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GENERALIZED CHEBYSHEV MAPS OF $\mathbf{C}^2$ AND THEIR PERTURBATIONS

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Abstract

The Chebyshev map is a typical chaotic map. We consider generalized Chebyshev maps T_k of $\mathbf{C}^2$. The support of the maximal entropy measure of T_k is connected. We perturb T_k in a certain direction. Then we can show the support of the maximal entropy measure of this map is a Cantor set.

1. Introduction

The Chebyshev map is a typical chaotic map. Compared with the dynamics of the Chebyshev map in one variable very few things are known about the dynamics of generalized Chebyshev maps in higher dimensions. Generalized Chebyshev maps were studied by several researchers, Koornwinder [9], Lidl [10], Veselov [15], Hoffman and Withers [8]. Their constructions are based on the theory of complex Lie algebras.

Veselov [15] defined generalized Chebyshev maps as follows.

Let G be a simple complex Lie algebra of rank n, H be its Cartan subalgebra, H^* be its dual space, $\mathcal{L}$ be a lattice of weights in H^* generated by the fundamental weights $\omega_1, \dots, \omega_n$ and L be the dual lattice in H (see [3]). One defines the mapping $\phi_G\colon H/L \to \mathbf{C}^n$, $\phi_G = (\varphi_1, \dots, \varphi_n)$, $\varphi_k = \sum_{w\in W} \exp[2\pi i w(\omega_k)]$, where W is the Weyl group, acting on the space H^*. Chevalley asserts that $\varphi_1, \dots, \varphi_n$ generate the algebra of exponential invariants freely.

With each simple complex Lie algebra G of rank n is associated an infinite series of integrable polynomial mappings P_G^k from $\mathbf{C}^n$ to $\mathbf{C}^n$, $k = 2, 3, \dots$, determined by the condition:

$$\phi_G(kx) = P_G^k(\phi_G(x)).$$

For $n = 1$ there is a unique simple algebra A_1. Here $\phi_{A_1} = 2\cos(2\pi x)$ and the $P_{A_1}^k$ are, within a linear substitution, Chebyshev polynomials. Here A_n is the Lie algebra of $SL(n + 1, \mathbf{C})$.

In this paper we will study the mappings P_G^k in the case $G = A_2$. When $G = B_2 \simeq C_2$, from [11] we know that the extended polynomial maps P_G^k from $\mathbf{P}^2$ to $\mathbf{P}^2$

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are represented as symmetric products of two maps of $\mathbf{P}^1$. So we study P_G^k in the case $G = A_2$. We denote $P_{A_2}^k$ by T_k.

The generalized Chebyshev maps T_k from $\mathbf{C}^2$ to $\mathbf{C}^2$ ($k \in \mathbf{Z}$), are given by $T_k(x, y) = (g^{(k)}(x, y), g^{(k)}(y, x))$. Here $g^{(k)}(x, y)$ is a generalized Chebyshev polynomial defined by Lidl [10]. Let $x = t_1 + t_2 + t_3$, $y = t_1t_2 + t_1t_3 + t_2t_3$, $1 = t_1t_2t_3$. Then we set $g^{(k)}(x, y) := t_1^k + t_2^k + t_3^k$. So $g^{(k)}(y, x) = (1/t_1)^k + (1/t_2)^k + (1/t_3)^k = g^{(-k)}(x, y)$. For instance, $T_2(x, y) = (x^2 - 2y, y^2 - 2x)$, $T_3(x, y) = (x^3 - 3xy + 3, y^3 - 3xy + 3)$, $T_4(x, y) = (x^4 - 4x^2y + 2y^2 + 4x, y^4 - 4xy^2 + 2x^2 + 4y)$. Recurrence relations for these polynomials are given by (see [10])

$$g^{(k)}(x, y) = xg^{(k-1)}(x, y) - yg^{(k-2)}(x, y) + g^{(k-3)}(x, y). \tag{1.1}$$

In this paper we consider the dynamics of generalized Chebyshev maps T_k of $\mathbf{C}^2$. We show they have similar properties to those of Chebyshev maps of $\mathbf{C}$. The support of the invariant probability measure μ of maximal entropy of T_k is connected. We will study external rays for the support of μ and foliations of the Julia set J_1. We also show the external rays have relations with the affine Weyl group of A_2.

Next we consider perturbations of the generalized Chebyshev maps. We perturb the generalized Chebyshev maps of $\mathbf{C}^2$ in a certain direction. Then we will show that the support of μ of the perturbed map is a Cantor set. In one variable case, this is parallel to the following fact. For typical quadratic maps $f_c(z) = z^2 + c$, when $c = -2$, $f_{-2}(z)$ is a Chebyshev map. Our result corresponds to the well known fact that when $c < -2$, the Julia set of f_c is a Cantor set.

The generalized Chebyshev maps T_k have relations with the classical Lie algebra A_2 and so the maps are very symmetric by nature. In this paper, we will show that these symmetric objects collapse under certain perturbations.

In Section 2, we will study the properties of generalized Chebyshev map of $\mathbf{C}^2$ and their dynamics. In Lemma 2.1, we show an exact formula for the critical set of $T_k(x, y)$. The set $K(T_k(x, y))$ of points with bounded orbits is a closed domain on the real plane $\{x = \bar{y}\}$.

In Proposition 2.3, we show an exact form of the invariant measure μ of maximal entropy for $T_k(x, y)$ and its support is equal to $K(T_k(x, y))$. We apply the definitions of external rays of polynomial endomorphism given by Bedford and Jonsson [1] to $T_k(x, y)$ and we give exact forms of external rays of $T_k(x, y)$ and show they have similar properties to those of one-dimensional Chebyshev maps $T_k(x)$ (see Proposition 2.4, Fig. 2.4 and Fig. 2.5). We also show the affine Weyl group of A_2 acts on a set of rays.

In Section 3, we will consider perturbations of $T_k(x, y)$ in a certain direction and find Cantor sets. We define c-Chebyshev maps $f_c(x, y)$ as follows: Let $T_k(x, y) = (u(x, y), u(y, x))$, where $u(x, y)$ is a polynomial of degree k in variables x and y. We introduce a new parameter c and make a homogeneous polynomial $u(x, y, c)$ of degree k with $u(x, y, 1) = u(x, y)$. Then we define c-Chebyshev maps by $f_c(x, y) :=$

$(u(x, y, c), u(y, x, c))$ (see Definition 3.1). The main result of this paper is that if $c > 1$, then the support of the maximal entropy measure μ of a c-Chebyshev map $f_c(x, y)$ is a Cantor set which lies on the plane $\{x = \overline{y}\}$ (see Theorem 3.1 and Proposition 3.6). The key observation of the proof of this result is the following: if $c > 1$, then the orbit $\{f_c^n(C)\}$ of the critical set C of $f_c(x, y)$ approaches the line at infinity uniformly (see Proposition 3.1). To prove this observation we use a topological argument principle, dynamics on the invariant plane $\{x = \overline{y}\}$ and the external rays of $T_k(x, y)$. When $c = 1$, the support of the maximal entropy measure μ of $f_c(x, y)$ is connected. But when $c > 1$, the support of μ is not connected. Then a bifurcation occurs at $c = 1$. The symmetric objects of generalized Chebyshev maps T_k collapse under the perturbations.

2. Generalized Chebyshev maps of $\mathbf{C}^2$

In this section we study some properties of the generalized Chebyshev maps $T_k(x, y)$ and their dynamics. From the definition of the generalized Chebyshev maps we can find a branched covering map. The following diagram is commutative,

$$\begin{array}{ccc} (\mathbf{C} - \{0\})^2 & \xrightarrow{g_k} & (\mathbf{C} - \{0\})^2 \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbf{C}^2 & \xrightarrow{T_k} & \mathbf{C}^2 \end{array}$$

where $g_k(t_1, t_2) = (t_1^k, t_2^k)$, and

$$(x, y) = \Psi(t_1, t_2) = \left(t_1 + t_2 + \frac{1}{t_1 t_2}, \frac{1}{t_1} + \frac{1}{t_2} + t_1 t_2\right). \tag{2.1}$$

The covering map

$$\Psi : \mathbf{C}^2 \setminus \Psi^{-1}(D) \to \mathbf{C}^2 \setminus D$$

is a 6-sheeted covering map. The branch locus D of Ψ is written as

$$x^2 y^2 - 4x^3 - 4y^3 + 18xy - 27 = 0.$$

$T_k(x, y)$ admits an invariant plane $\{x = \overline{y}\}$. $T_k(x, y)$ restricted to the real plane $\{x = \overline{y}\}$ may be regarded as a Chebyshev polynomial defined by Koornwinder [9]

$$P_{k,0}^{-1/2}(z, \overline{z}) = e^{ik\sigma} + e^{-ik\tau} + e^{i(k\tau - k\sigma)}, \quad \sigma, \tau \in \mathbf{R}.$$

Set $z(\sigma, \tau) := e^{i\sigma} + e^{-i\tau} + e^{i(\tau-\sigma)} = u + iv$. Based on [9], we review some known facts. Let R be a closed domain bounded by the triangle with vertices $O = (0, 0)$, $A = (\pi/\sqrt{2}, -\pi/\sqrt{6})$ and $B = (\pi/\sqrt{2}, \pi/\sqrt{6})$ in the (s, t) plane. Here we use the coordinate (s, t) which is related to the coordinate (σ, τ) by a coordinate transformation:

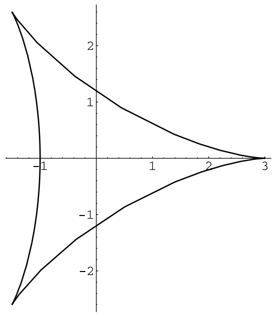

Fig. 2.1. The domain S.

