Guo, Y.-J.L.
Osaka J. Math.
47 (2010), 33–40

A NONLOCAL PARABOLIC PROBLEM ARISING IN LINEAR FRICTION WELDING

YUNG-JEN LIN GUO

(Received September 5, 2008)

Abstract

We study a nonlocal parabolic problem modeling the temperature in a thin region during linear friction welding for a hard material. We derive the structures of steady states of this nonlocal problem and its associated approximated problems. Moreover, some remarks on the parabolic problem are given.

1. Introduction

In this paper, we study the following initial boundary value problem:

$$u_t = u_{xx} - \frac{u^{-p}}{\left(\int_0^\infty u^{-p}(x,t)\,dx\right)^{1+1/p}}, \quad 0 < x < \infty,\ t > 0, \tag{1.1}$$

$$u_x(0,t) = 0, \quad u_x(\infty,t) = 1, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \geq 0, \tag{1.3}$$

where the parameter $p > 1$ and u_0 is a positive smooth function defined on $[0,\infty)$. This model arises in the study of linear friction welding for a hard material (cf. [8] and its references). In particular, in the real model the parameter p is close to 4.

To study this problem, in [8] they proposed to study the following approximated problem in bounded intervals:

$$u_t = u_{xx} - \left(\int_0^R u^{-p}(x,t)\,dx\right)^{-1-1/p} u^{-p}, \quad 0 < x < R,\ t > 0, \tag{1.4}$$

$$u_x(0,t) = 0, \quad u(R,t) = R, \quad t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad 0 \leq x \leq R, \tag{1.6}$$

2000 Mathematics Subject Classification. Primary 34K05, 34A34; Secondary 34K60, 34E05.

This work was partially supported by the National Science Council of the Republic of China under the grant NSC 96-2115-M-003-004.

where R is any positive constant. It is convenient to introduce the following transformation:

$$x = Ry, \quad t = R^2 s, \quad u(x, t) = Rv(y, s). \tag{1.7}$$

Then (1.4)–(1.6) is reduced to the following problem:

$$v_s = v_{yy} - \lambda \left(\int_0^1 v^{-p}(y, s)\, dy \right)^{-1-1/p} v^{-p}, \quad 0 < y < 1,\ s > 0, \tag{1.8}$$

$$v_y(0, s) = 0, \quad v(1, s) = 1, \quad s > 0, \tag{1.9}$$

$$v(y, 0) = v_0(y) := \frac{u_0(Ry)}{R}, \quad 0 \le y \le 1, \tag{1.10}$$

where $\lambda = \lambda(R) := R^{1-1/p}$.

It is well-known that the structure of steady states plays an important role in the study of parabolic problem. Therefore, the aim of this paper is to study the structure of steady states of (1.8)–(1.10) with $\lambda > 0$ as a free parameter. Moreover, we also study the structure of steady states of (1.1)–(1.3).

In Section 2, we first derive the structure of steady states of (1.8)–(1.10), by using a method motivated by the works [3, 2]. We show that the problem (1.8)–(1.10) has a unique steady state for each $\lambda > 0$. This is quite surprising in comparing with the local problem [9] and a similar nonlocal problem [5]. For some related works on nonlocal parabolic problems, we also refer the reader to [1, 4, 5, 6, 7, 8].

Then we prove in Section 3 that a unique steady state for the problem (1.1)–(1.3) exists for any $p > 1$. Also, no steady states exist if $p \in (0, 1]$. Indeed, by (1.2), when $p \in (0, 1]$ the equation (1.1) is reduced to the heat equation. Therefore, this case is not physically relevant. Finally, some remarks on the global (in time) existence of solution for (1.4)–(1.6) and some open problems are given in Section 4.

2. Steady states in finite interval

In this section, we shall study the structure of steady states of the approximated problem (1.4)–(1.6) in a finite interval. By the transformation (1.7), the problem (1.4)–(1.6) is reduced to the problem (1.8)–(1.10) with $\lambda = \lambda(R) = R^{1-1/p}$ for $R > 0$. In this section, we always assume that $p > 1$. With λ as a free parameter, the associated steady state of (1.8)–(1.10) is a solution of

$$w'' = \lambda \left(\int_0^1 w^{-p}(y)\, dy \right)^{-1-1/p} w^{-p}, \quad 0 < y < 1, \tag{2.1}$$

$$w'(0) = 0, \quad w(1) = 1. \tag{2.2}$$

It is easy to see that any solution of (2.1)–(2.2) must be a solution of

$$W'' = \sigma W^{-p}, \quad 0 < y < 1, \tag{2.3}$$

$$W'(0) = 0, \quad W(1) = 1, \tag{2.4}$$

for a certain positive constant σ. The problem (2.3)–(2.4) has been analyzed by Levine [9]. It is shown that there exists a positive constant σ^* such that the problem (2.3)–(2.4) has exactly two solutions when $\sigma \in (0, \sigma^*)$; has exactly one solution when $\sigma = \sigma^*$; and has no solutions when $\sigma > \sigma^*$. It is nature to expect that the structure of solutions of (2.1)–(2.2) is similar to that of (2.3)–(2.4). It turns out that this is not the case as shown below.

