAR POINTS AND INFLECTION POINTS ON WAVE FRONTS

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Abstract

In the previous paper, the authors gave criteria for A_{k+1}-type singularities on wave fronts. Using them, we show in this paper that there is a duality between singular points and inflection points on wave fronts in the projective space. As an application, we show that the algebraic sum of 2-inflection points (i.e. godron points) on an immersed surface in the real projective space is equal to the Euler number of M_-. Here M^2 is a compact orientable 2-manifold, and M_- is the open subset of M^2 where the Hessian of f takes negative values. This is a generalization of Bleecker and Wilson's formula [3] for immersed surfaces in the affine 3-space.

1. Introduction

We denote by $\boldsymbol{K}$ the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$. Let n and m be positive integers. A map $F\colon \boldsymbol{K}^n \to \boldsymbol{K}^m$ is called $\boldsymbol{K}$*-differentiable* if it is a C^∞-map when $\boldsymbol{K}=\boldsymbol{R}$, and is a holomorphic map when $\boldsymbol{K}=\boldsymbol{C}$. Throughout this paper, we denote by $P(V)$ the $\boldsymbol{K}$-projective space associated to a vector space V over $\boldsymbol{K}$ and let $\pi\colon V\to P(V)$ be the canonical projection.

Let M^n and N^{n+1} be $\boldsymbol{K}$-differentiable manifolds of dimension n and of dimension $n+1$, respectively. The projectified $\boldsymbol{K}$-cotangent bundle

$$P(T^*N^{n+1}) := \bigcup_{p\in N^{n+1}} P(T^*_pN^{n+1})$$

has a canonical $\boldsymbol{K}$-contact structure. A $\boldsymbol{K}$-differentiable map $f\colon M^n\to N^{n+1}$ is called a *frontal* if f lifts to a $\boldsymbol{K}$-isotropic map L_f, i.e., a $\boldsymbol{K}$-differentiable map $L_f\colon M^n\to P(T^*N^{n+1})$ such that the image $dL_f(TM^n)$ of the $\boldsymbol{K}$-tangent bundle TM^n lies in the contact hyperplane field on $P(T^*N^{n+1})$. Moreover, f is called a *wave front* or a *front* if it lifts to a $\boldsymbol{K}$-isotropic immersion L_f. (In this case, L_f is called a *Legendrian immersion*.) Frontals (and therefore fronts) generalize immersions, as they allow for singular points. A frontal f is said to be *co-orientable* if its $\boldsymbol{K}$-isotropic lift L_f can lift up to a $\boldsymbol{K}$-differentiable map into the $\boldsymbol{K}$-cotangent bundle T^*N^{n+1}, otherwise it is said to be *non-co-orientable*. It should be remarked that, when N^{n+1} is a Riemannian mani-

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fold, a front f is co-orientable if and only if there is a globally defined unit normal vector field ν along f.

Now we set $N^{n+1} = \boldsymbol{K}^{n+1}$. Suppose that a $\boldsymbol{K}$-differentiable map $F\colon M^n \to \boldsymbol{K}^{n+1}$ is a frontal. Then, for each $p \in M^n$, there exist a neighborhood U of p and a map

$$\nu\colon U \to (\boldsymbol{K}^{n+1})^* \setminus \{\mathbf{0}\}$$

into the dual vector space $(\boldsymbol{K}^{n+1})^*$ of $\boldsymbol{K}^{n+1}$ such that the canonical pairing $\nu \cdot dF(v)$ vanishes for any $v \in TU$. We call ν a *local normal map* of the frontal F. We set $\mathcal{G} := \pi \circ \nu$, which is called a (local) *Gauss map* of F. In this setting, F is a front if and only if

$$L := (F, \mathcal{G})\colon U \to \boldsymbol{K}^{n+1} \times P((\boldsymbol{K}^{n+1})^*)$$

is an immersion. When F itself is an immersion, it is, of course, a front. If this is the case, for a fixed local $\boldsymbol{K}$-differentiable coordinate system $(x^1, \dots, x^n)$ on U, we set

$$\nu_p\colon \boldsymbol{K}^{n+1} \ni v \mapsto \det(F_{x^1}(p), \dots, F_{x^n}(p), v) \in \boldsymbol{K} \quad (p \in U), \tag{1.1}$$

where $F_{x^j} := \partial F/\partial x^j$ $(j = 1, \dots, n)$ and 'det' is the determinant function on $\boldsymbol{K}^{n+1}$. Then we get a $\boldsymbol{K}$-differentiable map $\nu\colon U \ni p \mapsto \nu_p \in (\boldsymbol{K}^{n+1})^*$, which gives a local normal map of F.

Now, we return to the case that F is a front. Then it is well-known that the local Gauss map $\mathcal{G}$ induces a global map

$$\mathcal{G}\colon M^n \to P((\boldsymbol{K}^{n+1})^*) \tag{1.2}$$

which is called the *affine Gauss map* of F. (In fact, the Gauss map $\mathcal{G}$ depends only on the affine structure of $\boldsymbol{K}^{n+1}$.)

We set

$$h_{ij} := \nu \cdot F_{x^i x^j} = -\nu_{x^i} \cdot F_{x^j} \quad (i, j = 1, \dots, n), \tag{1.3}$$

where $\cdot$ is the canonical pairing between $\boldsymbol{K}^{n+1}$ and $(\boldsymbol{K}^{n+1})^*$, and

$$F_{x^i x^j} = \frac{\partial^2 F}{\partial x^i \partial x^j}, \quad F_{x^j} = \frac{\partial F}{\partial x^j}, \quad \nu_{x^i} = \frac{\partial \nu}{\partial x^i}.$$