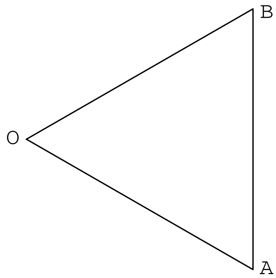

Fig. 2.2. The domain R.

$s = (\sigma + \tau)/(2\sqrt{2})$, $t = \sqrt{3}(\sigma - \tau)/(2\sqrt{2})$. Let J_1, J_2 and J_3 denote the reflections in the edges OA, OB and AB in the (s, t) plane, respectively. Then R is a fundamental domain for the group generated by J_1, J_2 and J_3.

In other words, R is similar to the closure of an alcove of the Lie algebra A_2. The set of simple root vectors is written as $\{\alpha_1 = (\sqrt{2}, 0),\ \alpha_2 = (-1/\sqrt{2}, \sqrt{3}/\sqrt{2})\}$ and the highest root vector is $\tilde{\alpha} = \alpha_1 + \alpha_2$. (See [3].)

We denote the images of R under the inverse of the coordinate transformation by R_1. Let S be a closed domain bounded by Steiner's hypocycloid

$$(u^2 + v^2 + 9)^2 + 8(-u^3 + 3uv^2) - 108 = 0$$

in the (u, v) plane. Then the mapping $z\colon (\sigma, \tau) \to (u, v)$ is a diffeomorphism from the interior R_1° of R_1 to S° and the boundary ∂R_1 is mapped one to one and onto the boundary ∂S. Then $S = \{e^{i\sigma} + e^{-i\tau} + e^{i(\tau-\sigma)}\colon 0 \le \sigma,\ \tau \le 2\pi\}$.

Combining the inverse of the map $z(\sigma, \tau)$ with the coordinate transformation, we get a continuous map φ from S to R such that φ is a diffeomorphism from S° onto R° and ∂S is mapped onto ∂R. See Figs. 2.1 and 2.2.

In [13], we show that for any segment l in R parallel to one of the three root vectors α_1, α_2 and $\tilde{\alpha}$, $\varphi^{-1}(l)$ is also a segment in S. Then such a segment $\varphi^{-1}(l)$ may be viewed as a “geodesic”. (See Fig. 2.4.)

In the first place we consider the critical set of $T_k(x, y)$ defined by

$$C(T_k) := \{(x, y) \in \mathbf{C}^2 : \det(DT_k) = 0\}.$$

Lemma 2.1. *Let $k \in \mathbf{Z}$. Assume that*

$$x = t_1 + t_2 + t_3, \quad y = t_1t_2 + t_1t_3 + t_2t_3, \quad t_1t_2t_3 = 1.$$

Then

$$\det(DT_k) = k^2 \frac{t_1^k - t_2^k}{t_1 - t_2} \cdot \frac{t_1^k - t_3^k}{t_1 - t_3} \cdot \frac{t_2^k - t_3^k}{t_2 - t_3}.$$

Proof. We note that

$$\det(DT_k) = \frac{\det(D(T_k \circ \Psi))}{\det(D\Psi)}.$$

By direct computations we have

$$\det(DT_k \circ \Psi) = \frac{k^2}{t_1t_2}(t_1^k - t_2^k)(t_1^k - t_3^k)(t_2^k - t_3^k),$$

and

$$\det(D\Psi) = \frac{1}{t_1t_2}(t_1 - t_2)(t_1 - t_3)(t_2 - t_3). \qquad \square$$

Dinh [4] shows that generalized Chebyshev maps are critically finite. From Lemma 2.1 we see that $C(T_k)$ can be parameterized as

$$x = t + \varepsilon t + \frac{1}{\varepsilon t^2}, \quad y = \frac{1}{t} + \frac{1}{\varepsilon t} + \varepsilon t^2 \quad (\varepsilon = e^{2j\pi\sqrt{-1}/k},\ j \in \mathbf{N}). \tag{2.2}$$

We will prove in Proposition 2.3 below that S is equal to the support of the maximal entropy measure for T_k.

Let f be a map from a complex manifold X to X. We define $K(f) := \{x \in X : \text{the orbit } \{f^n(x)\} \text{ is bounded}\}$. Then $K(T_k)$ is described in the following form.

Proposition 2.1 ([15]).

$$K(T_k) = \Psi(\{|t_1| = |t_2| = 1\}) = S \subset \{x = \overline{y}\}.$$

Proof. From the following commutative diagram we can prove this proposition.

$$\begin{array}{ccc} (t_1, t_2) & \xrightarrow{g_k} & (t_1^k, t_2^k) \\ \downarrow \Psi & & \downarrow \Psi \\ (x, y) & \xrightarrow{T_k} & (g^{(k)}, g^{(-k)}). \end{array}$$

□

Next we study the properties of periodic points of T_k.

Lemma 2.2. *Assume that $k \geq 2$. All the periodic points of T_k lie in S on the plane $\{x = \overline{y}\}$ and any periodic point in the interior S° is repelling.*

Proof. Clearly, any periodic point of T_k lies in the set $K(T_k)$. By the semi-conjugacy (2.1), we see that any periodic point in S° is repelling. □

We consider the function T_k restricted to $\{x = \overline{y}\}$, which is denoted by S_k.

$$S_k := T_k|_{\{x=\overline{y}\}} : \mathbf{R}^2 \to \mathbf{R}^2,$$

e.g.

$$S_2(z) = z^2 - 2\overline{z} : (u, v) \mapsto (u^2 - 2u - v^2, 2uv + 2v).$$

We use the bijection φ from S in the (u, v) space to R in the (s, t) space (see Figs. 2.1 and 2.2). We divide the closed triangular region R into k^{2n} congruent closed triangular regions $\triangle$.

Proposition 2.2. *Each region $\triangle$ has a periodic point of $\varphi \circ S_k \circ \varphi^{-1}$ of period n.*

Proof. We set $\kappa := \varphi \circ S_k \circ \varphi^{-1}$. Then $\kappa(s, t) = (ks, kt)$. We prove this lemma when $k = 2$. The proof in the general case is similar.

R is the closed domain bounded by an equilateral triangle $\triangle OAB$. See Fig. 2.3. We divide the triangle $\triangle OAB$ into four congruent equilateral triangles. Let the closed domain bounded by $\triangle DEF$, $\triangle OEF$, $\triangle ADF$, $\triangle BED$ denote $\triangle(0)$, $\triangle(1)$, $\triangle(2)$, $\triangle(3)$, respectively. Then the image of $\triangle(2)$ under the map κ is the closed domain bounded by $\triangle AA^*D^*$. This closed domain is equivalent to the fundamental domain R by reflections. Then we can define a homeomorphism k_2 from $\triangle(2)$ onto R. Let h_2 be the inverse of k_2. Hence h_2 is a continuous map from R to R. Then, by the fixed point theorem of a closed disk, we have a fixed point $p_2^{(1)}$ of h_2. Hence $p_2^{(1)}$ is a fixed point of k_2 in $\triangle(2)$. By the same arguments, we have a fixed point of κ on each $\triangle(j)$, ($j = 0, 1, 2, 3$). Further, we divide each $\triangle(j)$ into four smaller congruent equilateral triangular domains $\triangle(jl)$. In the same way, we can prove that there is a periodic point

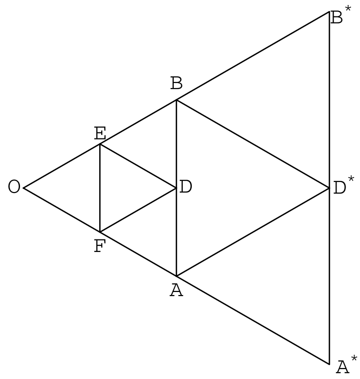

Fig. 2.3. Divisions of a regular triangle and their extensions.

of period 2 of κ on each $\triangle(jl)$. Repeating this procedure, we have a periodic point of period n on each $\triangle(j_1, j_2, \dots, j_n)$. We can show these are distinct k^{2n} periodic point of period n. Indeed. Note that $\kappa(\partial R) \subset \partial R$ and $\kappa^n(\partial \triangle (j_1, j_2, \dots, j_n)) = \partial R$. The point O is a fixed point and the points A and B are periodic points of period 2. □

Corollary 2.1. *The map $\kappa|_{\partial R}$ from ∂R onto ∂R is a k-sheeted covering map.*

Proof. Set $a = OB$, $b = AB$ and $c = OA$. Then $\kappa(\partial R)$ is represented as $(abc)^k$ in a counterclockwise orientation. □

Now we consider the invariant measure μ of maximal entropy for T_k.