For the structure of solutions of (2.1)–(2.2), we shall modify a method used in [2, 3]. Let w be a solution of (2.1)–(2.2). Then $w' > 0$ in $(0, 1]$ and the minimum $\mu := w(0) \in (0, 1)$. Multiplying (2.1) by w' and integrating it over $[0, y]$, using $w' > 0$ we deduce that

$$\frac{w'(y)}{\sqrt{\mu^{1-p} - w^{1-p}(y)}} = \sqrt{\frac{2\lambda}{p-1}}\, Y^{-(1+1/p)/2}. \tag{2.5}$$

Then, by integrating (2.5) over $y \in [0, 1]$, we end up with

$$G(\mu) = \sqrt{\frac{2\lambda}{p-1}}\, Y^{-(1+1/p)/2}, \tag{2.6}$$

where

$$Y := \int_0^1 w^{-p}(y)\, dy, \tag{2.7}$$

$$G(\mu) := \int_\mu^1 \frac{ds}{\sqrt{F(\mu) - F(s)}}, \quad F(s) := s^{1-p}. \tag{2.8}$$

In order to eliminate Y in (2.6), we introduce the following transformation:

$$z := \frac{\int_0^y w^{-p}(s)\, ds}{Y}, \quad 0 \le y \le 1, \quad h(z) := w^\gamma(y), \quad \gamma := 1 - p < 0.$$

Note that $y \in [0, 1]$ if and only if $z \in [0, 1]$ and h satisfies the equation

$$h''(z) = -\lambda(p-1)Y^{1-1/p}, \quad z \in [0, 1]. \tag{2.9}$$

Moreover, we have

$$h'(0) = 0, \quad h(1) = 1, \quad h'(z) < 0 \quad \text{for} \quad z \in (0, 1]$$

and $\nu := h(0) = \mu^\gamma \in (1, \infty)$ is the maximum of h. As before, we deduce from (2.9) the following relation

$$H(\nu) := \int_1^\nu \frac{dh}{\sqrt{\nu - h}} = \sqrt{2\lambda(p-1)}Y^{(1-1/p)/2}. \tag{2.10}$$

Note that $H(\nu) = 2(\nu - 1)^{1/2}$ for $\nu > 1$. Hence by (2.10) we obtain

$$Y^{1-1/p} = \frac{2(\nu - 1)}{\lambda(p-1)}. \tag{2.11}$$

Plugging (2.11) into (2.6), we end up with the following relation between the parameters λ and $\mu = w(0)$:

$$\lambda = \left[\frac{2}{p-1}\right]^{1/p} G(\mu)^{(p-1)/p}(\mu^\gamma - 1)^{(p+1)/(2p)} := \left[\frac{2}{p-1}\right]^{1/p} K(\mu)^{(p-1)/p}. \tag{2.12}$$

By the transformation

$$a^2 := 1 - \left(\frac{s}{\mu}\right)^{1-p}, \quad \theta = \theta(\mu) := \sqrt{1 - \mu^{p-1}},$$

we compute from (2.8) that

$$G(\mu) = \frac{2}{p-1}\mu^{(p+1)/2}\int_0^\theta (1 - a^2)^{-p/(p-1)}\, da. \tag{2.13}$$

It is trivial that $G(1^-) = 0$ and so $K(1^-) = 0$. From (2.12) and (2.13), we obtain

$$K(\mu) = \frac{2}{p-1}(1 - \mu^{p-1})^{(p+1)/[2(p-1)]}L(\mu), \tag{2.14}$$

where

$$L(\mu) := \int_0^\theta (1 - a^2)^{-p/(p-1)}\, da, \quad \theta = \sqrt{1 - \mu^{p-1}}.$$

We compute

$$L'(\mu) = -\frac{p-1}{2}\mu^{-2}(1 - \mu^{p-1})^{-1/2} < 0.$$

Therefore, $K'(\mu) < 0$ for all $\mu \in (0, 1)$. For $\mu > 0$ close to zero, we have

$$\begin{aligned} L(\mu) &\geq \int_{1/2}^{\theta} (1-a^2)^{-p/(p-1)}\, da \\ &\geq \int_{1/2}^{\theta} a(1-a^2)^{-p/(p-1)}\, da \\ &= \frac{p-1}{2}\left[\mu^{-1} - \left(\frac{3}{4}\right)^{-1/(p-1)}\right]. \end{aligned}$$

Hence $L(\mu) \to \infty$ as $\mu \to 0^+$.