Then

$$H := \sum_{i,j=1}^{n} h_{ij} dx^i\, dx^j \quad (dx^i\, dx^j := (1/2)(dx^i \otimes dx^j + dx^j \otimes dx^i)) \tag{1.4}$$

gives a $\boldsymbol{K}$-valued symmetric tensor on U, which is called the *Hessian form* of F associated to ν. Here, the $\boldsymbol{K}$-differentiable function

$$h := \det(h_{ij})\colon U \to \boldsymbol{K} \tag{1.5}$$

is called the *Hessian* of F. A point $p \in M^n$ is called an *inflection point* of F if it belongs to the zeros of h. An inflection point p is called *nondegenerate* if the derivative dh does not vanish at p. In this case, the set of inflection points $I(F)$ consists of an embedded $\boldsymbol{K}$-differentiable hypersurface of U near p and there exists a non-vanishing $\boldsymbol{K}$-differentiable vector field ξ along $I(F)$ such that $H(\xi, v) = 0$ for all $v \in TU$. Such a vector field ξ is called an *asymptotic vector field* along $I(F)$, and $[\xi] = \pi(\xi) \in P(\boldsymbol{K}^{n+1})$ is called the *asymptotic direction*. It can be easily checked that the definition of inflection points and the nondegeneracy of inflection points are independent of choice of ν and a local coordinate system.

In Section 2, we shall define the terminology that

- a $\boldsymbol{K}$-differentiable vector field η along a $\boldsymbol{K}$-differentiable hypersurface S of M^n is *k-nondegenerate at $p \in S$*, and
- *η meets S at p with multiplicity $k+1$.*

Using this new terminology, p $(\in I(F))$ is called an *A_{k+1}-inflection point* if ξ is k-nondegenerate at p but does not meet $I(F)$ with multiplicity $k+1$. In Section 2, we shall prove the following:

Theorem A. *Let $F\colon M^n \to \boldsymbol{K}^{n+1}$ be an immersed $\boldsymbol{K}$-differentiable hypersurface. Then $p \in M^n$ is an A_{k+1}-inflection point* $(1 \le k \le n)$ *if and only if the affine Gauss map $\mathcal{G}$ has an A_k-Morin singularity at p.* (*See the appendix of* [10] *for the definition of A_k-Morin singularities, which corresponds to A_{k+1}-points under the intrinsic formulation of singularities as in the reference given in* Added in Proof.)

Though our definition of A_{k+1}-inflection points are given in terms of the Hessian, this assertion allows us to define A_{k+1}-inflection points by the singularities of their affine Gauss map, which might be more familiar to readers than our definition. However, the new notion “k-multiplicity” introduced in the present paper is very useful for recognizing the duality between singular points and inflection points. Moreover, as mentioned above, our definition of A_k-inflection points works even when F is a front. We have the following dual assertion for the previous theorem. Let $\mathcal{G}\colon M^n \to P((\boldsymbol{K}^{n+1})^*)$ be an immersion. Then $p \in M^n$ is an *A_{k+1}-inflection point* of $\mathcal{G}$ if it is an A_{k+1}-inflection point of $\nu\colon M^n \to (\boldsymbol{K}^{n+1})^*$ such that $\pi \circ \nu = \mathcal{G}$. This property does not depend on a choice of ν.

Proposition A′. *Let $F\colon M^n \to \boldsymbol{K}^{n+1}$ be a front. Suppose that the affine Gauss map $\mathcal{G}\colon M^n \to P((\boldsymbol{K}^{n+1})^*)$ is a $\boldsymbol{K}$-immersion. Then $p \in M^n$ is an A_{k+1}-inflection point* *of $\mathcal{G}$ $(1 \le k \le n)$ if and only if F has an A_{k+1}-singularity at p.* (*See* (1.1) *in* [10] *for the definition of A_{k+1}-singularities.*)

In the case that $\boldsymbol{K} = \boldsymbol{R}$, $n = 3$ and F is an immersion, an A_3-inflection point is known as a *cusp of the Gauss map* (cf. [2]).

It can be easily seen that inflection points and the asymptotic directions are invariant under projective transformations. So we can define A_{k+1}-inflection points $(1 \le k \le n)$ of an immersion $f\colon M^n \to P(\boldsymbol{K}^{n+2})$. For each $p \in M^n$, we take a local $\boldsymbol{K}$-differentiable coordinate system $(U; x^1, \dots, x^n)$ $(\subset M^n)$. Then there exists a $\boldsymbol{K}$-immersion $F\colon U \to \boldsymbol{K}^{n+2}$ such that $f = [F]$ is the projection of F. We set

$$G\colon U \ni p \mapsto F_{x^1}(p) \wedge F_{x^2}(p) \wedge \cdots \wedge F_{x^n}(p) \wedge F(p) \in (\boldsymbol{K}^{n+2})^*. \tag{1.6}$$

Here, we identify $(\boldsymbol{K}^{n+2})^*$ with $\bigwedge^{n+1} \boldsymbol{K}^{n+2}$ by

$$\bigwedge^{n+1} \boldsymbol{K}^{n+2} \ni v_1 \wedge \cdots \wedge v_{n+1} \longleftrightarrow \det(v_1, \dots, v_{n+1}, *) \in (\boldsymbol{K}^{n+2})^*,$$

where 'det' is the determinant function on $\boldsymbol{K}^{n+2}$. Then G satisfies

$$G \cdot F = 0, \quad G \cdot dF = dG \cdot F = 0, \tag{1.7}$$

where $\cdot$ is the canonical pairing between $\boldsymbol{K}^{n+2}$ and $(\boldsymbol{K}^{n+2})^*$. Since, $g := \pi \circ G$ does not depend on the choice of a local coordinate system, the projection of G induces a globally defined $\boldsymbol{K}$-differentiable map

$$g = [G]\colon M^n \to P((\boldsymbol{K}^{n+2})^*),$$

which is called the *dual* front of f. We set

$$h := \det(h_{ij})\colon U \to \boldsymbol{K} \quad (h_{ij} := G \cdot F_{x^i x^j} = -G_{x^i} \cdot F_{x^j}),$$

which is called the *Hessian* of F. The inflection points of f correspond to the zeros of h.