Proposition 2.3. (1) supp $\mu = S$.
(2)

$$\mu = \frac{3}{\pi^2} \frac{dx_1 dx_2}{\sqrt{-x^2\bar{x}^2 + 4x^3 + 4\bar{x}^3 - 18x\bar{x} + 27}}, \quad x = x_1 + ix_2.$$

Proof. A theorem of [2] reads as follows. Let μ_n be the measure $\mu_n := (1/k^{2n}) \times \sum_{f^n(y)=y,\, y \text{ repelling}} \delta_y$. Then the sequence $\{\mu_n\}$ converges weakly to the invariant measure μ.

By Lemma 2.2, we see that all the periodic points in S° are repelling. We use the notations in the proof of Proposition 2.2. A small triangle $\triangle(j_1, j_2, \dots, j_m)$ in R° has exactly one repelling periodic point of κ of period m. Set $\tilde{\mu}_n := \varphi_* \mu_n$. Then, $\tilde{\mu}_m(\triangle(j_1, j_2, \dots, j_m)) = 1/k^{2m}$. Dividing $\triangle(j_1, j_2, \dots, j_m)$ into smaller equilateral triangles, we see that if $n \geq m$, $\tilde{\mu}_n(\triangle(j_1, j_2, \dots, j_m)) = 1/k^{2m}$. Thus, we deduce that the sequence $\{\tilde{\mu}_n\}$ converges weakly to $\sqrt{3}/\pi^2\, \tilde{\mu}$, where $\tilde{\mu}$ is the Lebesgue measure in the

(s, t) plane. Then $\mu = \varphi^*(2\sqrt{3}/\pi^2\, \tilde{\mu})$. Theorem 3.5 in [9] states that

$$ds\, dt = \frac{\sqrt{3}\, dx_1\, dx_2}{2\sqrt{-x^2\overline{x}^2 + 4x^3 + 4\overline{x}^3 - 18x\overline{x} + 27}}.$$

Hence this proposition follows. □

Next we study external rays of $T_k(x, y)$. We use the definitions of external rays given by Bedford and Jonsson [1] in our situation. We extend the map $T_k(x, y)\colon \mathbf{C}^2 \to \mathbf{C}^2$ to a holomorphic map from $\mathbf{P}^2$ to $\mathbf{P}^2$. Let $W^s(J_\Pi, T_k)$ be the stable set of the Julia set J_Π on the line at infinity. Bedford and Jonsson [1] show that there exists a homeomorphism Ψ (an inverse Böttcher coordinate) such that

$$\Psi\colon W^s(J_\Pi,\ f_k) \to W^s(J_\Pi, T_k)$$

conjugating f_k to T_k, where $f_k(t_1, t_2) = (t_1^k, t_2^k)$. They define a local stable manifold $W^s_{\mathrm{loc}}(a)$, $(a \in J_\Pi)$ and then a stable disk $W_a \supset W^s_{\mathrm{loc}}(a)$ and external rays $R(a, \theta)$.

Nakane [11] observed the following results on $T_2(x, y)$:

Nakane defined

$$\tilde{\Psi}(t_1, t_2) = \left(t_1 + \frac{1}{t_2} + \frac{t_2}{t_1},\ \frac{1}{t_1} + t_2 + \frac{t_1}{t_2}\right). \tag{2.3}$$

Two maps Ψ in (2.1) and $\tilde{\Psi}$ in (2.3) are essentially the same and $\tilde{\Psi}$ is an inverse Böttcher coordinate conjugating f_k to T_k. The stable disk W_a is the image of $\{(t, at)\colon |t| > 1\}$ under the map $\tilde{\Psi}$. Then the stable disk W_a can be written as the set of points $R(r, \sigma, \tau)$ in the form

$$\begin{aligned} &x = re^{-2\pi i\tau} + \frac{1}{r}e^{2\pi i(\tau-\sigma)} + e^{2\pi i\sigma}, \quad y = re^{2\pi i(\sigma-\tau)} + \frac{1}{r}e^{2\pi i\tau} + e^{-2\pi i\sigma}, \\ &a = e^{2\pi i\sigma}, \quad r > 1. \end{aligned} \tag{2.4}$$

Any external ray is written as $R(\sigma, \tau) := \{R(r, \sigma, \tau)\colon r > 1\}$. Each point z in S is a landing point of exactly 1, 3, or 6 external rays if z is a cusp point on ∂S, z is a non-cusp point on ∂S or $z \in S^\circ$, respectively.

We can show that Nakane's results are also true for any $T_k(x, y)$, $k \neq 0$. Further, we give a structure of foliations W_a of $W^s(J_\Pi, T_k)$ and show their relations to a Lie algebra.

Proposition 2.4. *For any point $z \in S^\circ$, there exist three stable disks W_a such that boundaries of these three disks lie on S and intersect at z. At that point, two external rays on each W_a land from opposite directions.*

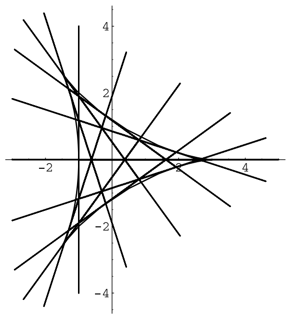

Fig. 2.4. Geodesics and external rays.

Metaphorically speaking, it is like three mouths (stable disks) biting a sandwich (supp $\mu = S$). When $r = 1$, the point $R(1, \sigma, \tau)$ lies on $S \subset \{x = \overline{y}\}$. The boundary of W_a is written as $\{R(1, \sigma, \tau) \colon 0 \le \tau \le 2\pi\}$ and covers a segment on S twice. The segment is a geodesic and is inscribed in the hypocycloid ∂S. See Fig. 2.4. We can extend the segment across the hypocycloid in both directions. The two half-lines are external rays of T_k in $\{x = \overline{y}\}$. We consider the affine Weyl group W of the Lie algebra A_2. The affine Weyl group W of A_2 is expressed by the following six transformations (see [9], p. 360):

$$J_1(\sigma, \tau) = (-\sigma + \tau, \tau), \quad J_2(\sigma, \tau) = (\sigma, \sigma - \tau), \quad J_3(\sigma, \tau) = (-\tau, -\sigma),$$
$$J_4(\sigma, \tau) = (-\tau, \sigma - \tau), \quad J_5(\sigma, \tau) = (\tau - \sigma, -\sigma), \quad J_0(\sigma, \tau) = (\sigma, \tau).$$

When $r = 1$ in (2.4), then a point $(x, y) = R(1, \sigma, \tau)$ lies in $S \subset \{x = \overline{y}\}$ and the point (x, y) is fixed under any transformation J_j.

When $r > 1$, any element J_j of affine Weyl group W at z in S° acts on a set of external rays. Two external rays $R(\sigma, \tau)$ and $R(\sigma, \sigma - \tau)$ corresponding to J_0 and J_2 lie on a stable disk W_a ($a = e^{2\pi i\sigma}$) and land at the same point z. Any two points $R(r, \sigma, \tau)$ and $R(r, \sigma, \sigma - \tau)$ are symmetrical about $\{x = \overline{y}\}$ in the following sense.
(1) The midpoint of the segment $R(r, \sigma, \tau)R(r, \sigma, \sigma - \tau)$ lies on the plane $\{x = \overline{y}\}$.
(2) The segment is perpendicular to the plane $\{x = \overline{y}\}$.
The same properties hold for external rays $R(-\tau, -\sigma)$ and $R(-\tau, \sigma - \tau)$ corresponding to J_3 and J_4 on a stable disk W_a ($a = e^{-2\pi i\tau}$) and also for external rays $R(\tau - \sigma, \tau)$ and $R(\tau - \sigma, -\sigma)$ corresponding to J_1 and J_5 on a stable disk W_a ($a = e^{2\pi i(\tau - \sigma)}$). These six external rays land on the same point z. □

We compare the external rays of $T_k(x, y)$ with those of a Chebyshev map $P^k_{A_1}(z) = T_k(z)$ on $\mathbf{C}$. Any external ray of $T_k(z)$ is written as

$$R(r, \phi) = re^{2\pi i\phi} + \frac{1}{r}e^{2\pi i(-\phi)}, \quad r > 1.$$

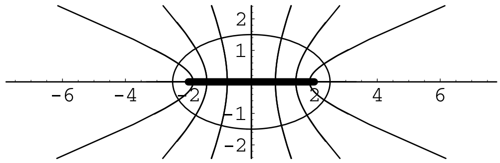

Fig. 2.5. External rays of $T_k(z)$.

Clearly, $R(r, -\phi) = re^{2\pi i(-\phi)} + (1/r)e^{2\pi i\phi}$. Then $R(r, \phi) = \overline{R(r, -\phi)}$. Hence $R(r, \phi)$ and $R(r, -\phi)$ are symmetrical about the real axis. See Fig. 2.5.