Since the above procedure in obtaining the relations between w, λ and μ is reversible, we thus have proved the following theorem.

Theorem 1. *Given $p > 1$ fixed. The problem* (2.1)–(2.2) *has a unique solution for each $\lambda > 0$.*

3. Steady states in half line

In this section, we study the positive steady states of (1.1)–(1.3), i.e., positive solutions of the following boundary value problem:

$$U'' = \frac{U^{-p}}{\left(\int_0^{\infty} U^{-p}\, dx\right)^{1+1/p}}, \quad 0 < x < \infty, \tag{3.1}$$

$$U'(0) = 0, \quad U'(\infty) = 1, \tag{3.2}$$

where p is a positive constant. To study this problem, we first study the related boundary value problem:

$$w'' = \lambda w^{-p}, \quad 0 < x < \infty, \tag{3.3}$$

$$w'(0) = 0, \quad w'(\infty) = 1, \tag{3.4}$$

where λ is any positive constant. Note that $w' > 0$ in $(0, \infty)$ and $w(x) \to \infty$ as $x \to \infty$ for any positive solution w of (3.3)–(3.4).

Multiplying the equation (3.3) by w', by an integration we obtain that

$$\frac{[w'(x)]^2}{2} = \frac{\lambda}{1-p}[w^{1-p}(x) - w^{1-p}(0)] \quad \text{for} \quad p \neq 1, \tag{3.5}$$

$$\frac{[w'(x)]^2}{2} = \lambda[\ln w(x) - \ln w(0)] \quad \text{for} \quad p = 1 \tag{3.6}$$

for all $x > 0$ for any positive solution w of (3.3)–(3.4).

If $p \in (0, 1)$, then, by letting $x \to \infty$ in (3.5), we obtain a contradiction. This implies that (3.3)–(3.4) has no positive solutions for any $\lambda > 0$, if $p \in (0, 1)$. Similarly,

using (3.6) we can show that (3.3)–(3.4) has no positive solutions for any $\lambda > 0$, if $p = 1$.

For $p > 1$, set $\mu := w(0)$. Then, by letting $x \to \infty$ in (3.5), we have the identity

$$\lambda = \frac{p-1}{2}\mu^{p-1}. \tag{3.7}$$

This implies that there exists at most one solution of the problem (3.3)–(3.4) for a given $\lambda > 0$. This unique solution is the solution of the initial value problem:

$$w'' = \lambda w^{-p}, \quad x > 0, \quad w'(0) = 0, \quad w(0) = \mu, \tag{3.8}$$

where $\mu = \mu(\lambda)$ is defined by (3.7). We shall denote this solution by w_μ for a given μ. Indeed, it is easily check that $w_\mu(x) = \mu w_1(x/\mu)$. In particular, w_1 is the solution of the initial value problem

$$w'' = \frac{p-1}{2}w^{-p}, \quad x > 0, \quad w'(0) = 0, \quad w(0) = 1. \tag{3.9}$$

Next, we show that the (local) solution w_1 of (3.9) exists globally in $[0, \infty)$. Indeed, if w_1 is not global, then there is $R < \infty$ such that $w_1(R^-) = \infty$. It follows that $w_1''(R^-) = 0$. Note that $w_1'' > 0$ in $[0, R)$. Then

$$w_1'(x) = \int_0^x w_1''(y)\,dy \le R \max_{z\in[0,R)} w_1''(z) < \infty$$

for all $x \in [0, R)$. This implies that $w_1'(x)$ is bounded in $[0, R)$, a contradiction. Therefore, we conclude that w_1 exists globally in $[0, \infty)$. Notice that $w_1'(\infty) = 1$.

We thus have proved the following proposition.

Proposition 3.1. *Let $p > 1$. For any given $\lambda > 0$, there exists a unique positive solution of the boundary value problem* (3.3)–(3.4). *This unique solution is given by w_μ which is the solution of the initial value problem* (3.8) *with μ satisfying the relation* (3.7).

Now, we return to the problem (3.1)–(3.2). Since $U'(\infty) = 1$ for a positive solution U of (3.1)–(3.2), we see that $U(x) \sim x$ as $x \to \infty$. Hence the integral $\int_0^\infty U^{-p}(x)\,dx$ is finite. This implies that $U = w_\mu$ for some positive μ satisfying

$$\frac{p-1}{2}\mu^{p-1}\left(\int_0^\infty w_\mu^{-p}(x)\,dx\right)^{1+1/p} = 1. \tag{3.10}$$

We compute that

$$\begin{aligned}\int_0^\infty w_\mu^{-p}(x)\,dx &= \mu^{-p}\int_0^\infty w_1^{-p}\left(\frac{x}{\mu}\right)dx \\ &= \mu^{1-p}\int_0^\infty w_1^{-p}(y)\,dy.\end{aligned}$$

It follows from (3.10) that μ satisfies

$$\frac{p-1}{2}\mu^{-1+1/p}\left(\int_0^\infty w_1^{-p}(y)\,dy\right)^{1+1/p} = 1. \tag{3.11}$$

The relation (3.11) defines μ uniquely. Hence we have the following existence and uniqueness theorem.