In Section 3, we prove the following

Theorem B. *Let $f\colon M^n \to P(\boldsymbol{K}^{n+2})$ be an immersed $\boldsymbol{K}$-differentiable hypersurface. Then $p \in M^n$ is an A_{k+1}-inflection point $(k \le n)$ if and only if the dual front g has an A_k-singularity at p.*

Next, we consider the case of $\boldsymbol{K} = \boldsymbol{R}$. In [8], we defined the *tail part* of a swallowtail, that is, an A_3-singular point. An A_3-inflection point p of $f\colon M^2 \to P(\boldsymbol{R}^4)$ is called *positive* (resp. *negative*), if the Hessian takes negative (resp. positive) values on the tail part of the dual of f at p. Let $p \in M^2$ be an A_3-inflection point. Then there exists a neighborhood U such that $f(U)$ is contained in an affine space A^3 in $P(\boldsymbol{R}^4)$. Then the affine Gauss map $\mathcal{G}\colon U \to P(A^3)$ has an elliptic cusp (resp. a hyperbolic cusp) if and only if it is positive (resp. negative) (see [2, p. 33]). In [13], Uribe-Vargas introduced a projective invariant ρ and studied the projective geometry of swallowtails. He proved that an A_3-inflection point is positive (resp. negative) if and only if $\rho > 1$ (resp. $\rho < 1$). The property that h as in (1.5) is negative is also independent of the choice of a local coordinate system. So we can define the set of negative points

$$M_- := \{p \in M^2;\ h(p) < 0\}.$$

In Section 3, we shall prove the following assertion as an application.

Theorem C. *Let M^2 be a compact orientable C^∞-manifold without boundary, and $f\colon M^2 \to P(\boldsymbol{R}^4)$ an immersion. We denote by $i_2^+(f)$ (resp. $i_2^-(f)$) the number of positive A_3-inflection points (resp. negative A_3-inflection points) on M^2 (see* Section 3 *for the precise definition of $i_2^+(f)$ and $i_2^-(f)$). Suppose that inflection points of f consist only of A_2 and A_3-inflection points. Then the following identity holds*

$$i_2^+(f) - i_2^-(f) = 2\chi(M_-). \tag{1.8}$$

The above formula is a generalization of that of Bleecker and Wilson [3] when $f(M^2)$ is contained in an affine 3-space.

Corollary D (Uribe-Vargas [13, Corollary 4]). *Under the assumption of* Theorem C, *the total number $i_2^+(f) + i_2^-(f)$ of A_3-inflection points is even.*

In [13], this corollary is proved by counting the parity of a loop consisting of flecnodal curves which bound two A_3-inflection points.

Corollary E. *The same formula* (1.8) *holds for an immersed surface in the unit* 3*-sphere S^3 or in the hyperbolic* 3*-space H^3.*

Proof. Let $\pi\colon S^3 \to P(\boldsymbol{R}^4)$ be the canonical projection. If $f\colon M^2 \to S^3$ is an immersion, we get the assertion applying Theorem C to $\pi \circ f$. On the other hand, if f is an immersion into H^3, we consider the canonical projective embedding $\iota\colon H^3 \to S^3_+$ where S^3_+ is the open hemisphere of S^3. Then we get the assertion applying Theorem C to $\pi \circ \iota \circ f$. $\square$

Finally, in Section 4, we shall introduce a new invariant for 3/2-cusps using the duality, which is a measure for acuteness using the classical cycloid.

This work is inspired by the result of Izumiya, Pei and Sano [4] that characterizes A_2 and A_3-singular points on surfaces in H^3 via the singularity of certain height functions, and the result on the duality between space-like surfaces in hyperbolic 3-space (resp. in light-cone), and those in de Sitter space (resp. in light-cone) given by Izumiya [5]. The authors would like to thank Shyuichi Izumiya for his impressive informal talk at Karatsu, 2005.

2. Preliminaries and a proof of Theorem A

In this section, we shall introduce a new notion "multiplicity" for a contact of a given vector field along an immersed hypersurface. Then our previous criterion for A_k-singularities (given in [10]) can be generalized to the criteria for k-multiple contactness of a given vector field (see Theorem 2.4).

Let M^n be a $\boldsymbol{K}$-differentiable manifold and S ($\subset M^n$) an embedded $\boldsymbol{K}$-differentiable hypersurface in M^n. We fix $p \in S$ and take a $\boldsymbol{K}$-differentiable vector field

$$\eta\colon S \supset V \ni q \mapsto \eta_q \in T_q M^n$$

along S defined on a neighborhood $V \subset S$ of p. Then we can construct a $\boldsymbol{K}$-differential vector field $\tilde{\eta}$ defined on a neighborhood $U \subset M^n$ of p such that the restriction $\tilde{\eta}|_S$ coincides with η. Such an $\tilde{\eta}$ is called *an extension of* η. (The local existence of $\tilde{\eta}$ is mentioned in [10, Remark 2.2].)

DEFINITION 2.1. Let p be an arbitrary point on S, and U a neighborhood of p in M^n. A $\boldsymbol{K}$-differentiable function $\varphi\colon U \to \boldsymbol{K}$ is called *admissible* near p if it satisfies the following properties

(1) $O := U \cap S$ is the zero level set of φ, and

(2) $d\varphi$ never vanishes on O.