Note that the affine Weyl group of A_1 acts on a set of external rays of $T_k(z)$. On the other hand, the affine Weyl group W of A_2 acts on a set of external rays of $T_k(x, y)$.

3. Perturbations of generalized Chebyshev maps of $\mathbf{C}^2$ and Cantor sets

In this section we perturb generalized Chebyshev maps $T_k(x, y)$ in a certain direction. Recall that $T_k(x, y) = (g^{(k)}(x, y), g^{(k)}(y, x))$, where $g^{(k)}(x, y)$ is a polynomial of degree $|k|$. We introduce a new parameter c in $T_k(x, y)$ as follows. We make a homogeneous polynomial $g^{(k)}(x, y, c)$ of degree $|k|$ by adding a new variable c such that $g^{(k)}(x, y, 1) = g^{(k)}(x, y)$. Then we define maps $f_c^{(k)}(x, y)$ from $\mathbf{C}^2$ to $\mathbf{C}^2$ by

$$f_c^{(k)}(x, y) = (g^{(k)}(x, y, c), g^{(k)}(y, x, c)).$$

DEFINITION 3.1. $f_c^{(k)}(x, y)$ is called a *c-Chebyshev map* of degree $|k|$.

When $c = 1$, $f_1^{(k)}(x, y) = T_k(x, y)$. From (1.1) we see that if $k \geq 1$, $g^{(k)}(x, y) = x^k + \pi_{k-1}(x, y)$, where $\pi_{k-1}(x, y)$ denotes a polynomial in x and y of degree $\leq k-1$. Then the map $f_c^{(k)}$ extends holomorphically to $\mathbf{P}^2$. We state the main result in this section.

Theorem 3.1. *Assume that $c > 1$. Then the support of the maximal entropy measure μ of the c-Chebyshev map $f_c^{(k)}(x, y)$ is a Cantor set for any $k \in \mathbf{Z} \setminus \{0, 1, -1\}$.*

The map $f_c^{(2)}(x, y)$ restricted to the line $\{x = y\}$ is the map $q_c^{(2)}(x) = x^2 - 2cx$ which is conjugate to the map $p_\lambda(x) = x^2 + \lambda$. The interval $1 \leq c < \infty$ corresponds to the interval $-\infty < \lambda \leq -2$ which is a half-line beginning at the top of antenna in the Mandelbrot set. By Proposition 2.3, we know that when $c = 1$, the support of the maximal entropy measure of $f_1^{(k)}(x, y)$ $(= T_k(x, y))$ is the connected set S on the plane $\{x = \overline{y}\}$. However if $c > 1$, the support of the maximal entropy measure of $f_c^{(k)}(x, y)$ is not connected. This shows that a bifurcation occurs at $c = 1$. This theorem is parallel to a well-known result that if $\lambda < -2$, then the Julia set of p_λ is a Cantor set.

When $k = 2$, the theorem is proved in [14]. This is a generalization of the result in [14].

We fix the value k and use an abbreviation $f_c(x, y)$ for $f_c^{(k)}(x, y)$. One of the reasons why we define a c-Chebyshev map in such a form is shown in the next lemma.

Lemma 3.1. *Let $x = c(t_1 + t_2 + t_3)$, $y = c(1/t_1 + 1/t_2 + 1/t_3)$ and $t_1 t_2 t_3 = 1$. Then*

$$f_c(x, y) = \left(c^{|k|}(t_1^k + t_2^k + t_3^k),\ c^{|k|}\left(\frac{1}{t_1^k} + \frac{1}{t_2^k} + \frac{1}{t_3^k}\right)\right).$$

Proof. Set $x' = t_1 + t_2 + t_3$ and $y' = 1/t_1 + 1/t_2 + 1/t_3$. By definition, $f_c(x, y) = (g^{(k)}(x, y, c), g^{(k)}(y, x, c))$.

Clearly, $g^{(k)}(cx', cy', c) = c^{|k|} g^{(k)}(x', y')$. Then

$$\begin{aligned} f_c(x, y) &= c^{|k|}(g^{(k)}(x', y'), g^{(k)}(y', x')) = c^{|k|} T_k(x', y') \\ &= c^{|k|}\left(t_1^k + t_2^k + t_3^k,\ \frac{1}{t_1^k} + \frac{1}{t_2^k} + \frac{1}{t_3^k}\right). \end{aligned}$$ □

Lemma 3.2. *The critical set $C(f_c)$ and the critical value set $f_c(C)$ are written as follows*:

$$\begin{aligned} C(f_c)&\colon x = c\left((1+\varepsilon)t + \frac{1}{\varepsilon t^2}\right), \quad y = c\left(\frac{1}{t} + \frac{1}{\varepsilon t} + \varepsilon t^2\right), \\ f_c(C)&\colon x = c^{|k|}\left(2t^k + \frac{1}{t^{2k}}\right), \quad y = c^{|k|}\left(\frac{2}{t^k} + t^{2k}\right), \end{aligned}$$

where $\varepsilon = e^{2j\pi\sqrt{-1}/k}$ and $t \in \mathbf{C} \setminus \{0\}$.

Proof. It can be easily observed that

$$\det(Df_c(x, y)) = c^{2(|k|-1)} \det\left(DT_k\left(\frac{x}{c}, \frac{y}{c}\right)\right).$$

Then by (2.2) we have the parameterization of $C(f_c)$. The parameterization of $f_c(C)$ is obtained from Lemma 3.1. □

The key observation in the proof of Theorem 3.1 is the following property.

Proposition 3.1. *If $c > 1$, $K(f_c) \cap C(f_c) = \emptyset$.*

This is equivalent to the statement if $c > 1$, then for any $(x, y) \in C(f_c)$, $\| f_c^n(f_c(x, y))\| \to \infty$ as $n \to \infty$ with respect to the Euclidean norm. By Lemma 3.2, we know that the critical value set $f_c(C)$ is parameterized by t in $\mathbf{C} \setminus \{0\}$. We will shrink the domain of definition $\mathbf{C} \setminus \{0\}$.

In the first place we assume that $k \geq 2$. Set

$$(u_n(t), v_n(t)) := f_c^n(u_0(t), u_0(1/t)), \quad \text{where} \quad u_0(t) = c^k(2t^k + 1/t^{2k}).$$

Since $u_0(t)$ is the first component of the critical value set $f_c(C)$ (see Lemma 3.2), $(u_n(t), v_n(t))$ represents an element of $f_c^n(f_c(C))$. We consider the maps $f_c^{(-k)}(x, y)$ with $k \geq 2$. In Section 1, we show that $g^{(-k)}(x, y) = g^{(k)}(y, x)$. Then $f_c^{(-k)}(x, y) = f_c^{(k)}(y, x)$. Hence we have $(f_c^{(-k)})^2(x, y) = (f_c^{(k)})^2(x, y)$. By Lemma 3.2, we know that the critical value of $f_c^{(-k)}(x, y)$ is parameterized as $x = c^k(2/t^k + t^{2k})$, $y = c^k(2t^k + 1/t^{2k})$. Note that $c^k(2t^k + 1/t^{2k})$ is the first component of the critical value of $f_c^{(k)}(x, y)$. Then

$$(f_c^{(-k)})^2(C(f_c^{(-k)})) = (f_c^{(k)})^2(C(f_c^{(k)})).$$

Hence

$$\text{if} \quad K(f_c^{(k)}) \cap C(f_c^{(k)}) = \emptyset, \quad \text{then} \quad C(f_c^{(-k)}) \cap K(f_c^{(-k)}) = \emptyset.$$

Thus it suffices to prove Proposition 3.1 when $k \geq 2$.

Lemma 3.3. (1) *We assume that* $k \geq 2$. *Then*

$$\textit{if} \quad K(f_c^{(k)}) \cap C(f_c^{(k)}) = \emptyset, \quad \textit{then} \quad C(f_c^{(-k)}) \cap K(f_c^{(-k)}) = \emptyset.$$

(2) *For any* $n \in \mathbf{N}$, $v_n(t) = u_n(1/t)$.

Proof. We can prove (2) by induction on n. □

By Lemma 3.3 (2) we see that proving Proposition 3.1 requires only proving the following proposition. Indeed. For $t \in \mathbf{C} \setminus \{0\}$, we consider two cases (1) $|t| \leq 1$ and (2) $|t| > 1$. In case (1), Proposition 3.2 implies Proposition 3.1. In case (2), we note that $u_n(t) = v_n(1/t)$.

Proposition 3.2. *For any* $t \in \overline{\mathbf{D}} \setminus \{0\}$, $|v_n(t)| \to \infty$ *as* $n \to \infty$ *where* $\mathbf{D}$ *denotes the unit disk.*

To prove this proposition we need two steps.

Proposition 3.3. *If* $c > 1$, *then* $|v_n(t)|$ *has its minimum value on a boundary* $\partial\mathbf{D}$ *of* $\overline{\mathbf{D}} \setminus \{0\}$ *for any* $n \in \mathbf{N}$.

Proposition 3.4. *If* $c > 1$, *then* $|v_n(e^{i\theta})| \to \infty$ *as* $n \to \infty$ *for any* θ *in* $[0, 2\pi)$.