Theorem 2. *For any $p > 1$, there exists a unique positive solution of the boundary value problem* (3.1)–(3.2).

For $p \in (0, 1]$, we have

$$\int_0^\infty U^{-p}(x)\,dx = \infty$$

for any solution U of (3.1)–(3.2). Hence a classical solution U of (3.1)–(3.2) exists only if $U'' \equiv 0$ in $[0, \infty)$. This is impossible, since $U'(0) = 0$ and $U'(\infty) = 1$. Therefore, we conclude that there is no solution of (3.1)–(3.2), if $p \in (0, 1]$.

4. Remarks and discussions

We recall from [9] that the solution of

$$V_s = V_{yy} - \sigma V^{-p}, \quad 0 < y < 1,\ s > 0, \tag{4.1}$$

$$V_y(0, s) = 0, \quad V(1, s) = 1, \quad s > 0, \tag{4.2}$$

$$V(y, 0) = v_0(y), \quad 0 \le y \le 1, \tag{4.3}$$

with $W_\sigma^-(y) \le v_0(y) \le 1$ for $y \in [0, 1]$ and $\sigma \in (0, \sigma^*)$, exists globally in time, where W_σ^- is the minimal solution of (2.3)–(2.4).

Now we suppose that v is a solution of (1.8)–(1.10) with $v_0 \in (0, 1]$. Then we have

$$g(s; v) := \left(\int_0^1 v^{-p}(y, s)\,dy\right)^{-1-1/p} \in (0, 1)$$

as long as v exists. Let V be the solution of (4.1)–(4.3) with $\sigma = \lambda$. It follows from the comparison principle that $v \ge V$ as long as v exists. Therefore, we conclude that

Theorem 3. *Suppose that $\lambda \in (0, \sigma^*)$. If $W_\lambda^-(y) < v_0(y) \le 1$ for $y \in [0, 1]$, then the solution v of* (1.8)–(1.10) *exists globally in time.*

Hence, when R is small enough and $W_\lambda^-(x/R) < u_0(x)/R \le 1$ for $x \in [0, R]$, the solution of (1.4)–(1.6) exists globally in time.

We also recall from [9] that the solution V of (4.1)–(4.3) tends to the maximal solution W_σ^+ of (2.3)–(2.4) as $t \to \infty$, if $v_0 > W_\sigma^-$. It should be very interesting to see if such result also holds for the nonlocal problem (1.4)–(1.6). In [8], numerical simulations indicate that, for $p > 1$, the solution u of (1.1)–(1.3) exists globally and tends to the unique steady state as $t \to \infty$. It is also very interesting to see, under certain condition, whether the solution of (1.1)–(1.3) exists globally and converges to the unique steady state as $t \to \infty$. We leave these two questions for the future study.

References

[1] G.-J. Chen and J.-S. Guo: *Critical length for a quenching problem with nonlocal singularity*, Methods Appl. Anal. **5** (1998), 185–194.

[2] K. Deng: *Dynamical behavior of solutions of a semilinear heat equation with nonlocal singularity*, SIAM J. Math. Anal. **26** (1995), 98–111.

[3] K. Deng, M.K. Kwong and H.A. Levine: *The influence of nonlocal nonlinearities on the long time behavior of solutions of Burgers' equation*, Quart. Appl. Math. **50** (1992), 173–200.

[4] J.-S. Guo: *Quenching behavior for the solution of a nonlocal semilinear heat equation*, Differential Integral Equations **13** (2000), 1139–1148.

[5] J.-S. Guo, B. Hu and C.-J. Wang: *A nonlocal quenching problem arising in micro-electro mechanical system*, preprint.

[6] J.-S. Guo and T.-M. Hwang: *On the steady states of a nonlocal semilinear heat equation*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **8** (2001), 53–68.

[7] J.-S. Guo and H.W. Su: *The blow-up behavior of the solution of an integrodifferential equation*, Differential Integral Equations **5** (1992), 1237–1245.

[8] N.I. Kavallaris, A.A. Lacey, C.V. Nikolopoulos, and C. Voong: *Behaviour of a non-local equation modelling linear friction welding*, preprint.

[9] H.A. Levine: *Quenching, nonquenching, and beyond quenching for solution of some parabolic equations*, Ann. Mat. Pura Appl. (4) **155** (1989), 243–260.

Department of Mathematics
National Taiwan Normal University
88, S-4 Ting Chou Road
Taipei 11677
Taiwan
e-mail: yjguo@math.ntnu.edu.tw