One can easily find an admissible function near p. We set $\varphi' := d\varphi(\tilde{\eta})\colon U \to \boldsymbol{K}$ and define a subset S_2 ($\subset O \subset S$) by

$$S_2 := \{q \in O\,;\ \varphi'(q) = 0\} = \{q \in O\,;\ \eta_q \in T_q S\}.$$

If $p \in S_2$, then η is said to *meet S with multiplicity* 2 *at* p or equivalently, η is said to *contact S with multiplicity* 2 *at* p. Otherwise, η is said to *meet S with multiplicity* 1 *at* p. Moreover, if $d\varphi'(T_p O) \neq \{\boldsymbol{0}\}$, η is said to *be* 2*-nondegenerate at* p. The k-th multiple contactness and k-nondegeneracy are defined inductively. In fact, if the j-th multiple contactness and the submanifolds S_j have been already defined for $j = 1, \dots, k$ ($S_1 = S$), we set

$$\varphi^{(k)} := d\varphi^{(k-1)}(\tilde{\eta})\colon U \to \boldsymbol{K} \quad (\varphi^{(1)} := \varphi')$$

and can define a subset of S_k by

$$S_{k+1} := \{q \in S_k;\ \varphi^{(k)}(q) = 0\} = \{q \in S_k;\ \eta_q \in T_q S_k\}.$$

We say that η *meets* S *with multiplicity* $k+1$ *at* p if η is k-nondegenerate at p and $p \in S_{k+1}$. Moreover, if $d\varphi^{(k)}(T_p S_k) \neq \{\mathbf{0}\}$, η is called $(k+1)$-*nondegenerate* at p. If η is $(k+1)$-nondegenerate at p, then S_{k+1} is a hypersurface of S_k near p.

REMARK 2.2. Here we did not define '1-nondegeneracy' of η. However, from now on, *any* $\boldsymbol{K}$*-differentiable vector field* η *of* M^n *along* S *is always* 1*-nondegenerate by convention*. In the previous paper [10], '1-nondegeneracy' (i.e. nondegeneracy) is defined not for a vector field along the singular set but for a given singular point. If a singular point $p \in U$ of a front $f: U \to \boldsymbol{K}^{n+1}$ is nondegenerate in the sense of [10], then the function $\lambda: U \to \boldsymbol{K}$ defined in [10, (2.1)] is an admissible function, and the null vector field η along $S(f)$ is given. When $k \geq 2$, by definition, k-nondegeneracy of the singular point p is equivalent to the k-nondegeneracy of the null vector field η at p (cf. [10]).

Proposition 2.3. *The k-th multiple contactness and k-nondegeneracy are both independent of the choice of an extension $\tilde{\eta}$ of η and also of the choice of admissible functions as in* Definition 2.1.

Proof. We can take a local coordinate system $(U; x^1, \dots, x^n)$ of M^n such that $x^n = \varphi$. Write

$$\tilde{\eta} := \sum_{j=1}^{n} c^j \frac{\partial}{\partial x^j},$$

where c^j $(j = 1, \dots, n)$ are $\boldsymbol{K}$-differentiable functions. Then we have that $\varphi' = \sum_{j=1}^{n} c^j \varphi_{x^j} = c^n$.

Let ψ be another admissible function defined on U. Then

$$\psi' = \sum_{j=1}^{n} c^j \frac{\partial \psi}{\partial x^j} = c^n \frac{\partial \psi}{\partial x^n} = \varphi' \frac{\partial \psi}{\partial x^n}.$$

Thus ψ' is proportional to φ'. Then the assertion follows inductively. □

Corollary 2.5 in [10] is now generalized into the following assertion:

Theorem 2.4. *Let $\tilde{\eta}$ be an extension of the vector field η. Let us assume $1 \leq k \leq n$. Then the vector field η is k-nondegenerate at p, but η does not meet S with* *multiplicity* $k+1$ *at* p *if and only if*

$$\varphi(p) = \varphi'(p) = \cdots = \varphi^{(k-1)}(p) = 0, \quad \varphi^{(k)}(p) \neq 0,$$

and the Jacobi matrix of $\boldsymbol{K}$*-differentiable map*

$$\Lambda := (\varphi, \varphi', \dots, \varphi^{(k-1)}) \colon U \to \boldsymbol{K}^k$$

is of rank k *at* p*, where* φ *is an admissible* $\boldsymbol{K}$*-differentiable function and*

$$\varphi^{(0)} := \varphi,\ \varphi^{(1)} (= \varphi') := d\varphi(\tilde{\eta}), \dots, \varphi^{(k)} := d\varphi^{(k-1)}(\tilde{\eta}).$$

The proof of this theorem is completely parallel to that of Corollary 2.5 in [10].

To prove Theorem A by applying Theorem 2.4, we shall review the criterion for A_k-singularities in [10]. Let U^n be a domain in $\boldsymbol{K}^n$, and consider a map $\Phi\colon U^n \to \boldsymbol{K}^m$ where $m \geq n$. A point $p \in U^n$ is called a *singular point* if the rank of the differential map $d\Phi$ is less than n. Suppose that the singular set $S(\Phi)$ of Φ consists of a $\boldsymbol{K}$-differentiable hypersurface U^n. Then a vector field η along S is called a *null vector field* if $d\Phi(\eta)$ vanishes identically. In this paper, we consider the case $m = n$ or $m = n+1$. If $m = n$, we define a $\boldsymbol{K}$-differentiable function $\lambda\colon U^n \to \boldsymbol{K}$ by

$$\lambda := \det(\Phi_{x^1}, \dots, \Phi_{x^n}). \tag{2.1}$$

On the other hand, if $\Phi\colon U^n \to \boldsymbol{K}^{n+1}$ $(m = n+1)$ and ν is a non-vanishing $\boldsymbol{K}$-normal vector field (for a definition, see [10, Section 1]) we set

$$\lambda := \det(\Phi_{x^1}, \dots, \Phi_{x^n}, \nu). \tag{2.2}$$

Then the singular set $S(\Phi)$ of the map Φ coincides with the zeros of λ. Recall that $p \in S(\Phi)$ is called *nondegenerate* if $d\lambda(p) \neq \boldsymbol{0}$ (see [10] and Remark 2.2). Both of two cases (2.1) and (2.2), the functions λ are admissible near p (cf. Definition 2.1), if p is non-degenerate. When $S(\Phi)$ consists of nondegenerate singular points, then it is a hypersurface and there exists a non-vanishing null vector field η on $S(\Phi)$. Such a vector field η determined up to a multiplication of non-vanishing $\boldsymbol{K}$-differentiable functions. The following assertion holds as seen in [10].