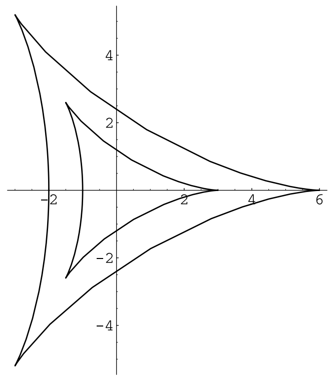

Fig. 3.1. cS and c^kS.

To prove these two propositions, we need to study the dynamics of f_c on an invariant plane. When c is real, $f_c(x, y)$ admits an invariant plane $\{x = \overline{y}\}$. Then we consider the map $g_c(z)$ on the plane $\{x = \overline{y}\}$. That is, $g_c(z) := f_c(z, \overline{z})$. The map $g_c(z)$ may be viewed as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$.

Lemma 3.4. *Assume that $c > 1$. The critical set of g_c is equal to the set*

$$\left\{c\left(e^{i\theta} + \varepsilon e^{i\theta} + \frac{1}{\varepsilon e^{2\theta i}}\right) : 0 \le \theta < 2\pi\right\}.$$

The critical value set of g_c is equal to the set

$$\{c^k(2e^{k\theta i} + e^{-2k\theta i})) : 0 \le \theta < 2\pi\}.$$

Proof. Let $z = z_1 + iz_2$, $(z_j \in \mathbf{R})$ and this be an element on the plane $\{x = \overline{y}\}$. By Proposition A.1 in Appendix A, we have $|Df_c(z, \overline{z})| = |Dg_c(z_1, z_2)|$. Hence the critical set of g_c is equal to $C(f_c) \cap \{x = \overline{y}\}$. The critical set $C(f_c)$ is described in Lemma 3.2. Clearly, $((1 + \varepsilon)t + 1/(\varepsilon t^2), 1/t + 1/(\varepsilon t) + \varepsilon t^2)$ belongs to the plane $\{x = \overline{y}\}$ if and only if $|t| = 1$. □

We consider the map $g_c(z)$ restricted to the closed domain $cS = \{cz : z \in S\}$, where S is the closed domain defined in Section 2. For any point cz in cS with $z = e^{i\sigma} + e^{-i\tau} + e^{i(\tau-\sigma)} \in S$, we have

$$T_k(z, \overline{z}) = (e^{ik\sigma} + e^{-ik\tau} + e^{ik(\tau-\sigma)}, e^{-ik\sigma} + e^{ik\tau} + e^{ik(\sigma-\tau)})$$

and so

$$f_c(cz, c\overline{z}) = c^k(e^{ik\sigma} + e^{-ik\tau} + e^{ik(\tau-\sigma)}, e^{-ik\sigma} + e^{ik\tau} + e^{ik(\sigma-\tau)}).$$

Then the map $g_c(z)$ from cS onto c^kS is similar to the map $T_k(x, y)$.

Lemma 3.5. (1) *For any point z in the interior c^kS°, $g_c^{-1}(z) \subset cS^\circ$ and $g_c^{-1}(z)$ consists of k^2 distinct points.*
(2) $g_c|_{\mathbf{C}\setminus cS}\colon \mathbf{C}\setminus cS \to \mathbf{C}\setminus c^kS$ *is a k-sheeted unbranched covering map and it is sense preserving.*
(3) $g_c|_{\partial cS}\colon \partial cS \to \partial c^kS$ *is a k-to-one map.*

Proof. (1) From the above remarks, we can consider $g_1(z)$ in place of $g_c(z)$. Clearly, $g_1(z) = S_k(z)$. (Note that $S_k = T_k|_{\{x=\overline{y}\}}$.) Then by similar arguments used in the proof of Proposition 2.2, we can prove this assertion. The map $\varphi \circ g_1(z) \circ \varphi^{-1}$ enlarges the fundamental domain R by k times. Then k^2 small subdivisions of R are mapped onto R under $\varphi \circ g_1(z) \circ \varphi^{-1}$.

(2) By Lemma 3.4, we know that the critical set of g_c is contained in the set cS since $S = \{e^{i\sigma} + e^{-i\tau} + e^{i(\tau-\sigma)}\colon 0 \le \sigma,\ \tau \le 2\pi\}$. Then the map $g_c|_{\mathbf{C}\setminus cS}$ does not have any critical points. Hence $g_c|_{\mathbf{C}\setminus cS}$ is an unbranched covering map. From the recurrence equation (1.1) of generalized Chebyshev polynomials, we know that $g_c(z) = z^k + \pi_{k-1}(z, \overline{z})$, where π_n denotes a polynomial in z and $\overline{z}$ of degree $\le n$. We will show $\det(Dg_c(z)) > 0$, for any $z \in \mathbf{C}\setminus cS$. Indeed. $\det(Dg_c(z)) = |\partial g_c/\partial z|^2 - |\partial g_c/\partial \overline{z}|^2$. When $|z|$ is large, $\det(Dg_c(z)) > 0$. Therefore from the fact the critical set of g_c is contained in cS we deduce that $\det(Dg_c(z)) > 0$, for any $z \in \mathbf{C}\setminus cS$. Then $g_c|_{\mathbf{C}\setminus cS}$ is sense preserving.

Consider a circle γ of center at the origin and with radius $R_0 \gg 1$. Then the image of γ under g_c lies outside γ and the winding number of $g_c(\gamma)$ around γ is k. We use a topological argument principle (see [12], p. 350); Let $h(z)$ be a continuous mapping such that only a finite number of its p-points lie inside a simple loop Γ. Then the total number of p-points inside Γ (counted with their topological multiplicities) is equal to the winding number of $h(\Gamma)$ around p. We apply this to our mapping g_c. Instead of the annulus $\mathbf{C}\setminus cS$ we consider a topological disk $\hat{\mathbf{C}}\setminus cS$ and use the usual substitution $z = \phi(\zeta) = 1/\zeta$ where $\hat{\mathbf{C}} = \mathbf{C}\cup\{\infty\}$. Let $G_c := \phi^{-1}\circ g_c \circ \phi$ be the function from $\hat{\mathbf{C}}\setminus \phi^{-1}(cS)$ to $\hat{\mathbf{C}}\setminus \phi^{-1}(c^kS)$. Since $\det(D\phi(\zeta)) > 0$ for any $\zeta \in \hat{\mathbf{C}}\setminus cS$, we have $\det(DG_c(\zeta)) > 0$.

We will select a point p inside $G_c(\phi^{-1}(\gamma))$ and near $\zeta = 0$ in the following manner. Let $g_c(z) = z^k + \pi_{k-1}(z, \overline{z}) = \Sigma a_{ij} z^i \overline{z}^j$. We set $h_c(z) := \Sigma |a_{ij}|\,|z|^{i+j}$. Then $|g_c(z)| \le h_c(z)$. Let $h_c(z) = |z|^k(1 + Q(|z|))$. There exists a large positive number R_1 satisfying $1 + Q(R_1) < 2$. Then we set $\phi(p) = 2R_1^k$. Thus we see that if $|z| \le R_1$ then $|g_c(z)| < \phi(p)$. Hence if u is any element of $g_c^{-1}(\phi(p))$ then $|u| > R_1$. We set $R_0 = R_1$. Then all the points of $g_c^{-1}(\phi(p))$ lie outside γ and $\phi(p)$ lies outside $g_c(\gamma)$.

Since G_c is sense preserving, the topological multiplicity of any p-point is 1. Since the winding number of $G_c(\phi^{-1}(\gamma))$ around p is k, the total number of p-points inside $\phi^{-1}(\gamma)$ is k. Hence the number of points of $G_c^{-1}(p)$ is k. Therefore that of $g_c^{-1}(p)$ is k. Since $\mathbf{C}\setminus c^kS$ is connected, all fibers of the covering $g_c|_{\mathbf{C}\setminus cS}$ have the same cardinality. Hence $g_c|_{\mathbf{C}\setminus cS}$ is a k-sheeted covering.

(3) To prove this, it suffices to consider the case when $c = 1$. In this case, the proof is the same as that used in Corollary 2.1. □

To prove Proposition 3.3, we will use the topological argument principle again. Let $W(\gamma, p)$ be the winding number of a closed curve γ around a point p. Let S^1 denote the unit circle $\{e^{i\theta}: 0 \le \theta < 2\pi\}$ with a counterclockwise orientation. We assume that a hypocycloid $c^k\partial S$ has a counterclockwise orientation.

Calculating the winding number $W(u_n(S^1), 0)$ is not easy because the relation from $u_n(t)$ to $u_{n+1}(t)$ is not given by a simple mapping. But when $t = e^{i\theta}$, $(u_n(t), v_n(t))$ lies on the plane $\{x = \overline{y}\}$. Then we can use the map g_c when $|t| = 1$.

Lemma 3.6. *Let n be any positive integer and $c > 1$.*
(1) $u_n(e^{i\theta}) = g_c^n(c^k(2e^{k\theta i} + e^{-2k\theta i}))$.
(2) $W(u_n(S^1), 0) = k \times W(g_c^n(c^k\partial S), 0)$.