Fact 2.5. *Suppose* $m = n$ *and* Φ *is a* C^∞*-map* (*resp.* $m = n+1$ *and* Φ *is a front*). *Then* Φ *has an* A_k*-Morin singularity* (*resp.* A_{k+1}*-singularity*) *at* $p \in M^n$ *if and only if* η *is* k*-nondegenerate at* p *but does not meet* $S(\Phi)$ *with multiplicity* $k+1$ *at* p. (*Here multiplicity* 1 *means that* η *meets* $S(\Phi)$ *at* p *transversally, and* 1*-nondegeneracy is an empty condition.*)

As an application of the fact for $m = n$, we now give a proof of Theorem A: Let $F\colon M^n \to \boldsymbol{K}^{n+1}$ be an immersed $\boldsymbol{K}$-differentiable hypersurface. Recall that a point $p \in$

M^n is called a *nondegenerate inflection point* if the derivative dh of the local Hessian function h (cf. (1.5)) with respect to F does not vanish at p. Then the set $I(F)$ of inflection points consists of a hypersurface, called the *inflectional hypersurface*, and the function h is an admissible function on a neighborhood of p in M^n. A nondegenerate inflection point p is called an A_{k+1}-*inflection point* of F if the asymptotic vector field ξ is k-nondegenerate at p but does not meet $I(F)$ with multiplicity $k+1$ at p.

Proof of Theorem A. Let ν be a map given by (1.1), and $\mathcal{G}\colon M^n \to P((\boldsymbol{K}^{n+1})^*)$ the affine Gauss map induced from ν by (1.2). We set

$$\mu := \det(\nu_{x^1}, \nu_{x^2}, \dots, \nu_{x^n}, \nu),$$

where 'det' is the determinant function of $(\boldsymbol{K}^{n+1})^*$ under the canonical identification $(\boldsymbol{K}^{n+1})^* \cong \boldsymbol{K}^{n+1}$, and $(x^1, \dots, x^n)$ is a local coordinate system of M^n. Then the singular set $S(\mathcal{G})$ of $\mathcal{G}$ is just the zeros of μ. By Theorem 2.4 and Fact 2.5, our criteria for A_{k+1}-inflection points (resp. A_{k+1}-singular points) are completely determined by the pair $(\xi, I(F))$ (resp. the pair $(\eta, S(\mathcal{G}))$). Hence it is sufficient to show the following three assertions (1)–(3).

(1) $I(F) = S(\mathcal{G})$.

(2) *For each $p \in I(F)$, p is a nondegenerate inflection point of F if and only if it is a nondegenerate singular point of $\mathcal{G}$.*

(3) *The asymptotic direction of each nondegenerate inflection point p of F is equal to the null direction of p as a singular point of $\mathcal{G}$.*

Let $H = \sum_{i,j=1}^n h_{ij}\, dx^i\, dx^j$ be the Hessian form of F. Then we have that

$$\begin{pmatrix} h_{11} & \dots & h_{1n} & * \\ \vdots & \ddots & \vdots & \vdots \\ h_{n1} & \dots & h_{nn} & * \\ 0 & \dots & 0 & \nu \cdot {}^t\nu \end{pmatrix} = \begin{pmatrix} \nu_{x^1} \\ \vdots \\ \nu_{x^n} \\ \nu \end{pmatrix} (F_{x^1}, \dots, F_{x^n}, {}^t\nu), \tag{2.3}$$

where $\nu \cdot {}^t\nu = \sum_{j=1}^{n+1} (\nu^j)^2$ and $\nu = (\nu^1, \dots, \nu^n)$ as a row vector. Here, we consider a vector in $\boldsymbol{K}^n$ (resp. in $(\boldsymbol{K}^n)^*$) as a column vector (resp. a row vector), and ${}^t(\cdot)$ denotes the transposition. We may assume that $\nu(p) \cdot {}^t\nu(p) \neq 0$ by a suitable affine transformation of $\boldsymbol{K}^{n+1}$, even when $\boldsymbol{K} = \boldsymbol{C}$. Since the matrix $(F_{x^1}, \dots, F_{x^n}, {}^t\nu)$ is regular, (1) and (2) follow by taking the determinant of (2.3). Also by (2.3), $\sum_{i=1}^n a_i h_{ij} = 0$ for all $j = 1, \dots, n$ holds if and only if $\sum_{i=1}^n a_i \nu_{x^i} = \mathbf{0}$, which proves (3). $\square$

Proof of Proposition A′. Similar to the proof of Theorem A, it is sufficient to show the following properties, by virtue of Theorem 2.4.

(1′) $S(F) = I(\mathcal{G})$, *that is, the set of singular points of F coincides with the set of inflection points of the affine Gauss map.*

(2′) *For each* $p \in I(\mathcal{G})$, p *is a nondegenerate inflection point if and only if it is a nondegenerate singular point of* F.
(3′) *The asymptotic direction of each nondegenerate inflection point coincides with the null direction of* p *as a singular point of* F.
Since $\mathcal{G}$ is an immersion, (2.3) implies that

$$\begin{aligned} I(\mathcal{G}) &= \{p;\ (F_{x^1}, \dots, F_{x^n}, {}^t\nu) \text{ are linearly dependent at } p\} \\ &= \{p;\ \lambda(p) = 0\} \quad (\lambda := \det(F_{x^1}, \dots, F_{x^n}, {}^t\nu)). \end{aligned}$$

Hence we have (1′). Moreover, $h = \det(h_{ij}) = \delta\lambda$ holds, where δ is a function on U which never vanishes on a neighborhood of p. Thus (2′) holds. Finally, by (2.3), $\sum_{j=1}^n b_j h_{ij} = 0$ for $i = 1, \dots, n$ if and only if $\sum_{j=1}^n b_j F_{x^j} = \mathbf{0}$, which proves (3′). □

EXAMPLE 2.6 (A_2-inflection points on cubic curves). Let $\gamma(t) := {}^t(x(t), y(t))$ be a $\boldsymbol{K}$-differentiable curve in $\boldsymbol{K}^2$. Then $\nu(t) := (-\dot{y}(t), \dot{x}(t)) \in (\boldsymbol{K}^2)^*$ gives a normal vector, and

$$h(t) = \nu(t) \cdot \ddot{\gamma}(t) = \det(\dot{\gamma}(t), \ddot{\gamma}(t))$$

is the Hessian function. Thus $t = t_0$ is an A_2-inflection point if and only if

$$\det(\dot{\gamma}(t_0), \ddot{\gamma}(t_0)) = 0, \quad \det(\dot{\gamma}(t_0), \dddot{\gamma}(t_0)) \neq 0.$$

Considering $\boldsymbol{K}^2 \subset P(\boldsymbol{K}^3)$ as an affine subspace, this criterion is available for curves in $P(\boldsymbol{K}^3)$. When $\boldsymbol{K} = \boldsymbol{C}$, it is well-known that non-singular cubic curves in $P(\boldsymbol{C}^3)$ have exactly nine inflection points which are all of A_2-type. One special singular cubic curve is $2y^2 - 3x^3 = 0$ in $P(\boldsymbol{C}^3)$ with homogeneous coordinates $[x, y, z]$, which can be parameterized as $\gamma(t) = [\sqrt[3]{2}t^2, \sqrt{3}t^3, 1]$. The image of the dual curve of γ in $P(\boldsymbol{C}^3)$ is the image of γ itself, and γ has an A_2-type singular point $[0, 0, 1]$ and an A_2-inflection point $[0, 1, 0]$.

These two points are interchanged by the duality. (The duality of fronts is explained in Section 3.)

EXAMPLE 2.7 (The affine Gauss map of an A_4-inflection point). Let $F\colon \boldsymbol{K}^3 \to \boldsymbol{K}^4$ be a map defined by

$$F(u, v, w) = {}^t\!\left(w, u, v, -u^2 - \frac{3v^2}{2} + uw^2 + vw^3 - \frac{w^4}{4} + \frac{w^5}{5} - \frac{w^6}{6}\right) \quad (u, v, w \in \boldsymbol{K}).$$

If we define $\mathcal{G}\colon \boldsymbol{K}^3 \to P(\boldsymbol{K}^4) \cong P((\boldsymbol{K}^4)^*)$ by

$$\mathcal{G}(u, v, w) = [-2uw - 3vw^2 + w^3 - w^4 + w^5, 2u - w^2, 3v - w^3, 1]$$

using the homogeneous coordinate system, $\mathcal{G}$ gives the affine Gauss map of F. Then the Hessian h of F is

$$\det\begin{pmatrix} -2 & 0 & 2w \\ 0 & -3 & 3w^2 \\ 2w & 3w^2 & 2u+6vw-3w^2+4w^3-5w^4 \end{pmatrix} = 6(2u+6vw-w^2+4w^3-2w^4).$$

The asymptotic vector field is $\xi = (w, w^2, 1)$. Hence we have

$$h = 6(2u + 6vw - w^2 + 4w^3 - 2w^4),$$
$$h' = 12(3v + 6w^2 - w^3), \quad h'' = 144w, \quad h''' = 144,$$

where $h' = dh(\xi)$, $h'' = dh'(\xi)$ and $h''' = dh''(\xi)$. The Jacobi matrix of (h, h', h'') at $\mathbf{0}$ is

$$\begin{pmatrix} 2 & * & * \\ 0 & 36 & * \\ 0 & 0 & 144 \end{pmatrix}.$$

This implies that ξ is 3-nondegenerate at $\mathbf{0}$ but does not meet $I(F) = h^{-1}(0)$ at p with multiplicity 4, that is, F has an A_4-inflection point at $\mathbf{0}$. On the other hand, $\mathcal{G}$ has the A_3-Morin singularity at $\mathbf{0}$. In fact, by the coordinate change

$$U = 2u - w^2, \quad V = 3v - w^3, \quad W = w,$$

it follows that $\mathcal{G}$ is represented by a map germ

$$(U, V, W) \mapsto -(UW + VW^2 + W^4, U, V).$$

This coincides with the typical A_3-Morin singularity given in (A.3) in [10].

3. Duality of wave fronts

Let $P(\boldsymbol{K}^{n+2})$ be the $(n+1)$-projective space over $\boldsymbol{K}$. We denote by $[x] \in P(\boldsymbol{K}^{n+2})$ the projection of a vector $x = {}^t(x^0, \dots, x^{n+1}) \in \boldsymbol{K}^{n+2} \setminus \{\mathbf{0}\}$. Consider a $(2n+3)$-submanifold of $\boldsymbol{K}^{n+2} \times (\boldsymbol{K}^{n+2})^*$ defined by

$$\tilde{C} := \{(x, y) \in \boldsymbol{K}^{n+2} \times (\boldsymbol{K}^{n+2})^*;\ x \cdot y = 0\},$$

and also a $(2n+1)$-submanifold of $P(\boldsymbol{K}^{n+2}) \times P((\boldsymbol{K}^{n+2})^*)$

$$C := \{([x], [y]) \in P(\boldsymbol{K}^{n+2}) \times P((\boldsymbol{K}^{n+2})^*);\ x \cdot y = 0\}.$$