Proof. (1) Induction on n.

(2) A closed path $\gamma: [0, 2\pi] \to \mathbf{C}$ given by $\gamma(\theta) = c^k(2e^{k\theta i} + e^{-2k\theta i})$ follows the hypocycloid $c^k\partial S$ k times. Then the assertion follows. □

In the proof of Lemma 3.5 (2), by analyzing a winding number near ∞ we see that the number of fibers of $g_c|_{\mathbf{C}\setminus cS}$ is k. Conversely from the number of fibers we can calculate the winding number $W(g_c^n(c^k\partial S), 0)$.

Lemma 3.7. *If $c > 1$, then $W(g_c^n(c^k\partial S), 0) = k^n$.*

Proof. Let Γ be any simple loop in the topological disk $\hat{\mathbf{C}} \setminus \phi^{-1}(cS)$ not passing through $\zeta = 0$ oriented in a counterclockwise orientation. By Lemma 3.5 (2) we can select a point p very near $\zeta = 0$ such that $G_c^{-n}(p)$ consists of k^n distinct points and all such p-points lie in the interior of the loop Γ. The total number of p-points of G_c^n inside Γ counted with topological multiplicities is equal to $W(G_c^n(\Gamma), p)$. Since G_c is sense preserving, the topological multiplicity of any p-point is equal to 1. Hence it follows that $W(G_c^n(\Gamma), p) = k^n$. Then we can easily see that $W(G_c^n(\Gamma), 0) = k^n$. From this we can easily obtain that $W(g_c^n(\phi(\Gamma)), 0) = k^n$. □

From this lemma and Lemma 3.3, we can deduce the following.

Corollary 3.1. *If $c > 1$, then $-W(v_n(S^1), 0) = W(u_n(S^1), 0) = k^{n+1}$.*

From the paragraph below Proposition 3.1, we see that $v_n(t)$ is a rational function in the variable t that has only a pole at $t = 0$.

Lemma 3.8. *The multiplicity of the pole at $t = 0$ of $v_n(t)$ is at most k^{n+1}.*

Proof. Since $v_n(t) = u_n(1/t)$, it suffices to show that

$$u_n(t) = a(k^{n+1})t^{k^{n+1}} + \cdots + a(0) + a(-1)t^{-1} + \cdots + a(-2k^{n+1})t^{-2k^{n+1}},$$

where $a(j) \in \mathbf{R}[c]$, $-2k^{n+1} \leq j \leq k^{n+1}$.

We may view c as a variable. Then we will prove the following:

(1) the maximum degree of t of $u_n(t)$ is at most k^{n+1},

(2) the minimum degree of t of $u_n(t)$ is equal to $-2k^{n+1}$.

Since $g^{(k)}(x, y) = x^k + \pi_{k-1}(x, y)$, we can easily prove (2) by induction on n. Next we will calculate the maximum degree of t in $u_{n+1}(t)$. We consider the weighted degree of $g^{(k)}(x, y, c)$. We define the weighted degree of a monomial $p(c)x^\alpha y^\beta$ to be $\alpha + 2\beta$. From the recurrence relation (1.1) for $\{g^{(m)}(x, y)\}$, we see that the maximum weighted degree of $g^{(k)}(x, y)$ is k and so that of $g^{(k)}(x, y, c)$ is k. Hence, when $k = \alpha + 2\beta$, the maximum degree of t of $u_n(t)^\alpha v_n(t)^\beta$ is at most $(\alpha + 2\beta)k^{n+1} = k^{n+2}$. □

Proof of Proposition 3.3. We apply the argument principle to the rational function $v_n(t)$. Thus $W(v_n(S^1), 0) = N - M$ where N is the number of zeros in the unit disk $\mathbf{D}$ and M is the number of poles in $\mathbf{D}$. From the proof of Lemma 3.7, we see that $u_n(S^1)$ and so $v_n(S^1)$ does not pass through the origin. Combining Corollary 3.1 and Lemma 3.8, we see that $v_n(t)$ does not have any zeros in $\mathbf{D}$ and it is holomorphic in $\mathbf{D} \setminus \{0\}$. If $|t| \ll 1$, $|v_n(t)|$ is large. Then $v_n(t)$ has its minimum-modulus on the boundary $\partial\mathbf{D}$. □

We begin with the proof of Proposition 3.4. Since $v_n(e^{i\theta}) = u_n(e^{-i\theta})$, to prove Proposition 3.4, it suffices to prove that if $c > 1$, then $|u_n(e^{i\theta})| \to \infty$ $(n \to \infty)$ for any θ. To prove this we will define a function $\|z\|$ with $\|z\| > 1$, for any $z \in \mathbf{C} \setminus cS$ and we will show that if $c > 1$, $\|g_c(z)\| > \|z\|^k$ and $|g_c^n(z)| \to \infty$ $(n \to \infty)$. If this is true, then from Lemma 3.6, Proposition 3.4 follows.

We restrict the map $\tilde{\Psi}$ in (2.3) to the set $\{t_1 = t,\ t_2 = \overline{t}\}$. We denote the map by ψ. Since $\tilde{\Psi}(t, \overline{t}) = (t + 1/\overline{t} + \overline{t}/t,\ 1/t + \overline{t} + t/\overline{t})$, $\psi(t) = t + 1/\overline{t} + \overline{t}/t$ and $\tilde{\Psi}(t, \overline{t})$ lies on the plane $\{x = \overline{y}\}$. Since $\psi(re^{i\theta}) = (r + 1/r)e^{i\theta} + e^{-2i\theta}$, $(r > 1)$, the map ψ from $\mathbf{C} \setminus \overline{\mathbf{D}}$ to $\mathbf{C} \setminus S$ is a homeomorphism.

The image of a radial line $\{re^{i\theta}: r > 1\}$ under the map ψ is also a half-line. Let h_λ be a function from $\mathbf{C} \setminus S$ to $\mathbf{C} \setminus \lambda S$ defined by $h_\lambda(z) = \lambda z$ with $\lambda \geq 1$. The composition $h_\lambda \circ \psi$ is a map from $\mathbf{C} \setminus \overline{\mathbf{D}}$ onto $\mathbf{C} \setminus \lambda S$. Then the image of a radial line under the map $h_\lambda \circ \psi$ is a half-line which is called a λ-external ray. We define $\|z\| := |(h_c \circ \psi)^{-1}(z)|$ for $z \in \mathbf{C} \setminus cS$.

Proposition 3.5. *We assume that $c > 1$. For any point z in $\mathbf{C} \setminus cS$, $\|g_c(z)\| > \|z\|^k$ and $|g_c^n(z)| \to \infty$ $(n \to \infty)$.*

Set $\|z\| = r_0$ and $\|g_c(z)\| = r_1$. To prove Proposition 3.5, we consider c-external rays and c^k-external rays. We first note the symmetry of c-external rays. Let ω be a

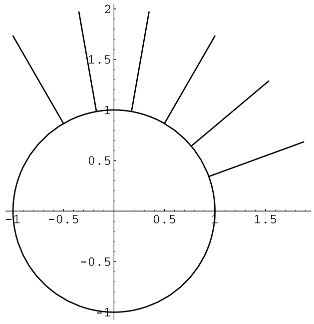

Fig. 3.2. Radial lines.

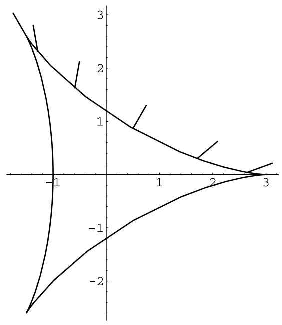

Fig. 3.3. External rays.

cubic root of unity. Then $\psi(\omega t) = \omega\psi(t)$ and $\psi(\omega^2 t) = \omega^2\psi(t)$. Then it suffices to consider only c-external rays $\{c\psi(re^{i\theta})\colon 0 \le \theta \le 2\pi/3,\ 1 < r\}$. Further for α with $0 \le \alpha \le \pi/3$, two c-external rays $\{c\psi(re^{(\pi/3-\alpha)i})\colon r > 1\}$ and $\{c\psi(re^{(\pi/3+\alpha)i})\colon r > 1\}$ are symmetric with respect to a c-external ray $\{c\psi(re^{\pi i/3})\colon r > 1\}$. See Fig. 3.4. Hence we consider only c-external rays $\{c\psi(re^{i\theta})\colon 0 \le \theta \le \pi/3,\ 1 < r\}$.

For a point $z \in \mathbf{C} \setminus cS$, we denote its image $g_c(z)$ by P. Let the landing point of the c-external ray through P be Q_1. Let Q_2 be the point of intersection of the segment PQ_1 and the curve $\partial c^k S$. Let Q_3 be the landing point of the c^k-external ray through P. See Fig. 3.5. Let $|AB|$ denote the Euclidian length of a segment AB. We will evaluate the length $|PQ_1| = |PQ_2| + |Q_1Q_2|$. Then, to prove Proposition 3.5, it suffices to prove the third assertion of the following lemma.