As C can be canonically identified with the projective tangent bundle $PTP(\boldsymbol{K}^{n+2})$, it has a canonical contact structure: Let $\pi\colon \tilde{C} \to C$ be the canonical projection, and define a 1-from

$$\omega := \sum_{j=0}^{n+1} (x^j\, dy^j - y^j\, dx^j),$$

which is considered as a 1-form of $\tilde{C}$. The tangent vectors of the curves $t \mapsto (tx, y)$ and $t \mapsto (x, ty)$ at $(x, y) \in \tilde{C}$ generate the kernel of $d\pi$. Since these two vectors also belong to the kernel of ω and $\dim(\ker \omega) = 2n + 2$,

$$\Pi := d\pi(\ker \omega)$$

is a $2n$-dimensional vector subspace of $T_{\pi(x,y)}C$. We shall see that Π is the contact structure on C. One can check that it coincides with the canonical contact structure of $PTP(\boldsymbol{K}^{n+2})$ ($\cong C$). Let U be an open subset of C and $s\colon U \to \boldsymbol{K}^{n+2} \times (\boldsymbol{K}^{n+2})^*$ a section of the fibration π. Since $d\pi \circ ds$ is the identity map, it can be easily checked that Π is contained in the kernel of the 1-form $s^*\omega$. Since Π and the kernel of the 1-form $s^*\omega$ are the same dimension, they coincide. Moreover, suppose that $p = \pi(x, y) \in C$ satisfies $x^i \neq 0$ and $y^j \neq 0$. We then consider a map of $\boldsymbol{K}^{n+1} \times (\boldsymbol{K}^{n+1})^* \cong \boldsymbol{K}^{n+1} \times \boldsymbol{K}^{n+1}$ into $\boldsymbol{K}^{n+2} \times (\boldsymbol{K}^{n+2})^* \cong \boldsymbol{K}^{n+2} \times \boldsymbol{K}^{n+2}$ defined by

$$(a^0, \dots, a^n, b^0, \dots, b^n) \mapsto (a^0, \dots, a^{i-1}, 1, a^{i+1}, \dots, a^n, b^0, \dots, b^{j-1}, 1, b^{j+1}, \dots, b^n),$$

and denote by $s_{i,j}$ the restriction of the map to the neighborhood of p in C. Then one can easily check that

$$s_{i,j}^* \left[\omega \wedge \left(\bigwedge^n d\omega \right) \right]$$

does not vanish at p. Thus $s_{i,j}^*\omega$ is a contact form, and the hyperplane field Π defines a canonical contact structure on C. Moreover, the two projections from C into $P(\boldsymbol{K}^{n+2})$ are both Legendrian fibrations, namely we get a double Legendrian fibration. Let $f = [F]\colon M^n \to P(\boldsymbol{K}^{n+2})$ be a front. Then there is a Legendrian immersion of the form $L = ([F], [G])\colon M^n \to C$. Then $g = [G]\colon M^n \to P((\boldsymbol{K}^{n+2})^*)$ satisfies (1.6) and (1.7). In particular, $L := \pi(F, G)\colon M^n \to C$ gives a Legendrian immersion, and f and g can be regarded as mutually dual wave fronts as projections of L.

Proof of Theorem B. Since our contact structure on C can be identified with the contact structure on the projective tangent bundle on $P(\boldsymbol{K}^{n+2})$, we can apply the criteria of A_k-singularities as in Fact 2.5. Thus a nondegenerate singular point p is an A_k-singular point of f if and only if the null vector field η of f (as a wave front) is

$(k-1)$-nondegenerate at p, but does not meet the hypersurface $S(f)$ with multiplicity k at p. Like as in the proof of Theorem A, we may assume that ${}^tF(p)\cdot F(p)\neq 0$ and $G(p)\cdot {}^tG(p)\neq 0$ simultaneously by a suitable affine transformation of $\boldsymbol{K}^{n+2}$, even when $\boldsymbol{K}=\boldsymbol{C}$. Since $(F_{x^1},\dots,F_{x^n},F,{}^tG)$ is a regular $(n+2)\times(n+2)$-matrix if and only if $f=[F]$ is an immersion, the assertion immediately follows from the identity

$$\begin{pmatrix} h_{11} & \dots & h_{1n} & 0 & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ h_{n1} & \dots & h_{nn} & 0 & * \\ 0 & \dots & 0 & 0 & G\cdot{}^tG \\ * & \dots & * & {}^tF\cdot F & 0 \end{pmatrix} = \begin{pmatrix} G_{x^1} \\ \vdots \\ G_{x^n} \\ G \\ {}^tF \end{pmatrix}(F_{x^1},\dots,F_{x^n},F,{}^tG). \tag{3.1}$$ □

Proof of Theorem C. Let $g\colon M^2\to P((\boldsymbol{R}^4)^*)$ be the dual of f. We fix $p\in M^2$ and take a simply connected and connected neighborhood U of p.

Then there are lifts $\hat f,\hat g\colon U\to S^3$ into the unit sphere S^3 such that

$$\hat f\cdot\hat g=0,\quad d\hat f(v)\cdot\hat g=d\hat g(v)\cdot\hat f=0\quad (v\in TU),$$

where $\cdot$ is the canonical inner product on $\boldsymbol{R}^4\supset S^3$. Since $\hat f\cdot\hat f=1$, we have

$$d\hat f(v)\cdot\hat f(p)=0\quad (v\in T_pM^2).$$

Thus

$$d\hat f(T_pM^2)=\{\zeta\in S^3;\ \zeta\cdot\hat f(p)=\zeta\cdot\hat g(p)=0\},$$

which implies that $df(TM^2)$ is equal to the limiting tangent bundle of the front g. So we apply (2.5) in [9] for g: Since the singular set $S(g)$ of g consists only of cuspidal edges and swallowtails, the Euler number of $S(g)$ vanishes. Then it holds that

$$\chi(M_+)+\chi(M_-)=\chi(M^2)=\chi(M_+)-\chi(M_-)+i_2^+(f)-i_2^-(f),$$

which proves the formula. □

When $n=2$, the duality of fronts in the unit 2-sphere S^2 (as the double cover of $P(\boldsymbol{R}^3)$) plays a crucial role for obtaining the classification theorem in [6] for complete flat fronts with embedded ends in $\boldsymbol{R}^3$. Also, a relationship between the number of inflection points and the number of double tangents on certain class of simple closed regular curves in $P(\boldsymbol{R}^3)$ is given in [11]. (For the geometry and a duality of fronts in S^2, see [1].) In [7], Porteous investigated the duality between A_k-singular points and A_k-inflection points when $k=2,3$ on a surface in S^3.