Lemma 3.9. (1) *The slope of PQ_3 is greater than that of PQ_2.*
(2) $|PQ_2| \ge |PQ_3|$.
(3) $|PQ_1| = c(r_1 + 1/r_1 - 2) > |PQ_3| = c^k(r_0^k + 1/r_0^k - 2) > c(r_0^k + 1/r_0^k - 2)$.

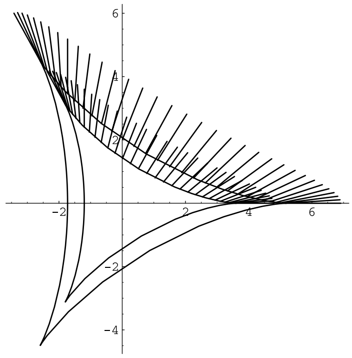

Fig. 3.4. c- and c^k-external rays.

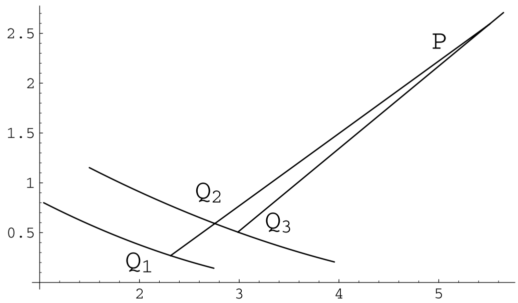

Fig. 3.5. Two external rays through P.

Proof. (1) Let Q_4 be the point of intersection of the segment Q_3O and ∂cS where O denotes the origin. Let $\{c((r+1/r)e^{i\sigma}+e^{-2\sigma i})\colon r>1\}$ be the c-external ray through Q_4. Let l denote this c-external ray. The c^k-external ray through Q_3 is parallel to this half line l. The slope of l is $\tan\sigma$. Since $Q_4 = c(2e^{i\sigma}+e^{-2\sigma i})$, the slope of the segment OQ_4 is $(2\sin\sigma-\sin 2\sigma)/(2\cos\sigma+\cos 2\sigma)$. We can easily verify that if $0\le\sigma\le\pi/3$, then $\tan\sigma\ge(2\sin\sigma-\sin 2\sigma)/(2\cos\sigma+\cos 2\sigma)$ by identities of trigonometric functions. Then the slope of OQ_4, which is equal to that of OQ_3, is less than or equal to the slope of l. Hence l lies above the c^k-external ray through Q_3. If the c-external ray l moves along the curve ∂cS downward, then it will touch the point P. Then we get the assertion (1).

(2) Let l_1 be the line through Q_3 that is perpendicular to PQ_3. To prove the above inequality, we see from (1) that it suffices to prove that the point Q_2 lies below the line l_1. The hypocycloid ∂c^kS is written as $\{c^k(2e^{i\tau}+e^{-2\tau i})\colon 0\le\tau<2\pi\}$. When $0\le\tau\le\pi/3$, it is convex. Hence it suffices to prove the above fact in the case when Q_2 is equal to the point $Q_5 := c^k(2e^{i\pi/3}+e^{-2\pi i/3})$. Hence, we will prove that the slope

of Q_3Q_5 is greater than that of the line l_1. This requires only showing that

$$-\cot\tau \le \frac{\sqrt{3}/2 - 2\sin\tau + \sin 2\tau}{1/2 - 2\cos\tau - \cos 2\tau}, \quad \text{when} \quad 0 \le \tau < \pi/3.$$

We denote the right hand side of the above inequality by $m(\tau)$. Then

$$\frac{dm}{d\tau} = \frac{-32\sin(\pi/6 - \tau/2)\cos(\pi/6 + \tau)\sin(3\tau/2)}{(-1 + 4\cos\tau + 2\cos 2\tau)^2}.$$

Then $m(\tau)$ is monotone decreasing in the range $[0, \pi/3)$ and $m(\tau)$ approaches $-1/\sqrt{3}$ as $\tau \to \pi/3$. On the other hand, $-\cot\tau$ is monotone increasing and $-\cot(\pi/3) = -1/\sqrt{3}$.

(3) Since $\|g_c(z)\| = r_1$, $g_c(z) = c((r_1 + 1/r_1)e^{i\phi} + e^{-2i\phi})$ and $Q_1 = c(2e^{i\phi} + e^{-2i\phi})$. Then $|PQ_1| = c(r_1 + 1/r_1 - 2)$. Since $\|z\| = r_0$, $z = c((r_0 + 1/r_0)e^{i\theta} + e^{-2i\theta})$. Then from Lemma 3.1, we know that $P = g_c(z) = c^k((r_0^k + 1/r_0^k)e^{k\theta i} + e^{-2k\theta i})$. Since $Q_3 = c^k(2e^{k\theta i} + e^{-2k\theta i})$, $|PQ_3| = c^k(r_0^k + 1/r_0^k - 2)$. □

From Lemma 3.4 we know that the critical value set of g_c is a compact set $\partial c^k S$ included in $\mathbf{C}\setminus cS$. From Proposition 3.5, we see that the sequence $\{g_c^n(C(f_c)\cap\{x = \overline{y}\})\}$ converges uniformly to ∞. This completes the proof of Proposition 3.4. Thus we have proved that $\{f_c^n(C(f_c))\}$ converges uniformly to the line at infinity. This completes the proof of Proposition 3.1. □

Next we will prove that if $c > 1$, then $K(f_c)$ is a Cantor set. To prove this we need some preparations. In the first place we assume that $k \ge 2$.

Lemma 3.10. *If $c > 1$, the number of periodic points of $g_c(z)$ of period n is k^{2n}.*

Proof. The proof of this lemma is almost the same as that of Proposition 2.2. □

From Corollary 3.2 in [5], we know that the number of periodic points of period n of $f_c(x, y)$ is k^{2n}. Then we have the following.

Corollary 3.2. *If $c > 1$, any periodic point of $f_c(x, y)$ lies on the plane $\{x = \overline{y}\}$ and belongs to the set $K(g_c)$ in the plane $\{x = \overline{y}\}$.*

Now we return to the proof of Theorem 3.1.

Note that the map $f_c^{(k)}$ is a regular endomorphism of $\mathbf{C}^2$. We use Theorem 3.8 in [6]. Then combining Theorem 3.8 in [6] and Proposition 3.1 yields $K(f_c^{(k)}) = \operatorname{supp}\mu$, for any $k \in \mathbf{Z} \setminus \{0, 1, -1\}$.

Before starting a proof of Theorem 3.1, we will state a precise version of Theorem 3.1 and prove it when $k \ge 2$.

Proposition 3.6. *Assume that $c > 1$ and $k \geq 2$. Then*:

(1) $K(f_c) = \operatorname{supp}\mu = K(g_c) \subset \{x = \overline{y}\}$.

(2) *$K(g_c)$ is a Cantor set.*

Proof. (1) From Theorem 3.8 in [6] and Corollary 3.2, we see that

$$\begin{aligned} K(f_c) &\subset \overline{\{\text{repelling periodic points of } f_c\}} \subset \overline{\{\text{periodic points of } f_c\}} \\ &\subset K(g_c) \subset K(f_c). \end{aligned}$$ □

The proof of (2) is essentially the same as that used in Theorem 5.1 in [13]. Recall that $g_c(z) = z^k + \pi_{k-1}(z, \overline{z})$, where π_{k-1} denotes a polynomial of degree $\leq k-1$. Then there is a constant $R \gg 1$ such that if $|z| > R$ then $g_c^n(z) \to \infty$ $(n \to \infty)$. Then we set

$$\mathbf{D}_R := \{z \in \mathbf{C} \colon |z| < R\}.$$

Lemma 3.11. *Assume that $c > 1$. Then there exists a nonnegative integer n such that $g_c^{-n}(\overline{\mathbf{D}}_R) \subset c^k S^\circ$.*

Proof. This follows from Proposition 3.5. □

Lemma 3.12. *Let $c > 1$. Assume that $g_c^{-n}(\overline{\mathbf{D}}_R)$ is not contained in $c^k S^\circ$ and $g_c^{-n}(\overline{\mathbf{D}}_R)$ is arcwise connected. Then $g_c^{-n-1}(\overline{\mathbf{D}}_R)$ is arcwise connected.*

Proof. Let P be any point of $g_c^{-n}(\overline{\mathbf{D}}_R) \setminus c^k S^\circ$. Then there is a path γ in $g_c^{-n}(\overline{\mathbf{D}}_R)$ connecting P and a fixed point Q of g_c. Let a point of intersection of $\underline{\gamma}$ and $\partial c^k S$ be M. Let P_{-1} be any point of $g_c^{-1}(P)$. We will construct a path in $g_c^{-n-1}(\overline{\mathbf{D}}_R)$ connecting P_{-1} and Q. Recall that the map $g_c(z)$ is the map $f_c^{(k)}(x, y)$ of degree k restricted to the plane $\{x = \overline{y}\}$.

From the recurrence relation (1.1) for Chebyshev polynomials, we have the following claim.