4. Cuspidal curvature on 3/2-cusps

Relating to the duality between singular points and inflection points, we introduce a curvature on 3/2-cusps of planar curves:

Suppose that (M^2, g) is an oriented Riemannian manifold, $\gamma\colon I \to M^2$ is a front, $\nu(t)$ is a unit normal vector field, and I an open interval. Then $t = t_0 \in I$ is a 3/2-cusp if and only if $\dot\gamma(t_0) = \mathbf{0}$ and $\Omega(\ddot\gamma(t_0), \dddot\gamma(t_0)) \neq 0$, where Ω is the unit 2-form on M^2, that is, the Riemannian area element, and the dot means the covariant derivative. When $t = t_0$ is a 3/2-cusp, $\dot\nu(t)$ does not vanish (if $M^2 = \boldsymbol{R}^2$, it follows from Proposition A$'$). Then we take the (arclength) parameter s near $\gamma(t_0)$ so that $|\nu'(s)| = \sqrt{g(\nu'(s), \nu'(s))} = 1$ $(s \in I)$, where $\nu' = d\nu/ds$. Now we define the *cuspidal curvature* μ by

$$\mu := 2\,\mathrm{sgn}(\rho)\sqrt{\left|\frac{ds}{d\rho}\right|}\,\Bigg|_{s=s_0} \qquad (\rho = 1/\kappa_g),$$

where we choose the unit normal $\nu(s)$ so that it is smooth around $s = s_0$ $(s_0 = s(t_0))$. If $\mu > 0$ (resp. $\mu < 0$), the cusp is called *positive* (resp. *negative*). It is an interesting phenomenon that the left-turning cusps have negative cuspidal curvature, although the left-turning regular curves have positive geodesic curvature (see Fig. 4.1). Then it holds that

$$\mu = \frac{\Omega(\ddot\gamma(t), \dddot\gamma(t))}{|\ddot\gamma(t)|^{5/2}}\Bigg|_{t=t_0} = 2\frac{\Omega(\nu(t), \dot\nu(t))}{\sqrt{|\Omega(\ddot\gamma(t), \nu(t))|}}\Bigg|_{t=t_0}. \tag{4.1}$$

We now examine the case that (M^2, g) is the Euclidean plane $\boldsymbol{R}^2$, where $\Omega(v, w)$ $(v, w \in \boldsymbol{R}^2)$ coincides with the determinant $\det(v, w)$ of the 2×2-matrix (v, w). A *cycloid* is a rigid motion of the curve given by $c(t) := a(t - \sin t, 1 - \cos t)$ $(a > 0)$, and here a is called the *radius* of the cycloid. The cuspidal curvature of $c(t)$ at $t \in 2\pi\mathbf{Z}$ is equal to $-1/\sqrt{a}$. In [12], the second author proposed to consider the curvature as the inverse of radius of the cycloid which gives the best approximation of the given 3/2-cusp. As shown in the next proposition, μ^2 attains this property:

Proposition 4.1. *Suppose that $\gamma(t)$ has a 3/2-cusp at $t = t_0$. Then by a suitable choice of the parameter t, there exists a unique cycloid $c(t)$ such that*

$$\gamma(t) - c(t) = o((t - t_0)^3),$$

where $o((t - t_0)^3)$ denotes a higher order term than $(t - t_0)^3$. Moreover, the square of the absolute value of cuspidal curvature of $\gamma(t)$ at $t = t_0$ is equal to the inverse of the radius of the cycloid c.

Proof. Without loss of generality, we may set $t_0 = 0$ and $\gamma(0) = \mathbf{0}$. Since $t = 0$ is a singular point, there exist smooth functions $a(t)$ and $b(t)$ such that $\gamma(t) = t^2(a(t), b(t))$.

Fig. 4.1. A positive cusp and a negative cusp.

Since $t = 0$ is a 3/2-cusp, $(a(0), b(0)) \neq \mathbf{0}$. By a suitable rotation of γ, we may assume that $b(0) \neq 0$ and $a(0) = 0$. Without loss of generality, we may assume that $b(0) > 0$. By setting $s = t\sqrt{b(t)}$, $\gamma(s) = \gamma(t(s))$ has the expansion

$$\gamma(s) = (\alpha s^3, s^2) + o(s^3) \quad (\alpha \neq 0).$$

Since the cuspidal curvature changes sign by reflections on $\boldsymbol{R}^2$, it is sufficient to consider the case $\alpha > 0$. Then, the cycloid

$$c(t) := \frac{2}{9\alpha^2}(t - \sin t,\, 1 - \cos t)$$

is the desired one by setting $s = t/(3\alpha)$. □

It is well-known that the cycloids are the solutions of the brachistochrone problem. We shall propose to call the number $1/|\mu|^2$ the *cuspidal curvature radius* which corresponds the radius of the best approximating cycloid c.

REMARK 4.2. During the second author's stay at Saitama University, Toshizumi Fukui pointed out the followings: Let $\gamma(t)$ be a regular curve in $\boldsymbol{R}^2$ with non-vanishing curvature function $\kappa(t)$. Suppose that t is the arclength parameter of γ. For each $t = t_0$, there exists a unique cycloid c such that a point on c gives the best approximation of $\gamma(t)$ at $t = t_0$ (namely c approximates γ up to the third jet at t_0). The angle $\theta(t_0)$ between the axis (i.e. the normal line of c at the singular points) of the cycloid and the normal line of γ at t_0 is given by

$$\sin\theta = \frac{\kappa^2}{\sqrt{\kappa^4 + \dot\kappa^2}}, \tag{4.2}$$

and the radius a of the cycloid is given by

$$a := \frac{\sqrt{\kappa^4 + \dot\kappa^2}}{|\kappa|^3}. \tag{4.3}$$

One can prove (4.2) and (4.3) by straightforward calculations. The cuspidal curvature radius can be considered as the limit.

ADDED IN PROOF. In a recent authors' preprint, "The intrinsic duality of wave fronts (arXiv:0910.3456)", A_{k+1}-singularities are defined intrinsically. Moreover, the duality between fronts and their Gauss maps is also explained intrinsically.

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