Claim 3.1. *Let ω be a cubic root of unity.*

(1) *If $k \equiv 0 \bmod 3$, $g_c(z) = g_c(\omega z) = g_c(\omega^2 z)$.*

(2) *If $k \equiv 1 \bmod 3$, $g_c(\omega z) = \omega g_c(z)$, $g_c(\omega^2 z) = \omega^2 g_c(z)$.*

(3) *If $k \equiv 2 \bmod 3$, $g_c(\omega z) = \omega^2 g_c(z)$, $g_c(\omega^2 z) = \omega g_c(z)$.*

Since $M \in \partial c^k S \cap g_c^{-n}(\overline{\mathbf{D}}_R)$, it follows that $\omega^j M \in \partial c^k S \cap g_c^{-n}(\overline{\mathbf{D}}_R)$. Then there is a path connecting $\omega^j M$ and Q. Next we consider the set $g_c^{-1}(g_c^{-n}(\overline{\mathbf{D}}_R))$. We may regard the closed domain cS bounded by a hypocycloid as the triangular region R. We prove this lemma when $k = 2$. (In other cases, the similar proofs hold.) We consider the closed domain bounded by $\triangle OAB$ in Fig. 2.3. The components of $g_c^{-1}(c^k S)$ may be regarded as small triangular regions $\triangle OEF$, $\triangle ADF$, $\triangle BDE$, $\triangle DEF$. Any element M_{jl}

of $g_c^{-1}(\omega^j M)$ lies on an edge of a small triangle. Let $g_c^{-1}(Q) = \{Q = Q_0, Q_1, Q_2, Q_3\}$. Each Q_i lies in an interior of a small triangle. Two points Q_i and Q_h are connected by a path in $g_c^{-n-1}(\overline{\mathbf{D}}_R)$ through some points M_{jl}. From Lemma 3.5, we know that there is a path in $g_c^{-n-1}(\overline{\mathbf{D}}_R)$ connecting P_{-1} and some point M_{jl}. Then we have a path in $g_c^{-n-1}(\overline{\mathbf{D}}_R)$ connecting P_{-1} and Q.

In the same way we can prove this lemma for any $P \in g_c^{-n}(\overline{\mathbf{D}}_R) \cap c^k S^\circ$. □

Proof of Proposition 3.6 (2). When $k = 2$, this lemma has already been proved in Theorem 5.1 in [13]. In the same way we can prove this lemma when $k \geq 3$. So we show only an outline of the proof. From Lemma 3.11 and Lemma 3.12, we see that there is a nonnegative integer N such that $g_c^{-N}(\overline{\mathbf{D}}_R)$ is contained in the interior of $c^k S$ and $g_c^{-N}(\overline{\mathbf{D}}_R)$ is connected. So, g_c does not have any critical values in $g_c^{-N}(\overline{\mathbf{D}}_R)$. We use inverse branches ϕ_j, $j = 1, \ldots, k^2$, of g_c. Set

$$K(i_1, \ldots, i_n) := \phi_{i_1} \circ \cdots \circ \phi_{i_n}(g_c^{-N}(\overline{\mathbf{D}}_R)).$$

Then in the same way used in the proof of Theorem 5.1 in [13], we can prove that for any given sequence (j_n), diameter$[K(j_1, \ldots, j_n)] \to 0$ as $n \to \infty$. Then

$$K(g_c) = \bigcap_{n=0}^{\infty} \left(\bigcup_{j_1,\ldots,j_n=1}^{k^2} K(j_1, \ldots, j_n) \right)$$

is a Cantor set. □

This completes the proof of Proposition 3.6. □

Proof of Theorem 3.1. When $k \geq 2$, Theorem 3.1 follows from Proposition 3.6. We will prove Theorem 3.1 for the maps $f_c^{(-k)}(x, y)$ with $k \geq 2$. In the proof of Lemma 3.3 (1) we prove that $(f_c^{(-k)})^2 = (f_c^{(k)})^2$. Then $K(f_c^{(-k)}) = K((f_c^{(-k)})^2) = K((f_c^{(k)})^2) = K(f_c^{(k)})$. Combining Proposition 3.1 and Theorem 3.8 in [6] yields $K(f_c^{(-k)}) = \operatorname{supp} \mu$. Then by Proposition 3.6, we conclude that supp μ is a Cantor set. □

Appendix A.

We show a relation between complex Jacobian matrices and real Jacobian matrices on $\{x = \overline{y}\}$ for symmetric polynomial endomorphisms.

Let $h(z_1, z_2) \in \mathbf{R}[z_1, z_2]$. We define a map f from $\mathbf{C}^2$ to $\mathbf{C}^2$ by

$$f(z_1, z_2) := (h(z_1, z_2), h(z_2, z_1)).$$

Then the holomorphic map f admits an invariant plane $\{z_1 = \overline{z_2}\}$. Let $g(z) := h(z, \overline{z})$. Then g may be viewed as a map from $\mathbf{R}^2$ to $\mathbf{R}^2$.

Proposition A.1. *Let $Df(z,\bar{z})$ and $Dg(z)$ be the complex Jacobian matrix of f at $(z,\bar{z})$ and the real Jacobian matrix of g at z, respectively. Then*

$$U^{-1}Df(z,\bar{z})U = Dg(z),$$

where U is a unitary matrix given by

$$U = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}.$$

Proof. Let $z_k = x_k + iy_k$ $(k = 1, 2)$. Let

$$f(z_1, z_2) = (p_1 + iq_1,\ p_2 + iq_2), \tag{A.1}$$

where $p_k(x_1, x_2, y_1, y_2)$ and $q_k(x_1, x_2, y_1, y_2)$ are real valued functions. It is well known (e.g. [7]) that if we define 2×2 matrices A and B by

$$A = \left(\frac{\partial p_j}{\partial x_k}\right) = \left(\frac{\partial q_j}{\partial y_k}\right) \quad \text{and} \quad B = \left(\frac{\partial q_j}{\partial x_k}\right) = -\left(\frac{\partial p_j}{\partial y_k}\right), \tag{A.2}$$
$$\text{then} \quad Df = A + iB.$$

By (A.1), we know that any term in p_k (resp. q_k) is represented as $a(x_1, x_2)y_1^m y_2^m$ where $m+n$ is nonnegative and even (resp. odd). On the plane $\{z_1 = \bar{z}_2\}$, $x_1 = x_2$ and $y_1 = -y_2$. Then we have the followings:
(1) $\partial p_1/\partial x_1 = \partial p_2/\partial x_2$ and $\partial p_1/\partial x_2 = \partial p_2/\partial x_1$,
(2) $\partial q_1/\partial x_1 = -\partial q_2/\partial x_2$ and $\partial q_1/\partial x_2 = -\partial q_2/\partial x_1$,
at any point $(z,\bar{z})$ in the plane $\{z_1 = \bar{z}_2\}$. Hence by (A.2) we see that

$$Df(z,\bar{z}) = \left.\begin{pmatrix} \dfrac{\partial p_1}{\partial x_1} & \dfrac{\partial p_1}{\partial x_2} \\ \dfrac{\partial p_1}{\partial x_2} & \dfrac{\partial p_1}{\partial x_1} \end{pmatrix}\right|_{(z,\bar{z})} + i \left.\begin{pmatrix} \dfrac{\partial q_1}{\partial x_1} & \dfrac{\partial q_1}{\partial x_2} \\ -\dfrac{\partial q_1}{\partial x_2} & -\dfrac{\partial q_1}{\partial x_1} \end{pmatrix}\right|_{(z,\bar{z})}.$$

Therefore

$$U^{-1}Df(z,\bar{z})U^{-1} = \left.\begin{pmatrix} \dfrac{\partial p_1}{\partial x_1} + \dfrac{\partial p_1}{\partial x_2} & -\dfrac{\partial q_1}{\partial x_1} + \dfrac{\partial q_1}{\partial x_2} \\ \dfrac{\partial q_1}{\partial x_1} + \dfrac{\partial q_1}{\partial x_2} & \dfrac{\partial p_1}{\partial x_1} - \dfrac{\partial p_1}{\partial x_2} \end{pmatrix}\right|_{(z,\bar{z})}.$$

Set $z := u + iv$, $p(u, v) := p_1(u, u, v, -v)$ and $q(u, v) := q_1(u, u, v, -v)$. Then

$$\frac{\partial p}{\partial u} = \frac{\partial p_1}{\partial x_1} + \frac{\partial p_1}{\partial x_2}, \quad \frac{\partial p}{\partial v} = \frac{\partial p_1}{\partial y_1} - \frac{\partial p_1}{\partial y_2} = -\frac{\partial q_1}{\partial x_1} + \frac{\partial q_1}{\partial x_2},$$
$$\frac{\partial q}{\partial u} = \frac{\partial q_1}{\partial x_1} + \frac{\partial q_1}{\partial x_2}, \quad \frac{\partial q}{\partial v} = \frac{\partial q_1}{\partial y_1} - \frac{\partial q_1}{\partial y_2} = \frac{\partial p_1}{\partial x_1} - \frac{\partial p_1}{\partial x_2}.$$

Since $g(u, v) = (p(u, v), q(u, v))$, it follows that

$$U^{-1}Df(u + iv, u - iv)U = Dg(u, v). \qquad \square$$

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